The CRM Infinity Project

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Supported by:

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Project participants:

2 full-time postdocs: Martin Koerwien (Model Theory)
Moritz Müller (Complexity Theory)

About 20 one-two week visitors (see below)

Barcelona logicians: Albert Atserias, Joan Bagaria, Maria-Luisa Bonet, Enrique Casanovas, Rafel Farré, Ignasi Jané, Juan-Carlos Martinez

Infinity Conference Programme Committee: John Baldwin, Maria-Luisa Bonet, SDF, Juan-Carlos Martinez, Michael Rathjen

The CRM Infinity Project: Idea of the Project

Combine different areas of logic to uncover new research directions

6 project themes:

Computations and Sets
Arnold Beckmann, Sam Buss, Yijia Chen, Jörg Flum, Moritz Müller

Models and Sets
John Baldwin, Fred Drueck, Rami Grossberg, Tapani Hyttinen, Martin Koerwien, Vadim Kulikov, Andrés Villaveces

Proofs and Sets
Lars Kristiansen, Michael Rathjen, Albert Visser, Andreas Weiermann

The CRM Infinity Project: Idea of the Project

Computations and Models
Ekaterina Fokina, Julia Knight, Russell Miller, Antonio Montalbàn

Computations and Proofs
Yijia Chen, Jörg Flum, Moritz Müller

History and Philosophy of Set Theory
Tatiana Arrigoni, Loren Graham, Jean-Michel Kantor

The CRM Infinity Project: Some Results

Computations and Sets
Arnold Beckmann, Sam Buss, Yijia Chen, Jörg Flum, Moritz Müller

1. (Buss-Chen-Flum-SDF-Müller) Isomorphism relations on finite structures

In Descriptive Set Theory:
Let $\mathcal{C}_0$, $\mathcal{C}_1$ be Borel, $\simeq$-closed classes of countable structures.
Then $\simeq_{\mathcal{C}_0}$ is *Borel reducible* to $\simeq_{\mathcal{C}_1}$ iff there is a Borel function $F : \mathcal{C}_0 \to \mathcal{C}_1$ such that $A \simeq B$ iff $F(A) \simeq F(B)$

In Complexity Theory:
Let $\mathcal{C}_0$, $\mathcal{C}_1$ be PTIME, $\simeq$-closed classes of finite structures.
Then $\simeq_{\mathcal{C}_0}$ is *srrong iso-reducible* to $\simeq_{\mathcal{C}_1}$ iff there is a PTIME function $F : \mathcal{C}_0 \to \mathcal{C}_1$ such that $A \simeq B$ iff $F(A) \simeq F(B)$

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Fact: If $\mathcal{C}_0$ is strong iso-reducible to $\mathcal{C}_1$ then $\#_{\mathcal{C}_0}$ is bounded by $\#_{\mathcal{C}_1} \circ p$ for some polynomial p, where:
$\#_{\mathcal{C}}(n) = \#$ isomorphism classes of models in $\mathcal{C}$ of size $\leq n$
(We say that $\mathcal{C}_0$ is *potentially reducible* to $\mathcal{C}_1$)

Results:

(a) The countable atomless Boolean Algebra embeds into the degrees of strong iso-reducibility.
(b) Assume N2EXP $\cap$ co-N2EXP $\neq$ 2EXP. Then reducibility and potential reducibility are distinct.
(c) Assume that reducibility and potential reducibility are distinct. Then P$\neq \#$P.

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2. (Beckmann-Buss-SDF) Polynomial-time Set Recursion

When is a function $F : V \to V$ *computable in "polynomial time"?*

Bellantoni-Cook: Schemes for generating the PTIME functions on finite strings

We develop a set-theoretic analogue of the Bellantoni-Cook schemes, the Safe-Recursive Set Functions

See Arnold Beckmann's talk

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3. (Atserias-Müller) Forcing and Complexity Theory

Forcing in ZF or ZFC Set Theory: Cohen et.al.

Forcing in Bounded Arithmetic: Paris, Wilkie, Riis, Ajtai, Takeuti, Krajicek et.al.

Is there a unifying framework for these forcing arguments?

Yes, see Moritz Müller's talk

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4. (SDF-Hyttinen-Kulikov) Shelah classification and Higher Descriptive Set Theory

T countable, complete, first-order theory

T is *classifiable* iff there is a "structure theory" for its models

Shelah's Characterisation (Main Gap): T is classifiable iff T is superstable without the OTOP and without the DOP

A classifiable T is *deep* iff it has the maximum number of models in all uncountable powers

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Another way of classifying theories: Descriptive Set Theory

Mod_T^ω = Models of T with universe ω

Isom_T^ω = The Equivalence Relation of Isomorphism on Mod_T

Classify T according to the complexity of Isom_T^ω

Bad news: The complexity of Isom_T^ω is not a good measure of the model-theoretic complexity of T:

Dense Linear Order is bad model-theoretically but Isom_T^ω is trivial

(Koerwien) There are very classifiable theories T such that Isom_T^ω is not even Borel

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Instead use Isom_T^κ *for an uncountable* κ

Results:

T is classifiable and shallow (i.e. not deep) iff Isom_T^κ is "Borel"

T is classifiable iff for all regular $\lambda < \kappa$, Isom_T^κ is not "Borel above" equality modulo the λ-nonstationary ideal

(for appropriate κ)

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5. (Koerwien-Todorcevic) $\aleph_1$-categoricity and Forcing Axioms

Shelah: A theory in $L(Q)$ which is $\aleph_1$-categorical under MA (Martin's Axiom) but not under CH

There is a theory in $L(Q)$ which is $\aleph_1$-categorical under PFA (Proper Forcing Axiom) but not under MA

The theory describes a partition of the reals into countable dense subsets

Related question: Is $\aleph_1$-categoricity for $L_{\omega_1\omega}$ absolute?

See Martin Koerwien's talk

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Proofs and Sets
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6. Proof-theoretic analogues of generic reals

Example from set theory:

Suppose that (f, g) is generic over L for Cohen forcing $\times$ Cohen forcing. Then neither f nor g is dominated by any real in L, but any real constructible in both f and g is constructible.

Analogue of this in proof theory?

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There are recursive functions f_0, f_1 (with natural representations) such that any function provably-recursive in both PA + "f_0 is total" and PA + "f_1 is total" is PA-provably recursive

Proof uses an idea of Kristiansen to "split" the Hardy function H_{ϵ_0}

Visser later reproved this using Rosser tricks

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7. Slow consistency

For honest-recursive f define:

Con_f^* : If $f(y)$ is defined for all $y \leq x$ then $\text{Con}(\text{PA}_x)$
$\text{Con}_f^\#$: If $f(y)$ is defined for all $y \leq x$ then $\text{Con}_x(\text{PA})$

where PA_x is PA with only Σ_x induction and
Con_x means consistency for proofs with Gödel $\#$ at most x

(a) $\text{PA} < \text{PA} + \text{Con}_f^* \leq \text{PA} + \text{Con}(\text{PA})$
(b) $\text{PA} \leq \text{PA} + \text{Con}_f^\# \leq \text{PA} + \text{Con}(\text{PA})$
(c) If f is the Paris-Harrington function then
$\text{PA} + \text{Con}_f^* < \text{PA} + \text{Con}(PA)$ but PA proves $\text{Con}_f^\#$

(c) uses the Solovay, Krajicek-Pudlak work on injecting inconsistencies as well as bounds on cut-elimination

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8. Proof-theoretic operators

In set theory, Steel showed that the definable "uniformly invariant" operators on the Turing degrees are wellordered with successor given by Turing jump

Is there an analogous result in proof theory, using Con as the jump operator? The answer is No:

There is a recursive F such that for T_0, T_1 of the form $\mathrm{PA} + \varphi$:
(a) If $T_0 < T_1$ then $T_0 < F(T_0, T_1) < T_1$.
(b) If T_0, T_0' have the same theorems and T_1, T_1' have the same theorems then $F(T_0, T_1)$ and $F(T_0', T_1')$ have the same theorems.

The proof uses Feferman provability and Orey sentences.
It is not known if one can require $F(T)$ to be Π^0_1 for all T

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Computations and Models
Ekaterina Fokina, Julia Knight, Russell Miller, Antonio Montalbàn

9. The computable model theory of the uncountable

A result of Fokina-SDF-Harizanov-Knight-McCoy-Montalban is that the isomorphism relation on computable graphs is as complex as any Σ^1_1 equivalence relation on numbers. See Julia Knight's talk

We get the same result for computable structures on ω_1 assuming $V = L$:

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For any Σ^1_1 equivalence relation E on ω_1 there is a uniformly ω_1-computable sequence of graphs $(M_\alpha \mid \alpha < \omega_1)$ (with universe ω_1) such that $\alpha\ E\ \beta$ iff M_α is isomorphic to M_β

"Graphs" can be replaced by "fields" or "linear orders".

We also obtained some results about ω_1-computable categoricity of fields. For more on ω_1-computable categoricity see Jesse Johnson's talk.

Another perspective on ω_1-computability ("local computability") will be presented in Russell Miller's talk

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Computations and Proofs
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10. Characterising consistency via algorithms

The theory $T + \mathrm{Con}\,T$ is not computationally stronger than T in the sense that any function provably total in this theory is already provably total in T (via some representation)

However is there some sense in which this theory can be characterised via its computational power? Chen-Flum-Müller show:

Suppose that P$\neq$NP and T is a strong enough theory of arithmetic. Then $T + \mathrm{Con}\,T$ is the least extension of T proving that some algorithm decides SAT as fast as any algorithm that T-provably decides SAT.

For more on this see Yijia Chen's talk

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11. Optimal proof systems and logics for PTIME.

An old question is whether there is a nice logic for PTIME on unordered finite structures.

A proof system for TAUT is a polytime function whose range is TAUT. It is optimal if it can simulate any other such proof system in polynomial time. Chen-Flum-Müller show:

There is an optimal proof system for TAUT iff the Blass-Gurevich logic captures PTIME

(It is conjectured that the equivalent statements above are false)

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History and Philosophy of Set Theory
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Graham and Kantor wrote a fascinating book about the impact of religion on the development of the Moscow school of mathematics: *Naming Infinity*. In the Infinity Project they continued their work on the concept of "naming" with a special focus on Luzin and Grothendieck.

Tatiana and I brought my Inner Model Hypothesis into the current debate on the foundations of set theory. She will discuss this in her Infinity Conference talk

What is the future of the Infinity Project?

I'll discuss this on Friday