Two observations regarding infinite time Turing machines

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Abstract

We observe: (I) There is a *"Theory Machine"* that can write down the Σ_2-Theories of the levels of the J-hierarchy up to Σ (the least Σ such that some smaller L_ζ is Σ_2 elementary in L_Σ) in a uniform way. Moreover, below Σ these theories are all distinct. This yields information about the halting times of ITTM's. (II) The ITTM degrees of the semi-recursive singletons are well-ordered in order type the least stable, i.e., the least σ such that L_σ is Σ_1 elementary in L.

The *Theory Machine* generates theories of initial segments of the J-hierarchy. This machine can be used to prove the "ζ-Σ theorem" and analyse the halting times of ITTM's.

The idea of the Theory Machine is to write down the theory of $(J_\alpha, \in)$ (appropriately Gödel-numbered) on the output tape at computation stage $\omega^2 \cdot (\alpha+1)$, for as long as possible. This will be easy to arrange for successor α, as long as a code for the structure $(J_\alpha, \in)$ can be read off from its theory. For limit α, the machine performs a liminf operation, resulting in a theory T_α; we show that the Σ_2 theory of $(J_\alpha, \in)$ is recursive in the Turing jump of T_α, uniformly in α. Provided a code for the structure $(J_\alpha, \in)$ can be read off from its Σ_2 theory, this will enable the machine to write down the theory of $(J_\alpha, \in)$ at stage $\omega^2 \cdot \alpha + \omega^2$. A fine-structural analysis shows that as long as α is less than the least Σ such that J_ζ is Σ_2 elementary in J_Σ for some $\zeta < \Sigma$, a code for $(J_\alpha, \in)$ can indeed be read off from its Σ_2 theory, uniformly. Therefore the machine will produce distinct theories of structures

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$(J_\alpha, \in)$ for $\alpha < \Sigma$, and then at stage Σ repeat what it wrote on the output tape at stage ζ.

A corollary is that, up to a "small" error, the halting times of ITTM's are exactly the ordinals $\alpha < \Sigma$ where sentences become true for the first time in the J-hierarchy, i.e., such that some sentence φ of set theory holds in $(J_\alpha, \in)$ but not in $(J_\beta, \in)$ for any $\beta < \alpha$.

The following two claims are crucial to verifying properties of the theory machine.

Lemma 1 *For a limit λ, let T denote the set of Σ_2 sentences that are true in $(J_\alpha, \in)$ for sufficiently large $\alpha < \lambda$. Then the Σ_2 theory of $(J_\lambda, \in)$ is RE in T. Moreover an index for this RE reduction is uniform in λ.*

Proof. Let φ be a Σ_2 sentence and write φ as $\exists x \psi(x)$ where $\psi(x)$ is Π_1. Also let $h_1(n, x)$ denote the canonical Σ_1 Skolem function; h_1 has a parameter-free Σ_1 definition and for any α, h_1 interpreted in J_α is a partial function from $\omega \times J_\alpha$ into J_α whose range on $\omega \times [A]^{<\omega}$ is the Σ_1 Skolem hull of A in $(J_\alpha, \in)$ (i.e., the universe of the least Σ_1 elementary submodel of $(J_\alpha, \in)$ containing A), for any $A \subseteq J_\alpha$. We say that an ordinal α is Σ_1 *stable* (in the universe) iff every true Σ_1 sentence with parameters from J_α is true in $(J_\alpha, \in)$.

We have the following equivalence:

$(J_\lambda, \in)$ satisfies φ iff
For some n, the following holds in $(J_\alpha, \in)$ for large enough $\alpha < \lambda$: There is a β which is either 0 or Σ_1 stable such that either φ holds in $(J_\beta, \in)$ or $h_1(n, \beta)$ is defined and $\psi(h_1(n, \beta))$ holds.

This equivalence is verified as follows:

Suppose that $(J_\lambda, \in)$ satisfies φ. If $(J_\beta, \in)$ satisfies φ for some β which is Σ_1 stable in λ (i.e., $\beta < \lambda$ and $(J_\beta, \in)$ is Σ_1 elementary in $(J_\lambda, \in)$), then for all α between β and λ, φ will also hold in $(J_\alpha, \in)$, as β is also Σ_1 stable in α. So the right half of the equivalence holds in this case. Otherwise let β be the largest β which is Σ_1 stable in λ (or 0 is there is no β which is Σ_1 stable in λ). Then every element of J_λ is of the form $h_1(n, \beta)$ for some n (as the Σ_1 Skolem hull of $\{\beta\}$ in $(J_\lambda, \in)$ is all of J_λ). Choose n so that $\psi(h_1(n, \beta))$ holds in J_λ. Then for sufficiently large $\alpha < \lambda$, $h_1(n, \beta)$ is defined in $(J_\alpha, \in)$, and $\psi(h_1(n,\beta))$ holds in $(J_\alpha, \in)$ as ψ is Π_1. So the right half of the equivalence also holds in this case.

Conversely, suppose that the right half of the equivalence holds and choose n to witness that. First suppose that the Σ_1 stables in λ are cofinal in λ. Then apply the right half of the equivalence to some α which is Σ_1 stable in λ. Then either φ holds in $(J_\beta, \in)$ for some β which is Σ_1 stable in α or $\psi(h_1(n,\beta))$ holds in $(J_\alpha, \in)$ for some β; in the former case φ holds in $(J_\lambda, \in)$ as β is Σ_1 stable in λ and in the latter case this holds as α is Σ_1 stable in λ. Now suppose that the Σ_1 stables in λ are bounded in λ and let β be the largest Σ_1 stable in λ (or 0 is there is no β which is Σ_1 stable in λ). Choose α to be sufficiently large in the sense of the right hand side of the equivalence and also such that there are no α-stables greater than β. (For example, choose n so that $h_1(n,\beta)$ is large enough and let α be least so that $h_1(n,\beta)$ is defined in $(J_\alpha, \in)$.) Then applying the right hand side of the equivalence to α, there is a β' which is either 0 or Σ_1 stable in α such that either φ holds in $(J_{\beta'}, \in)$ or $\psi(h_1(n,\beta'))$ holds in $(J_\alpha, \in)$. In the former case, β' is at most β and therefore is Σ_1 stable in λ; it follows that φ holds in $(J_\lambda, \in)$. In the latter case, argue as follows: If β' is less than β, then $h_1(n,\beta')$ in fact belongs to J_β and $\psi(h_1(n,\beta'))$ holds in $(J_\beta, \in)$, implying that φ holds in $(J_\lambda, \in)$. If β' equals β then $\psi(h_1(n,\beta))$ holds in J_α, and as α can be chosen arbitrarily large, $\psi(h_1(n,\beta))$ holds in $(J_\lambda, \in)$; it follows that φ holds in $(J_\lambda, \in)$, as desired.

The equivalence shows that the Σ_2 theory of $(J_\lambda, \in)$ is RE in T. And this RE definition is independent of λ. □

Lemma 2 *Let Σ be least so that some $\zeta < \Sigma$ is Σ_2 stable in Σ (i.e., $(J_\zeta, \in)$ is Σ_2 elementary in $(J_\Sigma, \in)$). Let T^α be the Σ_2 theory of the structure $(J_\alpha, \in)$. Then there is a real code for this structure which is recursive in T^α. Moreover, the reduction of this code to T^α is uniform in α.*

Proof. It suffices to show that there is a partial function f from ω onto J_α which is Σ_2 definable over $(J_\alpha, \in)$ without parameter (uniformly in $\alpha < \Sigma$). For given this, consider the set of n such that $f(n)$ is defined, modulo the equivalence relation $n \sim m$ iff $f(n) = f(m)$, together with the binary relation nEm iff $f(n) \in f(m)$. This yields an isomorphic copy of $(J_\alpha, \in)$.

Let $\psi_n(x)$ be the n-th Π_1 formula with free variable x. Define:

$f'(n) = (m, \beta)$ iff the following hold in $(J_\alpha, \in)$:
i. β is 0 or Σ_1 stable.
ii. $h_1(m, \beta) = x$ is defined.
iii. $\psi_n(x)$ holds.
iv. $\psi_n(x')$ fails for any x' of the form $h_1(m', \beta')$, $\beta' < \beta$, $m' < \omega$.
v. $m' < m \to h_1(m', \beta)$ is undefined or $\psi_n(h_1(m', \beta))$ fails.

Clauses i-iii are easily seen to be either Σ_1 or Π_1. Clause v is equivalent to a disjunction of a Σ_1 formula and a Π_1 formula. Clause iv is vacuous if β is 0 and otherwise holds in $(J_\alpha, \in)$ iff it holds in $(J_\beta, \in)$; it follows that clause iv is Σ_1. And f' is single-valued and therefore a partial function from ω into J_α which is Σ_2 definable over $(J_\alpha, \in)$ without parameter.

Now define $f(n) = h_1(f'(n))$ and let A be the Σ_1 Skolem hull of the range of f. Then A is the range of a partial function g from ω into J_α which is Σ_2 definable over $(J_\alpha, \in)$ without parameter. (Define $g(n) = h_1(n_0, \langle f(n_1), \ldots, f(n_k) \rangle)$, if n codes the sequence $(n_0, \ldots, n_k)$.)

We claim that $(A, \in)$ is Σ_2 elementary in $(J_\alpha, \in)$: Clearly $(A, \in)$ is Σ_1 elementary in $(J_\alpha, \in)$, as it is the Σ_1 Skolem hull of the range of f. Write $g(n) = x$ iff $(J_\alpha, \in) \models \exists y \psi(n, x, y)$, where ψ is Π_1. Now suppose that there exists x in J_α such that $(J_\alpha, \in) \models \gamma(x, g(n_1), \ldots, g(n_k))$, where γ is Π_1. Then there exists $\langle x, x_1, y_1, \ldots, x_k, y_k \rangle$ in J_α such that the following Π_1 formula holds in $(J_\alpha, \in)$:

$$\gamma(x, x_1, \ldots, x_k) \wedge \psi(n_1, x_1, y_1) \wedge \ldots \wedge \psi(n_k, x_k, y_k).$$

By the definition of f, there exists such a sequence $\langle \bar{x}, \bar{x}_1, \bar{y}_1, \ldots, \bar{x}_k, \bar{y}_k \rangle$ in the range of f. Also $\bar{x}$ belongs to A and $\bar{x}_i$ equals $g(n_i)$ for each i, $1 \leq i \leq k$, and therefore $\gamma(\bar{x}, g(n_1), \ldots, g(n_k))$ holds in $(J_\alpha, \in)$. As $(A, \in)$ is Σ_0 elementary in $(J_\alpha, \in)$, it follows that $\gamma(\bar{x}, g(n_1), \ldots, g(n_k))$ holds in $(A, \in)$ for some $\bar{x}$, proving that $(A, \in)$ is Σ_2 elementary in $(J_\alpha, \in)$.

Finally, every element of J_α is countable in $(J_\alpha, \in)$, as otherwise there would be a Σ less than α such that some $\zeta < \Sigma$ is Σ_2 stable in Σ. It follows that A is transitive, as by Σ_1 elementarity, A contains an injection of any of its elements into ω. We have assumed that α is less than Σ, so in fact A equals all of J_α, and therefore there is a partial function g from ω onto J_α which is Σ_2 definable over $(J_\alpha, \in)$ without parameter, as desired. $\square$

Now we are ready to describe the Theory Machine. When we say that the machine writes a theory T on its output tape at stage α, we mean that at stage α, the n-th cell of the output tape has a 1 written in it iff the n-th sentence (via a fixed Gödel numbering) belongs to T. Now the Theory Machine runs as follows: On input 0, the machine uses the first ω^2 stages to ensure that the theory of $(J_0, \in)$ is written on the output tape at stage ω^2. (In fact the machine could arrange this in fewer stages, but we prefer for this to occur at stage ω^2). Inductively, suppose that the theory of $(J_\alpha, \in)$ is written on the output tape at stage $\omega^2 \times (\alpha + 1)$. If α is less than Σ, then by Lemma 2, the machine can compute a code for $(J_\alpha, \in)$ by stage $\omega^2 \times (\alpha + 1) + \omega$. Then the machine uses the theory of $(J_\alpha, \in)$ to compute a code for $(J_{\alpha+1}, \in)$ by stage $\omega^2 \times (\alpha + 1) + \omega + \omega$ and the next ω^2 steps to write the theory of $(J_{\alpha+1}, \in)$ on its output tape at stage $\omega^2 \times (\alpha+1)+\omega+\omega+\omega^2 = \omega^2 \times (\alpha+2)$. The machine must however never write a 0 in the n-th cell of its output tape (at a stage between $\omega^2 \times (\alpha+1)$ and $\omega^2 \times (\alpha+2)$) if the n-th sentence is true in both $(J_\alpha, \in)$ and $(J_{\alpha+1}, \in)$.

The last requirement ensures that at a stage $\omega^2 \times \lambda$, λ limit, what is written on the output tape is the liminf of the theories of the $(J_\alpha, \in)$, $\alpha < \lambda$, i.e. the theory $T = \{\varphi \mid \varphi \text{ is true in } (J_\alpha, \in) \text{ for sufficiently large } \alpha < \lambda\}$. By Lemma 1, the machine can compute the Σ_2 theory of $(J_\lambda, \in)$ by stage $(\omega^2 \times \lambda) + \omega$ and by Lemma 2 it can compute a code for $(J_\lambda, \in)$ by stage $(\omega^2 \times \lambda) + \omega + \omega$, if λ is less than Σ. Then the machine uses the next ω^2 stages to write the theory of $(J_\lambda, \in)$ on its output tape, again never writing a 0 in the n-th cell of its output tape if the n-th sentence belongs both to T and to the theory of $(J_\lambda, \in)$.

This completes the description of the Theory Machine. The machine is capable of writing the theory of $(J_\alpha, \in)$ on its output tape at stage $\omega^2 \times (\alpha+1)$ provided α is less than Σ. The following corollaries easily follow, where λ is the least Σ_1 stable in Σ and ζ is the least Σ_2 stable in Σ:

On input 0:

Every ITTM either halts or repeats itself by stage Σ.
There is a machine that first repeats itself at stage Σ.
The supremum of the halting times of ITTM's is λ.
The reals that appear on the output tape of an ITTM are the reals in $J_\Sigma = L_\Sigma$.
The reals that appear on the output tape of a halting ITTM are the reals in $J_\lambda = L_\lambda$.
The reals that appear on the output tape of an ITTM from some stage onwards are the reals in $J_\zeta = L_\zeta$.

Also, if Σ^x, ζ^x, λ^x are the relativisations of Σ, ζ, λ to the real x:

A is an ITTM-semirecursive set of reals iff for some Σ_1 formula φ, we have: x belongs to A iff $L_{\lambda^x}[x] \vDash \varphi(x)$.

One can say a bit more about the halting times of ITTM's. Say that α is an *infinite power of* ω iff it is of the form ω^β, where β is infinite. An *infinite power of* ω *interval* is an interval $[\alpha, \beta)$ where $\alpha < \beta$ are adjacent infinite powers of ω. For any sentence φ of set theory let $\alpha(\varphi)$ denote the least α, if any, such that $(J_\alpha, \in)$ satisfies φ.

Corollary 3 *Let I be an infinite power of ω interval. Then the following are equivalent.*
i. I contains the halting time of an ITTM.
ii. I is below Σ and contains $\alpha(\varphi)$ for some sentence φ.

Proof. Suppose that α is the halting time of an ITTM. Then this can be expressed in $(J_{\alpha+1}, \in)$ and therefore $\alpha + 1 = \alpha(\varphi)$ for some φ. Conversely, suppose that $\alpha = \alpha(\varphi)$ for some φ and α is less than Σ. Then there is an ITTM that imitates the Theory Machine but halts when it sees that φ is true in $(J_\alpha, \in)$, at a stage less than $\omega^2 \times (\alpha + 1) + \omega^2 = \omega^2 \times (\alpha + 2)$. As the latter is less than the least infinite power of ω greater than α, it follows that $\alpha(\varphi)$ and α belong to the same infinite power of ω interval. $\square$

The previous corollary easily yields results about gaps in the set of halting times of ITTM's.

Our second observation concerns Γ-singletons, where Γ is the lightface pointclass of semirecursive sets of reals.

Theorem 4 *Suppose that x is a Γ-singleton, i.e., $\{x\}$ belongs to Γ. Then x is an element of L_{λ^x}.*

Proof. Let x be the unique x such that $L_{\lambda^x}[x] \vDash \varphi(x)$, where φ is Σ_1. Let c be a real which is generic over L_{Σ^x} for the Lévy collapse of λ^x to ω. By absoluteness, there is a real y in $L_{\Sigma^x}[c]$ such that $\varphi(y)$ holds in $L_{\lambda^x}[y]$ and λ^x is less than Σ^y. It follows that $\varphi(y)$ holds in $L_{\Sigma^y}[y]$, therefore in $L_{\lambda^y}[y]$ and therefore y equals x. As c is an arbitrary generic code for λ^x, x belongs to L_{Σ^x} and therefore to L_{λ^x}. $\square$

Corollary 5 *The ITTM-degrees of Γ-singletons are wellordered in order-type δ^1_2, the supremum of the lengths of Δ^1_2 wellorderings of ω, with successor given by ITTM-jump.*

Proof. If $\lambda^x \leq \lambda^y$ and x is a Γ-singleton then x belongs to L_{λ^y} and therefore is recursive in y. If $\lambda^x < \lambda^y$ then as the ITTM-jump of x is definable over $L_{\lambda^x}[x] = L_{\lambda^x}$, it follows that the ITTM-jump of x is recursive in y. The Γ-singletons include the Π^1_1-singletons, which are cofinal in $L_{\delta^1_2}$, and therefore the length of the wellordering of the ITTM-degrees of Γ-singletons is also δ^1_2. $\square$

Remarks. i. In fact the ITTM-degrees of Δ-singletons are cofinal in those of the Γ-singletons, where Δ is the lightface pointclass of recursive sets of reals. This is because each Π^1_1-singleton is a Δ-singleton.
ii. There are reals with ITTM-degree incomparable with $0'$ = the ITTM-jump of 0; for example, consider a real Cohen generic over L_Σ. But this cannot happen for reals in L_Σ, as such a real x belongs to L_{λ^x} and therefore is either ITTM-recursive or ITTM above $0'$. By using Sacks forcing one obtain a continuum of minimal ITTM-degrees over 0.