# Hypermachines

Sy-David Friedman (KGRC, Vienna)
Philip Welch (Bristol)

*13.xi.2009*

**Abstract**

The Infinite Time Turing Machine model [8] of Hamkins and Kidder is, in an essential sense, a "$\Sigma_2$-machine" in that it uses a $\Sigma_2$ *Liminf Rule* to determine cell values at limit stages of time. We give a generalisation of these machines with an appropriate $\Sigma_n$ rule. Such machines either halt or enter an infinite loop by stage $\zeta(n) =_{\mathrm{df}} \mu\zeta(n)\,[\ \exists\Sigma(n) > \zeta(n) \quad L_{\zeta(n)} \prec_{\Sigma_n} L_{\Sigma(n)}]$, again generalising precisely the ITTM case.[1]

## 1 Introduction

The Infinite Time Turing Machine (ITTM) model described in [8] is an attractive model of *transfinite time computation* based on the standard Turing machine with an infinite one way tape, and a finite transition table or instruction set. The latter specifies how the machine behaves at successor steps as is usual, and one needs really only to specify precisely how such a machine behaves at limit steps in time to give a complete description. The model in [8] resets the read/write head position on the first cell (or rather triplet of first cells on each of three parallel tapes for *input, output,* and *scratch work*) and assigns 0/1 values to a cell contents by means of a *limsup* rule: the value at time $\lambda$ is the limsup of the previous values. The use of limsup as opposed to *liminf* is immaterial in terms of computational functionality, and we tend to use *liminf* (as in this paper). The basic properties of these machines were explored in [8]; in [11] the *halting problem* and correspondingly, the *decidable*, and *semi-decidable* sets of integers were characterised. The companion structure to this notion of computation turned out to be the least level of the Gödel $L$-hierarchy, $L_\zeta$, which has a proper $\Sigma_2$-elementary end extension. The decidable sets of integers are those $\Delta_1$ definable (without parameters) over $\langle L_\zeta, \in\rangle$. (In the terminology of an earlier age, the ITTM decidable sets of integers are the "*abstract 1-section*" of the admissible set $\langle L_\zeta, \in\rangle$, and thus, by a theorem of Sacks [10], form the *1-section* of some *type-2 functional* $G$: 1-sc($G$). It is therefore possible to view this form of computation as a particular example of higher type recursion in the manner of generalised recursion theory).

With hindsight this is perhaps not unsurprising: either of the rules mentioned exhibits the essential $\Sigma_2$-nature of the machines: *e.g.* under *liminf*, the value of a cell at a limit time $\lambda$ is 1 if and only if "$\exists\beta < \lambda\forall\gamma \in (\beta, \lambda)[....]$". It is therefore apparent that if $L_\zeta \prec_{\Sigma_2} L_\Sigma$, then thinking of the machines running inside $L$ (which we may, as their construction and operation is absolute to $L$), we should have that the machine has either halted or is entering an infinite loop at time $\zeta$.

---

[1] The first author would like to thank the FWF (Austrian Science Fund) for its generous support through Grant # P19375-N18. The second author would similarly like to acknowledge the generous support of the Templeton Foundation and the EPSRC of the UK (for Grant No. EP/G020841/1), and the Kurt Gödel Research Centre, Vienna for its hospitality.

This naturally leads to the question: "Is there a notion of computation, presumably based on a $\Sigma_3$ style rule, that has as corresponding companion structure the level of the $L$-hierarchy at the least $\zeta(3)$ with some $\Sigma(3) > \zeta(3)$ and $L_{\zeta(3)} \prec_{\Sigma_3} L_{\Sigma(3)}$?" Such a notion would then have its "$\Sigma_3$-ITTM-*decidable sets*" as those of the abstract 1-section of $\langle L_{\zeta(3)}, \in \rangle$, *i.e.* those that are $\Delta_1(\langle L_{\zeta(3)}, \in \rangle)$, and would then presumably either be halted or be entering an infinite loop by time $\zeta(3)$.

It is the purpose of this note to provide such a notion. We further indicate how to generalise this to give a positive answer to the question for larger $n$ with $\Sigma_n$-limit rules. We only assume some familiarity with [8], with the basic facts of the constructible hierarchy $L_\alpha$ and the Jensen version, the $J_\alpha$-hierarchy (see [2], [5] or Jensen's original paper [9]). For $H$ any class of ordinals we let $H^*$ be the class of limit points of $H$. We remark that all of these arguments could be formulated within second order number theory.

In a concluding section we mention some open problems, but also connections with other notions of quasi-induction in the literature, and connections with determinacy questions.

## 2 The $\Sigma_3$-Machine construction

The $\Sigma_3$-machine has four tapes: besides the three parallel tapes of [8] for *input, scratch* and *output*, there is an additional parallel *rule* tape that the Read/Write head surveys, and reads and writes to, just as for the other three. (This is a convenience only: we could imagine the rule tape as being absorbed as a simple infinite recursive subtape of the official scratch tape, and stipulations about the rule tape which we are about to formulate could be made in terms of some priorly fixed recursive function $F$: ${}^\omega 2 \longrightarrow {}^\omega 2$ applied to the scratch tape cells $\langle C_{3i+1} \mid i < \omega \rangle$.) A *computation* is defined much as before: if $P_e(x)$ is the $e$'th program acting on input $x$, the successor stages are governed by the program instructions just as for a standard Turing machine. If we enumerate the cells' values (for all the tapes) at time $\nu$ by $\langle C_i(\nu) \mid i < \omega \rangle$ then at limit stages $\lambda$ the value of $C_i(\lambda)$ is given by a generalised limit rule controlled dynamically by the rule tape. (This is not the only method one can envisage for developing a new limit rule: one could by exterior fiat describe a rule in an absolute manner independent of which computation is being run - as is the case for $n = 2$. However this would result in a somewhat arbitrary action for programs not computing *universal computations*. We therefore prefer the dynamic approach to be given below.)

Before describing this action we make some preliminary definitions.

**Definition 2.1.** *An ordinal $\alpha$ is called* good *if $\omega\alpha = \alpha$. We write $\alpha \in G$ for $\alpha$ good or $\alpha = 1$.*

**Remark 2.2.** $\alpha$ is good iff it is a multiple of $\omega^\omega$; if $\gamma \in \mathrm{On}$, we write $\gamma^+$ for the least limit of good ordinals greater than $\gamma$. This is $\gamma + \omega^\omega \cdot \omega$.

Instead of taking *liminf*'s over all ordinals $\nu < \lambda$ for $\mathrm{Lim}(\lambda)$ in order to determine $C_i(\lambda)$ we do this for a restricted set of ordinals below $\lambda$ which correctly reflect certain patterns of occurrence on the rule tape. These ordinals we shall call 1-*correct in* $\lambda$. This we shall define in a moment, but we state the limit rule now:

**Definition 2.3.** *Given a Turing machine program $P_e(y)$ with input $y$, $C_i(\nu)$ is defined as follows:*

*If* $\nu=\bar{\nu}+1$ *then* $C_i(\nu)$ *is determined by the usual program action from* $\vec{C_i}(\bar{\nu})$*.*
*If* $\nu=\lambda$ *a limit, then*
$C_i(\lambda) =_{\mathrm{df}} \liminf_{\substack{\nu\to\lambda\\ \nu\,1-\text{correct at }\lambda}} C_i(\nu)$*. As shorthand we shall write:*
$C_i(\lambda)= \liminf^*_{\nu\to\lambda} C_i(\nu)$*.*

To put this another way:
$C_i(\lambda)=1 \Longleftrightarrow \exists\nu_0<\lambda\forall\nu\in(\nu_0,\lambda)[\nu \text{ 1-correct at } \lambda \longrightarrow C_i(\nu)=1]$. *Otherwise* $C_i(\lambda)=0$.

To make sense of this we need to define "1-correct in $\lambda$". Very broadly, a computation may work with snapshots, or reals $x$, that appear on a scratch tape (or a recursive sub-tape) and may produce certain information about $x$. Below $\lambda$ there may be stages, some sort of stable point, at which the computation has said all it is going to say about $x$ (before stage $\lambda$). We might then define a *stable* point (below $\lambda$) as some $\alpha$, where any snapshot/scratch tape real $x$, if it appears before stage $\alpha$, also has that all information about $x$ has also appeared before stage $\alpha$: that is, nothing new is said about such $x$ in the interval $(\alpha,\lambda)$. Such points are those then of *stable informational content*. A first approximation to our new $\liminf^*_{\nu\to\lambda}$ rule is that we could take liminf's along these points of stable content. In fact we do something slightly more refined, and also more generous in terms of points: we shall take liminf's along those points $\alpha$ where $\alpha$ sees the same stable points (in the above sense) below it, that $\lambda$ sees are below $\alpha$. Such an $\alpha$ will be called "1-*correct in* $\lambda$."

We use the following notation: for $y\in 2^{\mathbb{N}}$ let $1\frown y\in {}^{\omega}2$ be defined by: $1\frown y(k+1)=y(k)$ and $1\frown y(0)=1$. Then $1\frown y$ is the sequence $y$ prefixed by a '1'. Let $n*y\in {}^{\omega}2$ be similarly $y$ prefixed by $n$ zeroes. We assume the rule tape values are listed as: $\langle R_i(\nu)\mid i<\omega\rangle =_{\mathrm{df}} \langle C_{4i}(\nu)\mid i<\omega\rangle$. We abbreviate $\langle R_i(\nu)\mid i<\omega\rangle$ by $R(\nu)$ *etc.* In the following $\lambda$ will always denote a limit ordinal.

**Definition 2.4.** $S^1_\lambda$ *(the* 1-stable in $\lambda$ *ordinals) Set:*
$\alpha\in S^1_\lambda \Longleftrightarrow \alpha\in G\cap\lambda\wedge$
$\forall x\in {}^{\omega}2\forall\nu<\alpha\forall n\forall\beta<\lambda[(1\frown x=R(\nu)\wedge R(\beta)=n*1\frown x)\longrightarrow\exists\beta'<\alpha(R(\beta)=R(\beta'))]$.

Thus: any pattern of the form $n*1\frown x$ (where $1\frown x$ itself occurred before $\alpha$) which occurs before $\lambda$ must also have occurred before $\alpha$. The following may be established.
- If $\beta\in S^1_\lambda\cap\alpha$ then $\beta\in S^1_\alpha$.
- If $\alpha\in S^1_\lambda$ then $S^1_\lambda\cap\alpha=S^1_\alpha$.
- $S^1_\lambda\subseteq\lambda$ and is closed below $\lambda$.

**Definition 2.5.** $E^1_\lambda$ *(the* 1-correct in $\lambda$ ordinals*)*
*If* $\lambda\notin G^*$ *then* $E^1_\lambda=\lambda$*;*
*If* $\lambda\in G^*$ *then* $E^1_\lambda =_{\mathrm{df}} \{\alpha<\lambda\mid S^1_\lambda\cap\alpha=S^1_\alpha\}$.

We also note:

- $\alpha\in S^1_\lambda\longrightarrow\alpha\in E^1_\lambda$; if $\lambda\in G^*$ then $E^1_\lambda\subseteq G$;
- $E^1_\lambda$ is closed and unbounded in $\lambda$.

(Proof of this latter remark: closure is clear; assume $\lambda\in G^*$ but the unboundedness failed. Then $\alpha_0=\max S^1_\lambda<\lambda$. Thus on a tail of $\alpha\in\lambda$ there is a least $\beta(\alpha)\in S^1_\alpha\backslash S^1_\lambda$. Such a $\beta(\alpha)$ is greater than $\alpha_0$. However if $\alpha'>\alpha$ is least with $\beta(\alpha)\notin S^1_{\alpha'}$ (and such must exist as $\beta(\alpha)\notin S^1_\lambda$). one can see that $\max(S^1_{\alpha'})=\alpha_0$. Thus $\alpha'\in E^1_\lambda$.)

Notice that this definition means that for ordinals $\lambda \notin G^*$, the modified *liminf** rule is just the previous standard *liminf* rule. It is only for the limits of good ordinals (the ones we are principally interested in) that the rule may be different. This gives some substance to our previously described motivation that we take *liminf*'s at limits of good ordinals by considering only those earlier stages which are *"correctly reflecting of stable informational content."* This finishes the description of the general $\Sigma_3$-ITTM acting for a general program $P_e$. We next perform the more difficult task of demonstrating that there is a program that first commences a loop at stage $\zeta(3)$.

## 3 The $\Sigma_3$-Theory Program

We now describe an algorithm programmable as some $P_e$ on a $\Sigma_3$-ITTM-machine, and demonstrate that it first enters an infinite loop at the lexicographically least pair ($\zeta(3)$, $\Sigma(3)$) where $\Sigma(3) > \zeta(3) \wedge\ L_{\zeta(3)} \prec_{\Sigma_3} L_{\Sigma(3)}$. During its run it will produce codes $l(\alpha)$ for levels of the hierarchy $\langle J_\alpha, \in \rangle$ for $\alpha < \Sigma(3)$ in parallel with their complete theories. The reader may have noticed the similarity between the above definitions of "1-stability" to those obtained in the $L$-hierarchy. This of course is no accident. We now define the $L$-counterparts to the above definitions. We recall also that in the definition of the Jensen $J$-hierarchy, that $\mathrm{On} \cap J_\alpha = \omega.\alpha$ (and thus $J_1 = \mathrm{HF} = L_\omega$), and that $L_\alpha = J_\alpha$ if and only if $\alpha$ is good. We let the language of set theory be $\mathcal{L}_{\dot\in}$ and we assume for any $0 < n \leq \omega$ that we have a recursive enumeration of the following $\Sigma_n$-formulae of that language $\langle \varphi^n_i(v_0, v_1) \mid i < \omega \rangle$ with the two variables displayed. For $\alpha \leq \beta$ $J_\alpha \prec_{\Sigma_n} J_\beta$ has its usual meaning: given $\varphi^n_i(v_0, v_1)$ and $x_1 \in J_\alpha$, if $J_\beta \models \exists v_0 \varphi^n_i(v_0, , x_{1,})$ then $J_\alpha \models \exists v_0 \varphi^n_i(v_0, x_{1,})$.

**Definition 3.1.** $\widehat{S}^n_\lambda$ *(the* $\Sigma_n$-stable in $\lambda$ ordinals*) We set:*
$\alpha \in \widehat{S}^n_\lambda \Longleftrightarrow \alpha < \lambda \wedge J_\alpha \prec_{\Sigma_n} J_\lambda$.

- "$\alpha \in \widehat{S}^n_\lambda$" is uniformly $\Pi^{J_\lambda}_n$-definable. For any $\delta$, $\widehat{S}^1_\delta \subseteq (G^*)^*$.
- $\widehat{S}^n_\lambda$ is a closed subset of $\lambda$.

We remark the following: suppose $\beta_0$ is the least ordinal so that $L_{\beta_0} \models \mathrm{ZF}^-$. Then every ordinal $\beta < \beta_0$ satisfies $J_\beta \models$"$V = \mathrm{HC}$" (that is, every set is hereditarily countable). Moreover for such $\beta$ if $\gamma < \beta$ is $\Sigma_1$-definable in $J_\beta$ by some parameter free formula, then since there is a $\Delta^{J_\beta}_1$ definable map (in the parameter $\gamma$) of $\omega$ onto $\gamma$, every ordinal $\gamma' \leq \gamma$ is also so definable. This ensures that if $X \prec_{\Sigma_1} J_\beta$ then $X \cap \omega.\beta \in \omega.\beta + 1$. Moreover, if $\gamma < \beta$ is additionally closed under the Gödel pairing function then standard methods show there is a uniform parameter free map $\omega.\gamma \leftrightarrow J_\gamma$ which is also $\Delta^{J_\gamma}_1$. In particular any finite tuple $\vec{x}$ from $J_\gamma$ is enumerated by some ordinal $\xi < \gamma$ under this map. It follows that for such $\gamma$, an $X$ as above containing $\gamma$, also contains $J_\gamma$ as a subset.

For $\beta < \beta_0$ arguments from [6] can be used to establish that for each $n > 0$ $J_\beta$ has a *uniform parameter free* $\Sigma_n$*-Skolem function* $h^n_\beta(v_0, v_1)$. (In general this fails for levels $J_\gamma$ and $n > 1$.) The function $h^n_\beta$ is itself $\Sigma^{J_\beta}_n$-definable without parameters, with (as an inspection of the argument of [6] shows) the same definition uniformly for any $\beta < \beta_0$. The Skolem function *uniformises* the $\Sigma^{J_\beta}_n$ relations: given an $x \in J_\beta$ if there is a $y \in J_\beta$ such that $J_\beta \models \varphi^n_i[x, y]$, then $h^n_\beta[i, x]$ is a witness such that $J_\beta \models \varphi^n_i[x, h^n_\beta[i, x]]$. These Skolem functions readily yield $\Sigma_n$*-Skolem hulls*: if $A \subseteq J_\beta$ then $h^n_\beta$ " $\omega \times [A]^{<\omega}$ is the smallest $\Sigma_n$-elementary submodel of $J_\beta$ containing $A$ and is thus the $\Sigma_n$-Skolem hull of $A$ in $J_\beta$.

For such $\beta<\beta_0$ this has the ready consequence, for example, that if $\gamma_0=\sup\widehat{S}^n_\beta<\beta$ then the $\Sigma_n$-Skolem hull of $\{\gamma_0\}$ in $J_\beta$ is all of $J_\beta$. (For, as remarked above, every $\gamma<\gamma_0$ must be in the $\Sigma_1$-Skolem hull of $\{\gamma_0\}$, and hence $h^n_\beta$ " $\omega\times\{\gamma_0\}$ is a transitive $\Sigma_n$-elementary submodel of $J_\beta$ of the form $J_\delta\prec_{\Sigma_n}J_\beta$; as $\gamma_0\in J_\delta$ then we conclude $\delta=\beta$.) In particular if $\gamma_0=0$ (because $J_\beta$ has no proper $\Sigma_n$-elementary submodels) $h^n_\beta$ is then a partial $\Sigma_n^{J_\beta}$-definable function of $\omega$ onto $J_\beta$, a fact we shall use in the sequel. More particularly still we shall use this if $\beta<\Sigma(n)$ where $(\zeta(n),\Sigma(n))$ are the lexicographically least pair $(\zeta',\Sigma')$ so that $\zeta'<\Sigma'$ and $L_{\zeta'}\prec_{\Sigma_n}L_{\Sigma'}$.

We particularise the discussion now to $n=3$. Our machine will be a "theory machine" writing out now the $\Sigma_3$-theory of levels of the $J_\alpha$-hierarchy, for $\alpha<\Sigma(3)$, just as the "$\Sigma_2$-Theory Machine" of [7] did, as a standard ITTM.

We shall assume that our scratch tape is divided recursively into infinite sub–tapes $D_0$, $D_1$, .... We set aside the first cell on the $D_0$ tape as a "flag cell" - and designate it "$F$".

We shall describe a process that, *inter alia*, allows a code, $l(\alpha)$ for $\langle J_\alpha,\in\rangle$, written out as a characteristic function of a subset of $\omega$, to be uniformly obtainable from $S(\alpha')=_{\mathrm{df}}\langle C_i(\alpha')\,|\,i<\omega\rangle$, the *snapshot* at stage $\alpha'$, where $\alpha'=_{\mathrm{df}}\omega^3.(\alpha+1)$.

To do this we need some further nomenclature. We let $\mathcal{L}_{\dot\in,\dot p}$ be the language augmented by an extra constant symbol $\dot p$ and again assume a recursive enumeration of the $\Sigma_n$-sentences of this language. We use these enumerations in the following definitions.

**Definition 3.2.** *For $\alpha\in\mathrm{On},n\leq\omega$ (i) let $T^n_\alpha\subseteq\mathcal{L}_{\dot\in}$ be the $\Sigma_n$-Th$(\langle J_\alpha,\in\rangle)$;*
*(ii) for $p\in J_\alpha$ we let $T^n_\alpha(p)\subseteq\mathcal{L}_{\dot\in,\dot p}$ be the theory $\Sigma_n$-Th$(\langle J_\alpha,\in,p\rangle)$.*

As the base case for an induction, we shall assume that for the least $\alpha_0\in G$ (*i.e.* $\alpha_0=1$), our program has written $T^\omega_{\alpha_0}$ on $D_3$ and a code $l(\alpha_0)$ for $\langle J_1,\in\rangle$ on $D_2$ by stage $\omega^3.(\alpha_0+1)=\omega^3.2$.

We shall now describe the process that at the point in time $\omega^3.(\alpha+1)$: (I) has written a code $l(\alpha)$ for $J_\alpha$ to $D_2$ using (II) the complete $\Sigma_\omega$-theory $T^\omega_\alpha$ which has been written on $D_3$. The above is the base case for the least good ordinal $\alpha_0$. In the sequel we may refer to *stages* of the process. Each single stage may require infinitely many machine *steps* (each of the latter takes a single unit of 'time'). We do not wish to give all the details of the machine steps, but shall try and describe the stages that are translatable into steps, and shall endeavour to apply these two words in this way. However note that for $\lambda$ good it will take $\lambda$ steps to perform $\lambda$ stages, so at such points the numerations have caught up.

Suppose inductively that for $\alpha\in G$ we have derived (I) and (II) on the tape. We use the following notation:

$$\bar\alpha=\sup(\widehat{S}^1_\alpha);\qquad \alpha'=\alpha'(\alpha)\text{ is the largest }\alpha'\leq\alpha\text{ with }\alpha'\in G^*.$$

- Between $\alpha$ and $\alpha^+(=\alpha+\omega^\omega.\omega)$, the next limit of good ordinals after $\alpha$, the machine behaves as follows:

(A) (i) It computes a code for $\bar\alpha$ and writes this to $D_1$. We make the proviso that if at any time the value on $D_1$ is changed, then it is first preceded by a 'wipe clean' action that resets all the cells of $D_1$ first to zero, before overwriting the new value. We also, for convenience's sake, represent the ordinal '0' as '1000···' (or some such) to distinguish it from the 'empty' tape '0000···'.

Given $T^\omega_\alpha$, and $l(\alpha)$, as $\widehat{S}^1_\alpha$ is $\Pi_1^{J_\alpha}$ definable, it takes only $\omega + \omega$ steps to identify $\bar\alpha$ and write out a code for it.

(ii) From $T^\omega_\alpha$ it looks for $\alpha' = \alpha'(\alpha) \in G^*$, and sets

$F$ to 0 if $\alpha' \in \widehat{S}^1_\alpha$ ;
1 otherwise.

This takes only $<\omega$ steps.

Note for (iii) and (iv) to come, that given any set $x \in J_\alpha$ and any $n \le \omega$, $T^n_\alpha(x) \le_T T^\omega_\alpha$ since $T^n_\alpha(x)$ is the set of $\Sigma_n$-sentences true of $f_\alpha(k)$ for some $k$, where $f_\alpha\colon \omega \twoheadrightarrow J_\alpha$ is a canonical onto map defined from $T^\omega_\alpha$ and $f_\alpha(k) = x$. As our induction proceeds for all ordinals $\beta < \Sigma(3)$ we may assume that $f_\beta$ is always a $\Sigma_3^{J_\beta}$ parameter free map (uniformly defined for all $\beta$). We set $\bar f_\alpha = f_\alpha \cap (\omega \times {}^\omega 2)$.

(iii) From $\bar\alpha$ (which was identified at (i)) and using $T^\omega_\alpha$ it computes $T^\omega_{\bar\alpha}$ and writes it to $D_0$ (once $\bar\alpha$ is identifed, as just remarked, we may easily find the recursive function for the reduction $T^\omega_{\bar\alpha} \le_T T^\omega_\alpha$; this takes $\omega$ many steps).

(iv) We also require that $T^1_\alpha(\bar\alpha)$ be written out to $D_4$. Again a further $\omega$ steps, given $T^\omega_\alpha$ and $\bar\alpha$.

(v) From $l(\alpha)$ and $T^\omega_\alpha$ (on $D_3$) it can assemble a code for $l(\alpha + 1)$ and replace $l(\alpha)$ with this on $D_2$ ($\omega.2$ steps).

(vi) Using $l(\alpha+1)$ it writes the $\Sigma_\omega$-theory $T^\omega_{\alpha+1}$ to $D_3$. ($\omega^2$ steps).

(vii) On our rule tape, we may successively list

(a) $\bar f_\alpha(0)$, (if defined) in $\omega$ steps, then wipe clean all cells by resetting all $R_i$ values to 0; we then successively list all those $n * 1 \frown \bar f_\alpha(0)$, for which $J_\alpha \models \varphi^1_n[\bar f_\alpha(0)/\dot p]$ (where $\varphi^1_n$ is our prior fixed recursive enumeration of the $\Sigma_1$-sentences of $\mathcal{L}_{\dot\in,\dot p}$) again resetting all $R_i$ values to zero between each listing. When this is done we return to (a) looking in turn at those of $\bar f_\alpha(1), \bar f_\alpha(2), \ldots \bar f_\alpha(k), \ldots$ *etc.* which are defined, in a similar fashion.

This process will take care of our rule requirements. Note that it takes $\omega^3$ steps, and this in fact is the totality of all steps taken at this stage, which is now finished.

We now have $l(\alpha + 1)$ and $T^\omega_{\alpha+1}$ correctly written out and are at step $\omega^3.(\alpha + 1) + \omega^3 = \omega^3.(\alpha + 2)$. So now the machine may return to (A). To help summarise, we produce in the next $\omega^3$ steps:

$\overline{\alpha+1} \simeq \sup(\widehat{S}^1_{\alpha+1})$ on $D_1$ ;
$\alpha'(\alpha+1)\,(= \alpha'(\alpha))$, setting $F$ appropriately;
$T^\omega_{\overline{\alpha+1}}$ on $D_0$ ;
$T^1_{\alpha+1}(\overline{\alpha+1})$ on $D_4$ ;
$l(\alpha+2)$ on $D_2$ ;
$T^\omega_{\alpha+2}$ on $D_3$ ;
$1 * x$ and listings of $T^1_{\alpha+1}(x)$ on $R$ as above, for all $x \in {}^\omega 2 \cap J_{\alpha+1}$ in turn.

This fully describes the "successor" stage action relating the transitions from $J_\alpha$ to $J_{\alpha+1}$ to $J_{\alpha+2}$. Note that after the last stage the time clock is at $\omega^3.(\alpha+3)$.

At limit stages $\lambda \in (\alpha, \alpha^+)$ the liminf$^*$ rule is to be used. As $\lambda \notin G^*$, the rule is merely the former ITTM rule of straight *liminf*. This thus has the following effects at this stage:

(1) On $D_1$: as $\beta \longrightarrow \lambda$ $\bar{\beta}$ is eventually constant, with eventual value $\bar{\lambda} < \lambda$ say, with $\bar{\lambda}$=sup $\widehat{S}^1_\lambda$. This implies that $F$ is eventually constant, but moreover:

(2) On $D_0$: we have $T^\omega_\lambda$;

(3) On $D_4$: we have $T^1_\lambda(\bar{\lambda})$ (as both these are the simple union of earlier $\Sigma_1$-theories.)

(4) On $D_3$, $D_2$: we have the $\liminf^*_{\beta\to\lambda} T^\omega_\beta$ ; and $\liminf^*_{\beta\to\lambda} l(\beta)$.

(5) On $R$: every cell is zero, due to the resetting of all cells to zero between listings and the *liminf* rule.

- If $D_1$ is not empty, this is because $\bar{\lambda} < \lambda$. (This will always happen if $\lambda \notin G^*$.) In this case there is a $\Sigma_1^{J_\lambda}(\{\bar{\lambda}\})$ map uniform in $\lambda$ and the parameter $\bar{\lambda}$, $g_\lambda$: $\omega \twoheadrightarrow J_\lambda$ (meaning uniform for those $\lambda$ with $\bar{\lambda} < \lambda$ - note $\bar{\lambda} = 0$ is included as a possibility). Using $D_4$ this allows us to compute a code $l(\lambda)$ for $\langle J_\lambda, \in \rangle$ to be written to $D_2$ in $\omega$ many steps. From $l(\lambda)$ we may compute $T^\omega_\lambda$ and write this to $D_3$ (this takes $\omega^2$ time). Rather superfluously at this stage we run the Rule tape writing algorithm solely in order to keep the enumeration of steps in line.

We are now at time $\omega^3.(\lambda + 1)$. The machine then returns to (A) and continues as before. We now consider what to do when we arrive at a $\lambda \in G^*$.

**Definition 3.3.** *For* $\lambda \in G^*$ *let* $\widehat{E}^1_\lambda =_{\mathrm{df}} \{\alpha < \lambda \mid \widehat{S}^1_\alpha = \widehat{S}^1_\lambda \cap \alpha\}$.

The point of listing $\Sigma_1$-theories of the form $\langle J_\gamma, \in, x\rangle$ on the Rule tape $R$ is precisely to establish (6) below. This will have the consequence that if $\gamma \in G \cap \lambda$ then $\gamma$ is 1-correct in $\lambda$ in the sense of computation if and only if $\gamma$ is similarly '$\Sigma_1$-correct' about which ordinals are $\Sigma_1$-stables in the set-theoretic context:

$$\bar{\gamma} < \gamma \in G \cap E^1_\lambda \longrightarrow (J_{\bar{\gamma}} \prec_{\Sigma_1} J_\gamma \leftrightarrow J_{\bar{\gamma}} \prec_{\Sigma_1} J_\lambda).$$

We thus show:

(6) $\lambda \in G \longrightarrow \widehat{S}^1_\lambda = S^1_\lambda$. Hence $\lambda \in G^* \longrightarrow E^1_\lambda \cap G = \widehat{E}^1_\lambda \cap G$.

Proof: We show by induction on $\alpha < \lambda$ that $\widehat{S}^1_\lambda \cap \alpha = S^1_\lambda \cap \alpha$. Suppose this shown for $\alpha$. Let $\gamma = \min\, ((\widehat{S}^1_\lambda \cup S^1_\lambda) \backslash \alpha)$. Suppose $\gamma \in \widehat{S}^1_\lambda$. Then $J_\gamma \prec_{\Sigma_1} J_\lambda$ and $\gamma$ is highly closed, indeed admissible. However by our rule tape construction we have continually for any $\beta < \lambda$, listed $T^1_\beta(p)$ for any $p \in J_\beta$ of the form $p = x \in {}^\omega 2$. But any $\Sigma_1$ sentence $\varphi_n[x]$ with $x \in {}^\omega 2 \cap J_\gamma$ in this theory, is also in such a theory for a $\delta \in G \cap \gamma$, as $J_\gamma \prec_{\Sigma_1} J_\lambda$. Thus $n * 1 \frown x$ appears on $R$ before time $\gamma$. Hence $\gamma \in S^1_\lambda$.

Now suppose $\gamma \in S^1_\lambda$. Then $\gamma \in G$. We should like to have $J_\gamma \prec_{\Sigma_1} J_\lambda$. Note that it suffices to require $\Sigma_1$ elementarity for formulae $\varphi_n(\xi/v_0)$ for ordinals $\xi < \gamma$, and moreover as there is a real $x_\xi \in {}^\omega 2 \cap J_{\xi+1}$ coding $\xi$, it thus suffices to consider this for formulae $\varphi_n(x/v_0)$ with $x \in {}^\omega 2 \cap J_{\alpha'}$. By our construction for any $1 * x$ listed on the rule tape by stage $\gamma$, as $\gamma \in S^1_\lambda$ then $T^1_\lambda(x)$ is recursive in $\langle R(\nu) \mid \nu < \gamma\rangle$. Running the machine inside $J_\gamma$ this would make $T^1_\lambda(x)$ $\Sigma_1$-definable over $J_\gamma$ for any $x \in {}^\omega 2 \cap J_\gamma$. Hence $T^1_\lambda(x) = T^1_\gamma(x)$ for all such $x$. This suffices to imply that $J_\gamma \prec_{\Sigma_1} J_\lambda$. Q.E.D. (6)

Suppose now $\lambda$ equals $\alpha^+$. Note that $E^1_\lambda$ even in this case is simply a tail of $G\cap\lambda$. As $\beta\longrightarrow\lambda$ $\bar\lambda$ will settle down in value below $\lambda$, from some point on. (Recall that $\bar\lambda=\lambda$ would imply that $\lambda$ is in fact strongly admissible, and thus cannot be of the form $\alpha^+$.) The liminf$^*$ rule now comes into play in full, but we still have (1)-(5) holding just as for a limit which is a non-limit of good ordinals, as above, and the actions are the same. One point to note is:

- If $\beta=\alpha^++1$ then $F$ must then by the end of the operations at stage $\beta$ be set to 1: this is because $\alpha^+\notin\widehat{S}^1_\beta$ (an element of any $\widehat{S}^1_\delta$ is always in $(G^*)^*$).

However for a general $\lambda\in G^*$ $\bar\lambda$ may equal $\lambda$. Our Flag $F$ is designed to alert the machine when we are at such a point:

(7) $(\lambda\in(G^*)^*\wedge F(\lambda)=0)\longleftrightarrow\bar\lambda=\lambda$.

Proof: ( ← ) Suppose $\bar\lambda=\sup\widehat{S}^1_\lambda=\lambda$. Then for unboundedly many $\beta<\lambda$ we have $\beta\in\widehat{S}^1_\lambda$ and thus $\lambda\in(G^*)^*$. Further for $\beta\in\widehat{S}^1_\lambda$, we have that the Flag $F$ at stage $\beta+1$ is set to 0, as $\beta=(\beta+1)'\wedge\beta\in\widehat{S}^1_{\beta+1}$. This value of 0 persists for any of the steps $\eta\in(\beta+\omega^3,\beta+\omega^\omega]$. Thus at the next good ordinal $\beta+\omega^\omega$ the Flag is zero. Moreover $\beta+\omega^\omega\in E^1_\lambda$.

However by step $\beta^++\omega^3$ $F$ has been set to 1. But $\beta^+\notin\widehat{S}^1_\delta$ for any $\delta\in(\beta^+,\beta^++\omega^\omega]$. Hence the Flag stays set at 1 in this interval (and beyond). But both $\beta+\omega^\omega$ and $\beta^++\omega^\omega$ are in $G$ and are 1-correct in $\lambda$. This happens for unboundedly many $\beta<\lambda$. Hence $\liminf^*_{\nu\longrightarrow\lambda}F(\nu)=0$.

( → ) Suppose $\bar\lambda<\lambda$. Let $\gamma$ be the least element of $G^*$ greater than $\bar\lambda$. Then $F$ is at stage $\gamma+1$ set to value 1. It will only be 0 at any stage $\delta>\gamma$ if $\bar\lambda<\delta'\in\widehat{S}^1_\delta$. But such a $\delta$ is not in $E^1_\lambda$! Hence $\{\nu\in(\bar\lambda,\lambda)\,|\,F(\nu)=0\}\cap E^1_\lambda=\varnothing$. Hence $F(\lambda)=1$ by the liminf$^*$ rule. QED (7)

(8) *If* $\lambda\in G$ *and* $\bar\lambda<\lambda$ *then* $T^\omega_\lambda$ *is on* $D_0(\lambda)$.

Proof: By induction on $\bar\lambda<\beta\le\lambda$, for $\beta\in E^1_\lambda$ show that $T^\omega_\lambda$ is on $D_0(\beta)$, noting that $\bar\beta=\bar\lambda$ for such $\beta$. Q.E.D. (8)

(9) $\forall\lambda<\Sigma(3)($*a code for* $l(\lambda)$ *can be extracted from* $S(\lambda))$

Proof: We've seen how to do this using $T^1_\lambda(\bar\lambda)$ which is on $D_4$, when $\bar\lambda<\lambda$ (by using the method of an onto map explained just after (5)). So assume $\bar\lambda=\lambda$ (and *a fortiori* $\lambda\in(G^*)^*$). Then the Flag $F(\lambda)=0$ (see (7)) and hence the machine knows this fact. We show first how to determine $T^3_\lambda$ from $D_3(\lambda)=\liminf^*_{\alpha\longrightarrow\lambda}T^\omega_\alpha$ which is written on $D_3$.

Suppose $\varphi\equiv\exists u\psi(u)$ is $\Sigma_3$ with $\psi\in\Pi_2$ in $\mathcal{L}_{\dot\in}$. We use the following equivalence.

(10) *Assume* $\bar\lambda=\lambda$. $\varphi\in T^3_\lambda\longleftrightarrow\exists n\in\omega[\exists\gamma_0\forall\gamma>\gamma_0(\gamma\in\widehat{S}^1_\lambda\longrightarrow J_\gamma\models\sigma_n)]$
*where* $\sigma_n$ *is the following sentence*: “$\exists\beta[(\beta\in\widehat{S}^2_\gamma\cup\{0\}\wedge\exists k(k=h^2_\gamma(n,\beta)\wedge\psi(k)))]$”.

Proof of (10): ( → ). Suppose $\varphi\in T^3_\lambda$ and is of the form illustrated, with some $x\in J_\lambda$ witnessing $J_\lambda\models\psi[x]$. Then for some $\beta\in\widehat{S}^2_\lambda\cup\{0\}$, $x=h^2_\lambda(n,\beta)$. Choose $\gamma\in\widehat{S}^1_\lambda$ sufficiently large with $x\in J_\gamma\models x=h^2_\gamma(n,\beta)$. This is possible by our assumption on $\lambda$. However for any $\gamma'\ge\gamma$ with $\gamma'\in\widehat{S}^1_\lambda$ we also have that $x=h^2_{\gamma'}(n,\beta)$ and $\beta\in\widehat{S}^2_{\gamma'}$. Moreover $(\psi[x])_{J_\lambda}$ is $\Pi_2$ and so goes down to such $\gamma'$: $(\psi[x])_{J_{\gamma'}}$. Hence the right hand side holds.

$(\leftarrow)$: Suppose the RHS holds for some $n$, but the LHS failed for a contradiction. Then we first claim that $\widehat{S}^2_\lambda$ must be bounded in $\lambda$: for if it were unbounded then we could always take a $\gamma$ on the RHS to be from $\widehat{S}^2_\lambda$. However then the $\Pi_2$ statement $\psi[k]$ about $k$, included in $\sigma_n$, would go up to $\lambda$ and the LHS would hold. Hence $\sup(\widehat{S}^2_\lambda) = \beta_0 < \lambda$. Now there are unboundedly many $\gamma < \lambda$ with $\gamma \in \widehat{S}^1_\lambda$, $\gamma > \beta_0$, $\widehat{S}^2_\gamma = \widehat{S}^2_\lambda \cap \gamma$, and with $J_\gamma \vDash \sigma_n$. Let $\beta = \beta_\gamma$ be the least ordinal witnessing the existential quantifier of the quoted statement $\sigma_n$. By what we may call the "2-correctness" of $\gamma$, $\beta \in \widehat{S}^2_\lambda \cup \{0\}$. If for some such $\gamma$ satisfying these clauses, we had $\beta_\gamma < \beta_0$, we should have that "$k = h^2_\gamma(n, \beta_\gamma) \wedge \psi(k)$)" which is true in $J_\gamma$, would be absolute to $J_{\beta_0}$. But $\beta_0 \in \widehat{S}^2_\lambda \cup \{0\}$ so it is also true in $J_\lambda$ thus verifying $\psi$. On the other hand if for all such $\gamma$ satisfying the clauses we had $\beta_\gamma = \beta_0$, then $h^2_\gamma(n, \beta_\gamma)$ has constant value some $k$ for all such $\gamma$ on a tail of $\widehat{S}^1_\lambda$. Moreover as $(\psi[k])_{J_\gamma}$ in all sufficiently large $\gamma \in \widehat{S}^1_\lambda$ and as $\psi \in \Pi_2$, we conclude $(\psi[k])_{J_\lambda}$, another contradiction. Q.E.D. (10)

(11) *There is a* (1-1) *recursive* $F\colon \mathbb{N} \longrightarrow \mathbb{N}$ *so that for any* $\lambda < \Sigma(3)$, *if* $\bar{\lambda} = \lambda$ *then*

$$\varphi \in T^3_\lambda \longleftrightarrow \exists n \in \omega F(\prec n, \ulcorner\varphi\urcorner \succ) \in D_0(\lambda).$$

*Hence for such* $\lambda$, $T^3_\lambda$ *is unformly r.e. in* $S(\lambda) = \vec{C}_i(\lambda)$.

Proof: Assume $\bar{\lambda} = \lambda$ , then we recast (10) as:

$$\varphi \in T^3_\lambda \longleftrightarrow \exists n \in \omega\, [\, \exists\gamma_0 \forall\gamma > \gamma_0 (\gamma \in E^1_\lambda \longrightarrow \sigma_n \in D_0(\gamma))]$$

$(\rightarrow)$ This follows from (6) and (10).

$(\leftarrow)$ As $S^1_\lambda$ is unbounded in $\lambda$, and is contained in $E^1_\lambda$ this follows from the argument of (10).

From this our liminf* rule then shows $D_0(\lambda)$ has the correct information. Q.E.D. (11)

For $\lambda < \Sigma(3)$ we shall thus be able, by the comments before Definition 3.2, to use a $\Sigma^{J_\lambda}_3$ map $h$ of $\omega$ onto $J_\lambda$ (which by the above is at worst uniformly r.e in $S(\lambda)$ in the case that $\bar{\lambda} = \lambda$) to define a code $l(\lambda)$ for $J_\lambda$: if $h(k)$ is defined, we may form the equivalence class of $k'$ such that $h(k) = h(k')$, and define the binary relation $kEm \Longleftrightarrow h(k) \in h(m)$. This yields a code $l(\lambda)$ for $\langle J_\lambda, \in \rangle$ which may be written to $D_2$ in, say a further $\omega$ steps. Q.E.D. (9)

The above process lasts for as long as differing $\Sigma_3$ theories are produced for the different levels $\lambda < \Sigma(3)$. However, $(\zeta(3), \Sigma(3))$ is also the lexicographic least pair $(\pi, \chi)$ of ordinals with $T^3_\pi = T^3_\chi$. (Assume for a contradiction that $(\pi, \chi) <_{\text{lex}} (\zeta(3), \Sigma(3))$ have the same $\Sigma_3$-theories. Firstly if $\chi < \Sigma(3)$, using the onto $\Sigma_3$ function $h$, we have that the $\Sigma_3$ sentences $\sigma_{n,m} \equiv$ "$h(n) < h(m) \in \mathrm{On}$" are in $T^3_\chi$ and yield a wellorder of type $\chi$ (as $h_\chi$ is onto $\chi$). But it is impossible that such $\sigma_{n,m}$ are all in $T^3_\pi$ as the latter there yield a wellorder only of type $\pi$! Hence we must have $\chi = \Sigma(3) \wedge \pi < \zeta(3)$. But clearly $T^3_{\Sigma(3)} = T^3_{\zeta(3)}$, and the same argument shows that the sentences $\sigma_{n,m}$ in $T^3_{\zeta(3)}$ are in $T^3_\pi$ for the same contradiction.)

Hence the machine before stage $\Sigma(3)$ produces different theories, and at stage $\Sigma(3)$ produces only $T^3_{\zeta(3)}$ and then $l(\zeta(3))$ on $D_2$, and so commences to cycle.

**Remark 3.4.** Since it is (reasonably) clear that any machine with this limit rule will either halt before $\zeta(3)$ or enter a loop at this point (by considering such machines as running inside $L_{\Sigma(3)}$) we thus have a complete description of the *semi-decidable, decidable* predicates, *halting problem* and so on, just as the calculation determining the role of the ordinals $(\zeta(2),\ \Sigma(2))$ did for the original ITTM model. (See [12] for a somewhat cleaner development of this $\Sigma_2$-theory.) For example, the halting problem set will be recursively isomorphic to the $\Sigma_1$ truth set of $\langle L_{\lambda(3)}, \in\ \rangle$ where $\lambda(3)$ is least such that $L_{\lambda(3)} \prec_{\Sigma_1} L_{\zeta(3)}$. The other assertions concerning abstract 1-sections *etc.* from the Introduction then also follow.

## 4 $\Sigma_n$-Machines

We now consider machines related to levels of the $L$-hierarchy at the least $\zeta(n)$ with some $\Sigma(n) > \zeta(n)$ and $L_{\zeta(n)} \prec_{\Sigma_n} L_{\Sigma(n)}$ for larger $n < \omega$.

In developing the next level, $\Sigma_4$-machines, which halt or loop by $\zeta(4)$, we accordingly make use of the appropriate notions $E^2_\lambda$, $\widehat{E}^2_\lambda$, $S^2_\lambda$, $\widehat{S}^3_\lambda$,$\widehat{S}^4_\lambda$, *etc.* We assume now the rule tape $R$ is now recursively split into two infinite pieces, $Q$, $R$. (We keep the letter $R$ as we intend to expand on what the machine of the last Section does,) There is no difference between what is written to the $R$ tape in the current machine and that of the last section; we use $R$ to define $S^1_\lambda$ and $E^1_\lambda$ just as before. The $Q$-part will be used to define $S^2_\lambda$ *etc.* We further recursively split $Q$ into $2 \times \omega^2$ many infinite pieces $Q(i, k, m)$ with $i < 2, k, m \in \omega$. As before $Q(\beta, i, k, m)$ will denote that piece viewed at time $\beta$.

We adopt the notation that $\alpha_i =_{\text{df}} \sup S^i_\alpha$ for $i = 1, 2$.

**Definition 4.1.**

$$\begin{aligned} &\alpha \in S^2_\lambda \Longleftrightarrow \alpha = \alpha_1 \wedge \alpha \in S^1_\lambda \wedge \\ &\forall x \in {}^\omega 2 \forall \nu < \alpha \forall n [(1 \frown x = R(\nu) \wedge \exists \beta' \in E^1_\lambda \, Q(\beta', i, k, m) = 1 \frown x) \longrightarrow \\ &\qquad \longrightarrow \exists \beta' \in E^1_\alpha (Q(\beta', i, k, m) = 1 \frown x)]. \end{aligned}$$

**Definition 4.2.** $E^2_\lambda$ *(the* 2-correct in $\lambda$ ordinals*)*

$$\begin{aligned} E^2_\lambda &= \{\alpha \in E^1_\lambda \mid \alpha_{1,2} \, (=_{\text{df}} \sup S^2_{\alpha_1}) \in S^2_\lambda\} && \textit{if } \lambda_1 = \lambda; \\ &= E^1_\lambda && \textit{otherwise.} \end{aligned}$$

Our definitions imply:

**Remark 4.3.** $S^2_\beta \subseteq E^2_\beta$ ; $\beta = \beta_1 \longrightarrow S^1_\beta \cap E^2_\beta$ is closed and cofinal in $\beta$.

We now adopt:

$$\begin{aligned} \text{Limit Rule: } C_i(\lambda) =\ & 1 \Longleftrightarrow \exists \nu_0 < \lambda \forall \nu \in (\nu_0, \lambda) [\nu \text{ 2-correct at } \lambda \longrightarrow C_i(\nu) = 1]; \\ C_i(\lambda) =\ & 0 \quad \text{Otherwise.} \end{aligned}$$

The above then completes the description of the $\Sigma_4$-machine architecture. We turn now to a program that computes theories and codes for all levels of the constructible hierarchy below $\Sigma(4)$.

**$\Sigma_4$-theory machine description.** Let $\langle \varphi_n(v_0) \rangle$ effectively enumerate all formulae and $\langle \psi_n(v_0) \rangle$ all $\Sigma_1$ formulae in the free variable $v_0$. We continue to use the notation: $\bar{\alpha} =_{\text{df}} \sup \widehat{S}^1_\alpha$; and further adopt $\alpha^2 =_{\text{df}} \sup \widehat{S}^2_\alpha$. (Later we shall see that for the machine description to be specified we shall have $\alpha_1 = \bar{\alpha}$ and $\alpha_2 = \alpha^2$ *i.e.* with the notions of computable stability coinciding with set theoretical $\Sigma_n$-stability.) As in the previous section we have set $f_\alpha$: $\omega \twoheadrightarrow J_\alpha$ to be a canonical onto map defined from $T^\omega_\alpha$. As our induction proceeds for all ordinals $\beta < \Sigma(4)$ we may assume that $f_\beta$ is always a $\Sigma_4^{J_\beta}$ parameter free map (uniformly defined for all $\beta$). Again set $\bar{f}_\alpha = f_\alpha \cap (\omega \times {}^\omega 2)$. To specify the operation we simply augment the previous $\Sigma_3$-theory machine with an extra task at (6).

Let $\alpha \in G$ and let $\beta > \alpha$ be least with $\beta \in G$. Then at stage $\beta$ we shall now additionally require:

(6) Let $\bar{\beta}$ be as above and as already defined, with $\bar{\beta}^{2} = \sup \widehat{S}^{2}_{\bar{\beta}}$. Then the complete theories of $J_{\bar{\beta}}$ and $J_{\bar{\beta}^2}$ are written in the following way at stage $\beta$: if $y = \bar{f}_{\bar{\beta}}(k)$ is the $k$'th real in $J_{\overline{\beta}}$ (resp. $J_{\bar{\beta}^2}$ where then $y = \bar{f}_{\bar{\beta}^2}(k)$) and $J_{\bar{\beta}} \vDash \varphi_n[y]$ (resp. $J_{\bar{\beta}^2} \vDash \varphi_n[y]$ ) then it is required that $1 \frown y$ is on the the second rule $Q(\beta, 0, k, n)\,(Q(\beta, 1, k, n)$ resp.) tape segment at stage $\beta$; otherwise $Q(\beta, 0, k, n)(0) = 0$ $(Q(\beta, 1, k, n)\,(0) = 0$ resp.).

Starting from a code $l(\alpha)$ for $J_\alpha$ the machine writes codes for $J_\gamma$ for $\gamma \in [\alpha, \beta)$ writing, just as in the last section (i) - (vii). We further ensure:

(viii) Using $T^{\omega}_{\bar{\gamma}}$ which is written on $D_0$ and, $T^{\omega}_{\bar{\gamma}^2}$ (which is recursive $T^{\omega}_{\bar{\gamma}}$), we may compute values of $\bar{f}_{\bar{\gamma}}(k)$ and $\bar{f}_{\bar{\gamma}^2}(k)$) and write out the theories in the real parameters $\bar{f}_{\bar{\gamma}}(k) = y$ and $\bar{f}_{\bar{\gamma}^2}(k) = y$. Thus we set $Q(\gamma, 0, k, m) = 1 \frown \bar{f}_{\bar{\gamma}}(k)$ iff $J_{\bar{\gamma}} \vDash \varphi_m[\bar{f}_{\bar{\gamma}}(k)]$ and similarly $Q(\gamma, 1, k, m) = 1 \frown \bar{f}_{\bar{\gamma}^2}(k)$ iff $J_{\bar{\gamma}^2} \vDash \varphi_m[\bar{f}_{\bar{\gamma}^2}(k)]$. Mirroring the writing of $T^{1}_{\beta}(\bar{\beta})$ before on $D_1$, we use an additional piece of scratch tape $D_5$ and record here $T^{2}_{\bar{\gamma}}(\gamma^2)$. This can also be obtained from $T^{\omega}_{\bar{\gamma}}$ on $D_0$. All this can be done in $< \omega^2$ many steps. Recall that the previous $R$-writing process (vii) required $\omega^3$ many steps. So by dovetailing this process with that of (vii) we can still stick to the same ordinal arithmetic and have both $R$- and $Q$- writing done in $\omega^3$ steps.

Recall that (ii) ensures that the machine may start with a 0 on $F$ at stage $\alpha$ but will change it to a 1 as soon as it sees $\gamma' = \alpha' \notin \widehat{S}^{1}_{\gamma}$.

(6') $\lambda \in G \longrightarrow \widehat{S}^{1}_{\lambda} = S^{1}_{\lambda}$. (Thus $\bar{\lambda} = \lambda_1$.) Hence $\lambda \in G^* \longrightarrow E^{1}_{\lambda} \cap G = \widehat{E}^{1}_{\lambda} \cap G$.

(7') $(\lambda \in (G^*)^* \wedge F(\lambda) = 0) \longleftrightarrow \sup \widehat{S}^{1}_{\lambda} = \lambda)$.

We additionally have here the new:

(6") $\bar{\beta} = \beta \quad \longrightarrow \quad (\alpha \in S^{2}_{\beta} \longleftrightarrow \alpha \in \widehat{S}^{2}_{\beta})$. Thus $\beta^2 = \beta_2$.

For (6') note that the $\Sigma_4$-theory machine extends the action of the $\Sigma_3$-theory machine, with $\widehat{S}^{1}_{\lambda}, S^{1}_{\lambda}$, defined from before, so there is nothing to prove.

For (7'): ( $\leftarrow$ ) The argument is the same but here the only difference is that we are taking a liminf* along $E^{2}_{\lambda}$: but there are unboundedly many $\tau \in E^{2}_{\lambda}$ taking each of the values 0 and 1. For ( $\rightarrow$ ) if we assume, using (6'), $\bar{\lambda} < \lambda$ then in this case $E^{2}_{\lambda} = E^{1}_{\lambda}$ so the previous reasoning holds.

Proof of (6"). Suppose $\alpha \in S^{2}_{\beta}$. Suppose $J_\beta \vDash \varphi_n[y]$ with $\varphi$ expressing a $\Sigma_2$ property about $y$ where $y \in J_\alpha$, then we want $J_\alpha \vDash \varphi_n[y]$. However '$J_\beta \vDash \varphi_n[y]$' is equivalent to $J_{\overline{\delta}}$ modelling the truth of $\varphi_n[y]$ for some sufficiently large $\delta \in E^{1}_{\beta}$. Further this is recorded on the $Q(\delta, 0, k, n)$ part of the second rule tape at a sufficiently large stage $\delta \in E^{1}_{\beta}$ with $y \in J_{\bar{\delta}}$ and $y = \bar{f}_{\bar{\delta}}(k)$. (Note that $\bar{\delta} \in S^{1}_{\beta}$). Then $\alpha \in S^{2}_{\beta}$ guarantees that this $\Sigma_2$ property about $y$ has been recorded as holding for some sufficiently large $\delta_0 \in E^{1}_{\alpha}$ by $Q(\delta_0, 0, k, n)$, and thus $\varphi_n[y]$ holds in $J_{\bar{\delta}_0}$ with $\overline{\delta_0} \in S^{1}_{\alpha}$. By upwards $\Pi_1$ elementarity then, $\varphi_n[y]$ holds in $J_\alpha$. For the converse one may imagine the machine running in $J_\beta$ and then apply $\Sigma_2$-elementarity. Q.E.D.

We have the following addition to (8). Here we have assumed that $k_0$ has been chosen so that $\bar{f}_{\beta}(k_0) = 0$ for any $\beta$.

(8') *For any* $\lambda$, *if* $\bar{\lambda} = \lambda$ *and* $\lambda^2 < \lambda$ *then* $T^{\omega}_{\lambda^2}$ *is recursive in* $Q(\lambda, 1, k_0, n)$.

Proof (8'): Recall that on the $\langle 1, k, n\rangle$ parts of the $Q$-tape were recorded the complete theories $T^{\omega}_{\bar{\beta}^2}(y)$ in real parameters $y = \bar{f}_{\bar{\beta}^2}(k)$. $S(\lambda)$ is given by the liminf* taken over 2-correct ordinals $\beta < \lambda$. Suppose $\lambda^2 < \lambda$. Then for $\beta \in E^2_\lambda \backslash \lambda^2$ by definition we have that $\bar{\beta}^2 = \sup S^2_{\bar{\beta}} = \lambda^2$, and in particular as $\beta$ tends to $\lambda$ through $E^2_\lambda$, eventually $\bar{\beta}^2 = \lambda^2$. Hence the pure part of the theory $T^{\omega}_{\lambda^2}$ is eventually constant on the tape.

(9') $\forall\lambda < \Sigma(4)$(*a code for* $l(\lambda)$ *can be extracted from* $S(\lambda)$).

Proof: Using (7') the Flag $F$ tells the machine if $\bar{\lambda} = \lambda$. If $\bar{\lambda} < \lambda$ the construction of a code $l(\lambda)$ is just as for the $\Sigma_3$-theory machine. So assume $\bar{\lambda} = \lambda$. Suppose $\lambda^2 = \sup \widehat{S}^2_\lambda < \lambda$. Just as in the proof of (8') as $\beta$ tends to $\lambda$ through $E^2_\lambda$, eventually $\bar{\beta}^2 = \lambda^2$. Consequently at stages for a tail of $\beta \in E^2_\lambda$, on $D_5$ we have either the theory $T^2_{\bar{\beta}}(\lambda^2)$ itself, or a liminf of such theories $T^2_{\bar{\beta}}(\lambda^2)$ for an increasing chain of $\bar{\beta}$. However such $\bar{\beta}$ are in $\widehat{S}^1_\lambda$. By the upwards persistence of $\Sigma_2$ theories, these liminf's are simple unions and hence at stage $\lambda$ $D_5$ contains $T^2_\lambda(\lambda^2)$.

Our assumption is that there is a $\Sigma_2^{J_\lambda}(\{\lambda^2\})$ onto function $f\colon \omega \longrightarrow J_\lambda$. Moreover the machine does know that $\lambda^2 < \lambda$: $\lambda^2 = \lambda$ if and only if for arbitrarily large $\delta \in E^2_\lambda$ we have both that $J_{\bar{\delta}} \vDash$ "$\max S^2_{\bar{\delta}} = \max S^1_{\bar{\delta}}$" and also its negation. Consequently neither this sentence nor its negation is in $\liminf^*_{\beta \longrightarrow \lambda} T^{\omega}_{\bar{\beta}}$ on $D_0$. Hence the machine knows to construct a code for $l(\lambda)$ with that theory $T^2_\lambda(\lambda^2)$.

If $\lambda^2 = \lambda$ then the direct generalisation of the argument of (10) one level up yields:

(10') *Assume* $\lambda^2 = \lambda$. $\varphi \equiv \exists v_0 \psi(v_0) \in T^4_\lambda \longleftrightarrow \exists n \in \omega [\exists \gamma_0 \forall \gamma > \gamma_0 (\gamma \in \widehat{S}^2_\lambda \longrightarrow J_\gamma \vDash \sigma_n)]$
*where* $\sigma_n$ *is the following sentence*: "$\exists \beta[(\beta \in \widehat{S}^3_\gamma \cup \{0\} \wedge \exists k(k = h^3_\gamma(n, \beta) \wedge \psi(k)))]$".

And in turn now with some minor adjustments to some $F_0(\prec n, \ulcorner\varphi\urcorner \succ) := \ulcorner\sigma_n\urcorner$ for the latter $\sigma_n$:

(11') *There is a* (1-1) *recursive* $F\colon \mathbb{N} \longrightarrow \mathbb{N}$ *so that for any* $\lambda < \Sigma(4)$, *if* $\lambda^2 = \lambda$ *then*

$$\varphi \in T^4_\lambda \longleftrightarrow \exists n \in \omega\; Q(\lambda, 1, k_0, F(\prec n, \ulcorner\varphi\urcorner \succ))(0) = 1).$$

*Hence for such* $\lambda$, $T^4_\lambda$ *is unformly r.e. in* $S(\lambda) = \vec{C}_i(\lambda)$.

Consequently in this case we may obtain the $\Sigma_4$ theory of $J_\lambda$ in a r.e. manner from $S(\lambda)$. Just as in the previous section this allows the machine to construct a code $l(\lambda)$. Hence as long as we are below $\Sigma(4)$, we obtain new theories, and so codes $l(\alpha)$. Q.E.D.

We hope that the reader, having seen how to obtain $\Sigma_3$- and $\Sigma_4$-machines will be convinced as to how one can generalise the constructions to any $\Sigma_{n+2}$ by extending the above. The definitions of the higher $S^n_\beta$ and $E^n_\beta$ should reflect those of $E^2_\beta$ and $S^2_\beta$; the $Q$-rule tape should be further subdivided to write down the corresponding theories obtained from the last case, merely by raising complexities by 1. The specifications of the $\Sigma_{n+2}$-machine follows the same template. Additional slices of the worktape $D_6$, $D_7$, ...$D_{n+3}$ will record theories up to those of the form $T^n_{\beta^{n-2}}(\beta^n)$ with $\beta^{n-2} =_{\text{df}} \sup \widehat{S}^{n-2}_\beta$ *etc.* The proof of (10') is generalisable over $n$ *verbatim*. This leads to appropriate statements and proofs of (9') and (11').

## 5 Conclusions

The above shows that the notion of ITTM machine can be generalised in a satisfactory way to allow for limit rules that correspond to $\Sigma_n$ descriptions, and the resulting notions of *eventually decidable, eventually semi-decidable* and so on correspond to the appropriate levels of the constructible hierarchy, namely the first $\Sigma_n$-extendible ordinals $\zeta(n)$. As one of us a has observed elsewhere the output tape of the original ITTM running a standard program is an example of what Burgess ([1]) would call an *arithmetical quasi-inductive* (AQI) set. (In fact the conents of an eventually stable output tape of a Hamkins-Kidder ITTM is an example of a *recursive quasi-inductive* set (and indeed any AQI process can be simulated onan ITTM, thus showing that AQI processes can all be reduced to recursive ones). A number of questions could now be formulated:

**Question 1** *Can there be a perspicuous or sensible notion of* $n$-quasi-inductive *(where* $n=2$ *corresponds to AQI)?*

In stating this we are of course conscious of the fact that the limit rules for hypermachines we have proposed are explicitly designed to tie up with the $L$-hierarchy. Whether there are pleasing notions at levels above $n=2$ remains to be seen.

**Question 2** *Are there sensible notions of "machine" that transcend those of this paper?*

In the above one could think very speculatively of machines that go beyond the first model of second order number theory, or even have some form of in-built extra function allowing them to go outside of $L$ into say $K$ the core model?

More concretely arguments from [13] can be used to show that eventually stable output tapes of ITTM's are all $\Game$-$\Sigma^0_3$ (but not $\Game$-$\Sigma^0_2$) definable. Recent work of Montalban and Shore will show that those of the $\Sigma_{n+1}$-machines are in $\Game$-($n$-$\Sigma^0_3$) (where ($n$-$\Sigma^0_3$) represents the $n$-th level of the difference hierarchy on $\Sigma^0_3$). One can define quite naturally either using the machines here, or the $L$-hierarchy directly, Spector pointclasses of sets of reals corresponding to these machines. Let $\boldsymbol{\Gamma_n}$ denote the those classes of sets $A \subseteq \mathbb{R}$ so that for some $\Sigma_n$-machine program $P$, some real parameter $y$, that $x \in A$ iff $P(\langle x, y\rangle)$ eventually has a 1 on its output tape. (Thus $\boldsymbol{\Gamma_2}$ is a boldface pointclass corresponding to **AQI** sets of reals.) One can ask:

**Question 3** *How strong is* $\mathrm{Det}(\boldsymbol{\Gamma_n})$*?*

The second author has shown that $\mathrm{Det}(\boldsymbol{\Gamma_2})$ implies the existence of inner models with a proper class of strong cardinals; and assuming $\mathrm{Det}(\boldsymbol{\Gamma_3})$ that there are, in the nomenclature of Feng and Jensen [3], *type-2* premice.

Lastly we mention a direct application of the limit rules here to a define a class of models generalising one constructed by H. Field in his attempt to define a theory of truth with a conditional operator. The latter's *G-model* in [4] is defined using an $n=2$ quasi-induction (somewhat beyond arithmetic). By using limit rules at higher levels one can obtain all the desired effects of his $G$-solutions but with longer hierarchies of his 'determinateness operators'. We leave the interested reader to consult Field's book.

# Bibliography

**[1]** J.P. Burgess. The truth is never simple. *Journal for Symbolic Logic*, 51(3):663–681, 1986.

**[2]** K. Devlin. *Constructibility*. Perspectives in Mathematical Logic. Springer Verlag, Berlin, Heidelberg, 1984.

**[3]** Q. Feng and R.B. Jensen. Supercomplete extenders and type-1 mice. *Annals of Pure and Applied Logic*, 126(1-3):1–73, 2004.

**[4]** H. Field. *Saving Truth from Paradox*. Oxford University Press, 2008.

**[5]** S. D. Friedman. *Fine Structure and Class Forcing*, volume 3 of *Series in Logic and its Applications*. de Gruyter, Berlin, New York, 2000.

**[6]** S. D. Friedman. Parameter free uniformisation. *Proceedings of the American Mathematical Society*, 136:3327–3330, 2008.

**[7]** S. D. Friedman and P. D. Welch. Two observations concerning infinite time Turing machines. In I. Dimitriou, editor, *BIWOC 2007 Report*, pages 44–47, Bonn, January 2007. Hausdorff Centre for Mathematics. also at http://www.logic.univie.ac.at/sdf/papers/joint.philip.ps.

**[8]** J. D. Hamkins and A. Lewis. Infinite time Turing machines. *Journal of Symbolic Logic*, 65(2):567–604, 2000.

**[9]** R. B. Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 4:229–308, 1972.

**[10]** G.E. Sacks. The 1-section of a type-$k$ object. In J.E. Fenstad and P.G. Hinman, editors, *Proceedings of the 1972 Oslo Symposium*, Studies in Logic, pages 81–93, Amsterdam, 1974. North-Holland.

**[11]** P. D. Welch. Eventually infinite time Turing degrees: infinite time decidable reals. *Journal for Symbolic Logic*, 65(3):1193–1203, 2000.

**[12]** P. D. Welch. Characteristics of discrete transfinite Turing machine models: halting times, stabilization times, and normal form theorems. *Theoretical Computer Science*, 410:426–442, January 2009.

**[13]** P. D. Welch. Weak systems of determinacy and arithmetical quasi-inductive definitions. *arXiv: 0905.4412*, submitted.