The tree property at the $\aleph_{2n}$'s and the failure of SCH at $\aleph_\omega$

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The first author was supported by FWF Project $\boxed{??}$EDIT.

The second author was supported by the Program for Development of Sciences at Charles University in Prague no. 13 Rationality in humanities, section *Modern Logic, its Methods and its Applications.*

Abstract. We show – starting from a hypermeasurable-type large cardinal assumption – that one can force a model where $2^{\aleph_\omega} = \aleph_{\omega+2}$, $\aleph_\omega$ strong limit, and the tree property holds at all $\aleph_{2n}$, for $n > 0$. This provides a partial answer to the question whether the failure of SCH at $\aleph_\omega$ is consistent with many cardinals below $\aleph_\omega$ having the tree property.

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Keywords: Tree property, Sacks forcing.
AMS subject code classification: 03E35,03E55.

Contents

1 Introduction

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Assume that $\aleph_\omega$ is a strong limit cardinal. It is an open question whether one can have the tree property at every $\aleph_n$, $1 < n < \omega$, and simultaneously violate SCH at $\aleph_\omega$. The failure of SCH at $\aleph_\omega$ is a necessary condition for a positive answer to an even more difficult question, whether one can have the tree property also at $\aleph_{\omega+2}$ (together with the tree property below). Finally, one can wish to have the tree property at $\aleph_{\omega+1}$ as well.[1]

Many partial answers are known today. Cummings and Foreman showed in [3] that, from ω-many supercompacts, one have the tree property at every $\aleph_n$, $1 < n < \omega$, where $\aleph_\omega$ is a strong limit cardinal satisfying $2^{\aleph_\omega} = \aleph_{\omega+1}$. Neeman [13] recently extended this result and showed that the tree property can hold in the whole interval $[\aleph_2, \aleph_{\omega+1}]$ ($\aleph_\omega$ is again strong limit and $2^{\aleph_\omega} = \aleph_{\omega+1}$).

In [3], it is also proved from similar assumptions that one can get the tree property at κ^{++} for a strong limit cardinal κ with cofinality ω; it is claimed that κ can be as low as $\aleph_\omega$, but no proof of this result is given in [3]. The consistency of the tree property at $\aleph_{\omega+2}$ was first proved from an almost optimal hypothesis (existence of a weakly compact hypermeasurable cardinal) in [5]; in [5], the tree property below $\aleph_\omega$ is not discussed but one can show that the tree property holds at every fourth cardinal below $\aleph_\omega$.

Unfortunately, there seems to be little hope in combining the ideas from [3] and [5] to get the tree property at every $\aleph_n$, $1 < n < \omega$, together with the tree property at $\aleph_{\omega+2}$ (or at least the failure of SCH at $\aleph_\omega$). The reason is that the argument in [5] heavily uses the properties of extender ultrapower embeddings, while [3] uses supercompact cardinals (it is known that the tree property at successor cardinals requires very large cardinals).[2]

In this paper, we show that if we step back a little and ask for the tree property below $\aleph_\omega$ at *every other cardinal*, we can have the failure of SCH at $\aleph_\omega$, and moreover from very mild assumptions. The tree property at every $\aleph_{2n}$ for $0 < n < \omega$ is potentially problematic because the powersets "touch each other" (i.e. $2^{\aleph_{2n}} \ge \aleph_{2n+2}$), which causes interference. This interference is relatively simple to overcome locally for a fixed pair of cardinals, such as $\aleph_2$ and $\aleph_4$ (this result is implicit already in [12]), but obtaining the tree property at every other cardinal below $\aleph_\omega$ requires new ideas. We start – in Theorem 5.1 – by showing that if we are satisfied with $2^{\aleph_\omega} = \aleph_{\omega+1}$, then ω-many weakly compact cardinals suffice to get the tree property at every $\aleph_{2n}$, $0 < n < \omega$. In Theorem 6.1 we proceed to show that we can get in addition $2^{\aleph_\omega} = \aleph_{\omega+2}$.

The proof of the main Theorem 6.1 is technically demanding and heavily uses the properties of the κ-Sacks forcing, for a regular κ (not necessarily inaccessible). The fusion construction available for this forcing allows us to construct a guiding generic for the Prikry collapse which does not leave big gaps in collapsing a large

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[1] The ultimate goal is to have the tree property at every regular cardinal greater than $\aleph_1$, but this is another story; we will stay with $\aleph_\omega$ in this paper.

[2] At the first glance, it seems that a strong assumption featuring supercompact cardinals is at least as good as as the weaker one in [5], but this rule does not apply here: an extender embedding generated by system of ultrafilters has a simpler representation which allows some diagonal constructions which are not possible with supercompact embeddings.

cardinal κ to $\aleph_\omega$ (see Lemma 6.9). Moreover, the fusion construction allows us to lift certain generic elementary embeddings and thus show that the tree property is not destroyed by the Prikry collapse (see the easier Lemma 6.20, and the much harder Lemma 6.22).

The paper is organized as follows. In Section 2, we review basic forcing notation and notational conventions regarding the generalized Sacks forcing. In Section 3, we introduce a criterion for not adding new cofinal branches to trees; unlike similar criteria for forcings with nice chain conditions or nice closure, our criterion is based on fusion. In Section 4, we apply the criterion to the forcing iteration which we will use in the proof. In Section 5, we prove the first theorem which says that from ω-many weakly compact cardinals one can get a model where the tree property holds at every $\aleph_{2n}$ for $0 < n < \omega$. We use the Mitchell forcing for this result. In Section 6, we prove the main theorem which says that from the hypermeasurable-type assumptions, one can force the tree property at every $\aleph_{2n}$, $0 < n < \omega$, together with $2^{\aleph_\omega} = \aleph_{\omega+2}$, $\aleph_\omega$ strong limit. We end the paper by some open questions.

2 Preliminaries

2.1 Notation

We first fix notation which we use in the paper.

We use the symbol $|$ to denote restriction of a function. In particular, if $b \in 2^\alpha$ for some α and $\beta < \alpha$, then $b|\beta$ is the restriction of b to β.

Regarding forcing, we use the following notation. For a regular cardinal κ, we say that a forcing notion P is κ-closed (or κ-distributive) if every decreasing sequence of conditions of length $< \kappa$ has a lower bound (or every family of $< \kappa$ many dense open sets has a non-empty intersection). P has the κ-cc if every antichain has size less than κ; P is κ-Knaster if in every family of conditions of size at least κ one can find a subfamily of size at least κ of mutually compatible conditions.

For any forcing P and $p \in P$: if $p \Vdash \dot{x} \in V$, we say that p *decides*, or equivalently *determines* x if $p \Vdash \dot{x} = \check{y}$ for some $y \in V$.

If P is an iteration of length β, and $\gamma < \beta$, we write $P(< \gamma) * P(\geq \gamma)$ to denote the forcing equivalent to P, viewed as an iteration $P(< \gamma)$ indexed by $\delta < \gamma$, followed by the tail iteration $P(\geq \gamma)$. We use the analogous notation for conditions and generic filters: $p(< \gamma)$, and $g(< \gamma)$, for $p \in P$ and a generic filter g; sometimes we write $g_{<\gamma}$ instead of $g(< \gamma)$. The reason is to reduce the number of subscripts in formulas. We do use subscripts and write P_α instead of $P(< \alpha)$ if this is an established notation in the literature (as in $P = \langle (P_\alpha, \dot{Q}_\alpha) : \alpha < \kappa \rangle$, where P is an iteration).

Assume $P = \langle (P_\alpha, \dot{Q}_\alpha) : \alpha < \lambda \rangle$ is an iteration for some $\lambda > 0$. We say that P is a *κ-support iteration*, for a regular κ, if the support of the conditions in P has size at most κ (similarly for a product). The support of a condition p in P is denoted as $\mathrm{supp}(p)$.

By Cohen forcing at κ for a regular κ we mean the set of functions from κ to 2 of size $< \kappa$; ordering is by reverse inclusion. We denote this forcing $\mathrm{Add}(\kappa, 1)$. The product $\mathrm{Add}(\kappa, \alpha)$ is viewed as a set of functions from $\kappa \times \alpha$ to 2 of size $< \kappa$.

2.2 Generalized Sacks forcing

We often deal with the generalised Sacks forcing in this paper. We include basic definitions here; for more details see [10].

Definition 2.1 *Let $\kappa \geq \omega$ be a regular cardinal. By a perfect κ-tree, we mean a set $(T, \subseteq)$ such that*

(i) $T \subseteq 2^{<\kappa}$, T is closed under initial segments, i.e. if $t \in T$, $s \in 2^{<\kappa}$, and $s \subseteq t$, then $s \in T$;
(ii) Above every $t \in T$, there is a splitting node, i.e. $\forall t \in T \; \exists s \in T \; (t \subseteq s$ & $s^\frown 0 \in T$ & $s^\frown 1 \in T)$;
(iii) If $\langle s_\alpha : \alpha < \gamma \rangle$, $\gamma < \kappa$, is a $\subseteq$-increasing sequence of nodes in T, then the union $s = \bigcup_{\alpha<\gamma} s_\alpha$ is in T;
(iv) (Continuity). If there are unboundedly many splitting nodes below $s \in T$, then s splits, i.e. if $s \in T$, and for every $t \subsetneq s$ there exists a splitting node t', $t \subsetneq t' \subsetneq s$, then s splits in T.

Definition 2.2 *For a regular $\kappa \geq \omega$, Sacks forcing at κ, or κ-Sacks forcing, is the collection of all perfect κ-trees as in Definition 2.1. Extension is by inclusion.*

We denote the κ-Sacks forcing by $\mathrm{Sacks}(\kappa, 1)$. A κ-support product and iteration of κ-Sacks forcing is denoted $\mathrm{Sacks}(\kappa, \alpha)$ (according to the context).

We now review some basic definitions concerning trees. We will only consider trees $(T, \subseteq)$ where $T \subseteq 2^{<\kappa}$ for some regular κ.

If T is a κ-tree and t is in T, we write $T|t$ for the restriction of T to t:

$$T|t = \{s \in T \,:\, t \subseteq s \text{ or } s \subseteq t\}. \tag{2.1}$$

If $\langle T_i : i \in I \rangle$ is a sequence of trees and $\langle t_i : i \in I \rangle$ are such that $t_i \in T_i$ for $i \in I$, then we write $\langle T_i : i \in I \rangle | \langle t_i : i \in I \rangle$ to denote the coordinate-wise restriction of $\langle T_i : i \in I \rangle$ to $\langle t_i : i \in I \rangle$.

If p is s sequence of names for trees, i.e. p is a condition in the iteration $\mathrm{Sacks}(\kappa, \alpha)$, and $\langle t_i : i < \alpha \rangle$ is a sequence of elements in $2^{<\kappa}$, we define the restriction of p to $\langle t_i : i < \alpha \rangle$

$$p|\langle t_i \,:\, i < \alpha \rangle \tag{2.2}$$

only in the case it makes sense, i.e. by induction on $\beta < \alpha$, the following hold for every $\beta < \alpha$:

(i) $p|\langle t_i : i < \beta \rangle$ forces that t_β is in $p(\beta)$, and
(ii) $p|\langle t_i : i < \beta+1 \rangle$ is the condition $p|\langle t_i : i < \beta \rangle^\frown r$ where r is a name forced by $p|\langle t_i : i < \beta \rangle$ to be the tree $p(\beta)$ restricted to t_β.

If T and T' are two trees such that $T' \subseteq T$ and s is a stem of T', we say that S is an *amalgamation* of T and T' if the subtree $T|s$ is replaced by T' in T:

$$S = (T \setminus (T|s)) \cup T'. \tag{2.3}$$

One can amalgamate more than two trees by applying this definition successively.

If $s \in T$ is a splitting node, then we say that its *splitting rank* is α if the order type of the set $\{s' \subsetneq s : s' \text{ is a splitting node in } T\}$ is equal to α. We write $\mathrm{Split}_\alpha(T)$ to denote the collection of all nodes in T of the splitting rank α, and $\mathrm{Succ}_\alpha(T)$ to denote the set of all $s \in T$ such that $s = s'^\frown 0$ or $s'^\frown 1$ for some $s' \in \mathrm{Split}_\alpha(T)$ (the *successors* of the splitting nodes of rank α). Finally, we say that $s \in T$ has cofinality α if $s \in 2^\beta$ and $\mathrm{cf}(\beta) = \alpha$.

3 Fusion and the criterion for not adding new branches

Let Q be a forcing notion and G a Q-generic filter. We say that a sequence of ground-model objects $x = \langle a_i : i < \kappa \rangle$ in $V[G]$ is *fresh* if for every $\alpha < \kappa$, $x \restriction \alpha$ is in V, but x is in $V[G] \setminus V$. Note that x can be a sequence of 0's and 1's and can thus represent a characteristic function of a subset of κ – *a fresh subset of* κ; or more generally, x can be a sequence of nodes in a tree $T \in V$.

We give some examples to illustrate the notion of a fresh sequence.

(a) For any regular cardinal $\kappa > \omega$, the single Cohen forcing $\mathrm{Add}(\kappa, 1)$ adds a fresh subset of κ. Or more generally, if P is κ-distributive and adds a new subset of κ, then any such subset is fresh.

(b) If κ is regular, and P is κ-Knaster, then P does not add a fresh subset of κ ([2]). In particular, if $\kappa^{<\kappa} = \kappa$, then $\mathrm{Add}(\kappa, \alpha)$ for any α does not add a fresh subset of any regular $\lambda > \kappa$ because it is λ-Knaster for any such λ.

(c) If P is κ-closed, adds new subsets of κ, but is not κ^+-Knaster, then it may or may not add a fresh subset of κ^+.
  - If $\kappa^{<\kappa} = \kappa$ in the ground model, then $\mathrm{Sacks}(\kappa, 1)$ does not add a fresh subset of κ^+: Let g be $\mathrm{Sacks}(\kappa, 1)$-generic. If x is a set of ordinals in $V[g] \setminus V$, then g is actually in $V[x \cap a]$ for some a in V of size κ. If x were a fresh subset of κ^+, then $V[x \cap a]$ for any a of size κ is equal to V, and hence $V[x \cap a]$ cannot construct the generic g.
  - For any $\alpha \geq \kappa^+$, the product and iteration of the Sacks forcing $\mathrm{Sacks}(\kappa, \alpha)$ does add a fresh subset of κ^+. This holds because the support of the conditions in the product and iteration is of size $\leq \kappa$, and so the Cohen forcing $\mathrm{Add}(\kappa^+, 1)$ can be completely embedded.

(d) Interestingly, P may add fresh subsets of κ^+, and yet not add new cofinal branches to κ^+-trees. Let T be a κ^+-tree. Then if P is κ^+-closed, it cannot add a new cofinal branch to T ([2]). However, P can add a fresh subset of κ^+ (take for instance $\mathrm{Add}(\kappa^+, 1)$ for $\kappa \geq \omega$). A more difficult argument (see Theorem 4.3) shows that for regular κ, $\mathrm{Sacks}(\kappa, \alpha)$ for $\alpha \geq \kappa^+$ does not add new branches to κ^+-trees while it does add fresh subsets of κ^+.

In the course of the proof, we will be dealing with Sacks-like forcings with fusion and we will ask whether or not they add new cofinal branches to existing trees – we will isolate the concept of not "not deciding fresh sequences in a strong sense" as a criterion for not adding new branches (see Definition 3.3) . To make the discussion more transparent, we introduce in Definition 3.1 the notion of fusion with respect to some parameters (these parameters essentially define a way of building a "fusion sequence" with respect to the given forcing).

Definition 3.1 *Assume $\kappa^{<\kappa} = \kappa$. Let P be a κ-support iteration of length $\lambda > 0$ which has greatest lower bounds for descending sequences $\vec{p}$ of conditions of length $< \kappa$ (we denote these infimas as $\bigwedge \vec{p}$). Set $X = [\lambda]^{<\kappa} \setminus \{\emptyset\}$. We say that P together with relations $\leq_{\alpha,x}$ ($\alpha < \kappa$, $x \in X$)* satisfies κ-fusion *if and only if there exists a function f from the sequences of conditions in P to X such that:*

- *(i) $p \leq_{\alpha,x} q$ implies $p \leq q$ for all p, q in P.*
- *(ii) Whenever $\langle p_\alpha : \alpha < \kappa \rangle$ is a sequence which satisfies:*
  - *(a) For all $\alpha < \kappa$, $p_{\alpha+1} \leq_{\alpha,x_\alpha} p_\alpha$ and $x_\alpha \subseteq x_{\alpha+1}$.*
  - *(b) For all limit $\gamma < \kappa$, p_γ is the greatest lower bound of $\langle p_\alpha : \alpha < \gamma \rangle$ and $x_\gamma = \bigcup_{\alpha<\gamma} x_\alpha$.*
  - *(c) For all $\alpha < \kappa$, $x_\alpha = f(\langle p_\beta : \beta < \alpha \rangle)$.*

  Then the entire sequence $\langle p_\alpha : \alpha < \kappa \rangle$ has the greatest lower bound q. We say that $\langle p_\alpha : \alpha < \kappa \rangle$ is a fusion sequence *and q is its the* fusion limit*.*

For the usual Sacks iteration at ω of length ω_2, X consists of non-empty finite subsets of ω_2, $p \leq_{n,x} q$ says that $p \leq q$ and all splitting nodes of rank n on the coordinates in x still have rank n in q, and f requires that the x's be chosen in such a way that their union is equal to the whole support of the fusion limit. See Section 4 for more details and examples.

Remark 3.2 We introduced Definition 3.1 to prove Theorem 3.4 in a general form, with no reference to a particular forcing.

Definition 3.3 *Assume $\kappa^{<\kappa} = \kappa$. Assume P and $\leq_{\alpha,x}$ ($\alpha < \kappa$, $x \in X$) are as in Definition 3.1. We say that P together with $\leq_{\alpha,x}$ ($\alpha < \kappa$, $x \in X$)* does not decide fresh κ^+-sequences in a strong sense *if the following hold.*

Whenever $\dot{B}$ is a name for a fresh sequence of length κ^+, i.e.

$$1 \Vdash \text{“}\dot{B} \text{ is a fresh sequence of length } \kappa^+\text{,”} \tag{3.4}$$

then for every $p \in P$, every $\alpha < \kappa$, every $\delta < \kappa^+$, and every $x \in X$, there exist $p_0 \leq_{\alpha,x} p$ and $p_1 \leq_{\alpha,x} p$ and γ, with $\delta < \gamma < \kappa^+$, such that whenever $r_0 \leq p_0$ and $r_1 \leq p_1$ and

$$r_0 \Vdash \dot{B}|\gamma = \check{b}_0 \text{ and } r_1 \Vdash \dot{B}|\gamma = \check{b}_1, \tag{3.5}$$

then

$$b_0 \neq b_1. \tag{3.6}$$

That is, r_0 and r_1 force contradictory information about $\dot{B}$ restricted to γ.

Theorem 3.4 *Assume $\kappa^{<\kappa} = \kappa$ and let P be an iteration which together with relations $\leq_{\alpha,x}$ ($\alpha < \kappa$, $x \in X$) satisfies κ-fusion and does not decide fresh κ^+-sequences in a strong sense. Then P does not add new branches to κ^+-trees, and more generally, if $\kappa \leq \rho$ and $2^\kappa > \rho$, P does not add new branches to ρ^+-trees.*

Proof. Assume for contradiction that, without loss of generality, the weakest condition in P forces that $\dot{B}$ is a new branch through the ρ^+-tree T. We will build by induction a binary tree $\mathbb{T} = \{(p_s, x_s) : s \in 2^{<\kappa}\}$, where $p_s \in P$ and $x_s \in X$, of height κ indexed by sequences s in $2^{<\kappa}$ such that

(i) $p_s = \bigwedge\langle p_{s|\beta} : \beta < \delta\rangle$ and $x_s = \bigcup_{\beta<\delta} x_{s|\beta}$ for s in 2^δ whenever δ is a limit ordinal.

(ii) for any $b \in 2^\kappa$, and $\alpha < \kappa$,

$$p_{b|\alpha+1} \leq_{\alpha,x_\alpha} p_{b|\alpha}, \tag{3.7}$$

where $x_\alpha = f(\langle p_{b|\beta} : \beta < \alpha\rangle)$.

By (i) and (ii) in Definition 3.1, for any b in 2^κ, $\langle p_{b|\alpha} : \alpha < \kappa\rangle$ is a fusion sequence.

The tree $\mathbb{T}$ and an increasing sequence $\langle \gamma_\alpha : \alpha < \kappa\rangle$ of ordinals below κ^+ will be built by induction. At limit stage δ, for every $s \in 2^\delta$ set p_s and x_s to satisfy (i), and set $\gamma_\delta = \lim\langle\gamma_\beta : \beta < \delta\rangle$. Assuming $\mathbb{T}_\alpha$ and γ_α are given, we will describe how to construct $\mathbb{T}_{\alpha+1}$ and $\gamma_{\alpha+1}$. Enumerate all p in $\mathbb{T}_\alpha$ as $\langle p_\beta : \beta < 2^{|\alpha|}\rangle$; by our assumption $\kappa^{<\kappa} = \kappa$, $2^{|\alpha|} \leq \kappa$. By induction, define an increasing sequence $\langle\gamma_\alpha^\beta : \beta < 2^{|\alpha|}\rangle$ as follows. Given p_β and γ_α^β, apply the property in Definition 3.3 to find two incomparable extensions $p_0 \leq_{\alpha,x} p_\beta$ and $p_1 \leq_{\alpha,x} p_\beta$ forcing contradictory information about $\dot{B}$ below some $\gamma_\alpha^{\beta+1} > \gamma_\alpha^\beta$ in the sense of (3.5) and (3.6). Make sure that x is equal to f applied to the conditions below p_β, as given by the appropriate sequence $s \in 2^\alpha$ determining p_β. Finally, put these conditions at the next level $\mathbb{T}_{\alpha+1}$ above p_β; that is, if $p_\beta = p_s$ for some in $s \in 2^\alpha$, then set $p_{s^\frown 0} = p_0$ and $p_{s^\frown 1} = p_1$. Take suprema at limits, and also at the end to get $\gamma_{\alpha+1}$.

Let γ_∞ be the supremum of $\langle\gamma_\alpha : \alpha < \kappa\rangle$ and let $\langle p_b : b \in 2^\kappa\rangle$ be such that p_b is the fusion limit of $\langle p_{b|\alpha} : \alpha < \kappa\rangle$. Let $\langle r_b : b \in 2^\kappa\rangle$ be a sequence of any conditions such that

$$r_b \leq p_b \text{ and } r_b \text{ decides } \dot{B} \text{ up to } \gamma_\infty. \tag{3.8}$$

Let $\langle t_b : b \in 2^\kappa\rangle$ be the nodes of the tree T at level γ_∞ decided by these r_b's.

We claim that for every $b \neq b'$ in 2^κ, $t_b \neq t_{b'}$, and there therefore T_{γ_∞} has size $> \rho$ in V, a contradiction.

If $b \neq b'$, then for some $\alpha < \kappa$, b extends $s^\frown 0$ and b' extends $s^\frown 1$ for some $s \in 2^\alpha$. Then the claim follows by the construction of the tree $\mathbb{T}$ at stage $\mathbb{T}_{\alpha+1}$ because

$$r_b \leq p_b \leq p_{b|s^\frown 0} \text{ and } r_{b'} \leq p_{b'} \leq p_{b|s^\frown 1}. \tag{3.9}$$

This finishes the proof. □

By Theorem 3.4, for given P which satisfies κ-fusion, it suffices to check the property in Definition 3.3 to verify that P does not add branches to certain trees. The following folklore Lemma 3.5 is useful for this.

Let Q be a forcing notion, T a μ-tree for some regular μ, and $\dot{B}$ a Q-name for a branch in T. We say that p and q *force contradictory information about $\dot{B}$ at level* γ, or just *at* γ if p decides $\dot{B}|\gamma$ (the initial segment of $\dot{B}$ of height γ) and q decides $\dot{B}|\gamma$, and they decide this segment differently.

Lemma 3.5 *Let Q be a forcing notion, T a μ-tree for some regular μ, and let the weakest condition of Q force that $\dot{B}$ is a new branch through T (i.e. the branch is not in the ground model). Then for every p_1, p_2 in Q and every $\delta < \mu$, there are $r_1 \leq p_1$, $r_2 \leq p_2$ and $\gamma \geq \delta$ such that r_1 and r_2 force contradictory information about $\dot{B}$ at level γ.*

Proof. By contradiction argue that otherwise it is dense in $Q \times Q$ that the interpretation of $\dot{B}$ in V^Q is equal to the interpretation of $\dot{B}$ in $(V^Q)^Q = V^{Q\times Q}$. $\square$

4 Examples

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In the interest of clarity of the argument, we first show how Theorem 3.4 applies in the simplest case of a single κ-Sacks at the inaccessible (see Subsection 4.1). Then we proceed to state the theorem for the most complex case of an iteration of a κ-Sacks for a successor κ (see Subsection 4.2).

4.1 A single κ-Sacks at an inaccessibles

Theorem 4.1 *Let κ be inaccessible and S the κ-Sacks forcing* $\mathrm{Sacks}(\kappa, 1)$. *Then S satisfies κ-fusion according to Definition 3.1 and does not decide fresh κ^+-sequences in a strong sense. By Theorem 3.4 S does not add branches to κ^+-trees, and more generally, if $\kappa \leq \rho$ is such that $2^\kappa > \rho$, then S does not add branches to ρ^+-trees.*

Proof. Since $\lambda = 1$, define f to give constantly $\{\emptyset\}$ and define $p \leq_{\alpha,x} q$ so that $p \leq q$ and all splitting nodes of rank α in q have still rank α in p. By arguments in [10], this satisfies Definition 3.1. Since x is always equal to $\{\emptyset\}$ here, we write just $p \leq_\alpha q$ in what follows.

It remains to verify the property in Definition 3.3. Suppose $1 \Vdash$ "$\dot{B}$ is a new κ^+-branch." We wish to show that for any $\alpha < \kappa$, $\delta < \kappa^+$, and p, there are $p_0 \leq_\alpha p$, $p_1 \leq_\alpha p$ and γ, with $\delta < \gamma < \kappa^+$, such that whenever $r_0 \leq p_0$ and $r_1 \leq p_1$ and

$$r_0 \Vdash \dot{B}|\gamma = \check{b}_0 \text{ and } r_1 \Vdash \dot{B}|\gamma = \check{b}_1, \tag{4.10}$$

then

$$b_0 \neq b_1. \tag{4.11}$$

That is r_0 and r_1 force contradictory information about $\dot{B}$ at level γ.

Denote

$$A = \{(t, t') \,:\, t, t' \in \mathrm{Succ}_\alpha(p)\}. \tag{4.12}$$

Set $p^0_0 = p$ and $p^0_1 = p$; we will construct two $\leq_\alpha$-decreasing sequences continuous at limits $\langle p^i_0 \,:\, i < |A|\rangle$ and $\langle p^i_1 \,:\, i < |A|\rangle$; p_0 will be the infimum of $\langle p^i_0 \,:\, i < |A|\rangle$ and p_1 the infimum of $\langle p^i_1 \,:\, i < |A|\rangle$. We will also construct an increasing sequence of ordinals continuous at limits $\langle \gamma_i \,:\, i < |A|\rangle$, with $\gamma_0 > \delta$; the desired γ will be the supremum of this sequence.

Enumerate $A = \{(t, t')_i \,:\, i < |A|\}$. For $m < |A|$, assume p^m_j, for $j \in \{0, 1\}$, and γ_m were already constructed. To construct the $m + 1$-st element of the sequences, and also γ_{m+1}, consider $(t, t') = (t, t')_m$. Form the restrictions $p^m_0|t$ and $p^m_1|t'$ and by Lemma 3.5, find $s_0 \leq p^m_0|t$ and $s_1 \leq p^m_1|t'$ such that s_0 and s_1 force contradictory information about $\dot{B}$ at level η for some $\eta > \gamma_m$. Set p^{m+1}_0 to be the amalgamation of s_0 and p^m_0, p^{m+1}_1 the amalgamation of s_1 and p^m_1, and $\gamma_{m+1} = \eta$.

We now verify that $p_0 = \bigwedge\langle p^i_0 \,:\, i < |A|\rangle$, $p_1 = \bigwedge\langle p^i_1 \,:\, i < |A|\rangle$, and $\gamma = \sup\langle \gamma_i \,:\, i < |A|\rangle$ are as desired. Let $r_0 \leq p_0$ and $r_1 \leq p_1$ be given. We can assume that the stems of r_0 and r_1 are at least α' where α' is the supremum of the lengths of nodes in $\mathrm{Succ}_\alpha(p)$. Then there is some $(t, t')_m \in A$ such that $r_0 \leq p^m_0|t$ and $r_1 \leq p^m_1|t'$, and so r_0 and r_1 decide $\dot{B}$ differently at $\gamma_{m+1} < \gamma$. □

4.2 Iteration at a successor κ

Theorem 4.2 *Assume $\omega_1 < \kappa = \nu^+$, $2^\nu = \nu^+$ and $\lambda > 0$ is an ordinal number. Denote by $S = \mathrm{Sacks}(\kappa, \lambda)$ the κ-support iteration of λ-many copies of κ-Sacks forcing. Then S satisfies κ-fusion according to Definition 3.1 and does not decide fresh κ^+-sequences in a strong sense. By Theorem 3.4 it does not add branches to κ^+-trees, and more generally, if $\kappa \leq \rho$ is such that $2^\kappa > \rho$, then S does not add branches to ρ^+-trees.*

Proof. In preparation for the application of Theorem 3.4, set $X = [\lambda]^{<\kappa} \setminus \{\emptyset\}$ and choose f in any way to ensure that the union of the x_α's is equal to the union of the supports of the p_α's on the sequence as in Definition 3.1, and make f continuous at the limits. (For instance as follows: given $\langle p_\beta \,:\, \beta < \alpha\rangle$ for a successor $\alpha < \kappa$, define $f(\langle p_\beta \,:\, \beta < \alpha\rangle)$ to be equal to the union of the first α-many coordinates (under some fixed injective functions from supports of the p_β's to κ) in the supports of the p_β's.) Define $p \leq_{\alpha,x} q$ if and only if $p \leq q$ (i.e. for every $\xi < \lambda$, $p(< \xi)$ forces that $p(\xi)$ is a subtree of $q(\xi)$), and moreover for every $\xi \in x$, $p(< \xi)$ forces that $p(\xi) \cap 2^{\alpha+1} = q(\xi) \cap 2^{\alpha+1}$. Note that this is different from demanding that all splitting nodes of rank α are preserved as we did for the inaccessible case (the reason is that in the successor case, the lengths of the splitting nodes of rank $\alpha < \kappa$ may be unbounded in κ). With this definition of $\leq_{\alpha,x}$, the forcing is still κ-closed and satisfies κ-fusion. S preserves κ^+ because $2^\nu = \nu^+$ ensures we have a diamond sequence on κ, which is used for the argument (see [10] for details).[3]

Now we will prove that S does not decide fresh κ^+-sequences in a strong sense; by Theorem 3.4, this suffices to finish the proof.

Fix a diamond sequence on κ of the following form:

$$\langle S_\beta : S_\beta \subseteq 2 \times \beta \times \beta \;\&\; \beta < \kappa \rangle. \tag{4.13}$$

Let $p \in S$, $\alpha < \kappa$, $\delta < \kappa^+$, and $x \in X$ be given. We will construct the required $p_0 \leq_{\alpha,x} p$ and $p_1 \leq_{\alpha,x} p$ as the fusion limits of certain well chosen sequences:

$$p_0 = \bigwedge \langle p_0^\beta : \alpha \leq \beta < \kappa \rangle \text{ and } p_1 = \bigwedge \langle p_1^\beta : \alpha \leq \beta < \kappa \rangle. \tag{4.14}$$

We will also construct auxiliary sequences $\langle x_i^\beta : \alpha \leq \beta < \kappa \rangle$ and $\langle \pi_i^\beta : \alpha \leq \beta < \kappa \rangle$ for $i < 2$ (π_i^β is a bijection from x_i^β to some $\rho_i^\beta < \kappa$ which takes unions at limit β's). We will also construct a continuous sequence $\langle \gamma_\beta : \alpha \leq \beta < \kappa \rangle$ of ordinals below κ^+, with $\gamma_\alpha > \delta$.

Set $p_0^\alpha = p_1^\alpha = p$ and $x_0^\alpha = x_1^\alpha = x$. At limit stages, take infima of the sequences, and unions of the x_i's and π_i's constructed so far. Take also the supremum of the sequence of γ's constructed so far.

Assume stage β has been constructed. Find $p_0^{\beta+1} \leq_{\beta, x_0^\beta} p_0^\beta$ and $p_1^{\beta+1} \leq_{\beta, x_1^\beta} p_1^\beta$, and $\gamma_{\beta+1}$ as detailed below:

Do nothing unless the following conditions are satisfied in the order given – if one of the conditions is not satisfied, break the construction and set for $i < 2$, $p_i^{\beta+1} = p_i^\beta$ (and let $x_i^{\beta+1}$ be chosen by f).

(i) For $i < 2$, $\rho_i^\beta = \beta$.

For $i < 2$, set $\sigma_i^\beta = \langle \sigma_i^\beta(\xi) : \xi \in x_i^\beta \rangle$, where $\sigma_i^\beta(\xi) : \beta \to 2$ is defined at $\zeta < \beta$ as follows,

$$\sigma_i^\beta(\xi)(\zeta) = 1 \leftrightarrow \langle i, \pi_i^\beta(\xi), \zeta \rangle \in S_\beta. \tag{4.15}$$

(ii) Let us write $\sigma_i^\beta{}^\frown 0$ for $\langle \sigma_i^\beta(\xi)^\frown 0 : \xi \in x_i^\beta \rangle$.
For $i < 2$, there exists $u_i^\beta \leq p_i^\beta$ such that $u_i | \sigma_i^\beta{}^\frown 0 = u_i$ and for every $\xi \in x_i^\beta$,

$$u_i^\beta(< \xi) \Vdash \sigma_i^\beta(\xi) \text{ is splitting in } p_i^\beta(\xi). \tag{4.16}$$

If (i) and (ii) are true, use Lemma 3.5 to find extensions

$$t_i^\beta \leq u_i^\beta \tag{4.17}$$

which force contradictory information about $\dot{B}$ at some level $\eta > \gamma_\beta$.

Set $p_i^{\beta+1}$ to be the amalgamation of p_i^β and t_i^β, and $\gamma_{\beta+1} = \eta$ (see [10] for definition of amalgamation in case of names).

---

[3] It is well known that CH does not imply the existence of a diamond sequence at ω_1; to make the present theorem hold also for $\kappa = \omega_1$, we need to assume $\diamondsuit_{\omega_1}$ in addition to CH.

Set p_i for $i < 2$ to be the fusion limit of the respective sequences. Set $\gamma_\infty = \sup\langle\gamma_\beta : \alpha \le \beta < \kappa\rangle$. Note that $\gamma_\infty < \kappa^+$. Without loss of generality, assume for $i < 2$, $\pi_i = \bigcup_\beta \pi_i^\beta$ is a bijection from $\operatorname{supp}(p_i)$ onto κ.

For $i < 2$, let $w_i \le p_i$ decide $\dot{B}$ up to γ_∞. As in Sublemma 1 in [10], construct by induction sequences $\langle w_i^\beta : \beta < \kappa\rangle$ with $w_i^0 = w_i$ and functions s_i^β with domain x_i^β such that $s_i^\beta(\xi) : \rho_i^{\beta,\xi} \to 2$ for some $\rho_i^{\beta,\xi} \ge \beta$ such that for $i < 2$:

(i) $\beta \le \beta'$ implies $w_i^{\beta'} \le w_i^\beta$.
(ii) $\beta < \beta'$ implies $s_i^\beta(\xi)^\frown 0 \subseteq s_i^{\beta'}(\xi)$ for $\xi \in x_i^\beta$, and s_i^δ is the union at limit δ.
(iii) For every $\xi \in x_i^\beta$,

$$w_i^\beta(<\xi) \Vdash w_i^\beta(\xi) = (w_i^\beta(\xi) | s_i^\beta(\xi)^\frown 0) \text{ and } s_i^\beta(\xi) \text{ splits in } p_i(\xi). \tag{4.18}$$

Notice that $w_i^\beta = w_i | \langle s_i^\beta(\xi)^\frown 0 : \xi \in x_i^\beta\rangle$.

Denote $s_i = \bigcup_{\beta<\kappa} s_i^\beta$. Set:

$$\tilde{A} = \{\langle i, \xi, \zeta\rangle : s_i(\pi_i(\xi))(\zeta) = 1\}. \tag{4.19}$$

For $i < 2$, denote by C_i the closed unbounded set of all ordinals $\beta > \alpha$ such that $\rho_i^{\beta,\xi} = \beta$ for every $\xi \in x_i^\beta$ and $\pi_i^\beta : x_i^\beta \to \beta$. By the properties of the diamond sequence, there is some $\epsilon \in C_0 \cap C_1$ such that

$$\tilde{A} \cap (2 \times \epsilon \times \epsilon) = S_\epsilon. \tag{4.20}$$

It follows that $w_i^\epsilon = w_i^\epsilon | \langle s_i^\epsilon(\xi)^\frown 0 : \xi \in x_i^\epsilon\rangle$ extends w_i and moreover for every $\xi \in x_i^\epsilon$, $w_i^\epsilon(<\xi)$ forces that $s_i^\epsilon(\xi)$ splits in $p_i(\xi)$. Since ϵ is in $C_0 \cap C_1$, the construction of both $p_0^{\epsilon+1}$ and $p_1^{\epsilon+1}$ was non-trivial (with w_i^ϵ witnessing the required u_i^ϵ in the construction of $p_i^{\epsilon+1}$). It follows for $i < 2$:

$$w_i^\epsilon \le t_i^\epsilon, \tag{4.21}$$

where t_i^ϵ is as in (4.17). As $w_i^\epsilon \le w_i$ for $i < 2$ and w_i's decide $\dot{B}$ up to γ_∞, w_0 and w_1 force contradictory information about $\dot{B}$ at $\gamma_{\epsilon+1} < \gamma_\infty$. □

The following is a more general form of these theorems which will be useful for the construction later on.

Theorem 4.3 *Assume $\omega_1 < \kappa = \nu^+$, $2^\nu = \nu^+$ and $\lambda > 0$ is an ordinal. Denote by $S = \langle (S_\alpha, \dot{Q}_\alpha) : \alpha < \lambda\rangle$ a κ-support iteration of length λ such that for every α, $\dot{Q}_\alpha$ is a name for a forcing notion as follows:*

(i) Either $\dot{Q}_\alpha$ is a name for a κ^+-closed forcing notion, or
(ii) $\dot{Q}_\alpha$ is a name for the forcing $\mathrm{Sacks}(\kappa, 1)$.

Then S satisfies κ-fusion according to Definition 3.1 and does not decide fresh κ^+-sequences in a strong sense. By Theorem 3.4 it does not add branches to κ^+-trees, and more generally, if $\kappa \le \rho$ is such that $2^\kappa > \rho$, then S does not add branches to ρ^+-trees.

Proof. The definitions of X and f are as in Theorem 4.2. Define $p \le_{\alpha,x} q$ if and only if $p \le q$ and for all $\xi \in x$ such that $\dot{Q}_\xi$ is $\mathrm{Sacks}(\kappa, 1)$, $p(<\xi)$ forces

$p(\xi) \cap 2^{\alpha+1} = q(\xi) \cap 2^{\alpha+1}$. Note that the fusion limit takes fusion limits at the coordinates with the Sacks forcing and simple lower bounds at the coordinates with the κ^+-closed forcings.

The rest of the proof is an easy variant of the proof in Theorem 4.2. □

Remark 4.4 Mitchell [12] first showed how to collapse a weakly compact cardinal $\lambda > \kappa \geq \omega$, κ regular, to κ^{++} in such a way to force the tree property at κ^{++}. Key to the proof is that certain forcings do not add branches to existing trees. This can be used to argue that many other iterations, not just the one in [12], force tree property. Here is a quick review which shows the typical application of Theorem 4.2 (note that Mitchell used a different forcing). Suppose that GCH holds and $\kappa \geq \omega$ is regular and $\lambda > \kappa$ is weakly compact. We claim that the κ-support iteration S of Sacks forcing at κ of length λ forces the tree property at $\kappa^{++} = \lambda$. Let G be S-generic over V. Let T be a λ-tree in the generic extension by $V[G]$; we will show that T has a cofinal branch in $V[G]$. In V, let $j : M \to N$ be an elementary embedding with critical point λ, where M and N are transitive, $|M| = |N| = \lambda$, $M^{<\lambda} \subseteq M$, $N^{<\lambda} \subseteq N$, and λ, S and $\dot{T}$ are in M (such j exists by weak compactness of λ). Let H be a generic for $j(S)$ in the interval $[\lambda, j(\lambda))$ over $V[G]$. Then j lifts in $V[G][H]$ to $j : M[G] \to N[G][H]$. It is easy to see that $j(T)$ restricted to λ is equal to T and $T \in N[G]$. Notice that any node in $j(T)$ of length λ is a cofinal branch through T. It follows that T has a cofinal branch in $N[G][H]$. The key is to notice that any such cofinal branch must already be in $N[G]$ (and therefore in $V[G]$): by Theorem 4.2 applied in $N[G]$, H cannot add a new cofinal branch to T, and therefore any such branch must have been present already in $N[G]$.

Remark 4.5 Other forcings, not just Sacks forcing, can be used to obtain the tree property – it suffices to formulate the right kind of fusion which satisfies Definition 3.3 and apply the argument in the previous Remark 4.4. For instance Grigorieff forcing[4] at a regular $\kappa \geq \omega$ can be used to obtain the tree property.

4.3 A product lemma

In proofs which argue that the tree property can hold at two cardinals λ and λ^{++}, the relevant forcings which yield TP(λ) and TP(λ^{++}) are not entirely independent of each other, and some "interference" occurs. The general question is this: Assume S does not add branches to κ^+-trees (S can be any of the forcings in the previous fusion-based examples), and assume P has the κ-cc. Is it still true that S does not add branches to κ^+-trees in V^P?

Lemma 4.6 (Product lemma) *Let $\omega_1 < \kappa$ be regular, $\kappa = \nu^+$ and $2^\nu = \nu^+$, and let S be an iteration as in Theorem 4.3. Let P be a forcing which has the κ-cc, and let T be a κ^+-tree in V^P. Then any cofinal branch through T in $V^{P \times S}$ is already in V^P. Or more generally with the same assumptions on S, P, if $\kappa \leq \rho$ and $2^\kappa > \rho$, then for every ρ^+-tree T in V^P, any cofinal branch in $V^{P \times S}$ is already in V^P.*

---

[4] In the simplest setting, conditions are partial functions from κ to 2 with non-stationary domains.

Proof. We will follow closely the proof of Theorem 4.2, tacitly assuming that some of the coordinates we deal with are as in Theorem 4.3 (these κ^+-closed coordinates do not change the argument). We will explain what modifications must be made to the argument in the proof of Theorem 4.2, referring to the argument in Theorem 3.4 for the way to build a tree of conditions based on the basic step in Theorem 4.2.

Assume the following are given:

$$r \in S,\ x \in X,\ \alpha < \kappa, \text{ and } \delta < \kappa^+. \tag{4.22}$$

Let G be a P-generic filter and $\dot{T}$ a P-name for a κ^+-tree in $V[G]$. Let F be an S-generic filter over $V[G]$. Assume for contradiction that $\dot{B}$ is a $P \times S$-name for a cofinal branch through T in $V[G][F] \setminus V[G]$.

We will construct certain conditions $r_0, r_1 \leq_{\alpha,x} r$ in S and $\gamma^* > \delta$ which will *modulo* P (as will be apparent from the construction below) be such that whenever $\bar{r}_i \leq r_i$, $i < 2$, decide over V^P $\dot{B}$ up to γ^*, they decide it differently.

To start the construction, notice the following:

(*) The following set is dense in P for every r, r' in S and $\delta < \kappa^+$:

$$\begin{split}\{p \in P \,:\, \exists \bar{r} \leq r\ \exists \bar{r}' \leq r'\ \exists \gamma\ \delta < \gamma < \kappa^+\ \&\\ p \Vdash \text{“}\bar{r} \text{ and } \bar{r}' \text{ force contradictory information about } \dot{B} \text{ at } \gamma\text{”}\}.\end{split} \tag{4.23}$$

(*) can be used to argue for a more general property:

(**) Let r, r' in S be arbitrary and $\delta < \kappa^+$, then there exists a maximal antichain $A \subseteq P$, and $\bar{r} \leq r, \bar{r}' \leq r'$ in S and γ, $\delta < \gamma < \kappa^+$, such that for every $p \in A$,

$$p \Vdash \text{“}\bar{r} \text{ and } \bar{r}' \text{ force contradictory information about } \dot{B} \text{ at } \gamma.\text{”} \tag{4.24}$$

To see that (**) is true, just apply (*) successively, constructing an antichain in P, and taking lower bounds in S; the construction must stop after $< \kappa$ stages by the chain condition of P.

Fix in V a diamond sequence $\langle S_\alpha \,:\, \alpha < \kappa \;\&\; S_\alpha \subseteq (2 \times \alpha \times \alpha)\rangle$.

We will construct in V two fusion sequences $\langle r_i^\beta \,:\, \alpha \leq \beta < \kappa\rangle$ originating in r, but then splitting into two sequences as in the proof of Theorem 4.2 (together with sequences of functions mapping parts of supports into κ, and sequences of ordinals, etc. as in that proof). Assume that $\beta \geq \alpha$ is a nontrivial stage of the construction with r_i^β, $i < 2$, constructed, and assume there are $u_i \leq r_i^\beta$ which decide that it is possible to thin out r_i^β's according to S_β (details can be found in the proof of Theorem 4.2). Notice that this condition is decidable in V because it refers to S only.

Applying (**), construct a maximal antichain $A_\beta \subseteq P$ and decreasing sequences of conditions below u_i with the limit $t_i \leq u_i$, $i < 2$, such that for every $p \in A_\beta$:

$$p \Vdash \text{“}t_0 \text{ and } t_1 \text{ force contradictory information about } \dot{B} \text{ at } \gamma,\text{”} \tag{4.25}$$

where γ, $\delta < \gamma < \kappa^+$, is larger than the previous ordinals on the sequence.

Set $r_i^{\beta+1}$ to be the amalgamation of r_i^β and t_i so that $r_i^{\beta+1} \leq_{\alpha,x} r_i^\beta$. Let r_i be the fusion limit of the sequences $\langle r_i^\beta : \alpha \leq \beta < \kappa \rangle$ for $i < 2$, and let γ^* be the supremum of all the at most κ-many ordinals occurring in the construction.

Apply now the construction in Theorem 3.4 and construct in V a full binary tree $\mathbb{T}$ of conditions in S, where at each node of $\mathbb{T}$ carry out the construction detailed above (in particular, build all the relevant antichains, etc.). For every $b \in 2^\kappa$, let r_b be the fusion limit of the conditions determined by b in $\mathbb{T}$. Let γ_∞ be as in the proof of theorem 3.4.

Let G be a P-generic filter, and $\dot{T}^G = T$.

In $V[G]$, choose for each b in $2^\kappa \cap V$ a condition $\bar{r}_b \leq r_b$ which decides $\dot{B}$ up to γ_∞; denote the decided branch segment as B_b. We claim that in $V[G]$, $\{B_b : b \in 2^\kappa \cap V\}$ are pairwise distinct nodes on the level γ_∞ of T, which contradicts the fact that T is a κ^+-tree in $V[G]$.

Work in V now. Let $b_0 \neq b_1$ be distinct branches in 2^κ, and let w_0 and w_1 be the conditions in S deciding in $V[G]$ the branch segment of $\dot{B}$ up to γ_∞. Assume that b_i are first different at level $\alpha < \kappa$, and let us identify the node in $\mathbb{T}$ where b_0 and b_1 split with r in (4.22) above, and r_0 and r_1 with the nodes immediately above r in $\mathbb{T}$. Construct below w_i sequences determining the leftmost branches in these conditions on the relevant supports, just as in the construction in the proof of Theorem 4.2, leading up to (4.20). Let ϵ be the stage where $\tilde{A}$ is guessed. By the construction detailed in this proof above, there is a unique element p in $G \cap A_\epsilon$, where A_ϵ is the maximal antichain pertaining to the construction of r_0 and r_1 at stage ϵ; p forces that any extensions which are stronger the relevant t_0 and t_1 in (4.25) above decide $\dot{B}$ differently below γ_∞.

This ends the proof. $\square$

Note that Lemma 4.6 also holds for κ is inaccessible (the argument is easier because we do not need to use the diamond).

Remark 4.7 The proof is based on the idea which appears in the usual proof of Easton's lemma: if P has the κ-cc and Q is κ-closed, then any sequence of ordinals of length $< \kappa$ which appears in $V^{P \times Q}$ appears already in V^P (see [9]). A generalization of Easton's lemma to trees appeared already in [14]: if P has the κ^+-cc, and Q is κ^+-closed, then Q does not add cofinal branches to κ^+-trees in V^P. Our forcing S is not κ^+-closed, so a more complicated argument is needed. Also, unlike in the Easton's lemma, it seems essential – at least for the current proof – that P has the κ-cc, and not just the κ^+-cc (this is important in the key step (4.24)).

5 The tree property at every $\aleph_{2n}$, $0 < n < \omega$ (with SCH at $\aleph_\omega$)

As a warm-up, we show that the tree property at every $\aleph_{2n}$ for $0 < n < \omega$, with $\aleph_\omega$ strong limit, can be forced just from ω-many weakly compact cardinals. As our primary concern is to show that the failure of SCH can in addition hold at $\aleph_\omega$, and we use an iteration based on the Sacks forcing for that result, we will not give too many details in the proof of Theorem 5.1. The proof of Theorem 5.1 uses the Mitchell forcing and we assume some degree of familiarity with this forcing on the part of the reader (see [12] or a nice review in [1]).

Theorem 5.1 *(GCH) Assume there are ω-many weakly compact cardinals $\omega = \kappa_0 < \kappa_1 < \ldots$ with supremum λ. Then in the generic extension by the product of the Mitchell forcings at the κ_i's, the tree property holds at every $\aleph_{2n}$, $0 < n < \omega$.*

Proof. Let P be a reverse Easton iteration of the Cohen forcing $\mathrm{Add}(\alpha, 1)$ for every inaccessible $\alpha < \lambda$. Let $M(n, n+1)$ denote the Mitchell forcing which makes $2^{\kappa_n} = \kappa_{n+1}$ and forces TP at κ_{n+1}. Set Q to be the full support product

CHECK: the proof seems sketchy, do we need to add something?

$$Q = \prod_n M(n, n+1). \tag{5.26}$$

Remark 5.2 To define $M(n, n+1)$, first set for $\alpha \leq \kappa_{n+1}$, $P(\alpha) = \mathrm{Add}(\kappa_n, \alpha)$ (a condition in $P(\alpha)$ is a partial function from α to 2 of size $< \kappa_n$). A condition in $M(n, n+1)$ is a pair (p, q), where $p \in P(\kappa_{n+1})$, and q is a function with domain of size $\leq \kappa_n$ such that for every $\beta \in \mathrm{dom}(q)$, $q(\beta)$ is a $P(\beta)$-name for a condition in $\mathrm{Add}(\kappa_n^+, 1)$. $M(n, n+1)$ is κ_{n+1}-Knaster and κ_n-closed, and there is a κ_n^+-closed forcing $R(n, n+1)$ such that $M(n, n+1)$ is a projection of $P(\kappa_{n+1}) \times R$. This last also holds in the quotient $M(n, n+1)/M(n, n+1)(< \alpha)$ (where $M(n, n+1)(< \alpha)$ is the restriction of $M(n, n+1)$ to the first α stages).

Suppose $P * Q$ adds a κ_{n+1}-tree T. Then T is added by $P * \prod_{m \leq n+1} M(m, m+1)$. The forcing $\prod_{m \leq n+1} M(m, m+1)$ is κ_{n+2}-Knaster in V^P, and therefore T has a name $\dot{T}$ which can be taken to be a $< \kappa_{n+2}$-sequence of elements in V^P. This name is already present in $P(< \kappa_{n+2})$ (the iteration P below κ_{n+2}). It follows that $P(< \kappa_{n+2}) * \prod_{m \leq n+1} M(m, m+1)$ already adds T.

Let us write this forcing as

$$P(< \kappa_{n+2}) * \big(M(n+1, n+2) \times \prod_{m < n+1} M(m, m+1)\big). \tag{5.27}$$

This forcing is equivalent to the following forcing

$$P(< \kappa_{n+2}) * M(n+1, n+2) * \prod_{m < n+1} M(m, m+1) \tag{5.28}$$

because $M(n+1, n+2)$ does not change $H(\kappa_{n+1})$ where the product $\prod_{m < n+1} M(m, m+1)$ lives.

We claim that T is in fact added by

$$(5.29) \qquad P(<\kappa_{n+2}) * \mathrm{Add}'(\kappa_{n+1},1) * \prod_{m<n+1} M(m,m+1),$$

where $\mathrm{Add}'(\kappa_{n+1},1)$ is a subforcing of the first coordinate of $M(n+1,n+2)$ of size at most κ_{n+1}, and therefore isomorphic to $\mathrm{Add}(\kappa_{n+1},1)$. This is true because T has a name in the forcing $P(<\kappa_{n+2}) * \mathrm{Add}(\kappa_{n+1},\kappa_{n+2}) * \prod_{m<n+1} M(m,m+1)$ of size at most κ_{n+1} and therefore a name in the forcing $P(\kappa_{n+2}) * \mathrm{Add}'(\kappa_{n+1},1) * \prod_{m<n+1} M(m,m+1)$ for such an $\mathrm{Add}'(\kappa_{n+1},1)$.

$P(<\kappa_{n+2}) * \mathrm{Add}'(\kappa_{n+1},1)$ preserves the weak compactness of κ_{n+1} (since we prepared by the Cohen forcing below), and so we have the tree property at κ_{n+1} after further forcing with $\prod_{m<n+1} M(m,m+1)$ (the proof that $M(n,n+1)$ gives the tree property at κ_{n+1} also works for the product $\prod_{m<n+1} M(m,m+1)$). Therefore T has a cofinal branch. □

Remark 5.3 It seems natural to think that an analogous approach, based on a product forcing, can be used to show that ω-many Mahlo cardinals suffice to get a model where for every $\aleph_n$, $n>1$, there are no special Aronszajn trees at $\aleph_n$.

6 The tree property at every $\aleph_{2n}$, $0<n<\omega$ (with the failure of SCH at $\aleph_\omega$)

Assume GCH. We say that a measurable cardinal μ is *strongly measurable* if for every $\alpha \in (\mu^+,\mu^{++})$ there exists an embedding $j: V \to M$ with critial point μ, and M transitive, such that $j(\mu) > \alpha$.

Theorem 6.1 *(GCH) Assume $\kappa < \lambda$ are regular cardinal, and the following hold:*

- *(i) There is an embedding $j: V \to M$ with critical point κ, $H(\lambda)$ is included in M, and $M = \{j(f)(\alpha) : f : \kappa \to V \ \& \ \alpha < \lambda\}$.*
- *(ii) λ is the least strongly measurable above κ in both V and M.*

Then there exists a generic extension with $\aleph_\omega$ strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$, and the tree property holds at every $\aleph_{2n}$ for $0<n<\omega$.

Remark 6.2 Existence of such a j follows for instance from an embedding $j^*: V \to M$ with critical point κ such that $H(\lambda^{++})$ is included in M, where λ the least strongly measurable above κ. Then in M, λ is the least strongly measurable above κ. Let $N = \{j^*(f)(\alpha) : f : \kappa \to V \ \& \ \alpha < \lambda\}$; then N is an elementary submodel of M. If $\bar{N}$ is the transitive collapse of N via $\pi : N \to \bar{N}$, then because $\lambda + 1$ is included in N as a subset (note that $\lambda = j^*(f)(\kappa)$ for the f which picks the least strongly measurable above $\alpha < \kappa$), $\pi(\lambda) = \lambda$, and hence λ is the least strongly measurable cardinal above κ in $\bar{N}$. The embedding $j : V \to \bar{N}$, such that $j^* = j \circ \pi^{-1}$, satisfies the assumptions of Theorem 6.1. CHECK

The proof will be given in the rest of the section.

First we define a certain variant of the Sacks forcing which is convenient for our purposes.

Definition 6.3 *Suppose* $\omega_1 < \nu$ *and* $\nu^{<\nu} = \nu$. *For the rest of the present proof, we say that* T *is a perfect* ν, ω_1*-tree if it is a perfect* ν*-tree with the extra condition that only nodes of cofinality* ω_1 *are allowed to split (recall that a node has cofinality* ω_1 *if its length has that cofinality).* $\mathrm{Sacks}^{\omega_1}(\nu, 1)$ *is the forcing with these perfect* ν*-trees, and* $\mathrm{Sacks}^{\omega_1}(\nu, \beta)$ *for* $\beta > 0$ *is the* ν*-support iteration of such forcings.*

Remark 6.4 We have taken ω_1 for definiteness of the definition; ω_2 or ω_3 would work equally well. However, ν will be as small as ω_4 in later arguments, so the cardinal should not be larger than ω_3.

It is easy to see that this variant of ν-Sacks behaves much the same way as the usual ν-Sacks – in particular it is ν-closed, and has ν-fusion according to Definition 3.1 (this is used to argue that it preserves ν^+). In particular, Theorem 4.2 applies.

Definition 6.5 *Let*

$$P = \langle (P_\alpha, \dot{Q}_\alpha) \,:\, \alpha < \kappa + 1 \rangle \tag{6.30}$$

be a reverse Easton iteration of length $\kappa + 1$ *such that for each strongly measurable limit of strongly measurable cardinals* $\alpha \leq \kappa$, $\dot{Q}_\alpha$ *is an iteration of length* λ_α *with support* $\leq \alpha$, *where* λ_α *is the least strongly measurable above* α *and:*

$$\dot{Q}_\alpha = \langle ((\dot{Q}_\alpha)_\beta, \dot{R}_\beta) \,:\, \beta < \lambda_\alpha \rangle, \tag{6.31}$$

where for $\beta < \lambda_\alpha$, $\dot{R}_\beta$ *is* $\mathrm{Sacks}^{\omega_1}(\alpha, 1)$ *unless* β *is inaccessible in which case either of the following happens:*

- *(i) If* $P_\alpha * (\dot{Q}_\alpha)_\beta$ *forces that* β *is* α^{++}, *then* $\dot{R}_\beta$ *is the forcing* $\mathrm{Sacks}^{\omega_1}(\beta, \gamma)$, *for some* $\gamma < \lambda_\alpha$ *chosen generically.* CHECK: No small bound on γ.
- *(ii) Otherwise* $\dot{R}_\beta$ *is the trivial forcing.*

Some motivation for the definition of the forcing is in order. For the fixed α, $\dot{Q}_\alpha$ CHECK is a forcing which will force the tree property at $\lambda_\alpha = \alpha^{++}$ ($\dot{Q}_\alpha$ has the λ_α-cc, and by arguments in Theorem 4.3 and Remark 4.4, it forces the tree property at λ_α, which will become α^{++}). The forcing $\mathrm{Sacks}^{\omega_1}(\beta, \gamma)$ is a preparation for the lifting argument in Lemma 6.22. γ needs to be chosen generically because there is no bound – other than λ_α – on the length of the preparation in Lemma 6.22 (see also Remark 6.23). Since for large γ, $\mathrm{Sacks}^{\omega_1}(\beta, \gamma)$ collapses cardinals above β^+, it is not automatic that for every $\beta < \lambda_\alpha$ inaccessible, $P_\alpha * (\dot{Q}_\alpha)_\beta$ forces that β is α^{++} (or in general a regular cardinal); for this reason, we specifically verify that β is forced to be α^{++} before forcing with $\mathrm{Sacks}^{\omega_1}(\beta, \gamma)$.

Let $G * g$ be P-generic, where G is P_κ generic.

Lemma 6.6 *j lifts in $V[G*g]$ to*

$$j : V[G*g] \to M^* = M[G*g*H*h], \tag{6.32}$$

in particular κ is still measurable in $V[G*g]$.

Proof. The argument is a straightforward generalisation of the argument in [4] – in the difficult step of constructing h, the forcing in [4] is just the iteration of κ-Sacks while in our forcing $\dot{Q}_\kappa$, we have additional coordinates with a κ^+-closed forcing. A little reflection shows that these extra coordinates are easily dealt with – as in [4], to construct h, define a suitable fusion sequence on coordinates with the κ-Sacks forcing, and take simple lower bounds at the κ^+-closed coordinates. (A general treatment of such forcings with fusion with respect to preservation of measurability can be found in [7].) ☐

CHECK

The following lemma suggests that after the collapse, we have a chance of showing that $\aleph_{\omega+2}$ $(= \kappa^{++})$ still retains the tree property. However, we cannot prove this (see Section 7 with open questions). So Lemma 6.7 is stated for completeness but we will not make further use of it.

Lemma 6.7 *$\kappa^{++} = \lambda$ has the tree property in $V[G*g]$.*

Proof. This is again a simple generalisation of the argument in [4] – again we need to deal with extra κ^+-closed coordinates. The whole argument is sketched in Remark 4.4; the suitable generalisation of [4] is captured by Theorem 4.3 in the present paper. ☐

CHECK

Remark 6.8 It will be important that the embedding j in (6.32) is actually in $V[G*g]$ the normal measure ultrapower generated by $U = \{X \subseteq \kappa : X \in V[G*g] \ \& \ \kappa \in j(X)\}$. This follows from the fact that if we form the commutative triangle $j = j_U \circ k$, where $j_U : V[G*g] \to \mathrm{Ult}(V[G*g], U)$ is the normal measure ultrapower, then because the ultrapower $\mathrm{Ult}(V[G*g], U)$ contains all subsets of κ in $V[G*g]$, the embedding k is actually the identity.

Our strategy now is to carefully collapse κ to $\aleph_\omega$, forcing the failure of SCH at $\aleph_\omega$, and in addition ensuring that the tree property still holds at every $\aleph_{2n}$ for $0 < n < \omega$. In order to define the suitable collapse, we need a certain "guiding generic" – namely, a $\mathrm{Sacks}^{\omega_1}(\kappa^{++}, j(\kappa))$-generic filter over M^*. A substantial part of the argument is to show that it actually exists in $V[G*g]$.

Lemma 6.9 (Guiding generic lemma) *Let us denote $R = \mathrm{Sacks}^{\omega_1}(\kappa^{++}, j(\kappa))$ as defined in M^*. In $V[G*g]$, there exists an R-generic filter r over M^*.*

Proof. Recall that $\lambda = \kappa^{++}$ in M^* and that we have lifted j successively to

$$j : V[G] \to M[G*g*H], \text{ and } j : V[G*g] \to M^* = M[G*g*H*h], \tag{6.33}$$

where

$$M[G*g*H] = \{j(f)(\alpha) : f \in V[G] \ \& \ f : \kappa \to V[G] \ \& \ \alpha < \lambda\} \tag{6.34}$$

and

$$M^* = \{j(f)(\alpha) : f \in V[G*g] \;\&\; f : \kappa \to V[G*g] \;\&\; \alpha < \lambda\}, \tag{6.35}$$

with $2^\kappa = \kappa^+$ in $V[G]$, and $2^\kappa = \kappa^{++} = \lambda$ in $V[G*g]$.

By Remark 6.8, we actually have $M^* = \{j(f)(\kappa) : f \in V[G*g] \;\&\; f : \kappa \to V[G*g]\}$ although it will become important only later when we define the Prikry collapse forcing. CHECK

The representation in (6.34) has the advantage that there only κ^+ functions f considered here. We will show now that all maximal antichains of R^* (which exist in M^*) can be captured by these functions. We can view each $p \in R$ as an element of $H(j(\kappa))^{M^*}$. Moreover, every maximal antichain of R in M^* is an element of $H(j(\kappa))$ of M^* because R has the $j(\kappa)$-cc in M^*. Since h does not add new elements of $H(j(\kappa))$, it follows that A (as well as R) is in fact in $M[G*g*H]$. Thus we can represent A as $j(f)(\alpha)$, $\alpha < \kappa^{++}$, where $f : \kappa \to H(\kappa)$ is in $V[G]$ (note that there are only κ^+-many of such f in $V[G]$). By standard arguments, in order to find an R-generic over M^*, it suffices to find a filter which meets all dense open sets in M^* determined by maximal antichains, i.e. dense open sets of the form $D_A = \{q \in R : \exists a \in A \; a \le q\}$. In $V[G*g]$ we can write the collection of maximal antichains of R in M^* as the union of $\{\mathscr{A}_i : i < \kappa^+\}$ where for each $k < \kappa^+$, $\{\mathscr{A}_i : i < k\}$ is in M^* (by the closure of M^* under κ-sequences from $V[G*g]$) and for each $i < \kappa^+$, $\mathscr{A}_i$ is in M^* a collection of at most κ^{++}-many maximal antichains in R. Let $\mathscr{D}_i$ denote the set of dense open sets determined by the maximal antichains in $\mathscr{A}_i$; we write $\mathscr{D}_i(\xi)$ to denote the ξ-th set in $\mathscr{D}_i$ under some fixed enumeration.

Working in $V[G*g]$, we will define a decreasing sequence of conditions $\langle p_i : i < \kappa^+\rangle$ in R such that

(6.36)

(i) For each $i < \kappa^+$ limit, p_i is the infimum of the p_k's for $k < i$;
(ii) For each $i < \kappa^+$, p_{i+1} "reduces" $\mathscr{D}_i$ in the sense detailed below.

Fix in M^* a $\diamondsuit_{\kappa^{++}}(E^{\omega}_{\kappa^{++}})$ sequence $\langle S_\alpha : \alpha < \kappa^{++}\rangle$, where $E^{\omega}_{\kappa^{++}}$ is the set of ordinals below κ^{++} of cofinality ω. View this sequence as defined on $\kappa^{++} \times \kappa^{++}$; in particular for any $B \in M^*$, $B \subseteq (\kappa^{++} \times \kappa^{++})$, the following set is stationary:

$$\{\alpha < \kappa^{++} : \mathrm{cf}(\alpha) = \omega \;\&\; B \cap (\alpha \times \alpha) = S_\alpha\}. \tag{6.37}$$

For $\beta < \alpha$, we write $S_\alpha(\beta)$ to denote the projection of S_α to coordinate β viewed as a characteristic function of a subset of α, i.e. $S_\alpha(\beta)$ is a function with domain α such that for each $\gamma < \alpha$, $S_\alpha(\beta)(\gamma) = 1 \leftrightarrow \langle \beta, \gamma\rangle \in S_\alpha$. Note that the diamond sequence exists because $2^{\kappa^+} = \kappa^{++}$ in M^*.

Definition 6.10 *Let $\alpha < \kappa^{++}$ have cofinality ω, p_i^α be a condition in R, F_i^α a subset of $j(\kappa)$ of size $< \kappa^{++}$ (in M^*), and π_i^α a bijection from F_i^α onto an initial segment of κ^{++}. We say that α is* nontrivial *if $\pi_i^\alpha : F_i^\alpha \to \alpha$. For such α, let x be a function from*[5] *$\delta \le \alpha$ to 2^α. Via π_i^α, we can view x naturally as a sequence*

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[5] δ and x are used when by induction we want to show that $r_{<\xi}$ is generic for $R(<\xi)$. See Sublemma 6.15.

of characteristic functions of subsets of α*, indexed by the first* δ*-elements in* F_i^α*; since there is no danger of confusion, we will denote this sequence also* x*. For such an* x*, we say that* x *is* suitable *for* p_i^α *if either of the following hold:*

- *(i)* $x = \langle S_\alpha(\beta) : \beta < \delta \rangle$,
- *(ii) There exist a* ω*-sequence* $\alpha_0 < \alpha_1 < \cdots$ *with limit* α *such that for every* $\beta < \delta$*,* $x(\beta) = \bigcup_{0<n<\omega} S_{\alpha_n}(\beta)|\alpha_{n-1}$.

Remark 6.11 Suitability according to (ii) will be useful in guessing sets not in the current universe – we will be allowed to make mistakes (in the interval (α_{n-1}, α_n)), but after ω-many steps, we should get a correctly defined stage; see the end of proof of Sublemma 6.14 and Sublemma 6.15.

Fix $i < \kappa^+$ and p_i. The condition p_{i+1} is the limit of a decreasing fusion sequence $(\langle p_i^\alpha : \alpha < \kappa^{++} \rangle, \langle F_i^\alpha : \alpha < \kappa^{++} \rangle)$ continuous at the limits, built according to the relevant fusion parameters according to Definition 3.1; we explicitly include $\langle F_i^\alpha : \alpha < \kappa^{++} \rangle$ in the notation to denote the sequence of the subsets of $j(\kappa)$ chosen by f in Definition 3.1. Since the diamond sequence sits on $\kappa^{++} \times \kappa^{++}$, and our supports are in $j(\kappa)$, we will also keep track of functions π_i^α which map injectively F_i^α's to initial segments of κ^{++}; the sequence $\langle \pi_i^\alpha : \alpha < \kappa^{++} \rangle$ will be increasing under inclusion and continuous at limits.

To construct $p_i^{\alpha+1}$ from p_i^α, do nothing unless $\alpha < \kappa^{++}$ is nontrivial according to definition 6.10; in particular, α has cofinality ω. If α is nontrivial, the basic idea is to successively thin out to all suitable sequences and meet $\bigcap_{\gamma<\alpha} \mathscr{D}_i(\gamma)$. However, since we are dealing with names here, we first have to decide whether it makes sense to thin out a condition according to a suitable sequence.

Let $\langle x_\beta : \beta < \mu \rangle$, $\mu \leq \kappa^+$, be some enumeration of all suitable sequences (there at most $\kappa^+ \cdot (\kappa^+)^\omega = \kappa^+$-many of such sequences). Note that for every x_β, $\mathrm{dom}(x_\beta)$ is a subset of F_i^α. Construct a $\leq_{\alpha, F_i^\alpha}$ decreasing sequence $\langle q_\beta : \beta < \mu \rangle$ of conditions below p_i^α. Take infima at limits. Suppose q_β has been constructed; we wish to define $q_{\beta+1}$. First check whether it makes sense to thin out q_β according to x_β: by induction on $\xi \in \mathrm{dom}(x_\beta)$, extend $(q_\beta)(<\xi)|\langle x_\beta(\xi') : \xi' \in \mathrm{dom}(x_\beta) \cap \xi \rangle$ to a condition which forces that $x_\beta(\xi)^\frown 0$ or $x_\beta(\xi)^\frown 1$ is in $q_\beta(\xi) \cap 2^{\alpha+1}$; if no such stronger condition exists, stop the construction and set $q_{\beta+1} = q_\beta$. Suppose the construction does not stop; then it is possible to extend q_β to q_β^* so that for every $\xi \in \mathrm{dom}(x_\beta)$:

$$(q_\beta^*)(<\xi)|\langle x_\beta(\xi') : \xi' \in \mathrm{dom}(x_\beta) \cap \xi \rangle \Vdash x_\beta(\xi)^\frown i_\beta \in 2^{\alpha+1} \cap q_\beta^*(\xi), \tag{6.38}$$

for some $i_\beta \in \{0, 1\}$. Note that since α has cofinality ω, there is no splitting at the node $x_\beta(\xi)$, so i_β is either 0 or 1, but not both.

This means that the restriction $q_\beta^*|\langle x_\beta(\xi) : \xi \in \mathrm{dom}(x_\beta) \rangle$ is defined; set $q_{\beta+1}$ to be an extension of $q_\beta^*|\langle x_\beta(\xi) : \xi \in \mathrm{dom}(x_\beta) \rangle$ such that

- (i) If $\mathrm{dom}(x_\beta) = F_i^\alpha$, then $q_{\beta+1}|\langle x_\beta(\xi) : \xi \in F_i^\alpha \rangle$ meets $\bigcap_{\gamma<\alpha} \mathscr{D}_i(\gamma)$.
- (ii) If $\mathrm{dom}(x_\beta)$ is a proper initial segment of F_i^α, then build successively a decreasing sequence $\langle q_\beta^{*,\gamma} : \gamma < \alpha \rangle$ continuous at limits, successively meeting certain dense open sets in $\{\mathscr{D}_i(\gamma) : \gamma < \alpha\}$: if it is possible to extend $q_\beta^{*,\gamma}|\langle x_\beta(\xi) : \xi \in \mathrm{dom}(x_\beta) \cap F_i^\alpha \rangle$ to $q_\beta^{*,\gamma+1}$ which satisfies that $q_\beta^{*,\gamma+1}(\xi) =$

$q_\beta(\xi)$ for $\xi \in F_i^\alpha \setminus \mathrm{dom}(x_\beta)$ and $q_\beta^{*,\gamma+1}|\langle x_\beta(\xi) \,:\, \xi \in \mathrm{dom}(x_\beta) \cap F_i^\alpha \rangle$ meets $\mathscr{D}_i(\gamma)$, then do extend; otherwise, set $q_\beta^{*,\gamma+1} = q_\beta^{*,\gamma}$. Let $q_{\beta+1}$ be the infimum of $\langle q_\beta^{*,\gamma} \,:\, \gamma < \alpha \rangle$.

Remark 6.12 In (ii) above, we do not meet the intersection of all the sets in $\{\mathscr{D}_i(\gamma) \,:\, \gamma < \alpha\}$ at one step, but rather meet all those which can be met while keeping the coordinates outside $\mathrm{dom}(x_\beta)$ intact. This again anticipates the inductive construction of $r = r_{<j(\kappa)}$.

Finally, set $p_i^{\alpha+1}$ to be the infimum of $\langle q_\beta \,:\, \beta < \mu \rangle$.

Since M^* is closed under κ-sequences in $V[G * g]$, the sequence $\langle p_i \,:\, i < \kappa^+ \rangle$ built above satisfies (i) and (ii) of (6.36) as desired.

The idea now is to take "any sequence of branches" through all p_i for $i < \kappa^+$ and build the desired generic r from them. An obvious obstacle is that the conditions are made out of names, not ground model trees. Our strategy now will be to proceed inductively on $\xi < j(\kappa)$, define M^*-generics $r_{<\xi}$ for $R(<\xi)$, and argue by genericity that $r_{<\xi}$ determines a unique perfect κ^{++}, ω_1-tree T_ξ, which exists in $V[G * g]$ and which is in a well-defined sense the intersection of the trees $\{p_i(\xi) \,:\, i < \kappa^+\}$. The desired sequence of branches will be any sequence of branches through the T_ξ's, $\xi < j(\kappa)$ (although for definiteness, we will take the leftmost branches).

Recall that $R(0)$ is the forcing at the 0-th coordinate of the iteration R; in our definition $R(0)$ is the forcing $\mathrm{Sacks}^{\omega_1}(\kappa^{++}, 1)$ as defined in M^*. For every $i < \kappa^+$, $p_i(0)$ is a perfect κ^{++}, ω_1-tree in M^*, and also in $V[G*g]$ as $H(\kappa^{++})$ of $V[G*g]$ is included in M^*. In particular, $\bigcap_{i<\kappa^+} p_i(0)$ is a perfect κ^{++}, ω_1-tree in $V[G*g]$ (the intersection of a decreasing sequence of κ^{++}, ω_1-trees of length κ^+ is itself a perfect κ^{++}, ω_1-tree). Denote this tree T_0 and let b_0 be the leftmost cofinal branch through T_0.

Definition 6.13 *Set*

$$r_0 = \{p \in R(0) \,:\, \exists i < \kappa^+ \ \exists \alpha < \kappa^{++} \ p \geq p_i^\alpha(0)|(b_0|\alpha)\}. \tag{6.39}$$

Sublemma 6.14 *r_0 is $R(0)$-generic over M^*.*

Proof. It is clear from the definition that r_0 is a filter. It remains to verify that it meets every dense open set. Let D be a dense open set in M^* for $R(0)$. Then for some $i < \kappa^+$ and $\alpha < \kappa^{++}$, D contains some $\mathscr{D}_i(\alpha)$ restricted to the 0-th coordinate, where at the other coordinates $\mathscr{D}_i(\alpha)$ contains all conditions. We wish to show that for some $\bar\alpha \geq \alpha$, $p_i^{\bar\alpha}(0)|(b_0|\bar\alpha)$ is in D.

Build a sequence $\langle w_0^\alpha \,:\, \alpha < \kappa^{++} \rangle$ below $p_{i+1}|(b_0|\alpha)$ as follows:

(i) $w_0^0 = p_{i+1}|(b_0|\alpha)$,

(ii) w_0^γ for a limit γ is the infimum of $\langle w_0^\beta \,:\, \beta < \gamma \rangle$,

(iii) $w_0^{\gamma+1} \leq w_0^\gamma$ and there exists a sequence $\langle v_\xi^\gamma \,:\, \xi \in F_i^\gamma \rangle$, $v_\xi^\gamma \in 2^{\gamma+1}$, $\xi \in F_i^\gamma$, and $w_0^{\gamma+1}|\langle v_\xi^\gamma \,:\, \xi \in F_i^\gamma \rangle$ is defined and is equal to $w_0^{\gamma+1}$ (where the F_i^γ's are as in the construction of p_{i+1}).

Let $\langle c_0^\xi : \xi \in \mathrm{supp}(p_{i+1})\rangle$ be the sequence of the leftmost branches determined by $\langle w_0^\alpha : \alpha < \kappa^{++}\rangle$:

$$\text{For every } \xi \in \mathrm{supp}(p_{i+1}) \quad c_0^\xi = \bigcup_{\xi' < \zeta < \kappa^{++}} v_\xi^\zeta, \tag{6.40}$$

for ξ' least such that $\xi \in F_i^{\xi'}$.

Let C be a club of ordinals β where π_i^β maps F_i^β onto β. Apply diamond to $\langle c_0^\xi : \xi \in \mathrm{supp}(p_{i+1})\rangle$ (modulo the π_i^β's) and find $\alpha_0 > \alpha$ in C of cofinality ω such that the diamond sequences guesses $\langle c_0^\xi : \xi \in \mathrm{supp}(p_{i+1})\rangle$ at α_0. At α_0, the construction of $p_i^{\alpha_0+1}$ was nontrivial and the restriction of $p_i^{\alpha_0+1}$ to $\langle c_0^\xi|\alpha_0 : \xi \in F_i^{\alpha_0}\rangle$ (which is the same sequence as $\langle S_{\alpha_0}(\xi) : \xi \in F_i^{\alpha_0}\rangle$) is defined and meets the dense open sets detailed in (i) and (ii) below (6.38).

Now there are two cases.

Case 1. 0 is not in $F_i^{\alpha_0}$. Then in meeting D, no fusion restriction on levels is applicable, and $p_i^{\alpha_0+1}(0)$ meets D; in particular $p_i^{\alpha_0+1}(0)|(b_0|\alpha_0)$ meets D.

Case 2. 0 is in $F_i^{\alpha_0}$. Then $S_{\alpha_0}(0)|\alpha = b_0|\alpha$. Repeat the above argument, this time below $p_{i+1}|(b_0|\alpha_0)$, obtaining a decreasing sequence $\langle w_1^\alpha : \alpha < \kappa^{++}\rangle$ and a sequence of branches $\langle c_1^\xi : \xi \in \mathrm{supp}(p_{i+1})\rangle$. Let this sequence be guessed at a nontrivial stage of cofinality ω $\alpha_1 > \alpha_0$. This time we know that 0 is in $F_i^{\alpha_1}$; we also know that $S_{\alpha_1}(0)|\alpha_0 = b_0|\alpha_0$.

Repeat this ω-many times obtaining $\bar{\alpha}$ as the sup of $\alpha_0 < \alpha_1 < \cdots$. At stage $\bar{\alpha}$, there is a suitable x with domain equal to $\{0\}$ such that $x(0) = b_0|\bar{\alpha}$. Since $\mathscr{D}_i(\alpha)$ contains all conditions in coordinates larger than 0, the construction of $p_i^{\bar{\alpha}+1}$ ensures that $p_i^{\bar{\alpha}+1}$ restricted to $x(0)$ meets D. □

For every $i < \kappa^+$, $p_i(1)$ is in $M^*[r_0]$ realised by a perfect κ^{++}, ω_1-tree t_i. This tree is a perfect κ^{++}, ω_1-tree in $V[G * g]$. It follows that $T_1 = \bigcap_{i<\kappa^+} t_i$ is a perfect κ^{++}, ω_1-tree in $V[G * g]$.

This argument is generalised as an inductive construction of length $j(\kappa)$ as follows.

Sublemma 6.15 *Let $\gamma < j(\kappa)$ and as an induction assumption let $\langle T_\beta : \beta < \gamma\rangle$ be a sequence of trees constructed as in the previous paragraph, $\langle b_\beta : \beta < \gamma\rangle$ the sequence of leftmost branches through trees T_β, and let $r_{<\gamma}$ be a filter defined as follows:*

$$r_{<\gamma} = \{p \in R(<\gamma) : \exists i < \kappa^+ \; \exists \alpha < \kappa^{++} \; p \geq p_i^\alpha(<\gamma)|\langle b_\beta|\alpha : \beta \in F_i^\alpha \cap \gamma\rangle\}. \tag{6.41}$$

Then $r_{<\gamma}$ is $R(<\gamma)$-generic over M^*, and $p_i(\gamma)$ for every $i < \gamma$ is realised by a perfect κ^{++}, ω_1-tree t_i in $M^*[r_{<\gamma}]$; the intersection $\bigcap_{i<\kappa^+} t_i$ determines a tree T_γ.

Proof. We will proceed similarly as in Sublemma 6.14. Let D be as in Sublemma 6.14, this time obtained as a restriction of $\mathscr{D}_i(\alpha)$ to the first γ many coordinates of R. Consider the sequence $\langle b_\beta|\alpha : \beta < \gamma\rangle$, where b_β for $\beta < \gamma$ is the leftmost branch in T_β. Build the decreasing sequence $\langle w_0^\alpha : \alpha < \kappa^{++}\rangle$

below the condition p_{i+1}, with the associated branches $\langle c_0^{\xi} : \xi \in \mathrm{supp}(p_{i+1})\rangle$ as in Sublemma 6.14. As we deal with names here, choose $w_0^0 \leq p_{i+1}$ so that $w_0^0|\langle b_\beta|\alpha : \beta \in \gamma \cap F_i^{\alpha}\rangle = w_0^0$ is defined; this is possible by choice of $\langle T_\beta : \beta < \gamma\rangle$.

Let $\alpha_0 > \alpha$ be an ordinal of cofinality ω where the c_0^{β}'s for $\beta \in \mathrm{supp}(p_{i+1})$ are guessed. Assume $\gamma \cap F_i^{\alpha_0}$ is non-empty (otherwise we are done as in Sublemma 6.14). Repeat the construction leading to α_0 again with $w_1^0 = w_1^0|\langle b_\beta|\alpha_0 : \beta \in F_i^{\alpha_0} \cap \gamma\rangle$, branches $\langle c_1^{\xi} : \xi \in \mathrm{supp}(p_{i+1})\rangle$, and an ordinal α_1 of cofinality ω, where $\alpha_1 > \alpha_0$. Repeat this construction ω-many times. Let $\bar{\alpha}$ be the supremum of $\alpha < \alpha_0 < \alpha_1 < \cdots$. By the construction, $x = \langle b_\beta|\bar{\alpha} : \beta \in \gamma \cap F_i^{\bar{\alpha}}\rangle$ is a suitable sequence at stage $\bar{\alpha}$, and $p_i^{\bar{\alpha}+1}|\langle b_\beta|\bar{\alpha} : \beta \in \gamma \cap F_i^{\bar{\alpha}}\rangle$ is defined and meets D. □

Definition 6.16 *Set* $r = r_{<j(\kappa)}$.

By Sublemma 6.15 applied with $\gamma = j(\kappa)$, r is R-generic over M^* as required. This finishes the proof of Lemma 6.9. □

We can now define the Prikry-type collapsing of κ to $\aleph_\omega$, using r as a "guiding generic".

Let us first fix U, the normal ultrafilter on κ derived from the lifted embedding CHECK $j : V[G * g] \to M^*$ in (6.32). Clearly, U extends the original normal ultrafilter U_0 derived from $j : V \to M$. Moreover, by Remark 6.8, M^* is actually the normal ultrapower of $V[G * g]$ by U, and thus r is the guiding generic for a forcing in this ultrapower.

The set of strongly measurable limit of strongly measurable cardinals has mea- CHECK sure one not only in U_0, but also in U. Denote this set Z (Z is in $V[G * g]$).

Definition 6.17 *Define the collapsing order, C, as follows.*

A condition in C is of the form $(p_0, \kappa_1, p_1, \ldots, \kappa_n, p_n, H)$ where each κ_i is in Z,

- *(i)* p_0 *is in* $\mathrm{Sacks}(\omega, \kappa_1)$*;*
- *(ii) For $i > 0$, p_i is in* $\mathrm{Sacks}^{\omega_1}(\kappa_i^{++}, \kappa_{i+1})$*, and p_n is in* $\mathrm{Sacks}^{\omega_1}(\kappa_n^{++}, \kappa)$*;*
- *(iii) H is a function with* $\mathrm{dom}(H) \in U$, $H(\alpha) \in \mathrm{Sacks}^{\omega_1}(\alpha^{++}, \kappa)$*, and $[H]_U$ is in the guiding generic r, where U is the normal ultrafilter fixed above.*

Ordering is defined as follows: the condition $(q_0, \lambda_1, q_1, \ldots, \lambda_m, q_m, I)$ is stronger than the condition $(p_0, \kappa_1, p_1, \ldots, \kappa_n, p_n, H)$ if

- *(i)* $m \geq n$*,*
- *(ii) For every $i \leq n$, $\kappa_i = \lambda_i$, and $q_i \leq p_i$,*
- *(iii) For every i with $n < i \leq m$, $\lambda_i \in$* $\mathrm{dom}(H)$ *and* $q_i \leq H(\lambda_i)$*,*
- *(iv)* $\mathrm{dom}(I) \subseteq \mathrm{dom}(H)$ *and* $I(\lambda) \leq H(\lambda)$ *for every* $\lambda \in \mathrm{dom}(I)$*.*

Let c be C-generic over $V[G * g]$.

Lemma 6.18 *The forcing C makes κ into $\aleph_\omega$, forces $2^{\aleph_\omega} = \aleph_{\omega+2}$, and every κ_i for $0 < i < \omega$ (chosen by the generic c for C) becomes $\aleph_{4i-2}$.*

Proof. The proof uses the κ^+-cc of C (ensured by compatibility of elements in the guiding generic), and the standard properties of Prikry-type forcing intermixed with collapses. For details, see [8]. □

Remark 6.19 By the setup of P, for each κ_i, κ_i^{++} of $V[G*g]$ is in V the least strongly measurable cardinal above κ_i.

Lemma 6.20 *The tree property holds at each* $\aleph_{4i-2}$ *in* $V[G*g*c]$, $0 < i < \omega$

Proof. Work in $V[G*g]$, where you can fix $j_i : V[G*g] \to M_i$ with critical point κ_i.

Let T be a $\kappa_i = \aleph_{4i-2}$-tree in $V[G*g*c]$. Work below a condition p in c which says that κ_i is on the generically chosen sequence. In particular, C factors as $C_{<\kappa_i} \times C_{\geq\kappa_i}$, with the associated generics $c_{<\kappa_i} \times c_{\geq\kappa_i}$, where $C_{<\kappa_i}$ is the product (below a condition chosen by c) $\mathrm{Sacks}(\omega,\kappa_1) \times \mathrm{Sacks}^{\omega_1}(\kappa_1^{++},\kappa_2) \times \ldots \times \mathrm{Sacks}^{\omega_1}(\kappa_{i-1}^{++},\kappa_i)$. $C_{\geq\kappa_i}$ is the rest of the forcing; note that $C_{\geq\kappa_i}$ is κ_i^{++}-closed in the direct order relation $\leq^*$ (see [8]) and does not add new objects in $H(\kappa_i^{++})$ of $V[G*g*c_{<\kappa_i}]$; in particular no κ_i-trees. It follows that T exists in $V[G*g*c_{<\kappa_i}]$.

Let us write $C_{<\kappa_i}$ as $C_{<\kappa_{i-1}} \times C_{\kappa_{i-1}}$ where $C_{\kappa_{i-1}} = \mathrm{Sacks}^{\omega_1}(\kappa_{i-1}^{++},\kappa_i)$; and similarly for the generics, $c_{<\kappa_i} = c_{<\kappa_{i-1}} \times c_{\kappa_{i-1}}$.[6] We can write $j_i(C_{<\kappa_i})$ as $C_{<\kappa_{i-1}} \times (C_{\kappa_{i-1}} * Q)$, where $Q = \mathrm{Sacks}^{\omega_1}(\kappa_{i-1}^{++}, j_i(\kappa_i))$. Let q be a Q-generic over $M_i[c_{<\kappa_{i-1}} \times c_{\kappa_{i-1}}]$ (we need to force q over $V[G*g*c]$). Then we can lift in $V[G*g*c*q]$ to

$$j_i : V[G*g*(c_{<\kappa_{i-1}} \times c_{\kappa_{i-1}})] \to M_i[c_{<\kappa_{i-1}} \times (c_{\kappa_{i-1}} * q)]. \tag{6.42}$$

The tree T is in $V[G*g*(c_{<\kappa_{i-1}} \times c_{\kappa_{i-1}})$, and also in $M_i[c_{<\kappa_{i-1}} \times c_{\kappa_{i-1}}]$. By (6.42), $j_i(T)$ is in $M_i[c_{<\kappa_{i-1}} \times (c_{\kappa_{i-1}} * q)]$, and $T = j_i(T)$ restricted o κ_i has a cofinal branch in $M_i[c_{<\kappa_{i-1}} \times (c_{\kappa_{i-1}} * q)]$.

Sublemma 6.21 *Every cofinal branch in* T *which is in* $M_i[c_{<\kappa_{i-1}} \times (c_{\kappa_{i-1}} * q)]$ *is already in* $M_i[c_{<\kappa_{i-1}} \times c_{\kappa_{i-1}}]$.

Proof. Notice that the forcing $C_{<\kappa_{i-1}} \times (C_{\kappa_{i-1}} * Q)$ is equivalent to $C_{\kappa_{i-1}} * (C_{<\kappa_{i-1}} \times Q)$ because $C_{\kappa_{i-1}}$ is sufficiently closed and therefore does not change $C_{<\kappa_{i-1}}$. Now we are done by Product lemma 4.6, applied over $M_i[c_{\kappa_{i-1}}]$ to $C_{<\kappa_{i-1}}$ and Q: $C_{<\kappa_{i-1}}$ has the κ_{i-1}-cc, and Q is the iteration $\mathrm{Sacks}^{\omega_1}(\kappa_{i-1}^{++}, j_i(\kappa_i))$ which satisfies the κ_{i-1}^{++}-fusion. ☐

This ends the proof of Lemma 6.20. ☐

The hard part of the proof is to show that the tree property holds at every $\aleph_{4i}$; we will spend the rest of the section with the proof.

Lemma 6.22 *The tree property holds at each* $\aleph_{4i}$ *for* $i > 0$ *in* $V[G*g*c]$.

Proof. Fix $\mu = (\aleph_{4i})^{V[G*g*c]} = (\kappa_i^{++})^{V[G*g]}$ = the least strongly measurable above κ_i in V. CHECK: whole proof

Work below a condition in C which determines that $\kappa_1 < \ldots < \kappa_{i+1}$ are on the generically chosen sequence.

---

[6] If $i = 1$, we identify κ_{i-1}^{++} with the cardinal $\aleph_0$, and $\mathrm{Sacks}^{\omega_1}(\kappa_{i-1}^{++},\kappa_i)$ with $\mathrm{Sacks}(\omega,\kappa_1)$.

Let us write $C_{<\kappa_i} \times C_{\kappa_i}$ for the forcing $\mathrm{Sacks}(\omega, \kappa_1) \times \mathrm{Sacks}^{\omega_1}(\kappa_1^{++}, \kappa_2) \times \ldots \times \mathrm{Sacks}^{\omega_1}(\kappa_i^{++}, \kappa_{i+1})$, where C_{κ_i} denotes the last forcing in the product. Let us denote $c_{<\kappa_i} \times c_{\kappa_i}$ the associated generic.

Assume for contradiction that $P * (C_{<\kappa_i} \times C_{\kappa_i})$ forces there is a μ-Aronszajn tree in the generic extension. Recall that P factors as $P_{<\mu} = P_{\kappa_i} * \dot{Q}_{\kappa_i}$ followed by the tail forcing P_{tail}. The forcing $\dot{Q}_{\kappa_i}$ collapses μ to κ_i^{++}. Let us denote the associated generics $G_{<\mu} = G_{<\kappa_i} * g_{\kappa_i}$, and G_{tail}.

Notice that the offending tree is already in $V[G_{<\mu} * (c_{<\kappa_i} \times c_{\kappa_i})]$: by the κ_{i+1}-closure of P_{tail}, $C_{<\kappa_i} \times C_{\kappa_i}$ is the same in $V[G * g]$ as in $V[G_{<\mu}]$ and we can find a $C_{<\kappa_i} \times C_{\kappa_i}$-name for the tree which is already present in $V[G_{<\mu}]$ (because a nice name for the tree is determined by a sequence of conditions in $C_{<\kappa_i} \times C_{\kappa_i}$ of length less than κ_{i+1}, and all such sequences are already in $V[G_{<\mu}]$). It follows that already $P_{<\mu} * (C_{<\kappa_i} \times C_{\kappa_i})$ forces there is a μ-Aronszajn tree.

Remark 6.23 The proof would be much easier if we could assume that the Aronszajn tree is over $V[G_{<\mu}]$ actually added by some small subforcing of C_{κ_i} (times $C_{<\kappa_i}$); however it is not true: one can show that for every $\delta < \kappa_{i+1}$ (the length of the iteration of C_{κ_i}), there is a subset of μ not added before the stage δ. Therefore we need to use a LS-type argument and work with smaller models.

The fact that the forcing $P_{<\mu} * (C_{<\kappa_i} \times C_{\kappa_i})$ adds a μ-Aronszajn tree is reflected in an elementary submodel M of $H(\bar{\kappa})^V$, for some large regular $\bar{\kappa}$, such that M is closed under μ-sequences, and has size μ^+. In particular, there is some $\Delta < \mu^{++}$ such that $(P_{<\mu} * (C_{<\kappa_i} \times \mathrm{Sacks}^{\omega_1}(\mu, \Delta))^M$ forces inside M that there is a μ-Aronszajn tree (note that in M, Δ may be larger than $(\mu^{++})^M$). By strong measurability of μ in V, we can choose a measure U such that the canonical embedding derived from U sends κ above Δ.

No small bound on Δ in M

Consider the external ultrapower of M by U. Let $k : M \to N$ be the canonical ultrapower embedding. See Corollary 6.27 for more details about the properties of k and for the details concerning the rest of the paragraph. Let $G_{<\mu} * (c_{<\kappa_i} \times a')$ be a $P_{<\mu} * (C_{<\kappa_i} \times \mathrm{Sacks}^{\omega_1}(\mu, \Delta))$-generic over V (and hence also M; note also that by Corollary 6.27 the forcing is the same in V and M). By our assumption, there is an Aronszajn tree T on μ in $M[G_{<\mu} * (c_{<\kappa_i} \times a')]$. Now we will successively lift k and argue that the existence of such T in $M[G_{<\mu} * (c_{<\kappa_i} \times a')]$ is impossible.

The forcing $k(P_{<\mu})_{<\mu}$ is in N equal to $P_{<\mu}$. It follows that we can start lifting by considering the generic $G_{<\mu}$ for $k(P_{<\mu})_{<\mu}$. At stage μ, $k(\dot{Q}_{\kappa_i})_\mu = \dot{R}_\mu$ is the iteration $(\mathrm{Sacks}^{\omega_1}(\mu, \gamma))$, for some generically chosen $\gamma < k(\mu)$. Choose a condition which forces that γ is equal to Δ. Then by Corollary 6.27, a' is $(\dot{R}_\mu)^{G_{<\mu}}$ generic over $N[G_{<\mu}]$. For future use, denote $A' = (\dot{R}_\mu)^{G_{<\mu}}$.

Now consider the iteration $k(\dot{Q}_{\kappa_i})$ in the interval $(\mu, k(\mu))$ and denote it $\bar{A}$; let $\bar{a}$ be any generic for $\bar{A}$ over $V[G_{<\mu}][a']$. By standard arguments, one lifts in $V[G_{<\mu} * (c_{<\kappa_i} \times a')][\bar{a}]$ to

$$k : M[G_{<\mu} * c_{<\kappa_i}] \to N[G_{<\mu} * a' * \bar{a} * c_{<\kappa_i}]. \tag{6.43}$$

We wish to lift one step further to

$$k : M[G_{<\mu} * (c_{<\kappa_i} \times a')] \to N[G_{<\mu} * a' * \bar{a} * (c_{<\kappa_i} \times k(a'))], \tag{6.44}$$

where $k(a')$ contains the pointwise image of a' under k.

Sublemma 6.24 *There exists in $V[G_{<\mu} * (c_{<\kappa_i} \times a')][\bar{a}]$ a $k(A')$-generic over $N[G_{<\mu} * a' * \bar{a}]$, to be denoted as a'', which contains the pointwise image $k[a']$. It follows that k lifts as in (6.44), with $a'' = k(a')$.*

Proof. Work in $M[G_{<\mu}]$ where A' is defined. Recall that we have lifted to $k : M[G_{<\mu}] \to N[G_{<\mu} * a' * \bar{a}]$, and that every element of the target model is of the form $k(f)(\mu)$ for some $f : \mu \to M[G_{<\mu}]$, $f \in M[G_{<\mu}]$. Fix a diamond sequence $\langle S_\alpha \subseteq \alpha \times \alpha : \alpha \in \mathrm{cof}(\omega) \cap \mu\rangle$ on $\mu \times \mu$, concentrating on ordinals with countable cofinality.

We will define $a''_{<\gamma}$ by induction on $\gamma < k(\Delta)$, and finally set $a'' = a''_{<k(\Delta)}$. The technical details are very much like in Lemma 6.9 so we limit ourselves here to stating the main steps; for notation, refer to Lemma 6.9 as well.

We first define a''_0. For every $\alpha < k(\mu)$, there is some $q \in a'$ such that $k(q)(0)$ does not split in the interval $[\mu, \alpha)$ (i.e. nodes with length in the interval $[\mu, \alpha)$ do not split). To find such q, choose some $\nu : \mu \to \mu$, $k(\nu)(\mu) > \alpha$, and construct below any p a condition $q \leq p$ such that:

(i) q is the fusion limit of $(\langle q_i : i < \mu\rangle, \langle F_i : i < \mu\rangle)$, and
(ii) For every $i < \mu$, $q_i(0)$ does not split in the interval $(i, \nu(i))$.

Since such q's are dense, there is some such q in a'. By the choice of ν, $k(q)(0)$ does not split in the interval (μ, α); since splitting is allowed only at cofinality ω_1, there is no splitting at μ, either. So $k(q)(0)$ does not split in $[\mu, \alpha)$. Since this procedure works for every $\alpha < k(\mu)$, this construction – together with $a'(0)$ – determines a unique cofinal branch d_0 through $k(p)(0)$ for all $p \in a'$: let $q_\alpha \in a'$ denote the condition such that $k(q_\alpha)(0)$ does not split in $[\mu, \alpha)$, $\alpha > \mu$, then

$$d_0 = \bigcup_{\alpha < k(\mu)} t_\alpha, \tag{6.45}$$

where t_α is the unique node in $k(q_\alpha)(0)$ of height α such that $t_\alpha | \mu = a'(0)$.

As the second step in constructing a''_0, we need to show how dense open sets are met. Let D be a dense open set in $k(A')(0)$; then for some η with domain μ and range in the dense open sets of A', D is equal to $k(\eta)(\mu)$, restricted to the 0-coordinate (and we assume that $k(\eta)(\mu)$ at the remaining coordinates is equal to all conditions). For any r' in A', construct $p \leq r'$ as a fusion limit of $(\langle p_i : i < \mu\rangle, \langle E_i : i < \mu\rangle)$, such that p_{i+1} meets dense open sets $\langle \eta(i') : i' < i\rangle$ with respect to all suitable sequences, defined as in Definition 6.10. Proceed analogously as in the construction leading up to (6.38). Since such p's are dense, there is some such p in a'. Consider now $k(p)$, with the k-image of the related fusion sequence: $(\langle p^*_i : i < k(\mu)\rangle, \langle E^*_i : i < k(\mu)\rangle)$. By elementarity, one can apply the ω-construction detailed in Sublemma 6.14, with d_0 instead of b_0. In particular, at $\bar{\alpha}$, obtained as in Sublemma 6.14, it holds that $k(p)(0)|(d_0|\bar{\alpha})$ meets D.

It follows that

$$(6.46) \qquad a''_0 = \{p' \in k(A')(0) \,:\, \exists p \in a' \; p' \geq k(p)(0)|a'(0)\}$$

is $k(A')(0)$-generic over $N[G_{<\mu} * a' * \bar{a}]$.

An analogue of Sublemma 6.15 can now be formulated and proved. In particular, if $\gamma < k(\Delta)$ and $\langle d_\beta \,:\, i < \gamma\rangle$ are unique branches determined as d_0 above, one can define $a''_{<\gamma}$ and d_γ as follows:

- (i) Given a dense open D in $k(A')(< \gamma)$, carry out the fusion construction $(\langle p_i \,:\, i < \mu\rangle, \langle E_i \,:\, i < \mu\rangle)$ with the fusion limit p detailed above for $k(A')(0)$. By elementarity, apply the construction in Sublemma 6.15, this time with the sequence $\langle d_\beta \,:\, \beta < \gamma\rangle$ instead of $\langle b_\beta \,:\, \beta < \gamma\rangle$. At $\bar{\alpha}$, if $k(p)(< \gamma)$ is thinned out to $\langle d_\beta|\bar{\alpha} \,:\, \beta \in \gamma \cap E^*_{\bar{\alpha}}\rangle$, then it meets D.
- (ii) Let $q_{\bar{\alpha}}$ in a' be a condition such that for each $\beta \in \gamma \cap E^*_{\bar{\alpha}}$, $k(q_{\bar{\alpha}})(< \beta)$ forces that $k(q_{\bar{\alpha}})(\beta)$ does not split in the interval $[\mu, \bar{\alpha})$. Such $q_{\bar{\alpha}}$ exists by an argument similar to the construction of the q_α's above, paying attention to $\langle E_i \,:\, i < \mu\rangle$.

The common lower bound r^+ of $q_{\bar{\alpha}}$ and p, which is also in a', does satisfy that $k(r^+)(< \gamma)|\langle d_\beta|\bar{\alpha} \,:\, \beta \in \gamma \cap E^*_{\bar{\alpha}}\rangle$ is defined and meets D.

It follows that

$$(6.47) \quad a''_{<\gamma} = \{p' \in k(A')(< \gamma) \,:\, \exists p \in a' \; \exists E \in M', E \subseteq \gamma \text{ of size } < j(\mu) \\ \exists \bar{\alpha} < j(\mu) \; p' \geq k(p)(< \gamma)|\langle d_\beta|\bar{\alpha} \,:\, \beta \in E\rangle\}$$

is $k(A')(< \gamma)$-generic over $N[G_{<\mu} * a' * \bar{a}]$.

Finally, as in Sublemma 6.15, we argue that the genericity of $a''_{<\gamma}$ ensures that we can define d_γ. There is a tiny point here: if γ is in $k''\Delta$, then d_γ is the composition of $a'(k^{-1}(\gamma))$ with the unique continuation up to $\bar{\alpha}$; if γ is not in $k''\Delta$, then $k(q_{\bar{\alpha}})$ on γ actually determines a unique branch in $2^{\bar{\alpha}}$. For details, see for instance [6] which discusses lifting at a successor in the supercompactness setting. □

By Corollary 6.27 and the fact that T can be viewed as a subset of μ, since T is in $M[G_{<\mu} * (c_{<\kappa_i} \times a')]$, it follows that T is also in $N[G_{<\mu} * (c_{<\kappa_i} \times a')]$. By (6.44), T has a cofinal branch in $N[G_{<\mu} * a' * \bar{a} * (c_{<\kappa_i} \times k(a'))]$. We want argue now that a new cofinal branch cannot be added in the extension from the first model to the second – this would be the final contradiction because then the branch is already in $N[G_{<\mu} * (c_{<\kappa_i} \times a')]$, and hence in $M[G_{<\mu} * (c_{<\kappa_i} \times a')]$ (again because by Corollary 6.27, $N[G_{<\mu} * (c_{<\kappa_i} \times a')]$ and $M[G_{<\mu} * (c_{<\kappa_i} \times a')]$ have the same subsets of μ), contradicting that T is Aronszajn. CHECK

Sublemma 6.25 *Every cofinal branch in T which is in $N[G_{<\mu} * a' * \bar{a} * (c_{<\kappa_i} \times k(a'))]$ is already in $N[G_{<\mu} * (c_{<\kappa_i} \times a')]$.*

Proof. First notice that $k(A')$ is $k(\mu)$-distributive over $M[G_{<\mu} * a' * \bar{a} * c_{<\kappa_i}]$, so cannot add a new branch to T. So it suffices to argue that any branch in $M[G_{<\mu} * a' * \bar{a} * c_{<\kappa_i}]$ is already in $M[G_{<\mu} * (c_{<\kappa_i} \times a')]$. Note that the forcing $P_{<\mu} * A' * \bar{A} * C_{<\kappa_i}$ is equivalent to $P_{<\mu} * A' * (\bar{A} \times C_{<\kappa_i})$. Now the result follows by Product lemma 4.6, applied over $M[G_{<\mu} * a']$ to forcings $C_{<\kappa_i}$ and $\bar{A}$: $C_{<\kappa_i}$ has the κ_i-cc, and $\bar{A}$ is an iteration composed of $\mathrm{Sacks}^{\omega_1}(\kappa_i)$ and κ_i^+-closed forcings and therefore satisfies κ_i-fusion. □

This finishes the proof of Lemma 6.22. □

This finishes the proof of Theorem 6.1.

6.1 Some facts concerning elementary submodels and the Sacks forcing

The Lemmas stated in this section are used in the proof of the main Theorem 6.1. We have placed them in a separate section here to keep the proof of Theorem 6.1 as clear as possible.

CHECK: whole section

Lemma 6.26 *(GCH) Assume $\omega_1 < \mu = \mu^{<\mu}$ is a successor cardinal of a regular cardinal. Let S be the iteration* $\mathrm{Sacks}(\mu, \alpha)$ *for some $\alpha < \mu^{++}$. Let M be an elementary transitive submodel of some large $H(\theta)$ (e.g. $\theta > \mu^{+3}$) of size μ^+ which contains α as an element and is closed under μ-sequences. Denote S^M the iteration* $\mathrm{Sacks}(\mu, \alpha)$ *in the sense of M. Then S^M is a dense suborder of S and so S^M and S have isomorphic Boolean completions.*

Proof. The proof proceeds by induction on $\beta < \alpha$.

Since M is closed under μ-sequences, all perfect μ-trees are in M, so the lemma holds for $\alpha = 1$.

By our cardinal arithmetics assumptions, the principle $\diamondsuit_\mu(E^\omega_\mu)$ holds in V. Fix a diamond sequence $\langle S_\alpha : \alpha < \mu \rangle$ on $\mu \times \mu$ concentrating on the ordinals with countable cofinality; by the μ-closure of M, this diamond sequence is in M. We temporarily say that a fusion sequence in $S(\mu, \beta)$ is a *strong fusion sequence* if it is built precisely as the sequence $(\langle p_i^\alpha : \alpha < \mu \rangle, \langle F_i^\alpha : \alpha < \mu \rangle)$, whose construction is described form the paragraph after Remark 6.11 up to the Remark 6.12. (Note that our μ is denoted κ^{++} in that section.) The only difference is the choice of the dense open sets to meet − here, the dense open sets will be given by the conditions which decide the value of $\dot{q}(\alpha)$ for $\alpha < \mu$ (see the next paragraph).

Assume $S^M(\mu, \beta)$ is a dense suborder of $S(\mu, \beta)$, $\beta < \alpha$. We want to show that $S^M(\mu, \beta+1)$ is a dense suborder of $S(\mu, \beta+1)$. Let $(p, \dot{q})$ in $S(\mu, \beta+1)$ be given; by our induction assumption, we can w.l.o.g. assume that p is in $S^M(\mu, \beta)$. We wish to find $(p', \dot{q}')$ in $S^M(\mu, \beta + 1)$ below $(p, \dot{q})$. In the rest of the proof we identify the name $\dot{q}$ with a name for sequence of 0's and 1's in μ (any μ-tree can be viewed that way).

Work in V, and in the forcing $S^M(\mu, \beta)$, build a strong fusion sequence with the limit $p' \leq p$ which determines the name $\dot{q}$ in the sense that in constructing the sequence we want the meet the dense open sets $D_\alpha = \{p : p \text{ decides } \dot{q}(\alpha)\}$, for $\alpha < \mu$.[7] Build a name $\dot{q}'$ as follows:

$$\dot{q}' = \bigcup\{\{\alpha\} \times A_\alpha \,:\, \alpha < \mu\}, \tag{6.48}$$

where A_α is the set of all conditions of the form "p' restricted to a suitable sequence (see Definition 6.10, and the construction between Remark 6.11 and Remark 6.12) such that the restriction forces that $\dot{q}(\alpha)$ is equal to 1." Clearly, $\dot{q}'$ is in M because it can be viewed as a μ-sequence of elements in M, and is forced by the weakest condition in $S^M(\mu, \beta)$ to be a perfect μ-tree in M. We wish to show that p' forces that $\dot{q}$ is equal to $\dot{q}'$, and so $(p', \dot{q}') \leq (p, \dot{q})$ as desired.

Let G be $S^M(\mu, \beta)$ generic over V which contains p' and fix $\alpha < \mu$: we wish to show that $\alpha \in \dot{q}^G \leftrightarrow \alpha \in (\dot{q}')^G$. Assume $\alpha \notin \dot{q}^G$; then no r in A_α can be in G because all conditions in A_α force that α is in $\dot{q}$. Conversely, assume $\alpha \in \dot{q}^G$. Then there is some $p_0 \leq p'$ in G which forces it. We will argue that there is some p^* in G and some $r \in A_\alpha$ such that $p^* \leq r$ so that $\alpha \in (\dot{q}')^G$. The details here are very similar to the argument in Sublemma 6.15. The intuition for the proof is the following: in trying to find p^* in G which is stronger than some condition in A_α, we in fact need to guess in constructing the fusion sequence an initial segment of the $S(\mu, \beta)$-generic filter (on some support of size $< \mu$) (this initial segment is given by p^*, and the condition which guesses it is the desired $r \geq p^*$): since the diamond sequence can only guess sets in the ground model, we need to allow some mistakes in guessing – this is the reason for the suitable sequences (see Definition 6.10) and the more complicated form of the fusion argument.

Limit $\beta < \alpha$. Since the support of $S(\mu, \beta)$ has size μ, the density of $S^M(\mu, \beta)$ in $S(\mu, \beta)$ for a limit β follows easily by the μ-closure of M in V. ☐

Corollary 6.27 *(GCH) Let $\alpha < \mu^{++}$. Assume μ is measurable and this is witnessed by an embedding j that $j(\mu) > \alpha$. Assume M is a transitive submodel of size μ^+ of some large $H(\theta)$ which is closed under μ-sequences and contains α is an element. Let P be a reverse Easton iteration which is a subset of V_μ, has the μ-cc and forces "$\mu^{<\mu} = \mu > \omega_1$ is a successor of a regular cardinal." Let U be the normal measure derived from j. Then the following hold:*

- *(i) The ultrapower of M by the (external) measure U is well-founded. Let N be the transitive collapse of M, and $k : M \to N$ the embedding. It holds that $k(\mu) > \alpha$.*
- *(ii) Let G be P-generic over V (and hence also over M). Let $S = \mathrm{Sacks}(\mu, \alpha)$ be defined in $V[G]$, and let S^M and S^N be the relativizations of the definition to $M[G]$ and $N[G]$, respectively. It holds that S, S^M, S^N all have isomorphic Boolean completions (in $V[G]$).*
- *(iii) Moreover, if g is S-generic over $V[G]$, then all subsets of μ which are in $N[G][g]$ are also in $M[G][g]$, and conversely.*

Proof. (i). Please consult [11] for more details about external ultrapowers (i.e. ultrapowers by filters U's which are not elements of the respective models).

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[7] Note that by the μ-closure of M in V, the whole sequence built externally in V is actually in M; the reason for the external construction is that the name $\dot{q}$ itself – even if we took a nice name – may not be in M ($S^M(\mu, \beta)$ is only μ^{++}-cc and so nice names for perfect μ-trees are in general too big to be automatically in M).

Note that the pair (M, U) is amenable because M is closed under μ-sequences and so we can take the ultrapower of M by the (external) measure U; let N be the ultrapower. By the σ-closure of U in the real universe, N is well-founded (and we identify it with its transitive collapse). Note also that $N = \{k(f)(\mu) : f \in M^\kappa \cap M\}$, where k is the canonical embedding from M to N, and M and N have the same subsets of κ. By the μ-closure of M in the real universe, all function $f : \mu \to \mu$ which are in V are also in M, and so $k(\mu) = j(\mu) > \alpha$.

(ii). Since $P \in M$ and P has the μ-cc, $M[G]$ is still closed under μ-sequences in $V[G]$. As k is the identity on μ, we can identify $k(P)(< \mu)$, the iteration $k(P)$ up to μ, with P, and so $P \in N$. By the μ-cc of P, $N[G]$ is still closed under μ-sequences in $M[G]$, and hence also in $V[G]$. In particular, the subsets of μ in $M[G]$ and $N[G]$ coincide. Applying the argument in Lemma 6.26, which essentially uses the μ-closure of the respective models, one can see that S, S^M, S^N all have isomorphic Boolean completions.

(iii). Using the fusion properties of S, one can show, similarly as in Lemma 6.26, that any subset of μ can be coded modulo g with an S-name of size μ; by the μ-closure it follows that $M[G][g]$ and $N[G][g]$ have the same subsets of μ. □

7 Open questions

CHECK: whole section

It is natural to ask whether the argument in this paper does not give more than just the failure of SCH at $\aleph_\omega$. Perhaps in the model we have constructed the tree property holds at $\aleph_{\omega+2}$, especially when one takes into account Lemma 6.7. We think that this is probably true, but we cannot prove it. The problem is with the key (unnumbered) Claim in [5], top of page 487, and the relevant quotient analysis. The proof of the Claim seems to require the full κ^{+++}-closure of the guiding forcing, and not the weaker fusion closure of our κ^{++}-Sacks forcing (see Lemma 6.9 for the definition of the guiding forcing).[8]

Question 1. Is it consistent to have the tree property at every $\aleph_{2n}$, $0 < n < \omega$, and also at $\aleph_{\omega+2}$ ($\aleph_\omega$ strong limit)?

Note that an easy variant of the above proof – which is actually much simpler at certain places – ensures the tree property at $\aleph_{\omega+2}$ if we use a guiding forcing of the form $\mathrm{Sacks}^{\omega_1}(\kappa^{+++}, j(\kappa))$. Then they key Claim in [5] goes through. However, we pay the price of getting the tree property not at every other cardinal below $\aleph_\omega$, but at cardinals $\aleph_2, \aleph_5, \aleph_7, \aleph_{10}, \ldots$, i.e. we get the 2+3 pattern (the gap 3 is caused by the guiding forcing starting at κ^{+++}, and not at κ^{++}).

Perhaps less interesting is to ask whether we really need the strongly measurable cardinal above κ. After all, since we get only the failure of SCH, a (κ, κ^{++})-extender embedding might suffice. We did not attempt to use this assumption because the setup of iteration seems to force us to use the strongly measurable cardinal above κ. However, it is worth stating it as an open question:

Question 2. Is it possible to prove the theorem with a weaker starting assumption on κ?

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[8] We thank S. Unger for bringing this point to our attention.

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