Easton's theorem and large cardinals from the optimal hypothesis

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Abstract In this paper we prove the equiconsistency of the assumption that there is a measurable cardinal κ with the Mitchell order $o(\kappa) = \kappa^{++}$ with the statement that there exists a measurable cardinal κ violating GCH such that the continuum function on regular cardinals below κ can be anything not outright inconsistent with the measurability of κ (see Corollary 3.2).

This settles the question of the optimal large cardinal strength needed to not only violate GCH at a measurable cardinal κ, but also determine the continuum function below κ, generalizing [3] to a large-cardinal setting.

Keywords: Easton's theorem, large cardinals, Mitchell order.

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1 Introduction

As shown by W. Mitchell and M. Gitik, see [8] and [6], the following holds:

(*) The failure of GCH at a measurable cardinal is equiconsistent with the existence of a measurable cardinal of Mitchell order κ^{++}: $o(\kappa) = \kappa^{++}$.

It seems to be a reasonable assumption that (*) can be generalized for example to:

(**) The existence of κ such that $o(\kappa) = \kappa^{++}$ is equiconsistent with the existence of κ such that $2^\kappa = \kappa^{++}$, κ is measurable, and for all regular cardinals $\alpha < \kappa$, $2^\alpha = \alpha^{++}$.

Though (**) may seem to be only a minor generalization of (*), it is not so, and as a matter of fact (**) was not resolved in [6], nor [4]. (**) is a paradigmatical case of a still more general question which studies the possible behaviours of the continuum function with respect to large cardinals (large cardinals restrict the continuum function beyond the ZFC conditions identified in [3]). See the paper [4] for details.

The interesting direction in the equivalence (**), which does not follow from (*), is from the left to the right. We have shown the analogous implication in [4] from the slightly stronger assumption of κ being κ^{++}-strong (see Section 1.1 for definitions).

In Theorem 3.1, we show that (**) is true; in Corollary 3.2, we generalize (**) along the lines of [3]. The proof is structured as follows. By the construction in [6], the assumption $o(\kappa) = \kappa^{++}$ together with GCH implies that there exists a generic extension V^* satisfying GCH and an elementary embedding $j : V^* \to M$ with the critical point κ, which satisfies:

(i) M is closed under κ-sequences in V^*,
(ii) $(\kappa^{++})^M = \kappa^{++}$.

In this paper, we will call such a j a κ^{++}-correct embedding (to our knowledge, no terminology is as yet fixed for embeddings satisfying (i) and (ii)).

We start with GCH and a κ^{++}-correct embedding j and define a cofinality-preserving forcing $\mathbb{R}$ such that in $V^{\mathbb{R}}$:

(i) $2^\alpha = \alpha^{++}$ for every regular cardinal $\alpha \leq \kappa$, and
(ii) κ is measurable.

We achieve (ii) by lifting the embedding j to $V^{\mathbb{R}}$. The crucial step in the lifting argument, see Theorem 2.1, where the weaker assumption of κ^{++}-correctness needs to be taken into account, concerns the V-regular cardinal $(\kappa^{++})^M = \kappa^{++}$. Since κ^{++} is regular in M, the forcing iteration $j(\mathbb{R})$ is non-trivial at κ^{++}. Since the entire $H(\kappa^{++})$ may not be included in M, the forcing at κ^{++} in $j(\mathbb{R})$ (which typically uses conditions in $H(\kappa^{++})$) tends to behave erratically in the universe.

Inspired by U. Abraham's paper [1], we show that if we include in $\mathbb{R}$ some preparatory forcing, we can "force" $j(\mathbb{R})$ to behave properly at κ^{++}. In some sense, this preparatory forcing makes a κ^{++}-correct embedding look more like a κ^{++}-strong embedding, as far as the proximity of $H(\kappa^{++})^M$ to the real $H(\kappa^{++})$ is concerned. Once the problematic step of κ^{++} is resolved, the rest of the lifting to $V^{\mathbb{R}}$ is standard.

1.1 Terminology

We say that κ is a θ-strong cardinal, where $\kappa < \theta$ and θ is a cardinal, if there exists an elementary embedding j from V into some transitive class M with the critical point κ such that $j(\kappa) > \theta$, and $H(\theta)$ is included in M. If GCH is assumed, and θ is regular (this is sufficient for our purposes here), then the elementary embedding witnessing the θ-strength of κ can be taken to have the additional property that $M = \{j(f)(\alpha) \mid f : \kappa \to V, \alpha < \theta\}$, $\theta < j(\kappa) < \theta^+$, and M is closed under κ-sequences in V (such a j is called an extender ultrapower embedding). See [2] for details.

We will focus in this paper on the case when $\theta = \kappa^{++}$.

If we omit the condition that $H(\theta)$ is included in M, we obtain a weaker notion.

Definition 1.1 *Assume GCH. We say that $j : V \to M$ with the critical point κ is a κ^{++}-*correct embedding *if j satisfies:*

(i) $\kappa^{++} = (\kappa^{++})^M$,
(ii) M is closed under κ-sequences in V.

*If j is κ^{++}-correct, one can use the usual extender ultrapower construction to get an even better embedding. We call j a κ^{++}-*correct extender embedding *if j satisfies conditions (i)–(iii) above, and moreover satisfies:*

(iv) $M = \{j(f)(\alpha) \mid f : \kappa \to V, \alpha < \kappa^{++}\}$.

*We say that κ is κ^{++}-*correct *if there is a κ^{++}-correct embedding with the critical point κ.*

Fact 1.2 shows that in some sense a κ^{++}-strong embedding is substantially stronger than a κ^{++}-correct embedding.

Fact 1.2 *(GCH) While there are κ-many measurable cardinals below every κ^{++}-strong cardinal κ, a κ^{++}-correct cardinal κ may be the least measurable cardinal.*

Let us note that Fact 1.2 is implicit in the proof of the main theorem in [6].

The weaker notion of a κ^{++}-correct embedding is important because it has the same strength (in terms of consistency) as the assumption that κ has the Mitchell order κ^{++}: $o(\kappa) = \kappa^{++}$ (see [6]).

We know provide a quick review of the notion of lifting of embeddings.

Fact 1.3 *Let $\mathbb{P}$ be a forcing notion and $j : V \to M$ an embedding with the critical point κ. Then the following holds (for proofs, see [2]):*

(i) (Silver) Assume G is $\mathbb{P}$-generic over V and H is $j(\mathbb{P})$-generic over M such that $j[G] \subseteq H$. Then there exists an elementary embedding $j^* : V[G] \to M[H]$ such that $j^* \restriction V = j$, and $H = j^*(G)$. We say that j lifts to $V^{\mathbb{P}}$.

(ii) *If j is moreover an extender ultrapower embedding then if $\mathbb{P}$ is a κ^+-distributive forcing notion and G is $\mathbb{P}$-generic over V, then the filter G^* in $j(\mathbb{P})$ defined as*

$$G^* = \{q \mid \exists p \in G, j(p) \leq q\}$$

is $j(\mathbb{P})$-generic over M.

(iii) *If $j^* : V[G] \to M[H]$ is the lifting of j, then if j was an extender ultrapower embedding, so is j^*.*

2 The crucial step: κ^{++}

Theorem 2.1 captures the main idea of this paper. Theorem 3.1 and Corollary 3.2 are direct applications of Theorem 2.1 based on results in [5] and [4].

Theorem 2.1 *Assume GCH and let $j : V \to M$ be a κ^{++}-correct extender embedding with the critical point κ. Then there exists a cofinality-preserving forcing notion $\mathbb{P}$ such that if G is $\mathbb{P}$-generic, the following holds:*

(i) $2^\alpha = \alpha^{++}$ *for every regular cardinal $\alpha < \kappa$ which is a double successor of an inaccessible cardinal $\beta < \kappa$, where α is a double successor of β if $\alpha = \beta^{++}$.*

(ii) *The embedding j lifts to $j^* : V[G] \to M[j^*(G)]$, and j^* is a κ^{++}-correct extender embedding in $V[G]$.*

Proof. The proof of the theorem will be given in a sequence of lemmas.

For a regular cardinal α, we write $\mathrm{Add}(\alpha, 1)$ for the Cohen forcing which adds a single Cohen subset of α: a condition p is in $\mathrm{Add}(\alpha, 1)$ if and only if p is a function of size $< \alpha$ from α to 2. We also write $\mathrm{Add}(\alpha, 1)$ for the Cohen forcing viewed as adding a single Cohen function from α to α: a condition p is in $\mathrm{Add}(\alpha, 1)$ if and only if p is a function of size $< \alpha$ from α to α. These two descriptions are equivalent as far as forcing goes; we will use the one which is best suited for our purposes (we will indicate which representation we choose). If α is a regular cardinal and β an ordinal greater than 0, we write $\mathrm{Add}(\alpha, \beta)$ to denote the standard Cohen forcing which adds β-many subsets of α. A condition p belongs to $\mathrm{Add}(\alpha, \beta)$ if and only if p is a function from $\alpha \times \beta$ to 2 of size $< \alpha$. At times, we view $\mathrm{Add}(\alpha, \beta)$ as a β-product with $< \alpha$-support of the forcing $\mathrm{Add}(\alpha, 1)$. Again, these two descriptions of $\mathrm{Add}(\alpha, \beta)$ equivalent, and we will therefore use the one which best suits the given context (we will indicate which representation we have in mind).

Let us now define the forcing $\mathbb{P}$. $\mathbb{P}$ will be a two stage iteration $\mathbb{P}^0 * \dot{\mathbb{P}}^1$, where:

(1) $\mathbb{P}^0$ is an iteration of length κ with Easton support, $\mathbb{P}^0 = \langle(\mathbb{P}^0_\xi, \dot{Q}_\xi) \mid \xi < \kappa\rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless ξ is an inaccessible cardinal $< \kappa$, in which case

$$\mathbb{P}^0_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \mathrm{Add}(\xi^+, \xi^{++}) * \mathrm{Add}(\xi^{++}, \xi^{+4}),\text{”} \tag{2.1}$$

where $\mathrm{Add}(\xi^+, \xi^{++})$ is viewed as a product forcing which adds ξ^{++}-many Cohen functions from ξ^+ to ξ^+, and $\mathrm{Add}(\xi^{++}, \xi^{+4})$ is viewed as adding ξ^{+4}-many Cohen subsets of ξ^{++}.

(2) Notice that $\mathbb{P}^0$ is an element of M. $\dot{\mathbb{P}}^1$ is defined in M to be a $\mathbb{P}^0$-name which satisfies:

$$M \models \text{“}\mathbb{P}^0 \Vdash \text{“}\dot{\mathbb{P}}^1 \text{ is the forcing } \mathrm{Add}(\kappa^+, \kappa^{++}) * \mathrm{Add}(\kappa^{++}, 1),\text{””} \tag{2.2}$$

where $\mathrm{Add}(\kappa^+, \kappa^{++})$ is viewed as a product forcing which adds κ^{++}-many Cohen functions from κ^+ to κ^+, and $\mathrm{Add}(\kappa^{++}, 1)$ is viewed as adding a single Cohen subset of κ^{++}.

$\mathbb{P}$ is cofinality-preserving.

The forcing $\mathbb{P}^0$ is clearly cofinality-preserving by a standard argument. Let G_κ be a $\mathbb{P}^0$-generic filter over V. By κ^{++}-correctness of j, $\mathbb{P}^0$ is in M and G_κ is $\mathbb{P}^0$-generic over M. In order to verify that $\mathbb{P}$ is cofinality-preserving, it suffices to check that the forcing $(\dot{\mathbb{P}}^1)^{G_\kappa}$ defined in $M[G_\kappa]$ preserves cofinalities when forced over $V[G_\kappa]$. Notice first that $\mathrm{Add}(\kappa^+, \kappa^{++})$ of $M[G_\kappa]$ is the same set as $\mathrm{Add}(\kappa^+, \kappa^{++})$ of $V[G_\kappa]$: this is because $\mathbb{P}^0$ has κ-cc, and hence by standard arguments $M[G_\kappa]$ is still closed under κ-sequences in $V[G_\kappa]$. Let g be a $\mathrm{Add}(\kappa^+, \kappa^{++})^{V[G_\kappa]}$-generic over $V[G_\kappa]$. Then by the previous sentence, g is also $\mathrm{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $M[G_\kappa]$. Work in $M[G_\kappa * g]$ and let Q^* denote the forcing $\mathrm{Add}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$. Then the claim that $\mathbb{P}$ preserves cofinalities follows from the following lemma:

Lemma 2.2 *The forcing Q^* is still κ^{++}-distributive over $V[G_\kappa * g]$.*

Proof. First note that this lemma in non-trivial: if the original M missed some subsets of κ^+ in V, then Q^* is a proper subset of $\mathrm{Add}(\kappa^{++}, 1)$ of $V[G_\kappa * g]$, and hence Q^* is certainly not κ^{++}-closed over $V[G_\kappa * g]$.

We will argue that the preparatory forcing $\mathrm{Add}(\kappa^+, \kappa^{++})$ ensures that Q^* is still κ^{++}-distributive over $V[G_\kappa * g]$.

Let us work in $V[G_\kappa * g]$. Assume that $p \in Q^*$ is a condition and $\dot{f}$ is a name for a function from κ^+ to ordinals:

$$p \Vdash \dot{f} : \kappa^+ \to \mathrm{ORD}. \tag{2.3}$$

We will show that there exists $q \leq p$ which decides all values of $\dot{f}$.

Write $H(\kappa^{++})$ of $M[G_\kappa * g]$ as $L_{\kappa^{++}}[B]$ for some subset B of κ^{++}, B in $M[G_\kappa * g]$. This is possible because $H(\kappa^{++})$ of $M[G_\kappa * g]$ has size κ^{++} in $M[G_\kappa * g]$. Fix an elementary submodel N of some large enough $H(\theta)^{V[G_\kappa * g]}$ which has size κ^+, is transitive below κ^{++}, is closed under κ-sequences and contains as elements B, Q^*, p and $\dot{f}$. We will show that p has an extension $q \leq p$ which hits all dense subsets of Q^* which belong to N; this will imply that q decides all values of $\dot{f}$ as required.

Let β be the the ordinal $N \cap \kappa^{++}$ and let π be a transitive collapse of N to $\bar{N}$. Then $\pi(Q^*)$, which is equal to $Q^* \cap N$, belongs to $M[G_\kappa * g]$ because Q^* is definable in $L_{\kappa^{++}}[B]$, and so by π being an isomorphism, $\pi(Q^*)$ is definable in $L_{\pi(\kappa^{++})}[\pi(B)] = L_\beta[B \cap \beta]$. It suffices to extend $\pi(p) = p$ to a condition q which hits all dense subsets of $\pi(Q^*)$ which belong to $\bar{N}$.

For $\gamma < \kappa^{++}$, let us denote $g \restriction \gamma = \{q \in g \,|\, q \restriction \gamma = q\}$. Pick some $\gamma < \kappa^{++}$ such that $\bar{N}$ is in $V[G_\kappa * g \restriction \gamma]$, and $\pi(Q^*)$ and some enumeration $\langle p^*_\xi \,|\, \xi < \kappa^+\rangle$ of $\pi(Q^*)$ are in $M[G_\kappa * g \restriction \gamma]$. Such γ exists by κ^+-cc of the forcing $\mathrm{Add}(\kappa^+, \kappa^{++})$ and the fact that $\bar{N}$ is a transitive set of size κ^+. Let h be the generic function $\kappa^+ \to \kappa^+$ at the coordinate γ in g: $h = \{q(\gamma) \,|\, q \in g, \gamma \in \mathrm{dom}(q)\}$. So h is $\mathrm{Add}(\kappa^+, 1)$-generic over $V[G_\kappa * g \restriction \gamma]$. Note that $h \in M[G_\kappa * g]$.

Define inductively in $M[G_\kappa * g]$ a decreasing sequence of conditions $\langle p_\xi \,|\, \xi < \kappa^+\rangle$ with $p_0 = p$, $p_\lambda = \bigcup_{\xi<\lambda} p_\xi$ for λ a limit ordinal $< \kappa^+$, and:

$$p_{\xi+1} = \begin{cases} p^*_{h(\xi)} & \text{if } p^*_{h(\xi)} \text{ extends } p_\xi, \\ p_\xi & \text{otherwise.} \end{cases}$$

Since all the parameters used in this construction, i.e. the sequence $\langle p^*_\xi \,|\, \xi < \kappa^+\rangle$, and $h, \pi(Q^*), p$, are in $M[G_\kappa * g]$, so is the whole sequence $\langle p_\xi \,|\, \xi < \kappa^+\rangle$. Let q be the greatest lower bound of this sequence, $q = \bigcup_{\xi<\lambda} p_\xi$. Since $\langle p_\xi \,|\, \xi < \kappa^+\rangle$ is in $M[G_\kappa * g]$, $q \in Q^*$.

We will show in $V[G_\kappa * g \restriction \gamma][h]$ that the sequence $\langle p_\xi \,|\, \xi < \kappa^+\rangle$ is $(\bar{N}, \pi(Q^*))$-generic. This already implies that that q decides all the values of $\dot{f}$: For each $\xi < \kappa^+$, the set

$$D_\xi = \{p \in \pi(Q^*) \,|\, p \text{ decides } \pi(\dot{f})(\xi)\}$$

is a dense open set in $\pi(Q^*)$, which is an element of $\bar{N}$. If p_ζ for some $\zeta < \kappa^+$ meets D_ξ, then $p_\zeta = \pi^{-1}(p_\zeta)$ decides the value of $\dot{f}(\alpha)$, and so does $q \leq p_\zeta$.

The $(\bar{N}, \pi(Q^*))$-generocity is proved by using the generic h. Let D be a dense open set in $\pi(Q^*)$ which is an element of $\bar{N}$. We will show in $V[G_\kappa * g \restriction \gamma][h]$ that there is some p_ξ which meets D. To this end, it suffices to show that

$$\bar{D} = \{q \,|\, q \Vdash \text{“}\exists \xi < \kappa^+ \; p_\xi \in D\text{”}\}$$

is dense in $\mathrm{Add}(\kappa^+, 1)$ in $V[G_\kappa * g \restriction \gamma]$. Given a condition q, extend q first into some q' such that $\mathrm{dom}(q') = \delta$ for some $\delta < \kappa^+$; then q' decides the construction of $\langle p_\xi \mid \xi < \kappa^+ \rangle$ up to δ (because it decides h up to δ): for some $p' \in \pi(Q^*)$, $q' \Vdash p_\delta = p'$. Pick $p'' \leq p'$ in D. In the enumeration $\langle p^*_\xi \mid \xi < \kappa^+ \rangle$, p'' is some condition p^*_η. Set $q'' = q' \cup \{\langle \delta, \eta \rangle\}$. Then $q'' \Vdash$ "$p_{\delta+1}$ extends p_δ and meets D", and so $q'' \leq q$ is in $\bar{D}$. It follows that $\bar{D}$ is dense and the proof is finished. □

This shows that $\mathbb{P}$ is cofinality-preserving over V. We now show that the embedding j can be lifted to $V^{\mathbb{P}}$.

The embedding j lifts to j^*.

Let $G = G_\kappa * g * g'$ be a $\mathbb{P}$-generic over V, where G_κ is $\mathbb{P}^0$-generic, g is $\mathrm{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $V[G_\kappa]$, and g' is $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$-generic over $V[G_\kappa * g]$. We need to find a $j(\mathbb{P})$-generic H over M such that $j[G] \subseteq H$.

By κ^{++}-correctness of j, $j(\mathbb{P}^0)_\kappa = \mathbb{P}^0$, and so we start building H by plugging in G_κ as the $j(\mathbb{P}^0)_\kappa$-generic over M.

The next forcing in $j(\mathbb{P})$ above κ is $Q = \mathrm{Add}(\kappa^+, \kappa^{++}) * \mathrm{Add}(\kappa^{++}, \kappa^{+4})$ as defined in $M[G_\kappa]$. We need to find in $V[G]$ a Q-generic over $M[G_\kappa]$. It is here, where we make use of the preparatory forcing $\dot{\mathbb{P}}^1$: By definition of $\dot{\mathbb{P}}^1$, g is $\mathrm{Add}(\kappa^+, \kappa^{++})^{M[G_\kappa]}$-generic over $V[G_\kappa]$ (and hence over $M[G_\kappa]$). To complete the construction of a Q-generic, it remains to find some h, which will be $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $M[G_\kappa * g]$.

When we look at the generics at our disposal, the natural candidate for h is the generic filter g'. Clearly, g' will need to be tweaked a little because it is only $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$-generic over $V[G_\kappa * g]$, but not $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $V[G_\kappa * g]$. Note that there is a good reason for this apparent deficiency of g': While Lemma 2.2 shows that $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$ is sufficiently distributive over $V[G_\kappa * g]$, the forcing $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$ never is, in fact it collapses κ^{++}:

Observation 2.3 *Let γ be an ordinal $< j(\kappa)$ which has V-cofinality κ^+, and its cofinality in M is $> \kappa^+$. Then the forcing $\mathrm{Add}(\kappa^{++}, \gamma)^{M[G_\kappa * g]}$ collapses κ^{++} to κ^+ if forced over $V[G_\kappa * g]$.*

Proof. Notice that every M-regular cardinal in the interval $(\kappa^{++}, j(\kappa)]$ has V-cofinality κ^+ (this uses GCH in V, and the extender representation of M), and so setting $\gamma = (\kappa^{+4})^M$ achieves the desired claim. Fix X to be a cofinal subset of γ of order type κ^+. Now, for each $\zeta \in \kappa^{++}$ and every $p \in \mathrm{Add}(\kappa^{++}, \gamma)^{M[G_\kappa * g]}$, one can find $q \leq p$ and $\xi \in X$ such that q at the coordinate ξ codes ζ in the sense that it contains ζ-many 1's followed by 0. Hence it is dense that every $\zeta \in \kappa^{++}$ is coded at some element $\xi \in X$. □

Thus we have no choice but to try to use g' to obtain the desired generic filter h. A priori, this strategy may work because $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$ is small in $V[G_\kappa * g * g']$: it has size κ^{++} here. The following Lemma suggests that perhaps $\mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$ can be "stretched" to $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$.

Lemma 2.4 *Let $j : V \to M$ be a κ^{++}-correct extender ultrapower embedding. Let $\gamma = (\kappa^{+4})^M$. Then:*

(i) There exists in V a bijection $\pi : \gamma \to \kappa^{++}$ which is locally M-correct *in the sense that whenever $X \subseteq \gamma$ is in M and has in M size $\leq \kappa^{++}$, the restriction $\pi \restriction X$ is also in M.*

(ii) And more generally, if $\mathbb{R}$ is a forcing notion in M and $\mathbb{R}$ has κ^{+3}-cc in M, then the bijection π in (i) is $M^{\mathbb{R}}$-locally correct.

Proof. Ad (i). List all $f : \kappa \to \kappa$ in V as $\langle f_i \,|\, i < \kappa^+ \rangle$. For $\beta < \kappa^+$ let S_β denote the set of all $j(f_i)(\alpha)$, where $i < \beta$ and $\alpha < \kappa^{++}$, such that $j(f_i)(\alpha) < \gamma$:

$$S_\beta = \{ j(f_i)(\alpha) \,|\, i < \beta, \alpha < \kappa^{++}, j(f_i)(\alpha) < \gamma \}.$$

Then each S_β belongs to M, the S_β's form an increasing chain under inclusion, and the union $\bigcup \{ S_\beta \,|\, \beta < \kappa^+ \}$ is equal to γ. Moreover, if $X \in M$ is a subset of γ of size $\leq \kappa^{++}$ in M, then X is contained in some S_β. The sequence of S_β's can be used to construct a bijection $\pi' : \gamma \to \kappa^+ \times \kappa^{++}$ with the property that $\pi' \restriction X$ is in M for any X as above. If f is a bijection in M between $\kappa^+ \times \kappa^{++}$ and κ^{++}, then the composition $\pi = \pi' \circ f$ is the desired bijection.

Ad (ii). Let F be $\mathbb{R}$-generic over M. If X is a subset of γ in $M[F]$ which has size $\leq \kappa^{++}$ in $M[F]$, then by κ^{+3}-cc of $\mathbb{R}$ there is some $X' \supseteq X$ in M which has size $\leq \kappa^{++}$ in M. Then the claim follows by application of (i). $\square$

Note that the inverse function π^{-1} may not be "locally M-correct" in the sense of the previous Lemma. Indeed, if $\langle c_\xi \,|\, \xi < \kappa^+ \rangle$ is cofinal in $(\kappa^{+4})^M$, then for $X = \{ c_\xi \,|\, \xi < \kappa^+ \}$, the set $\pi[X]$ may be in M (for instance when κ is κ^{++}-strong), while $\pi^{-1}[\pi[X]] = X$ is certainly not in M.

We know show that the previous Lemma can be used to stretch the $\mathrm{Add}(\kappa^{++}, 1)$-generic g' over $V[G_\kappa * g]$ to a $\mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$-generic over $M[G_\kappa * g]$.

Let $Q^* = \mathrm{Add}(\kappa^{++}, 1)^{M[G_\kappa * g]}$, and $\tilde{Q} = \mathrm{Add}(\kappa^{++}, \kappa^{+4})^{M[G_\kappa * g]}$.

Lemma 2.5 *There exists in $V[G_\kappa * g * g']$ a $\tilde{Q}$-generic over $M[G_\kappa * g]$. Let us denote this generic as h.*

Proof. Let $\pi^* : \kappa^{++} \times (\kappa^{+4})^M \to \kappa^{++}$ be a bijection obtained by composing the bijection π from Lemma 2.4 with any bijection in M between $\kappa^{++} \times$

$(\kappa^{+4})^M$ and $(\kappa^{+4})^M$. Then π^* is locally $M[G_\kappa * g]$-correct in the sense of Lemma 2.4(ii). For $p \in \tilde{Q}$, write p^* to denote the image of p under π^*: $\mathrm{dom}(p^*) = \pi^*[\mathrm{dom}(p)]$, and for each (ξ, ζ) in the domain of p, $p^*(\pi(\xi, \zeta)) = p(\xi, \zeta)$. By the local $M[G_\kappa * g]$-correctness of π^*, each p^* is in $M[G_\kappa * g]$, and hence it is a condition in Q^*:

$$\{p^* \,|\, p \in \tilde{Q}\} \subseteq Q^*.$$

Note that the inclusion is proper because Q^* is κ^{++}-distributive over $V[G_\kappa * g]$, while $\tilde{Q}$ is not (see Observation 2.3).

Let us set

$$h = \{p \,|\, p^* \in g'\}.$$

We show that h is as required. Assume A lies in $M[G_\kappa * g]$ and it is a maximal antichain in $\tilde{Q}$, and so in particular A has size $\leq \kappa^{++}$ in $M[G_\kappa * g]$. Let us denote $\mathrm{dom}(A) = \bigcup\{\mathrm{dom}(p) \,|\, p \in A\}$. Let us write $A^* = \{p^* \,|\, p \in A\}$ and $\mathrm{dom}(A^*) = \bigcup\{\mathrm{dom}(p^*) \,|\, p^* \in A^*\}$; then A^* is an antichain in Q^* and $\pi^* \restriction \mathrm{dom}(A)$ is in $M[G_\kappa * g]$ by the local $M[G_\kappa * g]$-correctness of π^*. To show that h is as required, it suffices to show that A^* is a maximal antichain in Q^*. Let q be any condition in Q^*; since q is in $M[G_\kappa * g]$, the intersection $\mathrm{dom}(q) \cap \mathrm{dom}(A^*)$ is in $M[G_\kappa * g]$. Since $\pi^* \restriction \mathrm{dom}(A)$ is in $M[G_\kappa * g]$, the set $(\pi^*)^{-1}[\mathrm{dom}(q) \cap \mathrm{dom}(A^*)]$ is also in $M[G_\kappa * g]$. If q' denotes the condition in $\tilde{Q}$ with the domain $(\pi^*)^{-1}[\mathrm{dom}(q) \cap \mathrm{dom}(A^*)]$ defined by $q'(\xi, \zeta) = q(\pi^*(\xi, \zeta))$, then there exists by the maximality of A some $p \in A$ compatible with q'. It follows that $p^* \in A^*$ is compatible with q because it is compatible with q on $\mathrm{dom}(p^*) \cap \mathrm{dom}(q)$. Thus A^* indeed maximal, and h meets A as required. $\square$

Based on the previous Lemma, we see that $G_\kappa * g * h$ is $j(\mathbb{P}^0)_{\kappa+1}$-generic over M. The iteration $j(\mathbb{P}^0)$ in the interval $(\kappa + 1, j(\kappa))$ is κ^{+++}-distributive in $M[G_\kappa * g * h]$, and so all the relevant dense open sets in $M[G_\kappa * g * h]$ can be met in κ^+-many steps, using the extender representation of M (see [4] for details). Let the resulting generic be denoted as $\tilde{h}$. Then $G_\kappa * g * h * \tilde{h}$ is $j(\mathbb{P}^0)$-generic over M, and we can partially lift to

$$j' : V[G_\kappa] \to M[G_\kappa * g * h * \tilde{h}].$$

It remains to lift j' to $\mathbb{P}^1 = \mathrm{Add}(\kappa^+, \kappa^{++}) * \mathrm{Add}(\kappa^{++}, 1)$ of $M[G_\kappa * g]$. By Lemma 2.2, $\mathbb{P}^1$ is κ^+-distributive over $V[G_\kappa]$, and therefore by Fact 1.3, the filter $\tilde{\tilde{h}}$ generated by the j' image of $g * g'$ is $M[G_\kappa * g * h * \tilde{h}]$-generic over $j'(\mathbb{P}^1)$:

$$\tilde{\tilde{h}} = \{q \,|\, \exists p \in g * g', j'(p) \leq q\}.$$

If we define $H = G_\kappa * g * h * \tilde{h} * \tilde{\tilde{h}}$, then H is as required:

$$j^* : V[G_\kappa * g * g'] \to M[H].$$

This finishes the proof of Theorem 2.1. □

Remark 2.6 Note that the forcing $\mathbb{P}$ is trivial at κ, and hence the proof of Theorem 2.1 did not need to take special care to ensure the coherence condition $j[G] \subseteq H$ (except at G_κ, but this is satisfied automatically). See next section where the forcing at κ is non-trivial.

3 Easton's theorem and large cardinals from the optimal hypothesis

Theorem 3.1 *Assume GCH and let $j : V \to M$ be a κ^{++}-correct extender embedding with the critical point κ. Then there exists a cofinality-preserving forcing notion $\mathbb{R}$ such that if G is $\mathbb{R}$-generic, the following holds:*

(i) $2^\alpha = \alpha^{++}$ *for every regular cardinal* $\alpha \leq \kappa$.

(ii) The embedding j lifts to $j^* : V[G] \to M[j^*(G)]$, and j^* is a κ^{++}-correct extender embedding in $V[G]$. In particular, κ is still measurable.

Proof. Let $I(\kappa)$ denote the set of all inaccessible cardinals $< \kappa$, and $R(\kappa)$ the set of all regular cardinals $< \kappa$. Set $B = \{\alpha \in R(\kappa) \,|\, \exists \beta \in I(\kappa), \alpha = \beta \text{ or } \alpha = \beta^+\} \cup \{\kappa\}$, and $A = R(\kappa) \setminus B$. Then $A \cup B$ is the set of all regular cardinals $\leq \kappa$.

We define $\mathbb{R}$ as a two-stage iteration $\mathbb{R}_A * \dot{\mathbb{R}}_B$. $\mathbb{R}_A$ will be a cofinality-preserving forcing which will force the failure of GCH at every element in A. In $V^{\mathbb{R}_A}$, $\dot{\mathbb{R}}_B$ will be a cofinality-preserving forcing which will violate GCH at the remaining regular cardinals $\leq \kappa$, i.e. at the elements in B.

The definition of $\mathbb{R}_A$ is a modification of $\mathbb{P}$, as defined in Theorem 2.1. $\mathbb{R}_A$ is a two stage iteration $\mathbb{R}^0_A * \dot{\mathbb{R}}^1_A$, where:

(1) $\mathbb{R}^0_A$ is an iteration of length κ with Easton support, $\mathbb{R}^0_A = \langle (\mathbb{R}^0_A)_\xi, \dot{Q}_\xi) \,|\, \xi < \kappa \rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless ξ is a limit cardinal $< \kappa$, in which case there are two possibilities:

(a) If ξ is regular (and hence inaccessible), then

(3.4) $(\mathbb{R}^0_A)_\xi \Vdash$ "$\dot{Q}_\xi$ is the forcing
$$[\mathrm{Add}(\xi^+, \xi^{++}) * \mathrm{Add}(\xi^{++}, \xi^{+4})] \times \textstyle\prod_{\xi^{++} < \gamma < \xi^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++}),\text{"}$$

where $\mathrm{Add}(\xi^+, \xi^{++})$ is viewed as a product forcing which adds ξ^{++}-many Cohen functions from ξ^+ to ξ^+, $\mathrm{Add}(\xi^{++}, \xi^{+4})$ is viewed as adding ξ^{+4}-many Cohen subsets of ξ^{++}, and $\prod_{\xi^{++} < \gamma < \xi^{+\omega}} \mathrm{Add}(\gamma, \gamma^{++})$ is the standard Easton product, which adds γ^{++}-many Cohen subsets to each regular cardinal γ such that $\xi^{++} < \gamma < \xi^{+\omega}$ (where $\xi^{+\omega}$ is the least limit cardinal above ξ).

(b) If ξ is singular, then

$$(\mathbb{R}^0_A)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \prod_{\xi<\gamma<\xi^{+\omega}} \mathrm{Add}(\gamma,\gamma^{++}),\text{”} \tag{3.5}$$

where $\prod_{\xi^{++}<\gamma<\xi^{+\omega}} \mathrm{Add}(\gamma,\gamma^{++})$ is the standard Easton product, where γ ranges over regular cardinals.

(2) Notice that $\mathbb{R}^0_A$ is an element of M. $\dot{\mathbb{R}}^1_A$ is defined in M to be a $\mathbb{R}^0_A$-name which satisfies:

$$M \models \text{“}\mathbb{R}^0_A \Vdash \text{“}\dot{\mathbb{R}}^1_A \text{ is the forcing } \mathrm{Add}(\kappa^+,\kappa^{++}) * \mathrm{Add}(\kappa^{++},1),\text{””} \tag{3.6}$$

where $\mathrm{Add}(\kappa^+,\kappa^{++})$ is viewed as a product forcing which adds κ^{++}-many Cohen functions from κ^+ to κ^+, and $\mathrm{Add}(\kappa^{++},1)$ is viewed as adding a single Cohen subset of κ^{++}.

By standards argument, see [4], and Lemma 2.2 applied in the present context, the forcing $\mathbb{R}_A$ is cofinality-preserving. By [4], and an easy modification of Theorem 2.1, j lifts to a κ^{++}-correct extender embedding j' in $V^{\mathbb{R}_A}$: in the proof generalizing the proof of Theorem 2.1, one just needs to take into account the product $\prod_{\kappa^{++}<\gamma<\kappa^{+\omega}} \mathrm{Add}(\gamma,\gamma^{++})$ at the stage κ of the iteration $j(\mathbb{R}^0_A)$. However, since in $M^{j(\mathbb{R}^0_A)_\kappa}$, $\mathrm{Add}(\kappa^+,\kappa^{++}) * \mathrm{Add}(\kappa^{++},\kappa^{+4})$ has κ^{+3}-cc and the product $\prod_{\kappa^{++}<\gamma<\kappa^{+\omega}} \mathrm{Add}(\gamma,\gamma^{++})$ is κ^{+3}-closed, it follows by Easton's lemma that these two forcing are mutually generic. Accordingly, an $\mathrm{Add}(\kappa^+,\kappa^{++}) * \mathrm{Add}(\kappa^{++},\kappa^{+4})$-generic over $M^{j(\mathbb{R}^0_A)_\kappa}$ is obtained as in Theorem 2.1, while a $\prod_{\kappa^{++}<\gamma<\kappa^{+\omega}} \mathrm{Add}(\gamma,\gamma^{++})$-generic is obtained by a standard construction using the κ^{+3}-distributivity of the forcing.

Let G_A denote a $\mathbb{R}_A$-generic, then the following holds in $V[G_A]$:

(i) GCH holds in $V[G_A]$ at every inaccessible cardinal $\alpha \leq \kappa$ and at the successors of these inaccessible cardinals.
(ii) $2^\alpha = \alpha^{++}$ for every regular cardinal $< \kappa$ other than in (i) in the previous line.
(iii) There exists in $V[G_A]$ a κ^{++}-correct extender embedding $j' : V[G_A] \to M[j'(G_A)]$ which is a lifting of the original j.

In $V[G_A]$, we define $\mathbb{R}_B$ as follows.

$\mathbb{R}_B$ is an iteration of length $\kappa+1$ with Easton support, $\mathbb{R}_B = \langle (\mathbb{R}_B)_\xi, \dot{Q}_\xi) \mid \xi < \kappa+1\rangle$, where $\dot{Q}_\xi$ is a name for a trivial forcing unless ξ is an inaccessible cardinal $\leq \kappa$, in which case there are two cases:

(a) If $\xi < \kappa$, then

$$(\mathbb{R}_B)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing } \mathrm{Sacks}(\xi,\xi^{++}) \times \mathrm{Add}(\xi^+,\xi^{+3}),\text{”} \tag{3.7}$$

where $\mathrm{Sacks}(\xi^+,\xi^{++})$ is the generalized Sacks product forcing at ξ which adds ξ^{++}-many new subsets of ξ (see [7], and [5] for details), and $\mathrm{Add}(\xi^+,\xi^{++})$ is viewed as adding ξ^{+3}-many Cohen subsets of ξ^+.

(b) If $\xi = \kappa$, then

$$(\mathbb{R}_B)_\xi \Vdash \text{“}\dot{Q}_\xi \text{ is the forcing Sacks}(\xi, \xi^{++}) \times \text{Add}(\xi^+, 1).\text{”} \tag{3.8}$$

By standard results, see [4], $\mathbb{R}_B$ is cofinality-preserving over $V[G_A]$ (here, it is important that $\text{Add}(\xi^+, \xi^{+3})$ is still ξ^+-distributive over $\text{Sacks}(\xi, \xi^{++})$).

Let G_B be a $\mathbb{R}_B$-generic over $V[G_A]$. Using the “tuning-fork” argument in the original paper [5], together with [4], one can show that j' lifts to $V[G_A][G_B]$. Notice here that it is sufficient to add just one Cohen subset of κ^+, cf. (3.8), in order to lift, and so GCH holds in $V[G_A][G_B]$ above κ (if so desired).

If we set $G = G_A * G_B$, then $V[G]$ is as required. □

We can achieve even more generality, along the lines [3] and [4]. We say that a proper-class function F from regular cardinals into cardinals is an *Easton function*, if for all regular cardinals κ, λ:

(i) $\kappa < \lambda \rightarrow F(\kappa) \leq F(\lambda)$,
(ii) $\text{cf}(F(\kappa)) > \kappa$.

A cardinal μ is said to be a closure point of F if $F(\nu) < \mu$ for every regular cardinal $\nu < \mu$.

We say that F is realised in some cofinality-preserving extension $V^{\mathbb{R}}$ if F is the continuum function in $V^{\mathbb{R}}$ on regular cardinals.

Corollary 3.2 *Assume GCH and let κ be κ^{++}-correct cardinal. If an Easton function F satisfies:*

(i) κ is a closure point of F, $F(\kappa) = \kappa^{++}$, and
(ii) There exists a κ^{++}-correct embedding $j : V \rightarrow M$ with the critical point κ such that $j(F)(\kappa) \geq F(\kappa)$,

then there exists a cofinality-preserving forcing $\mathbb{R}$ such that the Easton function F is realised in $V^{\mathbb{R}}$, and j lifts to $V^{\mathbb{R}}$; in particular κ is still measurable in $V^{\mathbb{R}}$.

Proof. This is just like the relevant part of [4], with the arguments in Theorems 2.1 and 3.1 added to be able to prove this result from the optimal hypothesis of a κ^{++}-correct embedding (Lemma 2.4 must be generalized to $j(F)(\kappa^{++})$ instead to $(\kappa^{+4})^M$; this is straightforward since the cofinality of $j(F)(\kappa^{++})$ is strictly greater than κ^{++}). □

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