Definability of satisfaction in outer models[1]

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Abstract. Let M be a transitive model of ZFC. We say that a transitive model of ZFC, N, is an outer model of M if $M \subseteq N$ and $\mathrm{ORD} \cap M = \mathrm{ORD} \cap N$. The outer model theory of M is the collection of all formulas with parameters from M which hold in all outer models of M (which exist in a universe in which M is countable; this is independent of the choice of such a universe). Satisfaction defined with respect to outer models can be seen as a useful strengthening of first-order logic. Starting from an inaccessible cardinal κ, we show that it is consistent to have a transitive model M of ZFC of size κ in which the outer model theory is lightface definable, and moreover M satisfies $V = HOD$. The proof combines the infinitary logic $L_{\infty,\omega}$, Barwise's results on admissible sets, and a new forcing iteration of length strictly less than κ^+ which manipulates the continuum function on certain regular cardinals below κ. In the appendix, we review some unpublished results of Mack Stanley which are directly related to our topic.

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Contents

1 Introduction

Let V be the universe of sets and let $M \in V$ be a transitive model of ZFC. We say that $N \supseteq M$, $N \in V$, is an *outer model* of M if it is a transitive model of ZFC and the ordinals in N are the same as the ordinals in M.[2] Examples of outer models range from set and class forcing extensions to outer models obtained by means of large cardinal concepts (such as $0^\sharp$). In this paper, we study the outer models from the logical point of view and ask whether it is possible to define in M the satisfaction relation with respect to all outer models of M, thus strengthening the notion of first-order logic relative to M.

[2]We may also consider a stronger form of an outer model: with the notation as above, we say that N is a strong outer model of M if N satisfies ZFC with M as an additional predicate; alternatively, we can demand that M is a definable class in N. These stronger notions are not the main focus of this paper; we briefly comment on them in Section 4.4.

(Q1) Let V be the universe of sets. Suppose $M \in V$ is a transitive model of ZFC: is it possible to define in M the collection of all formulas with parameters in M which hold in some outer model of M in V? Can we also require that M is "nice" in the sense that it satisfies $V = HOD$?[3]

A related question is:

(Q2) Does the answer to (Q1) depend on the ambient universe V?

The existence of a model which gives a positive answer to (Q1) may seem improbable because the quantification over outer models is essentially higher-order over M (note that unlike in the case of forcing, there is no way to quantify over outer models by quantifying over elements in M). However, an analogy with first-order logic suggests that the definability level is more tractable than it at first appears:

Theorem 1.1 (First-order completeness) *Let V be the universe of sets. Suppose $M \in V$ is a transitive model of ZFC. Let φ be a first-order sentence with parameters in M. Then the following are equivalent:*

(i) $ZFC + \varphi + \mathrm{AtDiag}(M)$ *is consistent.*
(ii) $M \models$ *"*$ZFC + \varphi + \mathrm{AtDiag}(M)$ *is consistent".*
(iii) There is $N \in V$ such that N contains M as a substructure, and $N \models ZFC + \varphi$,

where $\mathrm{AtDiag}(M)$ *is the collection of all atomic sentences and their negations with parameters in M. In particular, the set of formulas with parameters in M satisfied in a model extending M in the inclusion relation is definable in M.*

By Theorem 1.1, we can refer to satisfaction in models containing M as a substructure by means of a syntactical property of having (or not having) in M a proof of a contradiction from a certain theory.

Is there some extension of first-order logic which provides an analogue of Theorem 1.1 for outer models? In principle there may be many, but one naturally looks for the weakest one because it may retain some of the desirable properties of first-order logic. It turns out that the infinitary logic $L_{\infty,\omega}$, which allows infinite conjunctions and disjunctions of any ordinal length, but only finitely many free variables in a formula, is the right framework.[4] Let

[3] If we drop the condition on $V = HOD$, the problem becomes easier; see Appendix 6.2. See also Remark 1.3 for more details.

[4] We identify the formulas in $L_{\infty,\omega}$ with sets under some reasonable coding; for instance if φ contains parameters from $H(\kappa)$ (the collection of sets whose transitive closure has size $< \kappa$) and has length less than κ, then we think of φ as an element of $H(\kappa)$. This convention makes it possible to refer to fragments of $L_{\infty,\omega}$; e.g. $L_{\infty,\omega} \cap M$ is the collection of all infinitary formulas which are elements of M.

us denote by $\text{Hyp}(M)$ the least *admissible* set containing M as an element, where a transitive set N as an admissible set if it satisfies the axioms of KP, Kripke-Platek set theory. $\text{Hyp}(M)$ is of the form $L_\alpha(M)$, where α is the least β such that $L_\beta(M)$ is a model of KP. Barwise developed the notion of proof (and therefore of syntactical consistency) for the infinitary logic $L_{\infty,\omega}$. An application of Barwise's Completeness theorem (see [1], Theorem 5.5) gives the following:

Theorem 1.2 (Barwise) *Let V be the universe of sets. Let $M \in V$ be a transitive model of ZFC, and let φ be an infinitary sentence in $L_{\infty,\omega} \cap M$ in the language of set theory. Then for a certain infinitary sentence φ^* in $L_{\infty,\omega} \cap \text{Hyp}(M)$ in the language of set theory, the following are equivalent:*

(i) $ZFC + \varphi^*$ *is consistent.*
(ii) $\text{Hyp}(M) \models$ *"$ZFC + \varphi^*$ is consistent".*
(iii) *In any universe W with the same ordinals as V which extends V and in which M is countable, there is an outer model N of M, $N \in W$, where φ holds.*

In particular, the set of formulas with parameters in M satisfied in an outer model M in an extension where M is countable is definable in $\text{Hyp}(M)$*.*

The statement of Theorem 1.2 is less elegant than of Theorem 1.1 because it concerns a logic with expressive strength greater than first-order logic. Most importantly, we do not get a first-order definition by means of the notion of consistency as in Theorem 1.1(ii): in Theorem 1.2(ii), we leave M, and refer to $\text{Hyp}(M)$, the smallest admissible set which contains M as an element. Thus the higher-order quantification over outer models is reduced to first-order quantification over $\text{Hyp}(M)$, but not over M.

Furthermore, N – the model of the consistent theory $ZFC + \varphi^*$ – may not exist in V, but only in some extension $W \supseteq V$ where M is countable. Theorem 1.2 therefore suggests an answer to (Q2): if we wish to answer (Q1) in the framework of $L_{\infty,\omega}$ and retain the straightforward correspondence between the consistency of a certain theory and existence of an outer model for that theory, M should be countable in V. However, the countability of M introduces technical issues in other respects, so we will not take this approach in the paper: instead, in Definition 2.1, we will define the notion of an outer model by referring to an extension $W \supseteq V$ where M is countable; or equivalently, by referring to consistency of a certain theory in $\text{Hyp}(M)$. See Section 2.1 for more discussion of outer models, and some comments regarding the proof of Theorem 1.2.

We prove in this paper that one can construct by forcing over L a model M of size κ, κ inaccessible in L, in which the satisfaction for outer models is definable not only in $\text{Hyp}(M)$ (which is ensured already by Theorem 1.2), but even in M, thus answering (Q1) positively for that M. Moreover M carries a definable wellorder (i.e. satisfies $V = HOD$). Our initial starting assumptions are minimal: we need just one inaccessible cardinal.

Remark 1.3 In his unpublished work [3], Mack Stanley proved that if M contains many Ramsey cardinals, then the answer to (Q1) for M is positive (see the Appendix for more details and the exact statement of the result). Later, he independently found a proof that only an inaccessible is enough to get a positive answer to the first part of (Q1) (see Appendix 6.2). However, his method does not seem to give our stronger result that M can satisfy $V = HOD$.[5]

The general outline of the paper is as follows:

In Section 2, we discuss the meaning of the notion of an outer model and its (apparent) dependence on the ambient universe. In Section 3, we give the proof of a weaker result which works for first-order formulas without parameters, or with a small number of parameters. In Section 4, we define the notion of a good iteration, prove the main theorem, and discuss some of its generalizations. In Section 5 we state some open questions. Finally, in the Appendix we briefly review the unpublished Stanley result.

2 Preliminaries

2.1 A theorem of Barwise and the notion of an outer model

Let V denote the universe of sets, and suppose $M \in V$ is a transitive model of ZFC. If we enlarge V to some V', for instance by forcing, then new outer models of M may appear in $V' \setminus V$. This process may be repeated indefinitely, with no natural stopping point.

However, perhaps the situation changes when we focus our attention on outer models which are relevant for *satisfaction* of formulas (with parameters). More precisely, there may be an extension V' of V (with the same ordinals) such that if there is no outer model of M in V' satisfying a given formula φ, then there will be no outer model satisfying φ in any further extension of V'. By a theorem of Barwise, see Theorem 1.2, this is in fact true; indeed V' can be taken to be a generic extension by the Levy collapsing forcing, which

[5]Notice that L can never define satisfaction in outer models otherwise it would be possible to define the satisfaction predicate in L. On the other hand with a proper class of Ramsey cardinals, M always defines satisfaction over its outer models. Thus the Dodd-Jensen K with large cardinals defines satisfaction in outer models. This presents a natural question whether M can satisfy $V = HOD$, and define satisfaction in outer models without relying on large cardinals.

collapses $|M|$ to ω.[6] In particular, if M is already countable in V, then V itself can be taken for V'. Thus Barwise's theorem allows us to define the notion of an outer model in a robust way which does not depend on the ambient universe.

We will not prove Theorem 1.2 (see [1], in particular Chapter III and Theorems 5.5, 5.6, and 5.7 for details), but shall at least make some comments.

First, it may be illustrative to give some details regarding the sentence φ^* in the theorem because it shows how $L_{\infty,\omega}$ captures the notion of an outer model. φ^* is built up of constants $\bar{a}$ for every $a \in M$, and can look for instance as follows (we view ZFC as a single infinitary sentence, and include it in φ^* for clarity):

$$(2.1)\quad \varphi^* = ZFC \;\&\; \bigwedge_{x \in M} (\forall y \in \bar{x})(\bigvee_{a \in x} y = \bar{a}) \;\&$$
$$\& \; [(\forall x)(x \text{ is an ordinal} \rightarrow \bigvee_{\beta \in M \cap \mathrm{ORD}} x = \bar{\beta})] \;\&\; \mathrm{AtDiag}(M) \;\&\; \varphi,$$

where $\mathrm{AtDiag}(M)$, the atomic diagram of M, is the conjunction of all atomic sentences and their negations which hold in M (when the constants are interpreted by the intended elements of M).

Second, the importance of countability in item (iii) of the theorem is caused by the use of model-theoretic inductive constructions which in general work only for countably many formulas (such as the Omitting Types Theorem); thus, for an uncountable M, the theory $ZFC + \varphi^*$ may be consistent, but we may not find a model for it in the current universe (this, of course, is a major difference between first-order logic and the infinitary logic $L_{\infty,\omega}$).

Third, the properties of $\mathrm{Hyp}(M)$ are important for the result: Barwise proved that with a suitable notion of proof, $\varphi \in M \cap \mathrm{Hyp}(M)$ is provable in the ambient universe iff it is provable in $\mathrm{Hyp}(M)$, thus making the notion of proof independent of the ambient universe. Behind this is of course the observation that $\mathrm{Hyp}(M)$ is absolute between all transitive models which contain M.

In view of Theorem 1.2, we define:

Definition 2.1 *Let V denote the ambient universe, and let $M \in V$ be a transitive model of size κ. We say that a first-order formula*[7] *φ with parameters from M is satisfied in an outer model of M if there is an outer model N of M in a generic extension of V where κ is countable, such that $N \models \varphi$.*

[6] This can also be seen by Lévy absoluteness, as the existence of an outer model of M satisfying φ is a Σ_1 statement with parameter R for any real R coding M.

[7] In general, φ can be an infinitary formula as well; we consider first-order formulas here for concreteness.

By Theorem 1.2, the definition is independent of the choice of the generic extension.

Remark 2.2 In principle, there may be other ways to formalize the notion of satisfaction in outer models. For instance we could require that an outer model N of M must exist in the current universe V, even if M is uncountable. However, we would lose the connection with the logic $L_{\infty,\omega}$ and the corresponding completeness Theorem 1.2, making the problem less tractable. For instance, if $M = V_\kappa$, then V sees no non-trivial outer models of M, and the outer model theory of M cannot be definable in M in this case. It seems that for a meaningful analysis, the collection of outer models of M which we consider must be reasonably large. Definition 2.1 is a canonical way of ensuring this largeness condition. See Section 5 for open questions.

2.2 The outer model theory

Let V be the universe of sets and let $M \in V$ be a transitive model of ZFC of size κ, for some regular infinite κ.

Definition 2.3 *We define the* outer model theory of M*, denoted* $\mathrm{OMT}(M)$*, as follows*

$$(2.2)\quad \mathrm{OMT}(M) = \{\varphi \mid \textit{there is no outer model } N \textit{ of } M,\ N \in W, \textit{ such that } N \models \neg\varphi\},$$

where φ is an infinitary formula in $L_{\infty,\omega} \cap M$ in the language of set theory with parameters in M and W is the model $V[G]$, where G is a generic filter for the Levy collapsing forcing which collapses κ to ω.

See Section 2.1, in particular Definition 2.1, for the legitimacy of W in this definition.

By Theorem 1.2, we can describe $\mathrm{OMT}(M)$ equivalently by means of the syntactical properties of $L_{\infty,\omega}$ and avoid talking about W:

$$(2.3)\qquad \mathrm{OMT}(M) = \{\varphi \mid ZFC + (\neg\varphi)^* \text{ is inconsistent in } \mathrm{Hyp}(M)\},$$

where φ^* is the sentence described in Section 2.1.

Definition 2.4 *Let M be as above. If* $\mathrm{OMT}(M)$ *is lightface definable in M, we say that M is* omniscient.

The term *omniscience* is meant to indicate that M "knows about truth in all of its outer models". We view omniscience as a maximality property of M (it maximizes expressive power). Perhaps surprisingly, this maximality property is not a large cardinal property (as Stanley's result 6.1 would seem to indicate). By Theorem 4.18, an upper bound on its consistency strength is just one inaccessible cardinal (see Section 5 with open questions).

Often, it is more convenient to consider the following collection of sentences (with the notation of (2.2)):

$$\text{(2.4)}\quad \mathrm{dOMT}(M) = \{\varphi \mid \text{there exists an outer model } N \in W \text{ of } M \text{ such that } N \models \varphi\}.$$

We call $\mathrm{dOMT}(M)$ the *dual of the outer model theory of* M. Since $\varphi \in \mathrm{OMT}(M)$ iff $\neg\varphi \notin \mathrm{dOMT}(M)$, the outer model theory $\mathrm{OMT}(M)$ and its dual $\mathrm{dOMT}(M)$ are mutually inter-definable (but the former is a consistent theory, while the latter not). When we refer to the "outer model theory" below, for the purposes of definability, we can refer to either of these two collections.

2.3 Notational conventions

Our notation is standard, following for instance [2]. In particular, if $P = \langle (P_i, \dot{Q}_i) \mid i < \mu \rangle$ is a forcing iteration of length μ and G is P-generic, let us write G_i, $i < \mu$, for G restricted to P_i. Further, if p is a condition in P_i, $i < \mu$, let us write $p^\frown 1$ for the condition in P which is the same as p at coordinates $j < i$, and at coordinates $j \in [i, \mu)$ is equal to the weakest condition in the respective forcing.

3 A simplified case

Let M be a transitive model of ZFC. Let us denote by $\mathrm{OMT}(M)^0$ (or $\mathrm{dOMT}(M)^0$) the intersection of $\mathrm{OMT}(M)$ (or $\mathrm{dOMT}(M)$) with the set of all first-order formulas with no parameters. As a warm-up, we construct a model in which the outer model theory for first-order formulas without parameters is lightface definable.

Theorem 3.1 *Assume κ is an inaccessible cardinal, $M = V_\kappa$. Then there exists a set-generic extension $V[G]$ of V by a forcing in V_κ such that the theory $\mathrm{OMT}(M[G])^0$ is lightface definable in $M[G]$.*

Proof. Denote, using the notation in (2.4):

$$A_0 = \mathrm{dOMT}(M)^0$$

and identify A_0 with a subset of ω. We know that A_0 is interdefinable with $\mathrm{OMT}(M)$, and is therefore an element of $\mathrm{Hyp}(M)$; by the inaccessibility of κ, A_0 is an element of M because M contains $\mathscr{P}(\omega)$. However, A_0 may not be lightface definable in M (if it is, then the proof is finished).

Let Q_0 be an Easton-product forcing which codes A_0 by the pattern of GCH at the first ω many uncountable cardinals; more precisely $Q_0 = \prod_{n\in A_0} \mathrm{Add}(\aleph_{n+1}, \aleph_{n+3})$. Thus

$$1_{Q_0} \Vdash (\forall n < \omega)(n \in A_0 \leftrightarrow 2^{\aleph_{n+1}} = \aleph_{n+3}), \tag{3.5}$$

which makes A_0 lightface definable in $M[G_0]$, where G_0 is a Q_0-generic. Let A_1 denote $\mathrm{dOMT}(M[G_0])^0$.

Crucially,

$$A_1 \subseteq A_0 \tag{3.6}$$

because every outer model of $M[G_0]$ is by definition an outer model of M. If we have $A_0 = A_1$, the proof is finished, and $M[G_0]$ is the desired model.

Suppose we have strict inclusion in (3.6), then we will continue by defining Q_1 in $M[G_0]$ to code A_1 by a GCH pattern on the cardinals in the interval $[\aleph_{\omega+1}, \aleph_{\omega+\omega})$. Continue in this fashion to define Q_α's and A_α's till the dual of the outer model theory stabilizes, i.e. until $A_\alpha = A_{\alpha+1}$ for some $\alpha < \omega_1$ (note that such an $\alpha < \omega_1$ must exist as otherwise we would shrink A_0 uncountably many times, which contradicts the countability of A_0).

Formally, define in the ambient universe V by induction a full support iteration

$$\mathbb{P}_{\omega_1} = \text{the inverse limit of } \langle (\mathbb{P}_\alpha, \dot{Q}_\alpha) \mid \alpha < \omega_1 \rangle$$

as follows:

(i) $\mathbb{P}_0 = \{\emptyset\}$.
(ii) If α is a limit ordinal, let $\mathbb{P}_\alpha$ be the inverse limit of $\langle (\mathbb{P}_\beta, \dot{Q}_\beta) \mid \beta < \alpha \rangle$.
(iii) If $\alpha = \beta + 1$, let $\dot{A}_\beta$ be a $\mathbb{P}_\beta$-name for $\mathrm{dOMT}(M[\dot{G}_\beta])^0$, where $\dot{G}_\beta$ is a name for the $\mathbb{P}_\beta$-generic filter. Set $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{Q}_\beta$, where $\dot{Q}_\beta$ is the name for $\prod_{n\in \dot{A}_\beta} \mathrm{Add}(\aleph_{\omega\beta+n+1}, \aleph_{\omega\beta+n+3})$.

Notice that the definition in (iii) makes sense because for every $\beta < \omega_1$, the forcing $\mathbb{P}_\beta$ preserves the inaccessibility of κ, and hence the outer model theory of $M[G_\beta]$ is an element of $M[G_\beta]$ and can therefore be coded.

Let G_{ω_1} be $\mathbb{P}_{\omega_1}$-generic. As we noted above there is some $\alpha < \omega_1$ such that the dual of the outer model theory of $M[G_\alpha]$ equals the dual of the outer model theory of $M[G_{\alpha+1}]$ because it cannot shrink properly uncountably many times. For any such α, $\bar{M} = M[G_{\alpha+1}]$ is the desired model: $\mathrm{dOMT}(\bar{M})^0$ can be read off from the continuum function on the last ω-segment of successor cardinals, where the GCH fails cofinally often. $\square$

Remark 3.2 Note that $\mathbb{P}_{\omega_1}$ may not be lightface definable in M, but by the inaccessibility of κ, $\mathbb{P}_{\omega_1}$ is an element of $M = V_\kappa$. $M[G]$ is therefore a legitimate generic extension of M, in particular $M[G]$ is a model of ZFC.

Remark 3.3 It is easy to see that the argument in the proof of Theorem 3.1 easily generalizes to situations where the relevant outer model theories are elements of the respective generic extensions. Thus the outer model theory of all first-order formulas with parameters from some $H(\mu)$, $\mu < \kappa$, can be coded by a variant of the construction in the proof of Theorem 3.1. We will not give the details because the relevant results are easy and moreover follow from the main Theorem 4.18. Finally notice that by further MacAloon coding, we can easily arrange that the resulting $M[G]$ carries a definable wellorder (i.e. satisfies $V = HOD$); see the main Theorem 4.18.

4 Main result

As before, let κ be an inaccessible cardinal and $M = V_\kappa$. Suppose now we wish to define the outer model theory of M with formulas (in the language of set theory) which allow all parameters from M, or with infinitary formulas in $M \cap L_{\infty,\omega}$. Then the coding method from Theorem 3.1 is no longer applicable because the outer model theory of M is not an element of M.

Instead we will define an iteration of length $< \kappa^+$ which will contain as its initial segments witnesses (i.e forcings), which will attempt to stabilize the membership or non-membership of a given formula to the outer model theory of the final generic extension. Since such witnesses need to be of length at least κ (because we need to decide the membership of κ many formulas), the whole iteration needs to be longer than κ as we are going to compose κ-many iterations of length at least κ. The length of the final iteration is not given in advance, but will be determined by an inductive definition. As in Theorem 3.1, we will code the theory – and also a definable wellorder of the universe (thus ensuring $V = HOD$ in the final model) – by the GCH pattern at some regular cardinals $< \kappa$. However, since the iteration is now longer than κ, we cannot choose these cardinals in increasing order.

We call such iterations *good iterations.*

4.1 Good iterations

Assume $V = L$. Let κ be the least inaccessible cardinal and let X be the set of all singular (i.e. uncountable limit) cardinals below κ. Fix a partition $\langle X_i \mid i < \kappa \rangle$ of X into κ pieces, each of size κ, such that $X_i \cap i = \emptyset$ for every $i < \kappa$.

Definition 4.1 *Let μ be an ordinal less than κ^+. We say that (P, f) is a* good iteration of length μ *if it is an iteration $P_\mu = \langle (P_i, \dot{Q}_i) \,|\, i < \mu \rangle$ with $< \kappa$ support of length μ, $f : \mu \to X$ is an injective function in L and the following hold:*

(i) $\mathrm{rng}(f) \cap X_i$ *is bounded in κ for every $i < \kappa$,*
(ii) For every $i < \mu$, P_i forces that $\dot{Q}_i$ is either $\mathrm{Add}(f(i)^{++}, f(i)^{+4})$ *or* $\mathrm{Add}(f(i)^{+++}, f(i)^{+5})$.

Remark 4.2 The properties of the good iterations discussed below, and in particular Theorem 4.13, would still be true if we specified in Definition 4.1(ii) that $\dot{Q}_i$ is one of a family of forcings which are all $f(i)^{++}$-closed, non-collapsing, and of size $< f(i)^{+\omega}$. There is nothing special about $\mathrm{Add}(f(i)^{++}, f(i)^{+4})$ and $\mathrm{Add}(f(i)^{+++}, f(i)^{+5})$ except that we use these forcings in the main Definition 4.24.

We are going to show that good iterations preserve cofinalities. For the usual reverse Easton iterations $\langle (\mathbb{P}_i, \dot{Q}_i) \,|\, i < \mu \rangle$, this is done by dividing the iteration at stage i into a lower part $\mathbb{P}_i$ which has a small chain condition, and the tail which is sufficiently closed. However, this easy division assumes that the cardinals are used in increasing order by $\mathbb{P}$. For good iterations, this is not the case: typically, $\mathbb{P}_i$ has just the κ^+-cc, and the tail may not be closed more than the first singular cardinal below κ. To overcome this problem, we need to define suitable notions of lower and upper part of a condition, which would enable us to carry out a similar kind of analysis as for the usual reverse Easton iteration. The analogy is not straightforward, though: we need to work with a quotient forcing (corresponding to the upper part) which is just distributive (see Lemma 4.8). The lower and upper part of a condition will be defined by means of a *good name* for an element of the forcing $\dot{Q}_i$ at stage i, which we define next.

Definition 4.3 *By induction on $i < \mu$, define the notion of a* good name *at i:*

(i) $LP(i)$, the lower part of P_i, *is the collection $\{j < i \,|\, f(j) < f(i)\}$. Similarly, let $UP(i)$, the* upper part of P_i, *be the complement of $LP(i)$: $UP(i) = i \setminus LP(i)$.*
(ii) σ is a good name at i *if σ is a P_i-name for an element of $\dot{Q}_i$, which satisfies:*
  (a) σ is a nice name for a subset of $f(i)^{+5}$; i.e. σ is of the form $\bigcup_{\alpha < f(i)^{+5}} (\{\alpha\} \times A_\alpha)$, where A_α is an antichain in P_i. Moreover, σ is forced by P_i to be in $\dot{Q}_i$.[8]

[8] We identify conditions in the Cohen forcings $\mathrm{Add}(f(i)^{++}, f(i)^{+4})$ and $\mathrm{Add}(f(i)^{+++}, f(i)^{+5})$ with subsets of $f(i)^{+5}$.

(b) The conditions $p \in A_\alpha$ satisfy that $p(j)$ for $j < i$ may be different from $1_{\dot{Q}_j}$ only at coordinates in $LP(i)$, and for all $j < i$, $p(j)$ is a good name at j (we regard $1_{\dot{Q}_j}$ as a good name).

The intuition behind the definition of a good name at i is to make sure that the interpretation of σ depends only on the generic at coordinates in $LP(i)$. Let us denote as Good(i) the collection of all good names at i.

Lemma 4.4 *(GCH) The number of good names at $i < \mu$ is less than $f(i)^{+\omega}$:*

$$|\mathrm{Good}(i)| < f(i)^{+\omega}. \tag{4.7}$$

Proof. The proof is by induction. Suppose it holds for $j < i$. Then the number of conditions p which satisfy (iib) in Definition 4.3 is at most $f(i)^{f(i)}$ because for every j such that $f(j) < f(i)$, the number of good names at j is by induction less than $f(j)^{+\omega}$, which is less than $f(i)$. The number of sets of these conditions (and so of antichains) is therefore at most $f(i)^{++}$, and hence the number of names satisfying (iia) in Definition 4.3 is then certainly less than $f(i)^{+\omega}$. □

Definition 4.5 *Let p and q be in P_i. For $\lambda \in X$ define*

$$p \leq_\lambda q \leftrightarrow p \leq q \text{ and } p(j) = q(j) \text{ for all } j < i \text{ such that } f(j) < \lambda. \tag{4.8}$$

Lemma 4.6 *Let $i \leq \mu$ be fixed. Suppose the following holds:*

$$(\forall p \in P_i)(\forall \lambda \in \mathrm{rng}(f \restriction i))(\exists q \leq_\lambda p)(q(f^{-1}(\lambda)) \in \mathrm{Good}(f^{-1}(\lambda))). \tag{4.9}$$

Then

$$(\forall p \in P_i)(\exists q \leq p)((\forall j < i) q(j) \in \mathrm{Good}(j)). \tag{4.10}$$

Proof. Denote by supp(p) the support of p, which by our definition has size $< \kappa$. Let $\langle \lambda_\alpha \,|\, \alpha < \nu \rangle$, $\nu < \kappa$, be the increasing enumeration of the range of f on supp(p). Using (4.9), define a decreasing sequence of conditions $p = q_0 \geq_{\lambda_0} q_1 \geq_{\lambda_1} \geq \cdots$ of length ν with limit q^0 such that $q_{\alpha+1} \leq_{\lambda_\alpha} q_\alpha$ has a good name at $f^{-1}(\lambda_\alpha)$. Note that the limit stages (including the last one) are defined because the conditions at the coordinate $f^{-1}(\lambda_\alpha)$, $\alpha < \nu$, are extended at most λ_α-many times, and the forcing at $f^{-1}(\lambda_\alpha)$ is forced to be λ_α^+-closed. By construction, q^0 satisfies (4.10) on supp(p) (and all $j < i$ outside the support of q^0). Repeat the construction ω-many times, obtaining a decreasing sequence $q^0 \geq q^1 \geq q^2 \geq \cdots$ with limit q. By the construction, q satisfies (4.10) as required because the support of q is the union of the supports of the q^i, $i < \omega$. $\square$

In Lemma 4.10, we will prove (4.9) by induction on $i \leq \mu$. For the argument at stage i, we will need to have some information about P_j for $j < i$ for which (4.9) already holds. This is the purpose of Lemma 4.8.

Definition 4.7 *Let $i < \mu$ be given. Set*

$$P_i^* = \{p \in P_i \,|\, (\forall j \in LP(i))(p(j) \in \mathrm{Good}(j)) \; \& \; (\forall k \in UP(i))(p(k) = 1_{\dot{Q}_k})\}. \tag{4.11}$$

Lemma 4.8 *Let $i < \mu$ be fixed. Assume (4.9), and therefore also (4.10), hold for P_i. Then:*

(i) There is a projection $\pi_i : P_i \to P_i^*$.
(ii) Let P_i^+ be the quotient P_i/P_i^*.

$$1_{P_i^*} \Vdash P_i^+ \text{ is } f(i)^{+\omega}\text{-distributive.} \tag{4.12}$$

Proof. (i). By (4.10), we can assume that $p \in P_i$ consists of good names at all $j < i$. Define $\pi(p)$ as the condition q such that $q(j) = p(j)$ for $j \in LP(i)$, and $q(j) = 1_{\dot{Q}_j}$ otherwise. It is easy to see that π is a projection.

(ii). Let p_0 in P_i force that $\dot{h}$ is a name for a function from $\eta < f(i)^{+\omega}$ to the ordinals; assume by (4.10) that p_0 contains just good names at all $j < i$. We wish to find $\tilde{r} \leq p_0$ in P_i such that over $V^{P_i^*}$, $\tilde{r}$ can be used to define the interpretation of h in V^{P_i} (for generics containing $\tilde{r}$). First note that by Lemma 4.4, the size of P_i^* is at most $f(i)^{f(i)} = f(i)^+$ because every $p \in P_i^*$ is determined by the sequence of good names in $LP(i)$. Also note that the forcings at $j \in UP(i)$ are at least $f(i)^{+\omega+2}$-closed because by the injectivity of f, $f(j) > f(i)$ for all $j \in UP(i)$.

Let Y be the family of all sequences of good names $p(j)$, where p is a condition in P_i^* and j is an index in $LP(i)$, i.e.

$$Y = \prod_{j \in LP(i)} \{p(j) \,|\, p \in P_i^* \text{ and } p(j) \text{ is a good name}\}.$$

If $p \in P_i$ and $y \in Y$, we write

$$p * y \tag{4.13}$$

to denote the condition which is obtained by replacing $p(j)$ by $y(j)$ for all $j \in LP(i)$. Note that $p * y$ is a legitimate condition because a good name at j is forced by 1_{P_j} to be in $\dot{Q}_j$.

To determine $\dot{h}(0)$, construct simultaneously a $\leq_{f(i)}$-decreasing sequence of conditions $\langle r^0_\alpha \mid \alpha < \nu_0 \rangle$, $\nu_0 < f(i)^{++}$, below p_0, and an $\subseteq$-increasing sequence of antichains $\langle A^0_\alpha \mid \alpha < \nu_0 \rangle$ of conditions in P^*_i below $\pi(p_0)$. Greatest lower bounds are taken to define r^0_α for α limit, and unions of A^0_β, $\beta < \alpha$, to define A^0_α, for α limit. To define $r^0_{\alpha+1}$, find $a \leq \pi(p_0)$ such that

$$\{a\} \cup A^0_\alpha \text{ is an antichain,} \tag{4.14}$$

and let $A^0_{\alpha+1} = A^0_\alpha \cup \{a\}$. Let $y \in Y$ be the sequence of good names in a at coordinates in $LP(i)$. Find

$$s_\alpha \leq r^0_\alpha * y \tag{4.15}$$

which decides the value of $\dot{h}(0)$. Define $r^0_{\alpha+1}$ by induction on $j < i$ as follows:

$$r^0_{\alpha+1}(j) = \begin{cases} p_0(j) & \text{if } j \in LP(i) \\ \sigma_j & \text{otherwise,} \end{cases} \tag{4.16}$$

where $(r^0_{\alpha+1} * y)|j$ forces that σ_j is equal to $s_\alpha(j)$, and simultaneously, $r^0_{\alpha+1}|j$ forces $\sigma_j \leq r^0_\alpha(j)$.[9] Thus in particular $r^0_{\alpha+1} \leq_{f(i)} r^0_\alpha$.

Since P^*_i has size at most $f(i)^+$, there is some $\nu_0 < f(i)^{++}$ such that there is no a satisfying (4.14). By the closure of the conditions at $UP(i)$, the sequence $\langle r^0_\alpha \mid \alpha < \nu_0 \rangle$ has a lower bound, which we denote as $\tilde{r}^{\dot{h}(0)}$.

Carry out the above construction for every $\gamma < \eta$ and construct sequences $\langle r^\gamma_\alpha \mid \alpha < \nu_\gamma \rangle$ and $\langle A^\gamma_\alpha \mid \alpha < \nu_\gamma \rangle$ to determine $\dot{h}(\gamma)$, and to obtain a decreasing sequence $\tilde{r}^{\dot{h}(0)} \geq_{f(i)} \tilde{r}^{\dot{h}(1)} \geq_{f(i)} \cdots$ of length η. By the closure of the coordinates at $UP(i)$, the sequence has a lower bound. Denote the lower bound as $\tilde{r}$.

Let G be a P_i-generic filter over V which contains $\tilde{r}$. Let g be the derived generic for P^*_i via the projection π. In $V[g]$ one can define $\dot{h}^G$ as follows: $\dot{h}^G(\gamma)$ is the unique ordinal ξ such that for a unique $a \in \bigcup\{A^\gamma_\alpha \mid \alpha < \nu_\gamma\}$, $a \in g$, $\tilde{r} * a$ forces $\dot{h}(\gamma) = \xi$. □

In Theorem 4.13, it will be useful to have the above Lemma 4.8 formulated for $i = \mu$ and some parameter $\zeta < \kappa$ (a regular cardinal). Given a regular cardinal $\zeta < \kappa$ and a condition p in P, let us define $LP_\zeta(\mu)$, the lower part of P with respect to ζ, as follows:

$$LP_\zeta(\mu) = \{j < \mu \mid f(j) < \zeta\}, \tag{4.17}$$

[9] This is the usual manipulation with names which can interpret differently below incompatible conditions.

and $UP_\zeta(\mu) = \mu \setminus LP_\zeta(\mu)$. The notion of the lower part with respect to ζ is used to define the analogue of P_i^*:

$$P_\zeta^* = \{p \in P \mid (\forall j \in LP_\zeta(\mu))(p(j) \in \mathrm{Good}(j)) \ \& \ (\forall k \in UP_\zeta(\mu))(p(k) = 1_{\dot{Q}_k})\}. \tag{4.18}$$

Lemma 4.8 can be directly generalized as follows:

Corollary 4.9 *Assume (4.9), and therefore also (4.10), hold for P. Suppose $\zeta < \kappa$ is a regular cardinal and*

(*) there is no unique $j < \mu$ such that $f(j) < \zeta < f(j)^{+\omega}$.

Then:

(i) There is a projection $\pi_\zeta : P \to P_\zeta^*$.

(ii) Let P_ζ^+ be the quotient P/P_ζ^*.

$$1_{P_\zeta^*} \Vdash P_\zeta^+ \text{ is } \zeta^+\text{-distributive.} \tag{4.19}$$

Proof. Since (4.10) is formulated also for $i = \mu$, the proof goes exactly as in Lemma 4.8. Regarding the distributivity in (ii), note that P_ζ^* has size at most ζ as by the condition (*), ζ is not an element of an ω-block of cardinals $[f(j), f(j)^{+\omega})$ for any j. Note that if (*) is not the case, we need to factor more carefully; see the proof of Theorem 4.13. □

Finally we can prove the key property (4.9).

Lemma 4.10 *For all $i \leq \mu$, (4.9) holds for P_i.*

Proof. The proof is by induction.

Assume that i is a limit ordinal and for every $j < i$, (4.9) holds for P_j. Let $p \in P_i$ and $\lambda \in X$ be given. Choose $j < i$ such that $f(j) = \lambda$. By the induction assumption applied to P_{j+1}, there is a $q' \leq_\lambda p|(j+1)$ such that $q'(j)$ is a good name. Clearly if we stretch q' to a condition q in P_i by substituting $p(k)$ for $k \in [j+1, i)$, we get the required $q \leq_\lambda p$.

Assume now that (4.9) holds for P_i, and we wish to show it for P_{i+1}. Let $p \in P_{i+1}$ and λ be given. Assume that $f(i) = \lambda$ (otherwise the lemma follows trivially). By Lemma 4.8 applied to P_i, all conditions in $\dot{Q}_i$ are added by P_i^*. It follows that $p|i$ can be extended to some $q_0 \leq p|i$ which forces that $p(i)$ is extended by some P_i^*-name σ_0, which can be taken to be a good name. The problem is that we only have $q_0 \leq p|i$, and not the required $q_0 \leq_\lambda p|i$. However, as in the proof of Lemma 4.8(ii), by diagonalizing over a maximal antichain in P_i^*, and taking lower bounds on the coordinates in $UP(i)$, one can find $q \leq_\lambda p|i$ and a good P^*_i-name σ such that q forces that σ extends $p(i)$. $\square$

Definition 4.11 *Let (P, f) be a good iteration of length $\mu < \kappa^+$. We call $p \in P$* a good condition *if for every $i < \mu$, $p(i)$ is a good name.*

Lemma 4.12 *Let (P, f) be a good iteration of length $\mu < \kappa^+$. Then the collection of good conditions forms a dense subset of P of size at most κ.*

Proof. By Lemma 4.10, the property (4.9) in Lemma 4.6 holds, and therefore there is a dense subset of P which contains conditions composed only of good names. By Lemma 4.4, the number of such sequences is at most κ. $\square$

The properties of good iterations identified in the previous lemmas provide a straightforward way to show that good iterations preserve cofinalities.

Theorem 4.13 *(GCH) Let μ be an ordinal less than κ^+. A good iteration (P, f) of length μ preserves cofinalities.*

Proof. By Lemma 4.12, P has the κ^+-cc, and so all cofinalities $\geq \kappa^+$ are preserved.

It remains to show that cofinalities $\leq \kappa$ are preserved.

Assume by contradiction that for some regular $\zeta < \kappa$ there is $p \in P$ such that

$$p \Vdash (\exists \lambda > \zeta \text{ regular in } V\ ) \operatorname{cf}(\lambda) = \zeta.$$

We need to distinguish two cases.

Case A). There is no unique $j < \mu$ such that $f(j) < \zeta < f(j)^{+\omega}$.

In this case, we reach contradiction by applying Corollary 4.9. P^*_ζ has size at most ζ so it cannot cofinalize λ; the quotient forcing P^+_ζ is forced to be ζ^+-distributive, so cannot cofinalize λ either.

Case B). There is a unique $j < \mu$ such that $f(j) < \zeta < f(j)^{+\omega}$.

Fix such j and write P as $P_0 * P_1 * P_2$, where P_0 is the forcing P_j, P_1 is the Cohen forcing (either at $f(j)^{++}$ or $f(j)^{+++}$), and P_2 is the tail of the iteration.

P_0 does not cofinalize λ by Lemma 4.8. P_1 is cofinality-preserving. For P_2, apply Corollary 4.9 over $V^{P_0 * P_1}$; note that Corollary 4.9 now applies because in $V^{P_0 * P_1}$, the tail iteration satisfies Case A). $\square$

Remark 4.14 If we defined a good iteration exactly as in Definition 4.3, but with full support, then (P, f) would still preserve all cofinalities. This is not useful for the present paper, but it might be of interest for future applications. The preservation of cofinalities $\leq \kappa$ is exactly the same as for the $< \kappa$ support. The preservation of κ^+ can be argued as follows:

We need to show that if $p \in P$ forces that $\dot{h}$ is a function from κ to κ^+, then for some condition $q \leq p$, q forces a bound on the range of $\dot{h}$ of size at most κ. The argument is a diagonal version of the proof of Lemma 4.8(ii), and uses an inductive construction of length κ as in Lemma 4.6. In particular, let $\langle \lambda_\alpha \,|\, \alpha < \kappa \rangle$ be the increasing enumeration of X. Define a decreasing sequence of conditions $p = q_0 \geq_{\lambda_0} q_1 \geq_{\lambda_1} q_2 \geq_{\lambda_2} \cdots$ of length κ with limit q such that $q_{\alpha+1}$ decides the value of $\dot{h}(\alpha)$ to be in some family Y_α of size $< \kappa$; to obtain this $q_{\alpha+1} \leq_{\lambda_\alpha} q_\alpha$, when q_α has already been constructed, carry out the argument in Lemma 4.8(ii) applied to P_i with $i = \mu$, and with the $LP(i)$ defined as $LP(i) = LP(i)_\alpha = \{j < i \,|\, f(j) < \lambda_\alpha\}$ (the definition of $LP(i)_\alpha$ using λ_α at stage α explains the use of the word "diagonal" in the description of the method). Note that the length of the inductive construction at stage α is at most $\lambda_\alpha^{\lambda_\alpha} = \lambda_\alpha^+$ (the number of sequences of good names at $j \in LP(i)_\alpha$), and the forcings at j, $f(j) \geq \lambda_\alpha$, are at least λ_α^{++}-closed, so $q_{\alpha+1}$ can be correctly defined.

4.2 Compositions of good iterations

Before we state the theorem we make a remark and state a lemma concerning a composition of good iterations.

Remark 4.15 Suppose (P, f) is a good iteration in the ground model V, and $(\dot{Q}, g)$ is forced by 1_P to be a good iteration in V^P.[10] If $\mathrm{rng}(f) \cap \mathrm{rng}(g) = \emptyset$, then $P * \dot{Q}$ is a good iteration with respect to a function h defined as follows: $h = f \cup g^{\mathrm{shift}}$ where g^{shift} is defined on $[\alpha, \beta)$ where $\alpha = \min(\kappa^+ \setminus \mathrm{dom}(f))$, $\beta = \alpha + \mathrm{dom}(g)$, and for $\xi \in \mathrm{dom}(g)$, $g^{\mathrm{shift}}(\alpha + \xi) = g(\xi)$. For simplicity of notation, if $\mathrm{rng}(f) \cap \mathrm{rng}(g) = \emptyset$, we will write

$$f \uplus g \text{ to denote } f \cup g^{\mathrm{shift}}. \tag{4.20}$$

In particular, we write $(P * \dot{Q}, f \uplus g)$ to denote the resulting composition.

Lemma 4.16 *Let $P = \langle (P_i, \dot{Q}_i) \,|\, i < \mu \rangle$, $P_0 = \{\emptyset\}$, be an iteration with $< \kappa$-support such that for every $i < \mu$, 1_{P_i} forces that $(\dot{Q}_i, f'_i)$ is a good iteration. Assume further that for every $i \neq j < \mu$, $\mathrm{rng}(f'_i) \cap \mathrm{rng}(f'_j) = \emptyset$. By induction define a sequence $\langle f_i \,|\, i < \mu \rangle$: $f_0 = f'_0$, $f_{i+1} = f_i \uplus f'_{i+1}$, and $f_i = \bigcup_{j<i} f_j$ for i limit. Denote $f = \bigcup_{i<\mu} f_i$.*

[10] Recall that g is a function in L by the definition of good iteration; further note that we use the convention that checked names are written without a dot: hence $(\dot{Q}, g)$ is the same as $(\dot{Q}, \check{g})$.

(i) If $\mu < \kappa$, then (P, f) is a good iteration.

(ii) If $\mu = \kappa$, and moreover for every $i < \kappa$, $\mathrm{rng}(f_i) \cap X_j = \emptyset$ for all $j < i$, then (P, f) is a good iteration.

Proof. Taking into account Remark 4.15, (P, f) satisfies the requirements for being a good iteration; the only property worth mentioning in (i) is that because $\mu < \kappa$, the range of f is bounded in every X_i. For (ii), the property of f being bounded in every X_i is ensured by demanding $\mathrm{rng}(f_i) \cap X_j = \emptyset$, $j < i$. □

Remark 4.17 Note that Lemma 4.16 is formulated only for $\mu \leq \kappa$ because we will not need it for larger values of μ. In particular μ will be the cofinality of the good iteration P.

4.3 Main theorem

4.3.1 Statement and motivation

Theorem 4.18 *Assume $V = L$. Let κ be the least inaccessible, and let $M = L_\kappa$. There is a good iteration $(\mathbb{P}, h)$ in V such that if G is $\mathbb{P}$-generic over V, then for some set $\tilde{G}$, which is defined from G, $M[\tilde{G}]$ is a model of ZFC in which $\mathrm{OMT}(M[\tilde{G}])$ is lightface definable. Moreover, $M[\tilde{G}]$ is a model of $V = HOD$.*

The set $\tilde{G}$ is defined from G as follows:

Definition 4.19 *Assume V satisfies GCH. If (P, f) is a good iteration and G is P-generic over V, let us write $\tilde{G}$ for the following object: $\tilde{G}$ is the collection of all generic subsets added by the generic G to cardinals ξ, where ξ is a double successor or a triple successor of a singular cardinal in the range of f.*

Notice in particular that if κ is inaccessible, $V_\kappa = M$, then $\tilde{G}$ is a subset of $H(\kappa)^{V[G]}$, and in $V[G]$, $M[\tilde{G}]$ is the smallest model of ZFC which contains $M \cup \tilde{G}$ and has height κ; in fact, $M[\tilde{G}] = H(\kappa)^{V[G]}$. Note also that the continuum function below κ is the same in $M[\tilde{G}]$ and $V[G]$.

We start by explaining the idea behind the proof of the main theorem to motivate rather technical definitions 4.22 and 4.24. First, we define the notion of "killing a formula".

Definition 4.20 *Let M be as above. We say that a condition p in a good iteration (P, f)* kills a formula *φ (with parameters which are P-names for* *sets belonging to $M[\tilde{G}]$) if for every P-generic G containing p, there is no outer model of $M[\tilde{G}]$ where φ holds, i.e. $\varphi \notin \mathrm{dOMT}(M[\tilde{G}])$.*

Note that if $p \in P$ kills φ, then for every good iteration $\dot{Q}$ in V^P and every $\dot{q} \in \dot{Q}$, $(p, \dot{q})$ also kills φ (or equivalently, $(p, 1_{\dot{Q}})$ kills φ). In other words: Any extension of an iteration which kills φ, also kills φ (killing of φ is upwards persistent). This simple observation allows us to compose together good iterations in Definition 4.24 below.

Remark 4.21 Recall that by Theorem 1.2, the existence of an outer model of $M[\tilde{G}]$ satisfying φ is equivalent to the consistency of a certain infinitary sentence φ^*, so the property of killing φ in Definition 4.20 is expressible as a property of the forcing (P, f).

Let us denote $M = V_\kappa = L_\kappa$. The main idea of the proof of Theorem 4.18 is as follows: we want to decide the membership or non-membership of κ-many formulas with parameters in the outer model theory of the final model. We are going to define an iteration of length κ, dealing with the i-th formula at stage $\mathbb{P}_i$. Suppose at stage i, it is possible to kill φ_i by a good iteration $\dot{W}_i$, i.e. ensure that in $V^{\mathbb{P}_i * \dot{W}_i}$ there is no outer model of φ_i. If such $\dot{W}_i$ exists, set $\mathbb{P}_{i+1} = \mathbb{P}_i * \dot{W}_i * \dot{C}_i$, where $\dot{C}_i$ codes this fact by means of a good iteration. In the final model $M[\tilde{G}]$, we can decide the membership of φ_i in $\mathrm{OMT}(M[\tilde{G}])$ by asking whether at stage i we have coded the existence a witness $\dot{W}_i$ which kills φ_i, arguing as follows: If there is no outer model of $M[\tilde{G}]$ where φ_i holds, then indeed we have coded this fact at stage i by using some $\dot{W}_i$ (because the tail of $\mathbb{P}$ – itself a good iteration – from stage i did kill φ_i so some such $\dot{W}_i$ must have existed). Conversely, if there is an outer model of $M[\tilde{G}]$ where φ_i holds, then we could not have found a witness $\dot{W}_i$ because if we did, then its inclusion in $\mathbb{P}$ would ensure that φ_i is killed. Note that there is no bound on the length of $\dot{W}_i$, except that it must be less than κ^+ (by the injectivity of the function f which makes $(\dot{W}_i, f)$ a good iteration).

Several points must be resolved to make this rough idea work. (A) Although technically $\mathbb{P}$ is an iteration of length κ, its length as a good iteration (using the composition Lemma 4.16) is some ordinal of cofinality κ below κ^+. This has two consequences: (i) $\mathbb{P}$ cannot choose the regular cardinals $< \kappa$ at which it forces in the increasing order, and (ii) $\mathbb{P}$ is not a subset of M. We solve (i) by considering an injective function f in the definition of a good iteration (P, f) which enumerates singular cardinals $< \kappa$ in a non-monotonic way; note also that because we need to code information as we progress, f must have some flexibility: therefore, f enumerates singular cardinals in whose "neighbourhood" we do the coding at regular cardinals. Regarding (ii), we solve the problem by considering $M[\tilde{G}]$ described in Definition 4.19.

(B) Whether $\mathbb{P}_i * \dot{W}_i$ does or does not kill φ_i may depend on a particular condition $p \in \mathbb{P}_i * \dot{W}_i$ which forces it (the forcings are not homogeneous); to deal with this problem, we will need to use bookkeeping functions e^i, $i < \kappa$, to enumerate all good conditions in the initial segments $\mathbb{P}_i$ and add witnesses $\dot{W}_i$ with respect to these conditions (we will not need to enumerate the conditions in $\dot{W}_i$ itself because by $< \kappa$ support, any condition which forces killing of φ_i in the final iteration $\mathbb{P}$ has its support bounded in some $\mathbb{P}_i$).

(C) Since the formulas φ_i contain parameters from the final extension $M[\tilde{G}]$, we will need to enumerate them as the iteration progresses by means of names (these are the (names for) enumerations $\dot{d}^i$ in the proof).

Finally (D) to achieve light-face definability, the well-ordering of the formulas given by the $\dot{d}^i$'s needs to be coded. We will also need to code which good conditions are elements of the generic filter (recall that our witnesses will be added with respect to all good conditions and we will need to look at the right conditions to decode the outer model theory correctly). This is the purpose of the forcing $\dot{C}_i$ in $\mathbb{P}_i * \dot{W}_i * \dot{C}_i = \mathbb{P}_i * \dot{Q}_i$, which does the coding by means of further good iterations. As a bonus, by coding the $\dot{d}^i$'s, the final model $M[\tilde{G}]$ will be a model of $V = HOD$.

4.3.2 Proof

In this section, we give the proof of Theorem 4.18.

Assume $V = L$, and let κ be the least inaccessible cardinal and X the set of all singular cardinals below κ; let $\langle \lambda_\alpha \,|\, \alpha < \kappa \rangle$ be the increasing enumeration of X. Fix an L_κ-definable partition

$$\langle X_i \,|\, i < \kappa \rangle \tag{4.21}$$

of X into κ pieces, each of size κ, such that $X_i \cap i = \emptyset$ for every $i < \kappa$.

As mentioned in (D) in Section 4.3.1, we need to code some information. Now, we describe one particular way of coding using good iterations. Let us say that a good iteration P has *pattern A* on $i \in X$ if it forces with $\mathrm{Add}(i^{++}, i^{+4})$; it has *pattern B* if it forces with $\mathrm{Add}(i^{+++}, i^{+5})$. Given $i < \kappa$ and $\alpha \in X_i$, we say that P has pattern $ABAB$ on α if P forces on four successive singular cardinals in X_i, starting with α, and on α, it has pattern A, on the successor of α in X_i, pattern B, etc. This notation extends to any finite combination of A's and B's. For any α we refer to pattern AA as *bit 0*, and pattern BB as *bit 1*.

Definition 4.22 *Let V be a ground model (a generic extension of L by a good iteration). Let a be in $H(\kappa)^V$. We say that a good iteration (P, f)* codes

a on top of X_i *if the range of* f *is an interval of cardinals in* X_i *starting with* $\alpha \in X_i$ *which is the least such that GCH holds in* V *for all* $\beta \geq \alpha$ *in* X_i*. In order to mark the beginning of the coding,* P *forces pattern* $ABAB$ *on* α*. The coding itself starts by first coding the transitive closure of* $\{a\}$ *by a subset* a' *of some ordinal.*[11] *The set* a' *is then coded by means of a good iteration, which codes* a' *by a sequence of bits 0 and 1, starting with the first singular cardinal in* X_i *after pattern* $ABAB$*.*

Definition 4.22 extends to coding a finite number[12] of sets $\{a_0, \ldots, a_n\}$: just code successively the sets a_i, $i \in \{0, \ldots, n\}$, and separate the coding intervals by, e.g., $ABABAB$. Note that one can decode the coded information as follows: given X_i, find the topmost occurrence of $ABAB$ and the next occurrence of $ABABAB$ (if any). Using the bits 0 and 1 in this interval, decode a_0, etc.[13]

In the course of the inductive definition of the iteration in Definition 4.24, we will use some bookkeeping of certain triples of parameters. The bookkeeping function will be an L_κ-definable surjective function $b : \kappa \to \kappa^3$ such that if

$$b(i) = \langle j, k, m \rangle, \tag{4.22}$$

then the following conditions are satisfied:

- $j \leq i$,
- If $i = 0$, then $m = 0$,
- If i is a singular cardinal, then $m < i$,
- If $i > 0$ is not a singular cardinal and $i < \lambda_0$, then $m < \lambda_0$ (where λ_0 is the least singular cardinal),
- If i is not a singular cardinal and is greater than the first singular cardinal, then $m < \bar{i}$, where $\bar{i}$ is the greatest singular cardinal below i.

We write $b(i)(0) = j$, $b(i)(1) = k$, and $b(i)(2) = m$ to express the components of $b(i)$.

Remark 4.23 The many cases of the definition of b are motivated by the fact that the coding of parameters in Definition 4.24 takes place only at singular cardinals i (which ensures that $H(i)$ of the relevant model has size i as under our assumptions every singular cardinal will always be strong limit); moreover, 0 will be treated similarly as the singular cardinals to prime the construction. At stages i which are not singular cardinals, we will

[11] For concreteness, we take a' to be a subset of some cardinal η, $|a'| = \eta$, which via a pairing function codes (η, R), $R \subseteq \eta^2$, such that (η, R) is isomorphic to the transitive closure of $\{a\}$ with the membership relation. By Mostowski collapse theorem, a' is enough to recover a.

[12] Finite is enough for us here.

[13] Any other reasonable notion of coding can be used here.

use a wellordering defined at the greatest previous singular stage (or 0 if there is no smaller singular cardinal).

The following definition specifies the main forcing $(\mathbb{P}, h)$ in Theorem 4.18.

Definition 4.24 $(\mathbb{P}, h) = \langle (\mathbb{P}_i, \dot{Q}_i) \mid i < \kappa \rangle$ *is going to be an iteration with* $< \kappa$ *support,* $h \in L$*, defined by induction together with names* $\dot{d}^i$*,* i *a singular cardinal, and sequences* e^i*,* $i < \kappa$*. The names* $\dot{d}^i$ *will interpret as wellorderings of rank-initial segments of the final model; it will be the case that* $1_{\mathbb{P}_i}$ *forces that* $\dot{d}^i$ *end-extends all* $\dot{d}^j$*,* $j < i$*, and that the ordertype of the ordering* $\dot{d}^i$ *is* i*. The sequences* e^i*,* $i < \kappa$*, will enumerate all good conditions in* $\mathbb{P}_i$*,* $i < \kappa$*.*

Stage 0**.** *To prime the construction let* $d^0 : \lambda_0 \to H(\lambda_0)^L$ *be an enumeration of* $H(\lambda_0)^L$ *(where* λ_0 *is the least singular cardinal). Set* $\mathbb{P}_0 = \{\emptyset\}$*,* $h_0 = \emptyset$*, and* $e^0 = \{\langle \emptyset, \emptyset \rangle\}$*.*

Successor stage.

Suppose the good iteration $(\mathbb{P}_i, h_i)$ *is defined, we wish to define* $(\mathbb{P}_{i+1}, h_{i+1})$*. Choose* e^i *to be an enumeration of all good conditions in* $\mathbb{P}_i$ *(there are at most* κ *many of these – for simplicity, we assume that for* $i > 0$*, the domain of* e^i *is always* κ*).*

If i *is a singular cardinal, fix a name* $\dot{d}^i$ *such that*

$$(4.23)\quad 1_{\mathbb{P}_i} \Vdash \text{“}\dot{d}^i : i \to \dot{H}(i)^{M[\tilde{G}_i]} \text{ is an enumeration of the elements of } \dot{H}(i)^{M[\tilde{G}_i]},\text{”}$$

and

$$(4.24)\qquad 1_{\mathbb{P}_i} \Vdash \text{“}(\forall j < i)\ \dot{d}^i \text{ end-extends } \dot{d}^j.\text{”}$$

We identify the range of $\dot{d}^i$ *with formulas in* $L_{\infty,\omega}^{M[\tilde{G}_i]} \cap \dot{H}(i)^{M[\tilde{G}_i]}$*.*

$\dot{Q}_i$ *will be of the form* $\dot{W}_i * \dot{C}_i$*, where* $\dot{W}_i$ *and* $\dot{C}_i$ *are both good iterations.*

Let i *be arbitrary (i.e. not necessarily a singular cardinal). Let* $b(i) = \langle j, k, m \rangle$*.* $\dot{W}_i$ *will be defined with respect to a good condition* p^j_k *which is the* k*-th good condition of* $\mathbb{P}_j$ *in the enumeration* e^j *and with respect to the* m*-th formula* $\dot{\varphi}$ *in* $\dot{H}(i^*)^{M[\tilde{G}_{i^*}]}$ *enumerated by* $\dot{d}^{i^*}$*:*

$$(4.25)\qquad 1_{\mathbb{P}_i} \Vdash \dot{d}^{i^*}(m) = \dot{\varphi},$$

where (recalling definition (4.22)):

- i^* *is* i *if* i *is a singular cardinal,*
- i^* *is* $\bar{i}$*, if* i *is not a singular cardinal and* $i > \lambda_0$*,*

- *i^* is 0 if $i < \lambda_0$. If $i^* = 0$, we take $H(i^*)^{M[\tilde{G}_{i^*}]}$ to denote $H(\lambda_0)^L$.*

Suppose first that there is a pair $(\dot{W}, f)$ forced by $1_{\mathbb{P}_i}$ to be a good iteration, which satisfies

$$\text{rng}(h_i) \cap \text{rng}(f) = \emptyset, \text{ and } \text{rng}(f) \cap X_j = \emptyset \text{ for all } j < i, \tag{4.26}$$

and such that the condition $p^j_k{}^\frown 1$ in $\mathbb{P}_i * \dot{W}$ kills $\dot{\varphi}$. Define

$$\dot{W}_i = (\dot{W}, f), \text{ and } h'_i = h \uplus f. \tag{4.27}$$

If no such $(\dot{W}, f)$ exists, set

$$\dot{W}_i = \{\emptyset\}, \text{ and } h'_i = h. \tag{4.28}$$

Finally, let $(\dot{C}_i, f')$ code the following up to three pieces of information on top of X_i (see Definition 4.22).[14]

(i) If $\dot{W}_i$ is nonempty, code the killing of $\dot{\varphi}$ by forcing pattern $ABAB$. If $\dot{W}_i$ is empty, code the non-killing of $\dot{\varphi}$ by forcing pattern $ABABABAB$.
(ii) Code the set of all $j < i$ such that the good condition $p^{b(j)(0)}_{b(j)(1)}$ is in $\dot{G}_{b(j)(0)}$.
(iii) If i is a singular cardinal, code the enumeration $\dot{d}^i$.

Let $h_{i+1} = h'_i \uplus f'$.

Limit stage.

Define: $h_i = \bigcup_{j<i} h_j$. If i is a limit of singular cardinals, let $\dot{d}^i$ be a name for $\bigcup_{j<i} \dot{d}^j$.

This completes Definition 4.24.

Note that by induction, $(\mathbb{P}, h)$ is a composition of κ-many good iterations, and by Lemma 4.16, it is a good iteration. The length of the good iteration $(\mathbb{P}, h)$ is some $\mu < \kappa^+$ of cofinality κ.

Let G be a $\mathbb{P}$-generic filter. As $i \cap X_i = \emptyset$ for every $i < \kappa$ (see (4.21)), and from stage i on $\mathbb{P}$ does not use cardinals in X_j, $j < i$, it follows that

$$H(i)^{M[\tilde{G}_i]} = H(i)^{M[\tilde{G}]}. \tag{4.29}$$

Also, because $X_i \cap \lambda_0$ is empty for every i, $H(\lambda_0)^L = H(\lambda_0)^{M[\tilde{G}]}$, which explains the definition of d^0 in Stage 0.

By design, the $\dot{C}_i$'s of the forcing code in a lightface way in $M[\tilde{G}]$ the following objects:

[14] By the definition of coding in Definition 4.22, f' automatically satisfies the conditions in (4.26), so the composition $\mathbb{P}_i * \dot{W}_i * \dot{C}_i$ is defined correctly.

- The set H of all $i < \kappa$ such that the condition $p^{b(i)(0)}_{b(i)(1)}$ is in $G_{b(i)(0)}$.
- The wellordering

(4.30)
$$D = \bigcup_{i \in X} (\dot{d}^i)^{G_i}$$

of all elements of $M[\tilde{G}] = \bigcup_{i \in X} H(i)^{M[\tilde{G}_i]}$.

By nature of the wellordering D, $M[\tilde{G}]$ is a model of $V = HOD$.

It remains to verify that in $M[\tilde{G}]$, the outer model theory with parameters $\mathrm{OMT}(M[\tilde{G}])$ is lightface definable.

Claim 4.25 *Let φ be in $L_{\infty,\omega} \cap M[\tilde{G}]$. The following are equivalent:*

(i) There is no outer model of $M[\tilde{G}]$ where φ holds.
(ii) There exists $i < \kappa$ such that i is in H and $b(i)(2)$ is the index of φ in the initial segment of the wellordering D coded at X_{i^*}, where i^* is defined from i as in the items below (4.25), and $\dot{C}_i$ codes the killing of φ.

Proof. Assume (i) holds. Then there is a condition $p_0 \in G$ which forces it. Let $\tilde{i} < \kappa$ be such that the support of p is bounded in $\tilde{i}$ in the sense that for every $j > \tilde{i}$, $j < \kappa$, $p(j)$ is the weakest condition in $\dot{Q}_j$. Choose i such that $b(i) = \langle j, k, m \rangle$ satisfies $\tilde{i} < j$, p^j_k is the restriction of p_0 to $\mathbb{P}_j$ and m is the index of $\dot{\varphi}$ in the enumeration $\dot{d}^{i^*}$.[15] Then at stage i, it was possible to choose $(\dot{W}, f)$ such that $p^j_k{}^\frown 1$ kills φ – namely the tail of the iteration $\mathbb{P}$ from $\mathbb{P}_i$ is an example of such $\dot{W}$. Accordingly, $\dot{C}_i$ codes the killing of φ.

Assume the negation of (i). Then there can be no such i as in (ii): if there is such i, then $\dot{Q}_i$ contains $\dot{W}_i$ which kills φ. But this is impossible if the negation of (i) holds. □

This ends the proof of Theorem 4.18.

4.4 Some generalizations

By definability of the outer model theory in $M[\tilde{G}]$ of Theorem 4.18, $M[\tilde{G}]$ is a model where one can define the generalized notion of satisfaction with respect to outer models.

Definition 4.26 *Let T be a theory extending ZFC, and φ a sentence, both T and φ in $L_{\infty,\omega} \cap M[\tilde{G}]$. We write $T \models_{OM} \varphi$ iff φ holds in every outer model of $M[\tilde{G}]$ satisfying T.*

[15] As the referee remarked, it may happen that φ (if considered as a concrete element of $H(i)^{M[\tilde{G}_i]}$ for some i, as we do) may not be in the range of the d^i's; however, a formula forced by the empty condition to be equivalent to φ will always be in the range of the d^i's.

Theorem 4.27 *The relation $\models_{OM}$ is definable in $M[\tilde{G}]$ of Theorem 4.18.*

Proof. Let T, φ be in $L_{\infty,\omega} \cap M[\tilde{G}]$. View ZFC as a single infinitary sentence and denote by ψ the infinitary sentence such that $T = ZFC \ \& \ \psi$. In $M[\tilde{G}]$, $T \models_{OM} \varphi$ iff there is no outer model of $\psi \ \& \ \neg\varphi$ iff $(\neg\psi \vee \varphi)$ is in $\mathrm{OMT}(M[\tilde{G}])$. □

Instead of working with the outer models of M, one might also study the *inner models of M.*

Definition 4.28 *Let V be the universe of sets and let $M \in V$ be a transitive model of ZFC. We say that $N \in V$ is an inner model of M if N is a transitive model of ZFC with the same ordinals as M, and $N \subseteq M$.*

Note that we do not require that N be definable in M.

As in the case of outer models, there may be more inner models of M as the universe V enlarges. However, there is a sentence $\varphi^* \in L_{\infty,\omega} \cap \mathrm{Hyp}(M)$ such that there is an inner model $N \subseteq M$ of φ in some extension of V where M is countable iff $ZFC + \varphi^*$ is consistent.

In analogy with Definition 2.3, let us define:

Definition 4.29 *Let $M \in V$ be a transitive model of ZFC of size κ. We define the* inner model theory of M, *denoted* $\mathrm{IMT}(M)$, *as follows*

$$\mathrm{IMT}(M) = \{\varphi \mid \textit{there is no inner model } N \textit{ of } M, \ N \in W, \textit{ such that } N \models \neg\varphi\}, \tag{4.31}$$

where φ is an infinitary formula in $L_{\infty,\omega} \cap N$ with parameters in N and W is the model $V[G]$, where G is a generic filter for the Levy collapsing forcing which collapses κ to ω.

Theorem 4.18 can be easily modified to yield:

Theorem 4.30 *Assume $V = L$. Let κ be the least inaccessible, and let $M = L_\kappa$. There is a good iteration $(\mathbb{P}, h)$ in V such that if G is $\mathbb{P}$-generic over V, then for some set $\tilde{G}$, which is defined from G, $M[\tilde{G}]$ is a model of ZFC in which* $\mathrm{IMT}(M[\tilde{G}])$ *is lightface definable. Moreover, $M[\tilde{G}]$ is a model of $V = HOD$.*

Proof. The proof proceeds exactly as the proof of Theorem 4.18, with the following exception. At stage i, if possible choose a good iteration $\dot{W}_i$ such that $p^j_k{}^\frown 1$ in $\mathbb{P}_i * \dot{W}_i$ forces that there is an inner model for $\dot{\varphi}$ (more precisely forces that $ZFC + \varphi^*$ is consistent, where φ^* is as in the paragraph preceding Definition 4.29). The property of having an inner model is upwards persistent, as is the property of being killed in the case Theorem 4.18. This persistence makes it possible to define the inner model theory in the final model $M[\tilde{G}]$ as in Claim 4.25. □

By routine modifications, one can get a model $M[\tilde{G}]$ where both $\mathrm{OMT}(M[\tilde{G}])$ and $\mathrm{IMT}(M[\tilde{G}])$ are lightface definable. Or even more generally, one can define the *compatible model theory* of M, $\mathrm{CMT}(M)$, which contains all formulas which hold in all models compatible with M, where M and N of the same ordinal height are compatible if there is N^* of the same ordinal height such that $M, N \subseteq N^*$. Again, Theorem 4.18 can be generalized so that in $M[\tilde{G}]$, $\mathrm{CMT}(M[\tilde{G}])$ is lightface definable.

5 Open questions

One property of omniscience which we have not discussed yet is the robustness of the notion in terms of its preservation. Suppose for instance that M is omniscient and we extend M by a set-forcing in the sense of M. Is $M[G]$ still omniscient? If the omniscience of M is witnessed by large cardinals, then $M[G]$ remains omniscient by Stanley's result (see Theorem 6.1). We do not know whether this holds in general:

Q1. Suppose M is an omniscient model. Is a set-generic extension of M still omniscient?

More specifically, we can ask whether one can modify our forcing construction to obtain an omniscient model which remains omniscient in forcing extensions of a certain type. We may reformulate it as asking for a model whose omniscience is indestructible for some non-empty collection of forcings.

Q2. Can one modify the present forcing to obtain an omniscient model indestructible for a certain non-empty collection of forcing notions?

Using Tarski's undefinability of truth, it is easy to see that L cannot be omniscient. However this does not extend to inner models for large cardinals by Theorem 6.1. An obvious question is therefore the following:

Q3. Suppose $V = K$ is omniscient, where K is the Dodd-Jensen core model. Does there exist a proper class of ω_1-Erdős cardinals in V?

Q4. The iteration $\mathbb{P}$ in Theorem 4.18 has some length $< \kappa^+$. Is it possible to show that $\mathbb{P}$ is actually a subset of $\mathrm{Hyp}(M)$? Or more generally, can one define an iteration $\mathbb{R}$ which achieves the results of Theorem 4.18 and is a subset of $\mathrm{Hyp}(M)$?

Note that with regard to Q4, our construction shows that $\mathbb{P}$ is contained in $\mathrm{Hyp}_2(M)$, the least Σ_2-admissible set containing M as an element (we just need an oracle for consistency, which is Π_1).

In Remark 2.2 we said that there may be other ways to define the collection of outer models to which we refer in defining $\mathrm{OMT}(M)$. We also noted that there are some obvious restrictions which should be considered to have the notion behave reasonably. The following question is relevant in this respect:

Q5. From similar assumptions as in Theorem 4.18, is there a good iteration, or at least a cardinal-preserving iteration, $\mathbb{P}$ and $M \subseteq M^*$ in $V[G]$, where G is a $\mathbb{P}$-generic filter, such that the outer model theory of M^* is definable, with the outer models restricted to be elements of $V[G]$?

Q6. What is the consistency strength of having M in which $\mathrm{OMT}(M)$ is lightface definable? By Theorem 4.18, the upper bound is ZFC plus "there is an inaccessible cardinal." Can this be improved to ZFC + "there is a standard model of ZFC"?

With regard to Q6, note that our construction actually gives a better upper bound than inaccessibility – for the proof of Theorem 4.18, it suffices that κ is inaccessible in $\mathrm{Hyp}_2(V_\kappa)$, the least Σ_2 admissible set containing V_κ as an element.

6 Appendix

6.1 Omniscience from large cardinals

In unpublished work [3], Mack Stanley proved that if M contains many Ramsey cardinals, then M is omniscient. The argument uses Barwise's Theorem 1.2 and the theory of iterated ultrapowers (for measurable cardinals) or sharps (for Ramsey cardinals) to "stretch" properties from rank-initial segments of M to the whole of M, thus making it possible to capture a higher-order property of M in a rank-initial segment of M.

With the permission of Stanley, we give here an outline of his argument that the existence of many measurable cardinals in M implies omniscience.

If M is a transitive set, let $\mathrm{Ord}(M)$ denote the ordinal $\mathrm{Ord} \cap M$. Also, let us denote by *M-logic* the fragment $L_{\infty,\omega} \cap \mathrm{Hyp}(M)$.

Theorem 6.1 (M. Stanley) *Suppose that M is a transitive set model of ZFC. Suppose that in M there is a proper class of measurable cardinals, and indeed this class is* $\mathrm{Hyp}(M)$*-stationary, i.e.* $\mathrm{Ord}(M)$ *is regular with respect to* $\mathrm{Hyp}(M)$*-definable functions and this class intersects every club in* $\mathrm{Ord}(M)$ *which is* $\mathrm{Hyp}(M)$*-definable. Then* $\mathrm{OMT}(M)$ *is M-definable.*

Proof. Using M-logic we can translate the statement that a first-order sentence φ (with parameters from M) holds in some outer model of M to the consistency of a sentence φ^* in M-logic, a fact expressible over $\mathrm{Hyp}(M)$ by a Π_1 sentence. Using this we show that the set of φ which hold in some outer model of M is M-definable, and from this it follows that $\mathrm{OMT}(M)$ is also M-definable.

As $\mathrm{Ord}(M)$ is regular with respect to $\mathrm{Hyp}(M)$-definable functions we can form a club C in $\mathrm{Ord}(M)$ such that for κ in C there is a Σ_1-elementary embedding from $\mathrm{Hyp}((V_\kappa)^M)$ into $\mathrm{Hyp}(M)$ (with critical point κ, sending κ to $\mathrm{Ord}(M)$). Indeed C can be chosen to be $\mathrm{Hyp}(M)$-definable.

For any κ in C let φ^*_κ be a sentence of $(V_\kappa)^M$-logic such that φ holds in an outer model of $(V_\kappa)^M$ iff φ^*_κ is consistent (a Π_1 property of $\mathrm{Hyp}((V_\kappa)^M)$). By elementarity, φ^*_κ is consistent iff φ^* is consistent.

Now suppose that φ holds in no outer model of M, i.e. φ^* is inconsistent. Then φ^*_κ is inconsistent for all κ in C and since the measurables form a $\mathrm{Hyp}(M)$-stationary class, there is a measurable κ such that φ^*_κ is inconsistent.

Conversely, suppose that φ^*_κ is inconsistent for some measurable κ. Now choose a normal measure U on κ and iterate $\mathrm{Hyp}((V_\kappa)^M)$ using U for $\mathrm{Ord}(M)$ steps to obtain a structure $\mathrm{Hyp}(M^*)$. By elementarity, the sentence φ^* which asserts that φ holds in an outer model of M^* is inconsistent. But M^* is an inner model of M, so also the sentence asserting that φ holds in an outer model of M is inconsistent.

Thus φ^* is consistent exactly if φ^*_κ is consistent for all measurable κ, and this is first-order expressible. □

6.2 Forcing omniscience

Mack Stanley independently discovered an easier construction of an omniscient model. However, his proof does not ensure $V = HOD$ in the final model (compare with Theorem 4.18). For the benefit of the reader, we state the result.

Theorem 6.2 (M. Stanley) *Work in L and let κ be inaccessible. There exists $\mathbb{P}(A) \subseteq L_\kappa$ such that if G is $\mathbb{P}(A)$-generic over L, then $L[G]$ is a cofinality preserving extension in which κ remains inaccessible, and in $L_\kappa[G]$ the set of all sentences of the language of set theory with parameters in $L_\kappa[G]$ that hold in all outer models of $L_\kappa[G]$ (calculated in a universe in which κ is countable) is definable without parameters in $L_\kappa[G]$.*

Proof. Set $M = L_\kappa$, where κ is inaccessible in L.

Working in L, define $\mathbb{P}(\kappa)$ to be the Easton support product of the Cohen forcing $\mathrm{Add}(\aleph_{2\alpha+1}, \aleph_{2\alpha+3})$ for $\alpha < \kappa$.

For $A \subseteq \kappa$ in L, set

$$\mathbb{P}(A) = \{p \in \mathbb{P}(\kappa) \mid p(\alpha) = \emptyset \text{ for all } \alpha \in \kappa \setminus A\}.$$

Note that if $A \subseteq B \subseteq \kappa$, then $\mathbb{P}(A) \subseteq \mathbb{P}(B)$ and $M^{\mathbb{P}(A)} \subseteq M^{\mathbb{P}(B)}$. Furthermore, if G is $\mathbb{P}(B)$-generic over L, then $G \cap \mathbb{P}(A)$ is $\mathbb{P}(A)$-generic over L.

Working in L, define $A_\alpha \subseteq \kappa$ by recursion on α. Start by setting $A_0 = \emptyset$. Then declare that β belongs to $A_{\alpha+1}$ when either $\beta \in A_\alpha$ or β codes a pair (p, φ) where $p \in \mathbb{P}(A_\alpha)$ and φ with parameters from $M^{\mathbb{P}(A_\alpha)}$ is such that $p \Vdash_{\mathbb{P}(A_\alpha)} \varphi \in \mathrm{OMT}(L[\dot{G}])$.

If α is a limit, set $A_\alpha = \bigcup_{\gamma<\alpha} A_\gamma$. Finally, set $A = A_\alpha$ where $A_\alpha = A_{\alpha+1}$. Note that A is definable over $L[G]$ for $\mathbb{P}(A)$-generic G.

Claim 6.3 *For $\mathbb{P}(A)$-generic G, $\varphi(x)$ is in $\mathrm{OMT}(L[G])$ iff in $L[G]$ there are a $\mathbb{P}(A)$-name σ, a generic $\bar{G}$ for $\mathbb{P}_A|\delta$ for some singular δ and a condition p in $\bar{G}$ such that (a code for) the pair (p, σ) is in the (definable) predicate A and $\sigma^{\bar{G}}$ equals x.*

Proof. The direction left-to-right is easy, as we can just take $\sigma^G = x$, $\bar{G}$ to be $G \cap \mathbb{P}_A|\delta$ for some large enough δ and p in G to force $\varphi(x)$ into $\mathrm{OMT}(L[\dot{G}])$. Conversely, the right-hand-side implies that $\varphi(x)$ belongs to $\mathrm{OMT}(L[G^*])$ where G^* agrees with $\bar{G}$ below δ and with G above δ (G^* is generic as G above δ does not add subsets of δ), and therefore to $\mathrm{OMT}(L[G])$ as $L[G]$ contains $L[G^*]$. □

The Claim shows that $\mathrm{OMT}(L[G])$ is definable in $L[G]$ for $\mathbb{P}(A)$-generic G, and therefore finishes the proof of Theorem 6.2. □

References

[1] Jon Barwise. *Admissible sets and structures.* Springer, 1975.

[2] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs.* North Holland, 1980.

[3] M. C. Stanley. Outer model satisfiability. An unpublished manuscript.