The internal consistency of Easton's theorem

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Let Card denote the class of infinite cardinals and Reg the class of infinite regular cardinals. The *continuum function on regulars* is the function $\kappa \mapsto 2^\kappa$, defined on Reg. This function C has the following two properties: $\alpha \leq \beta \to C(\alpha) \leq C(\beta)$ and $\alpha < \operatorname{cof}(C(\alpha))$. Easton [2] showed that, assuming GCH, any function $F : \text{Reg} \to \text{Card}$ with these two properties (any "Easton function") is the continuum function on regulars of a cofinality-preserving generic extension of the universe. We say that this generic extension "realises" the Easton function F. In particular, the statement "$2^\kappa = \kappa^{++}$ for all regular κ" is consistent, as by Easton's result it can be forced over Gödel's universe L.

The concept of *internal consistency* was introduced in [5], where one demands that consistency be witnessed in an *inner model*, under the assumption of large cardinals. The first result of this article is that any Easton function definable in L without parameters (or with parameters that are countable in $L[0^\#]$) can be realised in an inner model of $L[0^\#]$ with the same cofinalities as L. Thus the statement "$2^\kappa = \kappa^{++}$ for all regular κ" is not only consistent, but also internally consistent, under the assumption that $0^\#$ exists. The proof of this result makes us of a technique of "generic modification".

One can also consider Easton functions which are L-definable using parameters which are not necessarily countable in $L[0^\#]$. We show that such

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functions can also be realised by inner models of $L[0^\#]$ with the same cofinalities as L, provided these parameters are at most ω_1^V. The proof uses a "generic stretching" technique to transfer a generic for a given product forcing to a larger one.

One cannot hope to realise an arbitrary L-definable Easton function with parameter ω_2^V in an inner model, as $2^\omega = \omega_2^V$ will fail in all inner models if CH holds in V. A reasonable conjecture would be that any L-definable Easton function f with parameter ω_2^V satisfying $f(\alpha) < \omega_2^V$ for countable $\alpha \in \mathrm{Reg}^L$ can be realised in an inner model of $L[0^\#]$ with same cofinalities as L. We take a step in this direction by showing that in some inner model of $L[0^\#]$ with same cofinalities as L, ω_1^V is a strong limit cardinal and $2^{\omega_1^V} = \omega_2^V$. The proof uses a gap 1 morass.

Some preliminaries

We begin with some observations about I = the class of Silver indiscernibles, Skolem hulls and nice names.

The following is easily verified.

Lemma 1 *Let G be P-generic over L where P is a set in L and let X be a subclass of Ord. Let $Hull^{L[G]}(X)$ denote the smallest elementary submodel of $L[G]$ containing $X \cup \{G\}$. Then:*
(a) $Hull^{L[G]}(X) = \{\tau(\vec{x})^G \mid x_j \in X$ for each j, τ an L-term, $\tau(\vec{x})$ a P-name$\}$.
(b) If P belongs to L_i where $i \in I$ then $L_i[G] \prec L[G]$.

Let $\langle i_\alpha \mid \alpha \in Ord\rangle$ be the increasing enumeration of I and for each i in I let i^* denote the least indiscernible greater than i. For any α, $i^{*\alpha}$ is the α^{th} indiscernible greater than i.

Lemma 2 *Let i be an indiscernible, $j = i^{*\omega}$ and let G be P-generic over L where $P \in L_j$. Then $\bigcup_{n\in\omega} Hull^{L[G]}(i \cup \{i, i^*, \ldots, i^{*n}\}) = L_j[G]$*

Proof. Define $X = \bigcup_{n\in\omega} \mathrm{Hull}^{L[G]}(i \cup \{i, i^*, \ldots, i^{*n}\})$. Let σ be a name in L_j^P for an element of $L_j[G]$. Then $\sigma \in \mathrm{Hull}(i \cup \{i, i^*, \ldots, i^{*n}\})$ for some n. Therefore $\sigma^G \in X$; i.e., we have shown $X \supseteq L_j[G]$. Conversely, $L_j[G]$ is an elementary submodel of $L[G]$, so we have $X \subseteq L_j[G]$. □

Lemma 3 *Let $H_n = i^* \cap Hull^L(i \cup \{i, i^*, \dots, i^{*n}\})$, $X_0 = H_0$ and $X_{n+1} = H_{n+1} \setminus H_n$. Then $\|X_n\|^L = i$.*

Proof. For all n we have $\|X_n\|^L \leq \|H_n\|^L \leq i$. For the reverse inequality we first show $X_{n+1} \neq \emptyset$. Let $S_n = i \cup \{i, i^*, \dots, i^{*n}\}$. We have $L_{i^{*(n+1)}} \prec L$, so $Y_n \equiv \mathrm{Hull}^L(S_n) = \mathrm{Hull}^{L_{i^{*(n+1)}}}(S_n)$. So Y_n belongs to Y_{n+1}. Also $Y_n \cap i^{+L}$ is an ordinal $\alpha_n < i^{+L}$ and α_n is in Y_{n+1} but not in Y_n. So $\alpha_n \in X_{n+1}$.

Any $\alpha_n + \beta$, $\beta < i$, belongs to H_{n+1} and thus we have i-many elements in X_{n+1}. So $\|X_{n+1}\|^L = i$. As X_0 contains i, we also have $\|X_0\|^L = i$. □

Corollary 4 *Let $\alpha \in (i, i^{+L})$ and $H_n = \alpha \cap Hull^L(i \cup \{i, i^*, \dots, i^{*n}\})$, $X_0 = H_0$ and $X_{n+1} = H_{n+1} \setminus H_n$. Then there exists $m \in \omega$ such that*
(a) For all $n < m$: $\|X_n\|^L = i$.
(b) $\|X_m\|^L \leq i$.
(c) For $n > m$: $X_n = \emptyset$.

Proof. From the previous proof we have: $H_n = Y_n \cap \alpha = \alpha_n \cap \alpha$. If $\alpha_n < \alpha$, then $H_n = \alpha_n$ and $\|X_n\|^L = i$. If $\alpha_n \geq \alpha$, then $H_n = \alpha$. So we can choose m to be $\min\{n \mid \alpha_n \geq \alpha\}$. ($\{n \mid \alpha_n \geq \alpha\}$ is not empty because $\alpha < i^{+L}$.) □

Corollary 5 *Let $\alpha \in [i^{+L}, i^*)$ and $H_n = \alpha \cap (Hull^L(i \cup \{i, i^*, \dots, i^{*n}\})$, $X_0 = H_0$ and $X_{n+1} = H_{n+1} \setminus H_n$. Then $\|X_n\|^L = i$.*

Proof. The interval $[\alpha_n, \alpha_{n+1})$ from the proof of Lemma 3 is a subset of X_{n+1}. □

A *nice* P-name is a P-name of the form $\bigcup\{\{\alpha\} \times A_\alpha \mid \alpha \in S\}$, where S is a set of ordinals and each A_α is an antichain in P.

For a proof of the following, see [9], Lemma VII.5.12., page 208.

Lemma 6 *Let M be an inner model, $P \in M$ a partial ordering, $\sigma, \rho \in M^P$. Then there is a nice P-name $\xi \in M^P$ such that $1^P \Vdash (\rho \subseteq \sigma \rightarrow \rho = \xi)$.*

Distributivity is important for the existence of generic sets for partial orderings. This is formulated in the next two lemmas.

Lemma 7 *Suppose that P is a forcing in L, $\|P\|^L \leq i^*$ and P is i^{+L}-distributive in L. Then there exists a P-generic over L in $L[0^\sharp]$.*

Proof. We may assume that P is a subset of i^*. Let $\mathcal{D}$ be $\{D \mid D$ is open dense on P, $D \in L\}$. Write $\mathcal{D}$ as $\bigcup_{n\in\omega} X_n$, where $X_n = \mathrm{Hull}(i \cup \{i, i^*, \ldots, i^{*n}\}) \cap \mathcal{D}$. Then $X_n \in L$ and $\|X_n\|^L \leq i$ for each n.

Now choose p_n, $n \in \omega$, to meet all D in X_n, $p_{n+1} \leq p_n$. Then $G = \{p \mid p_n \leq p$ for some $n\}$ is P-generic. And the construction of G is possible in $L[0^\#]$. $\square$

Corollary 8 *Suppose $H \in L[0^\#]$ is Q-generic over L for some $Q \in L_{i^{*\omega}}$, where i is an indiscernible. Let P be a forcing in $L[H]$ such that $\|P\|^{L[H]} < i^{*\omega}$ and P is i^{+L}-distributive in $L[H]$. Then there exists a P-generic over $L[H]$ in $L[0^\#]$.*

Proof. We can use the previous proof, using Lemma 2 to guarantee $\mathcal{D} = \bigcup_{n\in\omega} X_n$. $\square$

Lemma 9 *If P is constructible, $\|P\|^L \leq \omega_1^V$, $0^\#$ exists and P preserves ω_1 over V then there exists a P-generic over L.*

Proof. [4], Theorem 1. $\square$

The next lemma explains why we cannot use a simple iterated forcing in this paper.

Lemma 10 *Suppose that $P = Add^L(\omega_1^L, \omega_3^L)$ and $Q = Add^{L[G]}(\omega_2^L, \omega_4^L)$, with generics G and H, respectively. Then $Card^{L[G][H]} \neq Card^L$.*

Proof. In $L[G]$, $2^{\omega_1^L} = \omega_3^L$. Let H_1 be the restriction of H to $\mathrm{Add}^{L[G]}(\omega_2^L, 1)$, a set generic for the latter forcing over $L[G]$. In $L[G][H_1]$ there is a function f from ω_2^L onto $(2^{\omega_1^L})^{L[G]}$, as any subset of ω_1^L in $L[G]$ gets "coded" into an interval of H_1. So in $L[G][H_1] \subseteq L[G][H]$ there is a surjection from ω_2^L onto ω_3^L, showing that cardinals are not preserved by the forcing $P * Q$. $\square$

The next lemma is standard.

Lemma 11 *Assume the forcing P is in the inner model M, λ is a regular cardinal in M and P is λ-distributive. Let G be P-generic over M and μ a cardinal in M less then λ. Then $\mathcal{P}(\mu)^M = \mathcal{P}(\mu)^{M[G]}$.*

Lemma 12 (Jensen's Covering Theorem) *Suppose there is an uncountable set of ordinals which is not covered in L, i.e., not a subset of a constructible set of the same V-cardinality. Then $0^{\#}$ exists.*

Lemma 13 *Let M be an inner model, P and Q set-forcings in M and suppose that for some cardinal λ of M, P has the λ^+-c.c. and Q is λ^+-closed in M. If $G = G_0 \times G_1$ is $P \times Q$-generic over M then $\mathcal{P}(\lambda) \cap M[G] = \mathcal{P}(\lambda) \cap M[G_0]$*

Proof. [7], proof of Lemma 15.19, page 234. (But note that [7] uses a different definition for "closed": κ^+-closed here is κ-closed in [7].) □

Lemma 14 *Suppose that $P = P_0 \times P_1$ where P_0 and P_1 are class forcings definable over the model M and P_0-forcing is definable.*

If G_0 is P_0-generic over M and G_1 is P_1-generic over $M[G_0]$ then $G_0 \times G_1$ is P-generic over M

If G is P-generic over M, then $G = G_0 \times G_1$ where G_0 is P_0-generic over M and G_1 is P_1-generic over $M[G_0]$.

Proof. [3], proof of Product Lemma 2.27, page 40. □

Lemma 15 *Suppose that $P = P_0 * P_1$ is a two-step iteration defined over the model M and P_0-forcing is definable.*

If G_0 is P_0 generic over M and G_1 is P_1-generic over $M[G_0]$ then $\{(p_0, \dot{p}_1) \mid p_0 \in G_0 \text{ and } (\dot{p}_1)^{G_0} \in G_1\}$ is P-generic over M.

If G is P-generic over M, then $G = \{(p_0, \dot{p}_1) \mid p_0 \in G_0 \text{ and } (\dot{p}_1)^{G_0} \in G_1\}$, where G_0 is P_0-generic over M and G_1 is P_1-generic over $M[G_0]$.

Proof. [3], proof of Product Lemma 2.30, page 44. □

Definition 16 *A family A of sets is called a* Δ-system *iff there is a fixed set r, called the* root, *such that $a \cap b = r$ whenever a and b are distinct members of A.*

Lemma 17 *Let κ be any infinite cardinal. Let $\phi > \kappa$ be regular and satisfy $\forall \alpha < \phi(\|\alpha^{<\kappa}\| < \phi)$. Assume $\|A\| \geq \phi$ and $\forall x \in A(\|x\| < \kappa)$. Then there is a $B \subseteq A$ such that $\|B\| = \phi$ and B forms a Δ-system.*

Proof. [9], proof of theorem II.1.6, page 49 □

We next discuss the type of iterated forcings we will use in this paper.

Definition 18 *Assume that M satisfies GCH and consider an M-definable sequence $\langle P(<\beta) \mid \beta \leq \alpha \rangle$, where $\alpha \in Ord$ or $\alpha = Ord$, with the following properties:*

1. *$P(<0)$ is the trivial forcing $\{\emptyset\}$ and each $P(<\beta)$ consists of functions $p : \beta \to M$ in M.*

2. *For $\gamma + 1 \leq \alpha$, $P(<\gamma+1) \simeq P(<\gamma) * P(\gamma)$ via the isomorphism $p \to (p(<\gamma), p(\gamma))$, where $p(<\gamma) = p \restriction \gamma$, $P(<\gamma)$ is a set in M and $P(<\gamma) \Vdash P(\gamma)$ is a cofinality-preserving set-forcing.*

3. *We have a continuous increasing sequence $\{c_\gamma \mid \gamma < \alpha\}$ of limit cardinals with the following properties:*

   (a) $P(<\gamma) \Vdash P(\gamma)$ is c_γ-closed

   (b) $P(\gamma)$ is a $P(<\gamma)$-name of cardinality at most $c^+_{\gamma+1}$.

4. *For a singular limit $\lambda \leq \alpha$: $P(<\lambda) =$ Inverse-Limit $\langle P(<\beta) \mid \beta < \lambda \rangle \equiv \{p : \lambda \to M \mid \forall \beta < \lambda\ (p(<\beta) \in P(<\beta))\}$. This is ordered by: $p \leq q$ iff $p(<\beta) \leq q(<\beta)$ for each $\beta < \lambda$.*

5. *For a regular limit $\lambda \leq \alpha$: $P(<\lambda) =$ Direct-Limit $\langle P(<\beta) | \beta < \lambda \rangle \equiv \{p : \lambda \to M \mid \exists \gamma < \lambda (p(<\gamma) \in P(<\gamma) \wedge \forall \beta \in [\gamma, \lambda),\ p(\beta) = 1^{P(\beta)})\}$. The ordering is the same as in the previous case.*

When $\alpha = Ord$, then $P =$ Direct-Limit $\langle P(\beta) \mid \beta \in Ord \rangle$.

We also define:

$P(\leq \beta) = P(<\beta+1)$
$P[\beta, \gamma) \equiv$ the iterated forcing in $M^{P(<\beta)}$ which starts from $P(\beta)$ and has length $\gamma - \beta$
$P(\geq \beta) \equiv P[\beta, \alpha)$
$P(\beta, \gamma) \equiv P[\beta+1, \gamma)$
$P(>\beta) \equiv P(\geq (\beta+1))$

For this iteration we have the Factoring property:

Lemma 19 *$P(<\beta)$ preserves cofinalities and for any $\alpha<\beta$, $P(<\beta)$ is isomorphic to $P(<\alpha)*P[\alpha,\beta)$.*

Proof. This is similar to [3], Lemma 2.34, page 46.

By induction on β. The result is trivial for $\beta=0$. If $\beta=\gamma+1\geq\alpha$ and we have defined the isomorphism $\theta: P(<\gamma)\simeq P(<\alpha)*P[\alpha,\gamma)$, then $P(\leq\gamma)\simeq P(<\gamma)*P(\gamma)\simeq(P(<\alpha)*P[\alpha,\gamma))*P(\gamma)\simeq P(<\alpha)*(P[\alpha,\gamma)*P(\gamma))\simeq P(<\alpha)*P[\alpha,\gamma+1)$. As by induction $P(<\gamma)$ preserves cofinalities and by hypothesis $P(<\gamma)\Vdash P(\gamma)$ preserves cofinalities it follows that $P(\leq\gamma)=P(<\beta)$ preserves cofinalities too.

Suppose that β is a limit. By induction $P(<\gamma)$ is canonically isomorphic to $P(<\alpha)*P[\alpha,\gamma)$ for $\gamma<\beta$. It follows that $P(<\beta)$ is canonically isomorphic to $P(<\alpha)*P[\alpha,\beta)$ provided we know that $P(<\alpha)$ preserves the regularity of β (in case β is regular). The latter follows from the fact that by induction, $P(<\alpha)$ preserves cofinalities. Finally, we show that $P(<\beta)$ preserves cofinalities. Suppose that γ is a regular cardinal; we show that $P(<\beta)$ preserves "cofinality greater than γ" (cof $>\gamma$). If γ is less than c_β then factor $P(<\beta)$ as $P(<\alpha)*P[\alpha,\beta)$ where γ is less than c_α. By induction, $P(<\alpha)$ preserves cof $>\gamma$ and by definition $P[\alpha,\beta)$ is c_α-closed and therefore γ^+-closed; it follows that $P(<\beta)$ preserves cof $>\gamma$. If γ is greater than c_β then as $P(<\beta)$ has a dense subset of cardinality c_β^+, $P(<\beta)$ preserves cof $>\gamma$. Finally if γ equals c_β then $P(<\beta)$ has a dense subset of cardinality $c_\beta=\gamma$, so again $P(<\beta)$ preserves cof $>\gamma$. □

Lemma 20 *P preserves cofinalities and for any $\alpha\in Ord$ is isomorphic to $P(<\alpha)*P(\geq\alpha)$.*

Proof. For $\beta=\infty$, we can use the same argument as for regular β in the previous proof. For smaller β, we have the previous proof. □

Lemma 21 *Let P be the direct limit of an iterated forcing over M as above, M a model of ZFC, and let G be P-generic. Then $M[G]\vDash ZFC$.*

Proof. See [3]. From the comment on page 47 we know, P is *tame*. Then we can use Lemma 2.21, page 36. □

Definition 22 *Let E be a subset of Ord and $\mathcal{P}=\{P_\alpha\mid\alpha\in E\}$ a family of posets. The* Easton product *$\prod_{\alpha\in E}^{Easton}P_\alpha$ of $\mathcal{P}$ consists of all $\langle p_\alpha\mid\alpha\in E\rangle$ in $\prod_{\alpha\in E}P_\alpha$ such that for all $\kappa\in Reg$ there is a $\beta<\kappa$ such that $\alpha\in(\beta,\kappa)\cap E\rightarrow p_\alpha=1^{P_\alpha}$.*

1 Easton functions with countable parameters

Definition 23 *An L-definable $f : Reg^L \to Card^L$ is an Easton function (for L) iff on its domain, $cof^L(f(\kappa)) > \kappa$ and $\mu < \lambda \to f(\mu) \leq f(\lambda)$.*

We first show:

Theorem 24 *Take any L-definable (without parameters) Easton function $f : Reg^L \to Card^L$. There is an inner model M of $L[0^\#]$ with the same cofinalities as L in which $2^\kappa = f(\kappa)$ for all $\kappa \in Reg^L$.*

We begin with some lemmas:

Lemma 25 *Let M be an inner model with the same cofinalities as L, κ a successor of a regular cardinal μ in M and assume $\lambda < \mu \to (\mu^\lambda)^M = \mu$. Let $f : Reg^L \to Card^L$ be an L-definable Easton function and $P = \prod^{Easton}_{\lambda<\kappa, \lambda \in Reg^L} Add(\lambda, f(\lambda))$ in M. Then P is κ-c.c. in M.*

Proof. Each a in P is a function from an "Easton subset" of $\{\lambda < \kappa \mid \lambda \in Reg^L\}$ which assigns to each λ in its domain a condition in $\mathrm{Add}(\lambda, f(\lambda))$, i.e., a function from a subset of $\lambda \times f(\lambda)$ of cardinality less than λ into 2. For each such a let $\mathrm{Dom}^*(a)$ denote the set of triples (λ, i, j) such that λ is in the domain of a and (i, j) is in the domain of $a(\lambda)$. Note that $\mathrm{Dom}^*(a)$ is a set of cardinality less than μ.

Now suppose $A = \{a_\beta \mid \beta < \kappa\} \in M$ were an antichain in P of cardinality κ (where the a_β's are distinct). Then apply Lemma 17 to the $\mathrm{Dom}^*(a_\beta)$'s (where κ is the current μ, ϕ is the current κ and A is $\{\mathrm{Dom}^*(a) | a \in A\}$; here we use the assumption $(\mu^\lambda)^M = \mu$ for $\lambda < \mu$). Let $B \subseteq \{\mathrm{Dom}^*(a) \mid a \in A\}$ have cardinality κ and core b. Then $\|b\|^M < \mu$ because for all $a \in A$, $\|\mathrm{Dom}^*(a)\| < \mu$. The number of functions from b to 2 is at most $2^{\|b\|} \leq \mu^{\|b\|} = \mu < \kappa$, so there are distinct $x, y \in A$ which are compatible (because $\mathrm{Dom}^*(x) \cap \mathrm{Dom}^*(y) = b$ and x, y agree on b). This contradicts the assumption that A is an antichain. $\square$

Definition 26 *Let α, β, γ be ordinals, $\alpha < \beta$ and i_γ the γ^{th} Silver indiscernible. Define:*

$\pi_{i_\alpha,i_\beta}(i_\gamma) = i_\gamma$ *for* $\gamma < \alpha$
$\pi_{i_\alpha,i_\beta}(i_{\alpha+\delta}) = i_{\beta+\delta}$.

π_{i_α,i_β} *extends uniquely to an elementary embedding $L \to L$, which we also denote by π_{i_α,i_β}.*

Lemma 27 *Suppose $i < j$ belong to I. If $x \in L$, $\|x\|^L = j$ then $x \cap Rng(\pi_{ij})$ belongs to L, $\pi_{ij}^{-1}\restriction(x \cap Rng(\pi_{ij}))$ belongs to L and $x \cap Rng(\pi_{ij})$ has L-cardinality at most i.*

Proof. Note that $\mathrm{Rng}(\pi_{ij}) = \mathrm{Hull}(i \cup I(\geq j))$, an elementary submodel of L.

First suppose that $x = j$. Then $x \cap \mathrm{Rng}(\pi_{ij}) = j \cap \mathrm{Rng}(\pi_{ij}) = i \in L$, $\pi_{ij}^{-1}\restriction(x \cap \mathrm{Rng}(\pi_{ij})) = \pi_{ij}^{-1}\restriction i = id_i \in L$.

Now write $x = \tau(\vec{\alpha}, \vec{\beta}, j, \vec{\gamma})$ where $\vec{\alpha} < i \leq \vec{\beta} < j < \vec{\gamma}$ are in I.

Suppose $\vec{\beta} = \emptyset$, an hypothesis equivalent to $x \in \mathrm{Rng}(\pi_{ij})$. Let ϕ be a bijection between x and $\|x\|^L = j$. As $\mathrm{Rng}(\pi_{ij})$ is an elementary submodel of L, we can choose $\phi \in \mathrm{Rng}(\pi_{ij})$. Then for $y \in x$, we have $y \in \mathrm{Rng}(\pi_{ij})$ iff $\phi(y) \in \mathrm{Rng}(\pi_{ij})$. So:

$x \cap \mathrm{Rng}(\pi_{ij}) = \phi^{-1}[j \cap \mathrm{Rng}(\pi_{ij})] = \phi^{-1}[i] \in L$.

For the second property, we use $\psi = \pi_{ij}^{-1}(\phi)$. ψ is a bijection between $\pi_{ij}^{-1}[x]$ and i. Let $y \in x \cap \mathrm{Rng}(\pi_{ij})$. From $\psi(y) = \pi_{ij}(\psi(y)) = (\pi_{ij}(\psi))(\pi_{ij}(y)) = \phi(\pi_{ij}(y))$ we have $\pi_{ij}(y) = \phi^{-1}(\psi(y))$. So $\pi_{ij}^{-1}\restriction(x \cap \mathrm{Rng}(\pi_{ij}))$ is a composition of two functions in L and therefore belongs to L.

Now suppose $\vec{\beta} \neq \emptyset$. Define $x^* = \bigcup\{\tau(\vec{\delta}, j, \vec{\gamma}) \mid \vec{\delta} < j \wedge \vec{\delta} \in \mathrm{Ord}^{<\omega} \wedge \|\tau(\vec{\delta}, j, \vec{\gamma})\|^L = j\}$. Then $x \subseteq x^*$, $\|x^*\|^L = j$ and $x^* \in \mathrm{Rng}(\pi_{ij})$. So by the above, $x^* \cap \mathrm{Rng}(\pi_{ij}) \in L$. We then have: $x \cap \mathrm{Rng}(\pi_{ij}) = x \cap (x^* \cap \mathrm{Rng}(\pi_{ij})) \in L$ and $\pi_{ij}^{-1}\restriction(x \cap \mathrm{Rng}(\pi_{ij})) = \pi_{ij}^{-1}\restriction(x^* \cap \mathrm{Rng}(\pi_{ij}))\restriction x \in L$, as desired. The statement about L-cardinality is implicit in the above. □

Lemma 28 *Take any L-definable Easton function $f : Reg^L \to Card^L$. Let M be an inner model such that $Card^L = Card^M$ and GCH holds in M for cardinals in $[\delta_1, \delta_2)$, where $\delta_1 < \delta_2$ are regular in M. Let P be the forcing $\prod^{Easton}_{\kappa \in [\delta_1,\delta_2) \cap Reg^L} Add(\kappa, f(\kappa))$ in M. Also assume $\kappa \in Reg^L \cap \delta_2 \to f(\kappa) \leq \delta_2$ and $\kappa \in Reg^L \cap \delta_1 \to 2^\kappa \leq \delta_1$. Let G be P-generic over M.*

Then $(2^\kappa)^{M[G]} = f(\kappa)$ for all $\kappa \in Reg^L \cap [\delta_1, \delta_2)$, $(2^\kappa)^{M[G]} = (2^\kappa)^M$ for $\kappa \in Reg^L \setminus [\delta_1, \delta_2)$ and G preserve cofinalities.

Proof. Let κ be a regular cardinal in L from $[\delta_1, \delta_2)$ which is either δ_1 or the successor of a regular cardinal. $P \simeq P(<\kappa) \times P(\geq \kappa)$. By Lemma 14 we have $G(<\kappa)$ and $G(\geq \kappa)$ such that $M[G] = M[G(<\kappa)][G(\geq \kappa)]$. $P(\geq \kappa)^M$ is κ-closed in M. By Lemma 25, $P(<\kappa)$ is κ-c.c. in M ($\kappa = \delta_1$ is a trivial case). From this we have: $M \vDash \text{cof}\,(\alpha) \geq \kappa \to M[G(<\kappa)] \vDash \text{cof}\,(\alpha) \geq \kappa \to M[G(<\kappa)][G(\geq \kappa)] \vDash \text{cof}\,(\alpha) \geq \kappa$ for successors of regular cardinals κ, and therefore for all regular cardinals. So G preserves all cofinalities.

We want $M[G] \vDash 2^\kappa = f(\kappa)$ for regular $\kappa \in [\delta_1, \delta_2)$ and $(2^\kappa)^M = (2^\kappa)^{M[G]}$ for other regular κ. For $\kappa < \delta_1$: P is δ_1-closed in M, so by Lemma 11 we have $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M[G]}$. P preserves cardinalities, so $(2^\kappa)^M = (2^\kappa)^{M[G]}$. For $\kappa \geq \delta_2$: $\|P\|^M \leq \delta_2$, so we have at most $(2^{\delta_2})^\kappa = 2^\kappa$ nice names for subsets of κ. So $\|2^\kappa\|^M = \|2^\kappa\|^{M[G]}$.

Suppose $\kappa \in [\delta_1, \delta_2)$, κ regular in M. We want $2^\kappa \geq f(\kappa)$ and $2^\kappa \leq f(\kappa)$.

In $M[G]$ we have a generic for $\text{Add}^M(\kappa, f(\kappa))$. So in $M[G]$ we have $2^\kappa \geq f(\kappa)$. For the other direction we use $P(<\kappa^+)$ and $P(\geq \kappa^+)$. $P(<\kappa^+)$ is κ^+-c.c. and $P(\geq \kappa^+)$ is κ^+-closed. Let λ denote κ^+. Then $\lambda \leq f(\kappa)$ and $\|P(<\lambda)\|^M = f(\kappa)$. So the number of antichains in $P(<\lambda)$ is $f(\kappa)^{<\lambda}$ and therefore we have only $(f(\kappa)^{<\lambda})^\kappa = f(\kappa)$ nice names for subsets of κ. From Lemma 13 we have $\mathcal{P}(\kappa)^{M[G(<\kappa^+)]} = \mathcal{P}(\kappa)^{M[G]}$. So $M[G] \vDash \|\mathcal{P}(\kappa)\| = 2^\kappa \leq f(\kappa)$. □

We turn now to the proof of the theorem.

Proof of theorem 24. We want to use reverse Easton forcing. But we cannot use iteration for f-crossings ($\kappa < \lambda < f(\kappa) < f(\lambda)$), because $\text{Add}(\kappa, f(\kappa)) * \text{Add}(\lambda, f(\lambda))$ collapses $f(\kappa)$ to λ (see the example in Lemma 10). So, we split the ordinals into the intervals determined by the closure points of f and take an Easton product within those intervals. Then we join these product forcings together to one iterated forcing.

Let C be $\{\alpha \in \mathrm{Card}^L \mid \omega \leq \alpha \wedge (\kappa \in \mathrm{Reg}^L \cap \alpha \to f(\kappa) < \alpha)\}$. We know that $I \subseteq C \subseteq \mathrm{LimCard}^L$, because f is L-definable without parameters and $L_i \prec L$ for $i \in I$. Let $\{c_\alpha\}$ be the increasing enumeration of C. Then $c_0 = \omega$ and $c_i = i$ for $i \in I$.

We define P:

1. For each β, $P(\beta) = \prod^{Easton}_{\alpha \in [c_\beta, c_{\beta+1}) \cap \mathrm{Reg}^L} \mathrm{Add}(\alpha, f(\alpha))$ over $L^{P(<\beta)}$.

   The rest is determined by Definition 18:

2. $P(<0) = \{\emptyset\}$, $P(<(\alpha+1)) = P(\leq \alpha)$
3. For regular λ in L, $P(<\lambda) =$ Direct-Limit $\{P(\leq \alpha) | \alpha < \lambda\}$
4. For singular λ in L, $P(<\lambda) =$ Inverse-Limit $\{P(\leq \alpha) | \alpha < \lambda\}$
5. $P(\leq \alpha) = P(<\alpha) * P(\alpha)$
6. $P =$ Direct-Limit $\{P(\leq \alpha) \mid \alpha \in Ord\}$

Because this definition fulfills definition 18, we can use Lemmas 19 and 21. For each α we have $P \simeq P(<\alpha) * P(\geq \alpha)$.

Our goal is to show that P preserves cofinalities, P forces $2^\kappa = f(\kappa)$ for all $\kappa \in \mathrm{Reg}^L$ and that there exists a P-generic over L in $V = L[0^\#]$.

Cofinality preservation

By Lemma 20 it suffices to verify that for each α, $P(<\alpha) \Vdash P(\alpha)$ preserves cofinalities. This follows from Lemma 28, setting $\delta_1 = c_\alpha$ if c_α is L-regular, $\delta_1 = c_\alpha^{+L}$ if c_α is L-singular and $\delta_2 = c_{\alpha+1}^{+L}$.

f is realised

We want: $2^\kappa = f(\kappa)$ for each L-regular κ. Write $P \simeq P(<\alpha) * P(\alpha) * P(>\alpha)$ where κ belongs to the interval $[c_\alpha, c_{\alpha+1})$. Then $P(<\alpha)$ has cardinality at most κ^+, $P(\alpha)$ adds exactly $f(\kappa)$ subsets if κ by Lemma 28 and $P(>\alpha)$ is κ^+-closed. It follows that $P \Vdash 2^\kappa = f(\kappa)$.

A P-generic class

We build a P-generic H definably in $L[0^\#]$. By induction on $i \in I$ we define a $P(\leq i)$-generic $H(\leq i)$ in $L[0^\#]$ with the property that for $j < k$ in I, the generics $H_0(j)$, $H_0(k)$ "fit together", where $H_0(j)$ is the restriction of $H(j) \subseteq P(j) = \prod^{Easton}_{\lambda \in [c_j, c_{j+1}) \cap \mathrm{Reg}^L} \mathrm{Add}(\lambda, f(\lambda))$ to $P_0(j) = \mathrm{Add}(j, f(j))$.

To make this precise, extend the π_{ij} of Definition 26 to an embedding $\tilde{\pi}_{ij} : L[H(<i)] \to L[H(<j)]$ by $\tilde{\pi}_{ij}(\sigma^{H(<i)}) = \pi_{ij}(\sigma)^{H(<j)}$. $\tilde{\pi}_{ij}$ is a well-defined elementary embedding:

$$\begin{array}{rcl} L[H(<i)] & \models & \psi(\sigma_1^{H(<i)}, \ldots, \sigma_n^{H(<i)}) \to \\ \exists p \in H(<i)\ (p & \Vdash & \psi(\sigma_1, \ldots, \sigma_n)) \to \\ \exists p \in H(<i)\ (\pi_{ij}(p) & \Vdash & \psi(\pi_{ij}(\sigma_1), \ldots, \pi_{ij}(\sigma_n))) \to \\ L[H(<j)] & \models & \psi(\pi_{ij}(\sigma_1)^{H(<j)}, \ldots, \pi_{ij}(\sigma_n)^{H(<j)})) \to \\ L[H(<j)] & \models & \psi(\tilde{\pi}_{ij}(\sigma_1^{H(<j)}, \ldots, \tilde{\pi}_{ij}(\sigma_n^{H(<j)})). \end{array}$$

In the third implication we use $\pi_{ij}[H(<i)] = H(<i) \subseteq H(<j)$. The above implications are in fact equivalences, as they also hold for $\neg\psi$. $\tilde{\pi}_{ij}$ also extends π_{ij}, as for $x \in L$, if $y = \pi_{ij}(x)$, then: $\tilde{\pi}_{ij}(x) = \tilde{\pi}_{ij}((\hat{x})^{H(<i)}) = (\pi_{ij}(\hat{x}))^{H(<j)} = (\hat{y})^{H(<j)} = y = \pi_{ij}(x)$, where $\hat{x} = \{(\hat{z}, 1) \mid z \in x\}$ is the canonical name for x.

For indiscernibles $i < j < k$ we have $\tilde{\pi}_{jk} \circ \tilde{\pi}_{ij} = \tilde{\pi}_{ik}$. This follows from the definition of $\tilde{\pi}_{ij}$ and $\pi_{jk} \circ \pi_{ij} = \pi_{ik}$. And Lemma 27 also holds for $\tilde{\pi}_{ij}$ (the same proof works, using Lemma 1).

Now we construct $H(\leq i)$ by induction on $i \in L$ so that: $j, k \in I$, $j < k \to \tilde{\pi}_{jk}[H_0(j)] \subseteq H_0(k)$.

We start the induction from $P(\leq i_0)$. As the set of constructible dense subsets of this forcing is countable in $L[0^\#]$, we may choose a generic $H(\leq i_0)$ for it.

Suppose that $H(\leq i)$ has been defined, and we now wish to define $H(\leq i^*)$, where i^* is the least indiscernible greater than i. Write $P(\leq i^*) \simeq P(\leq i) * P(i, i^*]$; we want to choose a generic for $P(i, i^*]^{H(\leq i)}$. The latter forcing is i^+-closed and has cardinality $f(i^*) < i^{**}$; it follows from Corollary 8 that we may choose a generic $H(i, i^*]$ for it in $L[0^\#]$. However we must ensure that $\tilde{\pi}_{ii^*}[H_0(i)] \subseteq H_0(i^*)$, where $H_0(i^*)$ is the restriction of $H(i^*)$ to $P_0(i^*) = \mathrm{Add}(i^*, f(i^*))$ (of $L[H(< i^*)]$). This will be guaranteed by modifying $H_0(i^*)$. Note that $H(i^*)$ can be written as $H_0(i^*) \times H_1(i^*)$, where $H_1(i^*)$ is generic over $L[H(< i^*)]$ for $P_1(i^*) = \prod^{Easton}_{\alpha \in (i^*, c_{i^*+1}) \cap \mathrm{Reg}^L} \mathrm{Add}(\alpha, f(\alpha))$, and for any $\mathrm{Add}(i^*, f(i^*))$-generic $H_0^*(i^*)$, the product $H_0^*(i^*) \times H_1(i^*)$ will still be generic for $P_0(i^*) \times P_1(i^*)$, as $P_0(i^*)$ has the $(i^*)^+$-c.c. and $P_1(i^*)$ is $(i^*)^*$-closed. So to define the desired $H(i^*)$, it suffices to find any $P_0(i^*)$-generic $H_0^*(i^*)$ such that $\tilde{\pi}_{ii^*}[H_0(i)] \subseteq H_0^*(i^*)$.

Suppose that p is a condition in $P_0(i^*) = \mathrm{Add}(i^*, f(i^*))$. Define a new condition $\Psi(p)$ as follows:

$\Psi(p)(w^*) = p(w^*)$ if $w^* \in \mathrm{Dom}(p) \wedge w^* \notin \mathrm{Rng}(\tilde{\pi}_{ii^*})$,
$\Psi(p)(w^*) = H_0(i)(w)$ if $w^* \in \mathrm{Dom}(p) \wedge \tilde{\pi}_{ii^*}(w) = w^*$.

$\Psi(p)$ is indeed a condition in $P_0(i^*)$ by Lemma 27 for $\tilde{\pi}_{ii^*}$, as $\|p\|^{L[H(<i^*)]} < i^*$ (actually we only need $\|p\|^{L[H(<i^*)]} \leq i^*$ here).

Now set $H_0^*(i^*) = \{\Psi(p) \mid p \in H_0(i^*)\}$, our desired modification of $H_0(i^*)$. We prove that $H_0^*(i^*)$ is $P_0(i^*)$-generic over $L[H(< i^*)]$. Clearly $H_0^*(i^*)$ is an upward-closed filter. Let $D \in L[H(< i^*)]$ be dense on $P_0(i^*)$. To show that $H_0^*(i^*)$ meets D, we define:

p' is a *small modification* of p iff $\mathrm{Dom}(p) = \mathrm{Dom}(p')$ and $\{x \mid p(x) \neq p'(x)\}$ has cardinality at most i.
p *strongly meets* D iff all small modifications p' of p meet D.

Note that the collection of all small modifications p' of p is a set in $L[H(< i^*)]$ of cardinality at most $(\|p\|^i \cdot 2^i)^{L[H(<i^*)]} < i^*$.

Claim. $\{p \mid p$ *strongly meets* $D\}$ *is dense in* $P_0(i^*)$.

Proof of Claim. Let $p_0 = q_0$ be some element of $P_0(i^*)$. We create a descending sequence of q_α's as follows: q_1 meets D and $q_1 \leq q_0$. q_1' is a small modification of q_1 on $\mathrm{Dom}(q_0)$, q_1'' is an extension of q_1' meeting D, and q_2 is q_1'' without the modification. We repeat this with another modification on $\mathrm{Dom}(p_0)$, etc., taking unions at limits, until all modifications on $\mathrm{Dom}(p_0)$ have been considered.

p_1, the union of all q_α, is an element of $P_0(i^*)$. For this p_1 we do the same as for p_0, obtaining p_2, and continue this for i^{+L} steps, taking unions at limits. Then $p_{i^{+L}}$ is an element of $P_0(i^*)$ which strongly meets D. $\square$ (*Claim*)

Choose $s \in H_0(i^*)$ which strongly meets D. $s' = \Psi(s) \in H_0^*(i^*)$ is a small modification of s, so we have $d \in D$ s.t. $s' \leq d$. As $H_0^*(i^*)$ is upward-closed, $d \in H_0^*(i^*)$, so we have shown that $H_0^*(i^*)$ meets D. This completes the construction of $H(\leq i^*)$.

Suppose that i is a limit indiscernible. By induction we have the following property: If $i_0 < i_1$ are indiscernibles less than i then $\pi_{i_0 i_1}[H(< i_0) * H_0(i_0)]$ is contained in $H(< i_1) * H_0(i_1)$ (where as before $H_0(j)$ is the restriction of $H(j)$ to $\mathrm{Add}(j, f(j))$). We take $H(< i) * H_0(i)$ to be the union of the $\pi_{\bar{i}i}[H(< \bar{i}) * H_0(\bar{i})]$, $\bar{i} \in I \cap i$. This is an upward-closed filter; we verify that it meets all dense sets in L for $P(< i) * P_0(i)$: Let $D = \tau(\vec{j}, i, \vec{\infty})$ be dense in $P_0(i)$. Choose an indiscernible k in $(\max(\vec{j}), i)$ and set $D_k = \tau(\vec{j}, k, \vec{\infty})$. Then D_k is dense in $P(< k) * P_0(k)$, so we have some $p_k \in D_k \cap H(< k) * H_0(k)$. Then $p = \pi_{ki}(p_k)$ belongs to $H(< i) * H_0(i)$ and belongs to D. So $H(< i) * H_0(i)$ is generic, as desired.

Finally, as $P_1(i) = \prod^{Easton}_{\alpha \in (i, c_{i+1}) \cap \mathrm{Reg}^L} \mathrm{Add}(\alpha, f(\alpha))$ is i^+-closed and of cardinality less than i^{**} in $L[H(< i)]$, we can use Corollary 8 to obtain a generic $H_1(i)$ for this forcing. Then $H(< i) * H_0(i) * H_1(i) = H(\leq i)$ is the desired $P(\leq i)$-generic. This completes the construction of the $H(\leq i)$, $i \in I$, and therefore of the desired P-generic $H = \bigcup_{i \in I} H(< i)$. Clearly H can be built definably in $L[0^\#]$. This completes the proof of Theorem 24. $\square$

Corollary 29 *Suppose that $i \in I$ and f is an Easton function which is L-definable with parameters $\leq i$. Then there is an inner model M of $L[0^\#]$ with the same cofinalities and the same subsets of i as L in which $2^\kappa = f(\kappa)$ for all $\kappa \in Reg^L$ greater than i.*

Proof. Consider the following variant of the iteration used to prove Theorem 24: $P(\beta)$ is trivial for $\beta < i$, $P(i)$ is $\prod^{Easton}_{\alpha\in(i,c_{i+1})\cap\mathrm{Reg}^L}\mathrm{Add}(\alpha, f(\alpha))$ over $L^{P(<i)}$ and $P(\beta)$ is $\prod^{Easton}_{\alpha\in[c_\beta,c_{\beta+1})\cap\mathrm{Reg}^L}\mathrm{Add}(\alpha, f(\alpha))$ over $L^{P(<\beta)}$ for $\beta > i$. Then as in the proof of the Theorem, this forcing preserves cofinalities and forces $2^\kappa = f(\kappa)$ for L-regular κ greater than i. Moreover, subsets of i are not added. □

Theorem 30 *Let $f : Reg^L \to Card^L$ be an Easton function which is L-definable with $L[0^\#]$-countable parameters. Then there is an inner model M of $L[0^\#]$ with the same cofinalities as L in which $2^\kappa = f(\kappa)$ for all regular κ.*

Proof. Let i be an $L[0^\#]$-countable indiscernible so that the parameters used to define f belong to L_i. By Corollary 29, we can preserve cofinalities over L, not add subsets of i and force $2^\kappa = f(\kappa)$ for all L-regular κ greater than i. Over this generic extension M, consider the forcing $Q = \prod^{Easton}_{\kappa\in\mathrm{Reg}^L\cap i^+}\mathrm{Add}(\kappa, f(\kappa))$. This forcing has only countably many dense subsets in M as i is countable, and therefore we can choose a Q-generic over M. The result is a model with the same cofinalities as L in which $2^\kappa = f(\kappa)$ for all L-regular κ. □

2 The parameter ω_1^V

Assume that $0^\#$ exists.

Definition 31 *Let M be an inner model, κ an M-cardinal and α an ordinal, $\kappa \le \alpha < \kappa^{+V}$. A bijection $f : \alpha \to \kappa$ is* M-good *iff $f \restriction x$ is in M whenever $x \in M$, $x \subseteq \alpha$ and $\|x\|^M \le \kappa$.*

Lemma 32 *(L-good bijections) For any uncountable cardinal κ in V and ordinal α, $\kappa \le \alpha < \kappa^{+V}$, there exists an L-good bijection $f : \alpha \to \kappa$.*

Proof. Let I be the class of Silver indiscernibles and i^* the I-successor to i. We prove by induction on $i \in I \cap [\kappa, \kappa^{+V})$ that there is an L-good bijection $f_\alpha : \alpha \to \kappa$ for any $\alpha \in [\kappa, i]$.

f_κ is the identity. If we have f_i, then we construct f_{i^*} as follows: For $n \in \omega$, define $H_n = i^* \cap \mathrm{Hull}^L((i+1) \cup \{i^*, i^{**}, \ldots, i^{*n}\})$. Each H_n has

L-cardinality i, $H_n \subseteq H_{n+1}$ for each n and $\bigcup H_n = i^*$ (Lemma 2). Let $X_0 = H_0$, $X_{n+1} = H_{n+1} \setminus H_n$. By Lemma 3, $\|X_n\|^L = i$. From f_i we create a bijection $f^* : \bigcup X_n \to (\kappa \times \omega)$ by choosing $g_n : X_n \to i$ to be a constructible bijection and setting $f^*(\gamma) = (f_i(g_n(\gamma)), n)$ for γ in X_n. We claim that $f^* \upharpoonright x$ is constructible for any constructible $x \subseteq i^*$ with $\|x\|^L = \kappa$: Any such x is a subset of some H_n, and therefore is contained in the union of finitely many X_n's. It follows that $f^* \upharpoonright x$ is the finite union of constructible functions and therefore constructible. We let f_{i^*} be the composition of f^* and a constructible bijection between $\kappa \times \omega$ and κ.

For $\alpha \in (i, i^{+L})$ we let f_α be the composition of a constructible bijection $g : \alpha \to i$ with f_i. For $\alpha \in [i^{+L}, i^*)$ we can use corollary 5 and the above argument for i^*.

Suppose that $i \in (\kappa, \kappa^+)$ is a limit indiscernible with V-cofinality γ. Then $\gamma \leq \kappa$. Let $S = \langle s_\alpha \mid \alpha \in \gamma \rangle$ be increasing and continuous with $s_0 = 0$, $s_1 = \kappa^*$ and $\bigcup_\alpha s_\alpha = i$. We split i into the intervals $[s_\alpha, s_{\alpha+1})$ and for every interval we can use $f_{s_{\alpha+1}}$ to create a good bijection between $[s_\alpha, s_{\alpha+1})$ and κ. The union of these good bijections is a good bijection f between i and $\kappa \times \gamma$: Any $x \in L$, $x \subseteq i$, $\|x\|^L = \kappa$ intersects only finitely many intervals $[s_\alpha, s_{\alpha+1})$, as otherwise some initial segment of x is cofinal in some indiscernible $> \kappa$, contradicting the L-regularity of indiscernibles. Now compose f with a constructible bijection between $\kappa \times \gamma$ and κ to obtain the desired L-good bijection f_i. □

Corollary 33 *(M-good bijections) Let M be any inner model with $Card^L = Card^M$. Then for any uncountable cardinal κ and ordinal α, $\kappa \leq \alpha < \kappa^{+V}$, there exists an M-good bijection $f : \alpha \to \kappa$.*

Proof. We can find some L-good bijection $f : \alpha \to \kappa$ using Lemma 32. This bijection is also M-good: Let $x \in M$, $x \subseteq \alpha$, $\|x\|^M = \kappa$. By the Covering theorem (Lemma 12) we have $y \in L$, $x \subseteq y$ and $\|y\|^M = \|y\|^L = \kappa$. As f is an L-good bijection, $f \upharpoonright y$ is constructible, and therefore $f \upharpoonright x = (f \upharpoonright y) \upharpoonright x \in M$, as desired. □

Lemma 34 *(Stretching at ω_1^V) Let M be an inner model with the same cofinalities as L such that $\alpha < \omega_1^V \to (2^\alpha)^M \leq \omega_1^V$ and let μ be an ordinal in the interval $[\omega_1^V, \omega_2^V)$. Suppose that there exists a generic over M for*

$P = Add(\omega_1^V, \omega_1^V)^M$ *in* V*. Then in* V *there is also a generic over* M *for* $Q = Add(\omega_1^V, \mu)^M$*.*

Proof. From corollary 33 we have an M-good bijection $\pi : \mu \to \omega_1^V$. We use π to construct $\pi' : \mathrm{Add}(\omega_1^V, \mu)^M \to \mathrm{Add}(\omega_1^V, \omega_1^V)^M$ as follows: Define $\tilde{\pi} : \omega_1^V \times \mu \to \omega_1^V \times \omega_1^V$ by $\tilde{\pi}(a, b) = (a, \pi(b))$. We know that $\tilde{\pi}\restriction x \in M$ for $x \subseteq \omega_1^V \times \mu$, $\|x\|^M \leq \omega_1^V$. Now for $p \in \mathrm{Add}(\omega_1^V, \mu)^M$ set $\mathrm{Dom}(\pi'(p)) = \tilde{\pi}[\mathrm{Dom}(p)]$ and $\pi'(p)(a, b) = p(\tilde{\pi}^{-1}(a, b))$. As $\tilde{\pi}\restriction\mathrm{Dom}(p)$ belongs to M, so does $\tilde{\pi}^{-1}\restriction\mathrm{Dom}(\pi'(p))$. It follows that $\pi'(p)$ is in M and is therefore a condition in $\mathrm{Add}(\omega_1^V, \omega_1^V)^M$. ($\pi'$ is *not* surjective, but this will not matter.)

Let G be $\mathrm{Add}(\omega_1^V, \omega_1^V)^M$-generic over M. Our candidate for a Q-generic over M is $H = \{q \in Q \mid \pi'(q) \in G\}$. We need only check that H intersects maximal antichains on Q which belong to M.

Let $A \in M$ be a maximal antichain on $\mathrm{Add}(\omega_1^V, \mu)^M$ and set $A' = \{\pi'(a) \mid a \in A\}$. A' is in M because Q is $(\omega_1^V)^{+L}$-c.c. and π is an M-good bijection. Clearly A' is an antichain; we want to show that A' is a *maximal* antichain in M.

Let $D(A) = \bigcup\{\mathrm{Dom}(a) \mid a \in A\}$ and $D(A') = \bigcup\{\mathrm{Dom}(a) \mid a \in A'\}$. The bijection $\mathrm{id} \times \pi$ maps $D(A)$ onto $D(A')$ and $\mathrm{id} \times \pi \restriction D(A)$ belongs to M. Therefore we have the following key property: For any $q \in \mathrm{Add}(\omega_1^V, \omega_1^V)^M$ with $\mathrm{Dom}(q) \subseteq D(A')$, there is $p \in \mathrm{Add}(\omega_1^V, \mu)^M$ with $\pi'(p) = q$. Now we can verify that A' is maximal: Let $q \in \mathrm{Add}(\omega_1^V, \omega_1^V)^M$. We want to find some element in A' compatible with q. Set $q_1 = q\restriction D(A')$. There is a p_1 with $\pi'(p_1) = q_1$ and p_1 is compatible with some $a \in A$ (because A is a maximal antichain). But then $\pi'(a)$ is in A' and compatible with q_1. As q_1 and q agree on $D(A')$, $\pi'(a)$ is also compatible with q.

As A' is a maximal antichain there exists $g \in A' \cap G$. But g is some $\pi'(h)$, with $h \in A \cap H$, showing that H meets A, as desired. □

Lemma 35 *(Uniformly L-good bijections) Let* $f \in L$*,* $\kappa \in Card^V$*,* $f : \kappa \to [\kappa, \kappa^{+V})$*. Then there exists a "uniformly L-good" bijection* $g : \bigcup_{\alpha \in \kappa}(\{\alpha\} \times f(\alpha)) \to \kappa \times \kappa$*, i.e., for each* $\alpha \in \kappa$*,* $g \restriction \{\alpha\} \times f(\alpha)$ *is a bijection between* $\{\alpha\} \times f(\alpha)$ *and* $\{\alpha\} \times \kappa$*, and for* $x \subseteq Dom(g)$*,* $\|x\|^L = \kappa$*, we have* $g\restriction x \in L$*.*

Proof. By induction on $\gamma = \sup\{f(\alpha) \mid \alpha < \kappa\}$. If $\gamma = \kappa$ then we choose g to be the identity.

Suppose that $\gamma \in (i, i^{+L})$ for some $i \in I$. Let $f'(\alpha) = \min\{f(\alpha), i\}$. For this f' we have the desired g' by induction. In L, canonically choose bijections $h_\beta : \beta \to i$ for $\beta \in [i, i^{+L})$. Then define:

$g(\alpha, y) = g'(\alpha, y)$ (if $f(\alpha) \leq i$),
$g(\alpha, y) = g'(\alpha, h_{f(\alpha)}(y))$ (if $f(\alpha) > i$).

As the sequence of h_α's is in L, this g is uniformly L-good.

Next suppose $\gamma \in [i^{+L}, i^*)$ for some $i \in I$. In this case we use the disjoint splitting of i^* into the X_n's from Lemma 3. Set:

$f'(\alpha) = f(\alpha)$ (if $f(\alpha) < i^{+L}$),
$f'(\alpha) = i$ (if $f(\alpha) \geq i^{+L}$).

For this f' we have the desired g' by induction. For each $n \in \omega$, choose in L canonical bijections $h^n_\beta : X_n \cap \beta \to i$, $\beta \in [i^{+L}, i^*)$. Then define:

$g(\alpha, y) = g'(\alpha, y)$ (if $f(\alpha) < i^{+L}$),
$g(\alpha, y) = g'(\alpha, h^n_{f(\alpha)}(y))$ (if $f(\alpha) \geq i^{+L}$ and $y \in X_n$).

We verify that g is uniformly L-good. Let $x \subseteq \mathrm{Dom}(g)$, $\|x\|^L = \kappa$. We need $g{\restriction}x \in L$. x is subset of $\kappa \times \gamma$ where $\gamma < i^*$. So $x \in L_{i^*}$ and therefore a subset of the union of finitely many X_n. So $g{\restriction}x$ is the union of finitely many constructible functions and is therefore in L.

Finally, suppose that γ is an indiscernible. We have that $\beta = \sup\{f(\alpha) \mid \alpha \in \mathrm{Dom}(f) \wedge f(\alpha) \neq \gamma\}$ is smaller than γ. By induction we have the desired g' for the following modification of f:

$f'(\alpha) = f(\alpha)$ (if $f(\alpha) < \gamma$),
$f'(\alpha) = \kappa$ (if $f(y) = \gamma$).

From lemma 32 we have an L-good bijection $b_\gamma : \gamma \to \kappa$. So define:

$g(\alpha, y) = g'(\alpha, y)$ (if $f(\alpha) < \gamma$),
$g(\alpha, y) = (\alpha, b_\gamma(y))$ (if $f(\alpha) = \gamma$).

If x is a subset of Dom(g) in L of L-cardinality κ then $g \restriction x$ is the union of two constructible functions, and is therefore in L. $\square$

Corollary 36 *(Uniformly M-good bijections) Let M be an inner model with $Card^L = Card^M$. Let $f \in M$ be from κ to $[\kappa, \kappa^{+V})$. Then there exists a uniformly M-good bijection $g : \bigcup_{\alpha \in \kappa}(\{\alpha\} \times f(\alpha)) \to \kappa \times \kappa$, i.e., for each $\alpha \in \kappa$, $g \restriction \{\alpha\} \times f(\alpha)$ is a bijection between $\{\alpha\} \times f(\alpha)$ and $\{\alpha\} \times \kappa$, and for $x \subseteq Dom(g)$, $\|x\|^M = \kappa$, we have $g{\restriction}x \in M$.*

Proof. As in Corollary 33. $\square$

Lemma 37 *(Stretching below ω_1^V) Let f be a constructible function from $Reg^L \cap \omega_1^V$ to $Card^L \cap \omega_2^V$ (obeying the Easton conditions). Define f' by $f'(\kappa) = \min\{f(\kappa), \omega_1^V\}$ for each $\kappa \in Reg^L \cap \omega_1^V$. Suppose that M is an inner model with the same cofinalities as L such that $\alpha < \omega_1^V \to (2^\alpha)^M \leq \omega_1^V$, and in V there is a generic over M for $P' = \prod^{Easton}_{\kappa \in \omega_1^V \cap Reg^L} Add(\kappa, f'(\kappa))$ of M. Then in V there is also a generic over M for $P = \prod^{Easton}_{\kappa \in \omega_1^V \cap Reg^L} Add(\kappa, f(\kappa))$ of M.*

Proof. By Corollary 36 we have a uniformly M-good bijection $g : \bigcup_{\alpha < \omega_1^V}(\{\alpha\} \times f''(\alpha)) \to \omega_1^V \times \omega_1^V$, where:

$f''(\alpha) = \max\{f(\alpha), \omega_1^V\}$ if $\alpha \in \mathrm{Reg}^L \cap \omega_1^V$,
$f''(\alpha) = \omega_1^V$ otherwise.

For $\alpha < \omega_1^V$ let $g_\alpha : f''(\alpha) \to \omega_1^V$ be defined by $g_\alpha(\beta) = g(\alpha, \beta)$.

Now for $\kappa \in \mathrm{Reg}^L \cap \omega_1^V$ define $\pi_\kappa : \mathrm{Add}(\kappa, f(\kappa))^M \to \mathrm{Add}(\kappa, f'(\kappa))^M$ by:

$\pi_\kappa(p)(a, b) = p(a, b)$ if $f(\kappa) = f'(\kappa)$,
$\pi_\kappa(p)(a, b) = p(a, g_\kappa^{-1}(b))$ if $f(\kappa) > f'(\kappa)$.

And define π from P to P' by: $\pi(p)(\kappa) = \pi_\kappa(p(\kappa))$ for each $\kappa \in \mathrm{Reg}^L \cap \omega_1^V$.

Choose G' in V to be P'-generic over M. Our candidate for a P-generic over M is $G = \{p \in P \mid \pi(p) \in G'\}$. This is a filter, so we need only show that it intersects maximal antichains on P which belong to M.

Suppose that A is a maximal antichain in P and define $A' = \{\pi(a) \mid a \in A\}$. A' is an antichain on P'; we will show that A' is in fact a *maximal* antichain on P'. Note that A' belongs to M because (by the hypothesis $\alpha < \omega_1^V \to (2^\alpha)^M \leq \omega_1^V$) P is $(\omega_1^V)^{+L}$-c.c. in M and g is a good bijection.

Let $D(A') = \bigcup\{\mathrm{Dom}(a') | a' \in A'\}$. Then for any $p' \in P'$ with $\mathrm{Dom}(p') \subseteq D(A')$ there is $p \in P$ such that $\pi(p) = p'$. This is again because $D(A')$ has M-cardinality at most ω_1^V and g is good. Now let p' belong to P'. We want to find some element in A' which is compatible with p'. Set $p'_1 = p' \restriction D(A')$. Then there is p such that $\pi(p) = p'_1$. p is compatible with some $a \in A$ (because A is a maximal antichain in P), and therefore $p'_1 = \pi(p)$ is compatible with $\pi(a) \in A'$. As p', p'_1 agree on $\mathrm{Dom}(\pi(a)) \subseteq D(A')$, it follows that p' is also compatible with $\pi(a) \in A'$. So A' is a maximal antichain.

Now as A' is a maximal antichain on P' we may choose some $g' \in A' \cap G'$. Then $g' = \pi(p)$ for some p and p belongs to both A and G, so we have shown that G is P-generic over M, as desired. $\square$

Lemma 38 *Let $P = \bigstar^{Easton}_{\kappa \in Reg^L} Add(\kappa, \kappa)$ in L. Then P preserves cofinalities and the GCH, and there exists a P-generic over L in $L[0^\#]$.*

Proof. Preservation of cofinalities and of the GCH are straightforward, using the factoring of P as $P(<\kappa) * P(\kappa) * P(>\kappa)$ for $\kappa \in \mathrm{Reg}^L$: "Cofinality greater than κ" is preserved as $P(\leq \kappa)$ has a dense subset of L-cardinality κ and $P(>\kappa)$ is κ^+-closed. The GCH still holds at the infinite cardinal λ as $P(\leq \lambda)$ has a dense subset of L-cardinality at most λ^+ and $P(>\lambda)$ is λ^+-closed.

To build a P-generic in $L[0^\#]$ we proceed as in the previous section (although the proof here is much easier). By induction on $i \in I$ we define a generic $G(\leq i)$ for $P(\leq i)$. We inductively ensure the following coherence property: For indiscernibles $i < j$, $G(< i)$ is a subset of $G(< j)$ and $G(i) \subseteq \mathrm{Add}(i,i)^{L[G(<i)]}$ is a subset of $G(j)$. If $i = \min I$ then we choose $G(< i)$ to be some $P(< i)$ generic, which exists due to the countability of i^{+L} in $L[0^\#]$. Our coherence property ensures that for i a limit indiscernible we can take $G(< i)$ to be $\bigcup_{\bar{i}<i} G(< \bar{i})$ and $G(i)$ to be $\bigcup_{\bar{i}<i} G(\bar{i})$. The resulting $G(\leq i)$ is $P(\leq i)$-generic as if $D = \tau(\vec{j}, i, \vec{\infty})$ is dense in $P(\leq i)$ we choose an

indiscernible k from $(\max(\vec{j}), i)$ and consider $D_k = \tau(\vec{j}, k, \vec{\infty})$, a dense subset of $P(\leq k)$. Then by induction there is $\bar{p}$ in $D_k \cap G(\leq k)$ and $\pi_{ki}(\bar{p}) = \bar{p}$ belongs to D. By coherence, $\bar{p}$ belongs to $G(\leq i)$, so $G(\leq i)$ meets D, as desired.

Finally suppose that $G(\leq i)$ has been defined and we wish to define $G(\leq i^*)$. Now $P(< i^*)$ factors as $P(< i^*) \simeq P(\leq i) * P(i, i^*)$, where $P(i, i^*)$ is i^+-closed and of cardinality i^* in $L^{[G(\leq i)]}$. It follows from Lemma 8 that we can choose a $P(i, i^*)$-generic in $L[0^\#]$, resulting in a $P(< i^*)$-generic $G(< i^*)$ including $G(< i)$. Similarly, we can choose a $P(i^*)$-generic in $L[0^\#]$ below the condition $G(i)$ in $P(i^*)$. The result is a $P(\leq i^*)$-generic $G(\leq i^*)$ obeying our coherence property. $\square$

Theorem 39 *Let $f : Reg^L \to Card^L$ be an L-definable Easton function with parameters $\leq \omega_1^V$. Then there is a cofinality preserving generic extension $M \subseteq V$ of L such that $M \vDash 2^\kappa = f(\kappa)$ for all $\kappa \in Reg^L$.*

Proof. Let $\gamma = \min\{\alpha \mid f(\alpha) \geq \omega_1^V\}$. We first use the following forcings:

1. $P_1 = \bigstar_{c_i \in C}[\prod^{Easton}_{\kappa \in [c_i, c_{i+1}) \cap \text{Reg}^L} \text{Add}(\kappa, f(\kappa))]$ in L, where $C = \{c_\alpha \mid \alpha \in \text{Ord}\}$ is the increasing enumeration of the class consisting of $(\omega_1^V)^{+L}$ together with the uncountable closure points of f.
2. $P_2 = \bigstar_{\kappa < \omega_1^V \cap \text{Reg}^L} \text{Add}(\kappa, \kappa)$ in L^{P_1}
3. $P_3 = \text{Add}(\omega_1^V, \omega_1^V)$ in $L^{P_1 * P_2}$
4. $P_4 = \prod^{Easton}_{\gamma \leq \kappa < \omega_1^V \wedge \kappa \in \text{Reg}^L} \text{Add}(\kappa, \omega_1^V)$ in $L^{P_1 * P_2 * P_3}$
5. $P_5 = \prod^{Easton}_{\kappa \in \gamma \cap \text{Reg}^L} \text{Add}(\kappa, f(\kappa))$ in $L^{P_1 * P_2 * P_3 * P_4}$.

Here, $\bigstar$ denotes reverse Easton iteration with Easton supports. By Lemma 28, the iteration $P_1 * P_2 * P_3 * P_4 * P_5$ preserve cofinalities over L and forces the generic extension to realise the following Easton function f':

$f'(\kappa) = f(\kappa)$, if $\kappa < \omega_1^V$ and $f(\kappa) < \omega_1^V$,
$f'(\kappa) = \omega_1^V$, if $\kappa < \omega_1^V$ and $f(\kappa) \geq \omega_1^V$,

$f'(\kappa) = (\omega_1^V)^{+L}$, if $\kappa = \omega_1^V$ and
$f'(\kappa) = f(\kappa)$, if $\kappa > \omega_1^V$.

We next find generics for the P_i's:

P_1 has a generic G_1 by Corollary 29.
P_2: By Lemma 38, there exists a P_2-generic G_2 over L. But as $\mathcal{P}(\omega_1^V)^L = \mathcal{P}(\omega_1^V)^{L[G_1]}$, it follows that G_2 is also P_2-generic over $L[G_1]$.
P_3: By Lemma 38, there is a P_3-generic G_3 over $L[G_2]$; again as $\mathcal{P}(\omega_1^V)^L = \mathcal{P}(\omega_1^V)^{L[G_1]}$, it follows that $G_2 * G_3$ is $P_2 * P_3$-generic over $L[G_1]$.
P_4: By Lemma 9, $P_2 * P_4$ is ω_1^V-cc in V and therefore has a generic $G_2 *' G_4$ over L. We can assume that $G_2' = G_2$. Then G_4 is also generic over $L[G_1 * G_2 * G_3]$, as all bounded subsets of ω_1^V of the latter model belong to $L[G_2]$.
P_5; If γ is less than ω_1^V then P_5 has only countably many subsets in $L[G_1 * G_2 * G_3 * G_4]$ and therefore there is a P_5-generic G_5 over that model. If γ equals ω_1^V, then we can apply Lemma 9 to $P_2 * P_4 * P_5$.

Now let M be the model $L[G_1 * G_2 * G_3 * G_4 * G_5]$ and define $P_6 = \mathrm{Add}(\omega_1^V, f(\omega_1^V))$ in M and $P_7 = \prod^{Easton}_{\kappa \in \mathrm{Reg}^L \cap \omega_1^V} \mathrm{Add}(\kappa, f(\kappa))$ in M^{P_6}. By Lemma 28 it suffices to find generics for P_6 and P_7. A P_6-generic G_6 over M exists by Lemma 34 (Stretching at ω_1^V). And a P_7-generic over $M[G_6]$ exists by Lemma 37 (Stretching below ω_1^V). This completes the proof. $\square$

3 The parameter ω_2

As mentioned in the introduction, we cannot expect every Easton function which is L-definable with parameter ω_2 to be realisable in an inner model, as CH implies that $2^\omega < \omega_2^V$ holds in all inner models. A reasonable conjecture would be that any L-definable Easton function f with parameter ω_2^V satisfying $f(\alpha) < \omega_2^V$ for countable $\alpha \in \mathrm{Reg}^L$ can be realised in an inner model of $L[0^\#]$ with same cofinalities as L. The following result is a step in that direction.

Theorem 40 *There is an inner model of $L[0^\#]$ with the same cofinalities as L in which ω_1^V is a strong limit cardinal and $2^{\omega_1^V} = \omega_2^V$.*

Proof. Assume $V = L[0^\#]$. We shall use the gap 1 morass at ω_1 whose construction is based on [1]. In particular, a *morass point* is an ordinal σ (with sufficient closure) such that $\sigma < \omega_2$ and $L_\sigma[0^\#] \vDash \omega_1$ is the largest cardinal. The *level* of the morass point σ, denoted $\alpha(\sigma)$, is the ω_1 of $L_\sigma[0^\#]$. A *morass level* is an ordinal of the form $\alpha(\sigma)$ for some morass point σ. If α is a countable morass level then $\sigma(\alpha)$ denotes the largest σ such that $\alpha(\sigma) = \alpha$. If α is countable then $\sigma(\alpha)$ is also countable. To certain pairs (σ, τ) of morass points with $\alpha(\sigma) < \alpha(\tau)$ is associated a Σ_1 elementary map $\pi_{\sigma\tau}$ from $L_\sigma[0^\#]$ to $L_\tau[0^\#]$ which is the identity on $\alpha(\sigma)$ and sends $\alpha(\sigma)$ to $\alpha(\tau)$. We write $\sigma <_1 \tau$ when $\pi_{\sigma\tau}$ is defined. Also write $\sigma <_0 \tau$ when $\alpha(\sigma) = \alpha(\tau)$ and σ is less than τ. All morass points, and all morass levels, are limit points of I.

The desired inner model M is a generic extension of L via the forcings described next.

First we add a function $f : \omega_1^V \to \omega_1^V$ using a reverse Easton iteration of length ω_1^V. At L-regular stage $\alpha \leq \omega_1^V$, force a function from α to α with initial segments of size less than α. A generic for this forcing P can be built using $0^\#$: By induction on $i \in I$ we define a generic for $P(\leq i)$. This is easy when $i = \min I$ by the countability of i^{+L} and also when $i = j^*$ is a successor indiscernible, using the i-closure of the forcing together with a decomposition of the collection of dense subsets of $P(j, i]$ into the union of ω-many subcollections in $L[G(\leq j)]$ of size j. For limit i, we take $G(< i)$ to be the union of the $G(< j)$, $j \in I \cap i$. To obtain $G(i)$ we need to know inductively that $j < k < i$ in $I \to G(j) \subseteq G(k)$. This we can easily arrange at the successor steps of the construction. The desired generic function is $f = G(\omega_1^V) : \omega_1^V \to \omega_1^V$.

Now notice that in the above construction we had complete freedom about how to define f at indiscernibles. We choose our f so that for a morass level i, $f(i) = \sigma(i)$, the largest morass point on level i, and for an indiscernible i which is not a morass level, $f(i) = 0$.

The desired inner model M is a generic extension of $L[f]$ via the reverse Easton iteration Q of $\mathrm{Add}(\alpha, f(\alpha))$ for L-regular $\alpha < \omega_1^V$ followed by $\mathrm{Add}(\omega_1^V, \omega_2^V)$. ($\mathrm{Add}(\alpha, 0)$ is the trivial forcing.) To obtain a generic for this iteration, we inductively build generics $G(\leq i)$ for $P(\leq i)$, $i \in I$, $i \leq \omega_1^V$, which bey the following condition:

(*) For a morass point σ let $G(\sigma)$ denote the restriction of $G(\alpha(\sigma)) \subseteq \text{Add}(\alpha(\sigma), \sigma(\alpha(\sigma)))$ to $\text{Add}(\alpha(\sigma), \sigma)$. Then if $\sigma <_1 \tau$, we have $\pi_{\sigma\tau}[G(\sigma)] \subseteq G(\tau)$.

Now we describe the inductive construction of the $G(\leq i)$, $i \in I$, $i \leq \omega_1^V$. If i is not a limit indiscernible then we take $G(< i)$ to be any $P(< i)$-generic extending the $G(< j)$, $j \in I \cap i$; otherwise, $G(< i)$ is the union of the $G(< j)$, $j \in I \cap i$. If i is not a morass level then we take $G(i)$ to be trivial and if i equals ω_1^V then we take $G(i)$ to be the union of the $\pi_{\bar{\sigma}\sigma}[G(\bar{\sigma})]$ for $\bar{\sigma} <_1 \sigma$, $\alpha(\sigma) = \omega_1$. So assume now that i is a countable morass level and recall that $\sigma(i)$ denotes the largest morass point σ such that $\alpha(\sigma) = i$.

Case 1: $\sigma(i)$ is $<_1$ minimal. For $\sigma <_0 \sigma(i)$ define $G(\sigma)$ to be the union of $\pi_{\bar{\sigma}\sigma}[G(\bar{\sigma})]$ for $\bar{\sigma} <_1 \sigma$. By an inductive use of $(*)$ and morass properties, $G(\sigma)$ is generic for $\text{Add}(i, \sigma)$ for $\sigma <_0 \sigma(i)$ and $G(\sigma) \subseteq G(\sigma')$ for $\sigma <_0 \sigma' <_0 \sigma(i)$. If $\sigma(i)$ is a $<_0$ limit, then take $G(i) = G(\sigma(i))$ to be the union of the $G(\sigma)$, $\sigma <_0 \sigma(i)$ (this is $\text{Add}(i, \sigma(i))$-generic), and otherwise take $G(i) = G(\sigma(i))$ to be any $\text{Add}(i, \sigma(i))$-generic containing the $G(\sigma)$, $\sigma <_0 \sigma(i)$.

Case 2: $\sigma(i)$ is the $<_1$-successor to some $\bar{\sigma}$. First suppose that $\sigma(i)$ is $<_0$ minimal. We must choose $G(i) = G(\sigma(i))$ to be $\text{Add}(i, \sigma(i))$-generic and to contain $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$. This brings us to the main step in the proof, based on the following generalisation of Lemma 27.

Lemma 41 *Suppose that i is an indiscernible and X is a set of indiscernibles greater than i of limit ordertype. Let j be the minimum of X and let H denote the Skolem hull of $X \cup i$ in L. Then if x is a constructible set of L-cardinality at most j, the intersection of x with H is a constructible set of L-cardinality at most i.*

Proof. We may assume that x is a set of ordinals. Let k denote the least indiscernible such that x is a subset of k. We may assume that x is a subset of $\sup X$ and therefore k is at most $\sup X$. In fact, k is less than $\sup X$ as the latter is regular in L. Now we prove the lemma by induction on k. If k is at most j, then the desired conclusion is immediate, as in this case $x \cap H = x \cap (H \cap j) = x \cap i$. If k is greater than j then it cannot be a limit indiscernible, as indiscernibles are L-regular. So assume that k is the least indiscernible greater than the indiscernible l, where l is at least j.

If l does not belong to H then $x \cap H = (x \cap l) \cap H$ so the desired conclusion follows by induction. If l does belong to H then as x has L-cardinality at most l, there is some finite n such that x is a subset of H_n = the Skolem hull in L of $l \cup \{l\} \cup \infty_n$, where ∞_n consists of n indiscernibles greater than l in H (recall that X has limit ordertype). But now let π be a bijection in H between l and H_n. We have $x \cap H = \pi[y \cap H]$, where $y = \pi^{-1}[x]$. By induction, $y \cap H$ is constructible of L-cardinality at most i and therefore so is $x \cap H$. $\square$

Using this lemma we proceed with the construction of $G(\sigma(i))$ in Case 2 as follows. First select $G'(\sigma(i))$ to be any $\mathrm{Add}(i, \sigma(i))$-generic. We must modify $G'(\sigma(i))$ to the desired $G(\sigma(i))$ containing $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$. By Lemma 41, we obtain a well-defined condition p^* if we modify a condition p in $\mathrm{Add}(i, \sigma(i))$ so as to agree with $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$ on the range of $\pi_{\bar{\sigma}\sigma(i)}$. Let $G(\sigma(i))$ consist of all modification p^* of conditions in $p \in G'(\sigma(i))$. Then as in the construction of the generic in the first section of this paper (see the reference to "small modifications"), this modified $G(\sigma(i))$ is also $\mathrm{Add}(i, \sigma(i))$-generic. This completes the construction of $G(\leq i)$ in Case 2 when $\sigma(i)$ is $<_0$ minimal. When $\sigma(i)$ is the $<_0$-successor to σ_0, then we choose $G'(\sigma(i))$ to be any $\mathrm{Add}(i, \sigma(i))$-generic extending $G(\sigma_0) = \bigcup_{\bar{\sigma}_0 <_1 \sigma_0} \pi_{\bar{\sigma}_0\sigma_0}[G(\bar{\sigma}_0)]$ and then modify it as in the $<_0$-minimal case to the desired $G(\sigma(i))$ which agrees with $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$ on the range of $\pi_{\bar{\sigma}\sigma(i)}$. If $\sigma(i)$ is a $<_0$-limit, then we set $G(\sigma(i))$ to be the union of the $\pi_{\bar{\sigma}_0\sigma_0}[G(\bar{\sigma}_0)]$ for $\bar{\sigma}_0 <_1 \sigma_0 <_0 \sigma(i)$. By an inductive use of $(*)$ and morass properties, it follows that the resulting $G(\sigma(i))$ is a well-defined $\mathrm{Add}(i, \sigma(i))$-generic which agrees with $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$ on the range of $\pi_{\bar{\sigma}\sigma(i)}$.

Case 3: $\sigma(i)$ is a $<_1$ limit. In this case we take $G(i)$ to be the union of the $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$. By an inductive use of $(*)$ together with morass properties, this yields a well-defined $\mathrm{Add}(i, \sigma(i))$-generic, which by definition contains $\pi_{\bar{\sigma}\sigma(i)}[G(\bar{\sigma})]$ for $\bar{\sigma} <_1 \sigma$.

This completes the construction of the $G(\leq i)$ for indiscernibles $i \leq \omega_1^V$. The model $M = L[f][G(\leq \omega_1^V)]$ is the desired inner model of $V = L[0^\#]$ with same cofinalities as L in which ω_1^V is a strong limit cardinal and $2^{\omega_1^V} = \omega_2^V$. $\square$

Question. Which L-definable Easton functions with parameters are realisable in an inner model of $L[0^{\#}]$ with the same cofinalities as L?

References

[1] Keith Devlin, *Constructibility*, Springer-Verlag, 1984.

[2] William B. Easton, Powers of regular cardinals, Annals of Mathematical Logic 1970, p.139-178

[3] Sy D. Friedman, *Fine structure and class forcing*, Vol. 3, de Gruyter Series in Logic and its Applications, Berlin-New York, 2000.

[4] Sy D. Friedman, A characterisation of $0^{\#}$ in terms of forcing, Proceedings of the American Mathematical Society, Vol.134, No.6, pp.1823–1824, June 2006.

[5] Sy D. Friedman, Internal consistency and the inner model hypothesis, Bulletin of Symbolic Logic, Vol.12, No.4, pp. 591–600, December 2006.

[6] Sy D. Friedman, Internal consistency and the inner model hypothesis, Bulletin of Symbolic Logic, Vol.12, No.4, December 2006, pp. 591–600.

[7] Thomas Jech, *Set Theory*, 3rd edition, Springer, 2002.

[8] Akihiro Kanamori *The Higher Infinite*, Springer, 1994.

[9] Kenneth Kunen, *Set Theory*, 9th impression, North Holland, 2005.