THE JOURNAL OF SYMBOLIC LOGIC
Volume 73, Number 2, June 2008

INTERNAL CONSISTENCY AND GLOBAL CO-STATIONARITY OF THE GROUND MODEL

NATASHA DOBRINEN AND SY-DAVID FRIEDMAN

Abstract. Global co-stationarity of the ground model from an $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ is internally consistent relative to an ω_1-Erdős hyperstrong cardinal and a sufficiently large measurable above.

§1. **Introduction.** Suppose $\mathbb{P}$ is a notion of forcing, κ is regular and uncountable in $V^{\mathbb{P}}$, and λ is a cardinal $> \kappa$ in $V^{\mathbb{P}}$. We say that *the ground model is co-stationary* or that $(\mathscr{P}_\kappa(\lambda))^V$ *is co-stationary in* $V^{\mathbb{P}}$ if $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus (\mathscr{P}_\kappa(\lambda))^V$ is stationary in $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}}$. Note that $(\mathscr{P}_\kappa(\lambda))^V = (\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \cap V$; hence, $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus (\mathscr{P}_\kappa(\lambda))^V = (\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus V$.

We say that a forcing $\mathbb{P}$ achieves *global co-stationarity* of the ground model if $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus V$ is stationary in $V^{\mathbb{P}}$ for *all* cardinals $\aleph_2 \le \kappa < \lambda$ in $V^{\mathbb{P}}$ with κ regular in $V^{\mathbb{P}}$. In [3], we showed that it is relatively consistent (from a proper class of ω_1-Erdős cardinals) that every $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ achieves global co-stationarity of the ground model.

THEOREM 1.1 (Dobrinen/Friedman [3]). *The following are equiconsistent*:

1. *There is a proper class of ω_1-Erdős cardinals.*
2. *If $\mathbb{P}$ is $\aleph_1$-Cohen forcing, then $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus V$ is stationary in $V^{\mathbb{P}}$ for all regular $\kappa \ge \aleph_2$ and all $\lambda > \kappa$.*
3. *If $\mathbb{P}$ adds a new subset of $\aleph_1$ and is $\aleph_2$-c.c.* (*or just satisfies the* $(\kappa^+, \kappa^+, <\kappa)$-*distributive law for all successor cardinals $\kappa \ge \aleph_2$ and is θ-c.c. for the least strongly inaccessible cardinal θ, if it exists*), *then $(\mathscr{P}_\kappa(\lambda))^{V^{\mathbb{P}}} \setminus V$ is stationary in $V^{\mathbb{P}}$ for all regular $\kappa \ge \aleph_2$ and all $\lambda > \kappa$.*

In this paper, we address some questions left open by Theorem 1.1. Can we get a model with large cardinals of global co-stationarity for $\aleph_2$-c.c. forcings which add a new subset of $\aleph_1$? The model we used to prove the consistency of Theorem 1.1 (3) does not necessarily even have measurable cardinals, although it can have inaccessibles.

The second question we investigate is the following. What is the internal consistency strength of global co-stationarity for $\aleph_2$-c.c. forcings which add a new subset

Received October 24, 2006.
This work was supported by FWF grants P 16334-N05 (Natasha Dobrinen) and P 16790-N04 (Sy-David Friedman).

0022-4812/08/7302-0009/$2.00

of $\aleph_1$? Recall that as defined in [4], a statement is *internally consistent* iff it holds in an inner model, assuming the existence of inner models with large cardinals. *Internal consistency strength* refers to the large cardinals required. There is reason to believe that the internal consistency strength of our global co-stationarity property is at least that of a Woodin cardinal. Indeed, our strategy to obtain global co-stationarity internally requires collapsing the successor of each regular cardinal below some measurable cardinal, a property whose (ordinary) consistency strength is at the level of a Woodin cardinal.

Our main results are Theorems 3.1 and 3.2. Integral to our results are the following large cardinals.

DEFINITION 1.2. Given a model M and an ordinal α, we let M_α denote the V_α of M. κ is *superstrong* if there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)} = M_{j(\kappa)}$. κ is *hyperstrong* if there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)+1} = M_{j(\kappa)+1}$. κ is ω_1-*Erdős hyperstrong* if there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_\lambda = M_\lambda$, where λ is an ω_1-Erdős cardinal of V above $j(\kappa)$.

THEOREM 3.1. *Let V be a model of* ZFC *with a proper class of ω_1-Erdős cardinals. Then there is a class forcing extension W of V such that every $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ achieves global co-stationarity of W. Moreover, every ω_1-Erdős hyperstrong cardinal in V is superstrong in W.*

THEOREM 3.2. *Assume V is a model of* ZFC *with an ω_1-Erdős hyperstrong cardinal κ and a measurable cardinal above $j(\kappa)$, where j is an elementary embedding witnessing the ω_1-Erdős hyperstrength of κ. Then there is an inner model $W \subseteq V$ with a proper class of Woodin cardinals such that every $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ achieves global co-stationarity of W. If moreover κ is an ω_1-Erdős hyperstrong cardinal which is a limit of ω_1-Erdős hyperstrong cardinals, then W has a proper class of superstrong cardinals.*

Remark. In Theorems 3.1 and 3.2, $\aleph_2$-c.c. can be weakened to $(\rho^+, \rho^+, < \rho)$-distributivity for all ρ less than the least regular limit cardinal plus the θ-c.c. where θ is the least regular limit cardinal (if it exists).

§2. Definitions and background.

§2. Definitions and background. Throughout this paper, standard set-theoretic notation is used. α, β, γ are used to denote ordinals, while $\kappa, \lambda, \mu, \nu, \rho, \theta$ are used to denote cardinals. $\mathscr{P}_\kappa(X) = [X]^{<\kappa} = \{x \subseteq X : |x| < \kappa\}$. Usually we use $[X]^{<\omega}$ instead of $\mathscr{P}_\omega(X)$ to denote the collection of finite subsets of X. $X^{<\kappa}$ and $(X)^{<\kappa}$ denote the collection of all functions from an ordinal less than κ into X; i.e., the collection of all sequences of length less than κ of elements of X. We will hold to the convention that if $V \subseteq W$ are models of ZFC with the same ordinals and $\kappa < \lambda$ are cardinals in W, then $\mathscr{P}_\kappa(\lambda)$ denotes $(\mathscr{P}_\kappa(\lambda))^W$.

Certain generalised distributive laws imply preservation of the stationarity of the $\mathscr{P}_\kappa(\lambda)$ of the ground model. In addition, they will aid us in obtaining extension models in which the ground model is co-stationary. We present the forcing-equivalent definitions of distributivity, referring the reader to [5] for the Boolean algebraic versions.

DEFINITION 2.1. Let κ, λ, μ be cardinals with $\mu \leq \lambda$. A partial ordering $\mathbb{P}$ is $(\kappa, \lambda, < \mu)$-*distributive* if for any function $\dot{f} : \check{\kappa} \to \check{\lambda}$, there is a function $g : \kappa \to [\lambda]^{<\mu}$ in V such that for each $\alpha < \kappa$, $\dot{f}(\alpha) \in g(\alpha)$ in $V^{\mathbb{P}}$. We will say that $\mathbb{P}$ is (κ, λ, μ)-*distributive* if it is $(\kappa, \lambda, < \mu^+)$-distributive.

One can think of (ρ, λ, κ)-distributivity as a weakening of the κ^+-c.c.

FACT 2.2.

1. *If $\mathbb{P}$ is κ^+-c.c., then $\mathbb{P}$ is (ρ, λ, κ)-distributive for all ρ and for all $\lambda > \kappa$.*
2. *The $(\kappa, \lambda, \kappa)$-d.l. holds iff every subset of λ of size κ in $V^{\mathbb{P}}$ can be covered by a subset of λ of size κ in V.*
3. *If $\lambda > \kappa$ and $\mathbb{P}$ is $(\lambda, \lambda, \kappa)$-distributive, then $\mathbb{P}$ preserves all cardinals ρ with $\kappa^+ \leq \rho \leq \lambda$. Moreover, every stationary subset of $(\mathscr{P}_{\kappa^+}(\lambda))^V$ in V is a stationary subset of $\mathscr{P}_{\kappa^+}(\lambda)$ in $V^{\mathbb{P}}$. Hence, $(\mathscr{P}_{\kappa^+}(\lambda))^V$ is stationary in $V^{\mathbb{P}}$.*

The following theorem due to Kueker will be employed throughout this paper.

THEOREM 2.3 (Kueker [6]). *Suppose $\kappa < \lambda$ and κ is regular. For each club $C \subseteq \mathscr{P}_\kappa(\lambda)$ there exists a function $f : [\lambda]^{<\omega} \to \mathscr{P}_\kappa(\lambda)$ such that $C_f \subseteq C$, where*

$$C_f = \{x \in \mathscr{P}_\kappa(\lambda) : \forall y \in [x]^{<\omega}\ f(y) \in x\}. \tag{1}$$

Moreover, C_f is club.

Next we state a well-known result of Menas.

THEOREM 2.4 (Menas [7]). *Let $A \subseteq B$ with $|A| \geq \kappa$. For $X \subseteq \mathscr{P}_\kappa(A)$, let $X^* = \{x \in \mathscr{P}_\kappa(B) : x \cap A \in X\}$. If $C \subseteq \mathscr{P}_\kappa(A)$ is club then C^* is club in $\mathscr{P}_\kappa(B)$. For $Y \subseteq \mathscr{P}_\kappa(B)$, let $Y \restriction A = \{y \cap A : y \in Y\}$. If $C \subseteq \mathscr{P}_\kappa(B)$ is club, then $C \restriction A$ contains a club set in $\mathscr{P}_\kappa(A)$.*

Two special facts follow from this theorem.

FACT 2.5. *Let $V \subseteq W$ be models of* ZFC *with the same ordinals and κ be regular and $\lambda > \kappa$ in W.*

1. *If $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in W, then for all $\mu \geq \lambda$, $(\mathscr{P}_\kappa(\mu))^V$ is also co-stationary in W.*
2. *If $(\mathscr{P}_\kappa(\lambda))^V$ is stationary in W and $\kappa \leq \mu < \lambda$, then $(\mathscr{P}_\kappa(\mu))^V$ is also stationary in W.*

Thus, to show that $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in W for all $\lambda \geq \kappa^+$, κ regular in W, it suffices to show that $(\mathscr{P}_\kappa(\kappa^+))^V$ is co-stationary in W.

DEFINITION 2.6. [2] Let $\alpha \leq \lambda$, α a limit ordinal. λ is α-*Erdős* if whenever C is club in λ and $f : [C]^{<\omega} \to \lambda$ is regressive $(f(a) < \min(a))$, then f has a *homogeneous* set of order type α; that is, a set $X \subseteq C$ such that for each $n \in \omega$, $|f''[X]^n| = 1$.

The following is a model-theoretic equivalent of being α-Erdős: λ is α-Erdős iff for any structure $\mathfrak{A}$ with universe λ (for a countable language) endowed with Skolem functions, for any club $C \subseteq \lambda$, there is an $I \subseteq C$ of order type α such that I is a set of indiscernibles for $\mathfrak{A}$ and in addition I is *remarkable*; i.e., whenever $\iota_0, \dots, \iota_n$ and $\eta_0, \dots, \eta_n$ are increasing sequences from I with $\iota_{i-1} < \eta_i$, τ is a term and $\tau^{\mathfrak{A}}(\iota_0, \dots, \iota_n) < \iota_i$, then $\tau^{\mathfrak{A}}(\iota_0, \dots, \iota_n) = \tau^{\mathfrak{A}}(\iota_0, \dots, \iota_{i-1}, \eta_i, \dots, \eta_n)$. (See [1].)

In [3], we proved the following lemma, which is a generalisation of part of the proof of Theorem 5.9 of Baumgartner in [1].

LEMMA 2.7. [3] *Suppose that in V, $|2^\omega| < \kappa < \lambda$, κ is regular, and λ is ω_1-Erdős. Let $\mathbb{Q} = Col(\kappa, < \lambda)$ and G be $\mathbb{Q}$-generic over V. Then in $V[G]$, given a function $g : [\kappa^+]^{<\omega} \to \mathscr{P}_\kappa(\kappa^+)$, there is a tree $T \subseteq (\kappa^+)^{<\omega_1}$ isomorphic to $2^{<\omega_1}$ such that for any two branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$.*

Notation. Let W be a model of ZFC and let κ be regular in W. We will say $*(W, \kappa)$ holds if the following is true: Given a function $g : [\kappa^+]^{<\omega} \to \mathscr{P}_\kappa(\kappa^+)$ in W, there is a tree $T \subseteq (\kappa^+)^{<\omega_1}$ in W isomorphic to $2^{<\omega_1}$ such that for any two branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$.

The next theorem is a distillation of what we will need from the proof of Theorem 3.5 in [3].

THEOREM 2.8. [3] *Suppose κ is regular in W and $*(W, \kappa)$ holds. In W, let $\mathbb{C}$ be $\aleph_1$-Cohen forcing (or any partial ordering which adds a new subset of $\aleph_1$ and satisfies the $(\kappa^+, \kappa^+, < \kappa)$-d.l. if κ is a successor cardinal, or the κ-c.c. if κ is a regular limit cardinal). Then $\mathscr{P}_\kappa(\mu) \setminus W$ is stationary in $W^{\mathbb{C}}$ for all $\mu \geq \kappa^+$.*

To prove Theorem 3.1 it suffices, by Theorem 2.8, to obtain a model W in which $*(W, \kappa)$ holds for all regular κ. To create this, we will iterate Lévy collapses, applying Lemma 2.7 to obtain $*(W', \kappa)$ for a given κ in some intermediate model W'. It will then be incumbent on us to preserve this property through later stages of the forcing construction. The next two Lemmas are designed to do just that.

LEMMA 2.9. *Let W be a model of ZFC and $\kappa \geq \aleph_2$ be regular in W. If $*(W, \kappa)$ holds and $\mathbb{F}$ is a κ-c.c., $\aleph_2$-closed forcing, then $*(W^{\mathbb{F}}, \kappa)$ holds.*

PROOF. Let $h : [\kappa^+]^{<\omega} \to (\mathscr{P}_\kappa(\kappa^+))^{W^{\mathbb{F}}}$ in $W^{\mathbb{F}}$ be given. Since $\mathbb{F}$ is κ-c.c., there is a $g : [\kappa^+]^{<\omega} \to (\mathscr{P}_\kappa(\kappa^+))^W$ in W such that for each $x \in [\kappa^+]^{<\omega}$, $h(x) \subseteq g(x)$. By $*(W, \kappa)$, there exists a tree $T \subseteq (\kappa^+)^{<\omega_1}$ in W such that $T \cong 2^{<\omega_1}$ and for all branches b, c through T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$. Since $\mathbb{F}$ is $\aleph_2$-closed, T has the same branches in W and $W^{\mathbb{F}}$. For each $x \in [\kappa^+]^{<\omega}$, $h(x) \subseteq g(x)$; so $b \cap \bigcup h''[c]^{<\omega} \subseteq b \cap \bigcup g''[c]^{<\omega}$. Therefore, $*(W^{\mathbb{F}}, \kappa)$ holds. ⊣

The next lemma is a slight generalisation of Lemma 3.7 in [3].

LEMMA 2.10. *Suppose in W that $\kappa \geq \aleph_2$, κ is regular, $*(W, \kappa)$ holds, and $\mathbb{F}$ is a κ^+-closed forcing. Then $*(W^{\mathbb{F}}, \kappa)$ holds.*

PROOF. In W, suppose $p \Vdash (\dot{h} : [\kappa^+]^{<\omega} \to \mathscr{P}_\kappa(\kappa^+))$, where $p \in \mathbb{F}$ and $\dot{h}$ is an $\mathbb{F}$-name. Fix an enumeration $\langle x_\zeta : \zeta < \kappa^+ \rangle$ of $[\kappa^+]^{<\omega}$ in W with the property that for each $\beta < \kappa^+$, $\langle x_\zeta : \zeta < \kappa \cdot \beta \rangle$ enumerates $[\kappa \cdot \beta]^{<\omega}$. In W, form a decreasing sequence $\langle p_\zeta : \zeta < \kappa^+ \rangle$ of elements of $\mathbb{F}$ with $p_0 \leq p$ such that for each $\zeta < \kappa^+$, p_ζ decides $\dot{h}(x_\zeta)$. $\langle p_\zeta : \zeta < \kappa^+ \rangle$ is in W, so it evaluates $\dot{h}$ to be some function in W, call it g. By the hypothesis, there is a tree $T \subseteq (\kappa^+)^{<\omega_1}$ in W with $T \cong 2^{<\omega_1}$ such that for all branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$. T has the same branches in W and $W^{\mathbb{F}}$ since $\mathbb{F}$ is $\aleph_2$-closed.

Let $\beta = \sup(T)$ and $\delta = \kappa \cdot \beta$. Then $p_\delta \Vdash (\dot{h} \upharpoonright [\delta]^{<\omega} = g \upharpoonright [\delta]^{<\omega})$. So given branches b, c in T, $b \cap \bigcup g''[c]^{<\omega} \subseteq b \cap c$, and $p_\delta \Vdash (b \cap \bigcup g''[c]^{<\omega} = b \cap \bigcup \dot{h}''[c]^{<\omega})$. Thus, for each $p \in \mathbb{F}$ there exist a $q \leq p$ and a tree $T \in W$ such that $q \Vdash (\forall$ branches b, c in T, $b \cap \bigcup \dot{h}''[c]^{<\omega} \subseteq b \cap c)$. Therefore, $*(W^{\mathbb{F}}, \kappa)$ holds. ⊣

The following lemma is a standard way of lifting elementary embeddings to generic extensions.

LEMMA 2.11. *Suppose $j : V \to M$ is an elementary embedding with $\kappa = crit(j)$. Let $\mathbb{P}$ be a definable forcing in V ($\mathbb{P}$ may be a class forcing), and let $\mathbb{P}^*$ denote the version of $\mathbb{P}$ in M. Let G be $\mathbb{P}$-generic over V and G^* be $\mathbb{P}^*$-generic over M. If $j[G] \subseteq G^*$, then the elementary embedding j lifts to an elementary embedding $j^* : V[G] \to M[G^*]$ with $crit(j^*) = \kappa$ and $j^*(\kappa) = j(\kappa)$.*

PROOF. Let j^* denote the mapping from $V[G]$ to $M[G^*]$ induced by j given by $j^*(\sigma^G) = (j(\sigma))^{G^*}$, where σ is a $\mathbb{P}$-name in V. Let φ be a formula and suppose $V[G] \models \varphi[\sigma_1^G, \dots, \sigma_n^G]$, where $\sigma_1, \dots, \sigma_n$ are $\mathbb{P}$-names. Then there is some $p \in G$ which decides $\sigma_1, \dots, \sigma_n$ and $V \models (p \Vdash \varphi[\sigma_1, \dots, \sigma_n])$. Therefore, $M \models (j(p) \Vdash \varphi[j(\sigma_1), \dots, j(\sigma_n)])$. $j(p) \in G^*$, so $M[G^*] \models \varphi[(j(\sigma_1))^{G^*}, \dots, (j(\sigma_n))^{G^*}]$. Therefore, $M[G^*] \models \varphi[j^*(\sigma_1^G), \dots, j^*(\sigma_n^G)]$. It follows that j^* is well-defined. For each $x \in V$, $j^*(x) = j^*(\hat{x}^G) = (j(\hat{x}))^{G^*} = (\widehat{j(x)})^{G^*} = j(x)$. ⊣

§3. Internal consistency of global co-stationarity. In this section, we show the internal consistency of global co-stationarity from an ω_1-Erdős hyperstrong cardinal and a measurable (far enough) above it, and the internal consistency of global co-stationarity with a proper class of superstrong cardinals from a proper class of ω_1-Erdős hyperstrong cardinals.

Recall: if the superstrength of κ is witnessed by $j : V \to M$, then M can be given as $\{j(f)(a) : f \in V, f : V_\kappa \to V, a \in V_{j(\kappa)}\}$. If κ is hyperstrong, then M can be given as $\{j(f)(a) : f \in V, f : V_{\kappa+1} \to V, a \in V_{j(\kappa)+1}\}$. If κ is ω_1-Erdős hyperstrong, then M can be given as $\{j(f)(a) : f \in V, f : V_{\bar{\lambda}} \to V, a \in V_\lambda\}$, where $\bar{\lambda}$ is the least ω_1-Erdős cardinal above κ in V and λ is the least ω_1-Erdős above $j(\kappa)$ in V.

Theorem 3.1 is the main ingredient in our internal consistency result.

THEOREM 3.1. *Let V be a model of* ZFC *with a proper class of ω_1-Erdős cardinals. Then there is a class forcing extension W of V such that every $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ achieves global co-stationarity of W. Moreover, every ω_1-Erdős hyperstrong cardinal in V remains at least superstrong in W.*

PROOF. We may assume that V satisfies CH, else work in $V^{\mathbb{F}}$, where $\mathbb{F}$ is the forcing which collapses the continuum to $\aleph_1$, i.e., the collection of all functions from an ordinal less than $\aleph_1$ into 2^ω. This preserves ω_1-Erdős cardinals as well as ω_1-Erdős hyperstrong cardinals.

The goal is to Lévy collapse ω_1-Erdős cardinals to successors of regular cardinals, thereby obtaining global co-stationarity as in the proof of Theorem 1.1 (3). However, the method used in that proof, namely the reverse Easton iteration of Lévy collapses, does not necessarily preserve large cardinals, for instance measurables. In order to preserve large cardinals, we will do two phases of Lévy collapse iteration, skipping some cardinals in the first phase in order to preserve some large cardinal strength. We must check that the second phase did not destroy the co-stationarity of the ground model in the places obtained by the first phase.

We begin with some useful notation. Let $\langle \varepsilon_\alpha : \alpha \in \mathrm{Ord} \rangle$ enumerate the ω_1-Erdős cardinals in V. Without loss of generality, we can assume that there is a proper class of cardinals which are both inaccessible and also a limit of ω_1-Erdős cardinals. (If not, then there is a least cardinal, say θ, above which there is no inaccessible limit of ω_1-Erdős cardinals in V. Then above θ, just do a reverse Easton iteration of Lévy collapses of the ω_1-Erdős cardinals to successors of regular cardinals exactly as in the proof of Theorem 1.1 (3).) Let $\langle \delta_\beta : \beta \in \mathrm{Ord}\rangle$ enumerate the cardinals in V which are both inaccessible and a limit of ω_1-Erdős cardinals. The δ_β's will be fixed points of our forcing.

Phase 1: *Construction of* $\mathbb{P}$. We define a reverse Easton iterated forcing $\mathbb{P}$ in V which collapses ω_1-Erdős cardinals to successors of regular cardinals except at successors of successors of inaccessible limits of ω_1-Erdős cardinals. Precisely, we mean the following.

At limit ordinals α, let $\mathbb{P}_\alpha$ be the direct limit of $\langle \mathbb{P}_\beta : \beta < \alpha\rangle$ if α is regular in V, and let $\mathbb{P}_\alpha$ be the inverse limit of $\langle \mathbb{P}_\beta : \beta < \alpha\rangle$ if α is singular in V. Let $\mu_\alpha = \sup_{\beta<\alpha} \mu_\beta$ in $V^{\mathbb{P}_\alpha}$. Note: The δ_α's will be fixed points under forcing with $\mathbb{P}$.

Let $\mathbb{P}_0 = \mathbb{P}_1 = \mathbb{P}_2$ be the trivial forcing. Let $\mathbb{P}_3 = \mathrm{Col}(\aleph_2, < \varepsilon_3)$. ε_3 becomes $\aleph_3$ in $V^{\mathbb{P}_3}$. Let μ_3 denote $(\aleph_3)^{V^{\mathbb{P}_3}}$. In $V^{\mathbb{P}_3}$, let $\mathbb{Q}_3 = \mathrm{Col}(\aleph_3, < \varepsilon_4)$, and let $\mathbb{P}_4 = \mathbb{P}_3 * \mathbb{Q}_3$. ε_4 becomes $\aleph_4$ in $V^{\mathbb{P}_4}$. Let μ_4 denote $(\aleph_4)^{V^{\mathbb{P}_4}}$. Let $4 \leq \alpha < \delta_0$. If α is a successor ordinal, let $\mathbb{Q}_\alpha = \mathrm{Col}(\mu_\alpha, < \varepsilon_{\alpha+1})$, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$, and $\mu_{\alpha+1} = (\aleph_{\alpha+1})^{V^{\mathbb{P}_{\alpha+1}}}$. If α is a limit ordinal, let $\mathbb{Q}_\alpha$ be the trivial forcing, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha \cong \mathbb{P}_\alpha$, and $\mu_{\alpha+1} = (\aleph_{\alpha+1})^{V^{\mathbb{P}_{\alpha+1}}}$.

In general, let $\alpha \geq \delta_0$ be an ordinal and suppose we have constructed $\mathbb{P}_\alpha$ and μ_α in $V^{\mathbb{P}_\alpha}$. We construct $\mathbb{Q}_\alpha$ and let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$. Let γ be such that $\delta_\gamma \leq \alpha < \delta_{\gamma+1}$.

1. If $\alpha = \delta_\gamma$, then $\mu_\alpha = \alpha = \aleph_\alpha = \delta_\gamma = \gamma$ in $V^{\mathbb{P}_\alpha}$, since $\mathbb{P}_\alpha$ is α-c.c. So μ_α is regular in $V^{\mathbb{P}_\alpha}$. Let $\mathbb{Q}_\alpha = \mathrm{Col}(\mu_\alpha, < \varepsilon_{\alpha+1})$. $\varepsilon_{\alpha+1}$ is collapsed to $(\mu_\alpha)^+$ in $V^{\mathbb{P}_{\alpha+1}}$, which is $\aleph_{\alpha+1} = (\delta_\gamma)^+$ in $V^{\mathbb{P}_{\alpha+1}}$. Let $\mu_{\alpha+1} = ((\delta_\gamma)^+)^{V^{\mathbb{P}_{\alpha+1}}}$.
2. If $\alpha = \delta_\gamma + 1$, let $\mathbb{Q}_\alpha$ be the trivial forcing. Then $\mathbb{P}_{\alpha+1}$ is equivalent to $\mathbb{P}_\alpha$. Let $\mu_{\alpha+1} = \varepsilon_{\delta_\gamma+2}$.
3. If $\alpha = \delta_\gamma + 2$, let $\mathbb{Q}_\alpha = \mathrm{Col}(\mu_\alpha, < \varepsilon_{\alpha+1})$. Let $\mu_{\alpha+1} = ((\mu_\alpha)^+)^{V^{\mathbb{P}_{\alpha+1}}}$, which is $((\varepsilon_{\delta_\gamma+2})^+)^{V^{\mathbb{P}_{\alpha+1}}}$.
4. If $\delta_\gamma + 3 \leq \alpha < \delta_{\gamma+1}$ and α is a successor ordinal, then let $\mathbb{Q}_\alpha = \mathrm{Col}(\mu_\alpha, < \varepsilon_{\alpha+1})$. Let $\mu_{\alpha+1} = ((\mu_\alpha)^+)^{V^{\mathbb{P}_{\alpha+1}}}$, which is what $\varepsilon_{\alpha+1}$ gets collapsed to by $\mathbb{Q}_\alpha$.
5. If α is a limit ordinal and μ_α is singular in $V^{\mathbb{P}_\alpha}$, then let $\mathbb{Q}_\alpha$ be the trivial forcing, and let $\mu_{\alpha+1} = ((\mu_\alpha)^+)^{V^{\mathbb{P}_{\alpha+1}}}$.
6. If α is a limit ordinal and μ_α is regular in $V^{\mathbb{P}_\alpha}$, then let $\mathbb{Q}_\alpha = \mathrm{Col}(\mu_\alpha, < \varepsilon_{\alpha+1})$ and let $\mu_{\alpha+1} = ((\mu_\alpha)^+)^{V^{\mathbb{P}_{\alpha+1}}}$.

Let $G \subseteq \mathbb{P}$ be generic over V. Suppose κ is ω_1-Erdős hyperstrong and $j : V \to M$ witnesses this.

Claim 1. κ remains hyperstrong in $V[G]$.

Proof. Let λ be the least ω_1-Erdős in M above $j(\kappa)$. $M_\lambda = V_\lambda$. Let $\mathbb{P}^*$ denote the forcing $\mathbb{P}$ defined in M. In order to show that $\mathbb{P}$ preserves the hyperstrength of κ, we need to create a generic $G^* \subseteq \mathbb{P}^*$ over M such that $j[G] \subseteq G^*$ and $V[G]_{j(\kappa)+1} = M[G^*]_{j(\kappa)+1}$.

Construction of G^*. Let $G^*_{j(\kappa)} = G_{j(\kappa)}$. Let $\bar{\lambda}$ denote $\varepsilon_{\kappa+1}$, the least ω_1-Erdős cardinal above κ in V. Note that $j(\bar{\lambda}) = \lambda$. We need to ensure that $j[G_{\kappa+1}] \subseteq G^*_{j(\kappa)+1}$. $\mathbb{P}_{\kappa+1} = \mathbb{P}_\kappa * \mathbb{Q}_\kappa$, where $\mathbb{Q}_\kappa$ is $\mathrm{Col}(\kappa, < \bar{\lambda})$ in $V[G_\kappa]$. Let j^*_κ denote the lifting of j to $V[G_\kappa]$. That is, $j^*_\kappa : V[G_\kappa] \to M[G^*_{j(\kappa)}]$ by $j^*_\kappa(\sigma^{G_\kappa}) = (j(\sigma))^{G_{j(\kappa)}}$, where σ is a $\mathbb{P}_\kappa$-name in V.

Let g_κ denote the $\mathbb{Q}_\kappa$-generic over $V[G_\kappa]$. $g_\kappa \in V_\lambda[G_{j(\kappa)}]$ and $g_\kappa \subseteq V_{\bar\lambda}[G_\kappa]$. $j \restriction V_{\bar\lambda} \in V_\lambda$, so $j^*_\kappa \restriction V_{\bar\lambda}[G_\kappa] \in V_\lambda[G_{j(\kappa)}]$. Thus, $j^*_\kappa[g_\kappa] \in V_\lambda[G_{j(\kappa)}] = M_\lambda[G^*_{j(\kappa)}]$, so there is a lower bound of $j^*_\kappa[g_\kappa]$ in $\mathrm{Col}(j(\kappa), < \lambda)$ in $M[G^*_{j(\kappa)}]$. Call it q. Choose the generic $g^*_{j(\kappa)}$ for $\mathbb{Q}^*_{j(\kappa)}$ over $M[G^*_{j(\kappa)}]$ inside $V[G_{j(\kappa)+1}]$ such that $q \in g^*_{j(\kappa)}$. This is possible using the homogeneity of $\mathbb{Q}^*_{j(\kappa)}$. Then $j^*_\kappa[g_\kappa] \subseteq g^*_{j(\kappa)}$.

Now we construct $G^{*,j(\kappa)+1}$, the generic for $\mathbb{P}^{*,j(\kappa)+1}$ where $\mathbb{P}^*$ factors as $\mathbb{P}^*_{j(\kappa)+1} * \mathbb{P}^{*,j(\kappa)+1}$. Assume $j : V \to M$ is given as an ultrapower embedding, i.e., $M = \{j(f)(a) : f \in V,\ f : V_{\bar\lambda} \to V,\ a \in V_\lambda\}$. Let D be a set-sized maximal antichain of $\mathbb{P}^{*,j(\kappa)+1}$ in $M[G^*_{j(\kappa)+1}]$. Then there is a $\mathbb{P}^*_{j(\kappa)+1}$-name τ in M such that $D = \tau^{G^*_{j(\kappa)+1}}$. Fix $f \in V$ and $a \in V_\lambda$ such that $f : V_{\bar\lambda} \to V$ and $\tau = j(f)(a)$. We seek a condition $p \in G^{\kappa+1}$ which extends an element of $(f(\bar a))^{G_{\kappa+1}}$ whenever $\bar a \in V_{\bar\lambda}$ and $(f(\bar a))^{G_{\kappa+1}}$ is predense in $\mathbb{P}^{\kappa+1}$. This is possible if the upper part of the forcing has enough closure. (It will not work if we let $\mathbb{Q}_{\kappa+1}$ be $\mathrm{Col}(\mu_{\kappa+1}, < \varepsilon_{\kappa+2})$, for then the upper part does not have enough closure to find such a p. This is precisely why we break our forcing into two parts, $\mathbb{P}$ and $\mathbb{R}$.) Conditions (2) and (3) in the construction of $\mathbb{P}$ ensure that $\mathbb{P}_{\kappa+1}$ is $\varepsilon_{\kappa+2}$-closed in $V[G_{\kappa+1}]$. This is enough closure to find a condition $p \in G^{\kappa+1}$ which extends an element of $(f(\bar a))^{G_{\kappa+1}}$ whenever $\bar a \in V_{\bar\lambda}$ and $(f(\bar a))^{G_{\kappa+1}}$ is predense in $\mathbb{P}^{\kappa+1}$.

Let $j^*_{\kappa+1} : V[G_{\kappa+1}] \to M[G^*_{j(\kappa)+1}]$ be given given by $j^*_{\kappa+1}(\sigma^{G_{\kappa+1}}) = (j(\sigma))^{G^*_{j(\kappa)+1}}$, for σ any $\mathbb{P}_{\kappa+1}$-name. $V[G_{\kappa+1}] \models (\forall \bar a \in V_{\bar\lambda}$, if $(f(\bar a))^{G_{\kappa+1}}$ is predense in $\mathbb{P}^{\kappa+1}$, then $\exists r \in (f(\bar a))^{G_{\kappa+1}}$ such that $p \le r)$. Hence, there is an $s \in G_{\kappa+1}$ such that $V \models (s \Vdash (\forall \bar a \in V_{\bar\lambda}$, if $f(\bar a)$ is predense in $\mathbb{P}^{\kappa+1}$, then $\exists r \in f(\bar a)$ such that $\dot p \le r))$. (Here $\dot p$ is a $\mathbb{P}_{\kappa+1}$-name for p.) By elementarity, $M \models (j(s) \Vdash (\forall b \in j(V_{\bar\lambda})$, if $j(f)(b)$ is predense in $j(\mathbb{P}^{\kappa+1})$, then $\exists r \in j(f)(b)$ such that $j(\dot p) \le r))$. Since $j(s) \in j[G_{\kappa+1}] \subseteq G^*_{j(\kappa)+1}$, $M[G^*_{j(\kappa)+1}] \models (\exists r \in D$ such that $j^*_{\kappa+1}(p) \le r))$. The same proof works for arbitrary definable maximal antichains. By elementarity, $j^*_{\kappa+1}[G^{\kappa+1}]$ is pairwise compatible. In $M[G^*_{j(\kappa)+1}]$, let $G^{*,j(\kappa)+1} = \{q \in \mathbb{P}^{*,j(\kappa)+1} : \exists r \in G^{\kappa+1}\ (j^*_{\kappa+1}(r) \le q)\}$, the upward closure in $\mathbb{P}^{*,j(\kappa)+1}$ of $j^*_{\kappa+1}[G^{\kappa+1}]$.

Thus, $j[G] \subseteq G^*$. Let $j^* : V[G] \to M[G^*]$ be given by $j^*(\sigma^G) = (j(\sigma))^{G^*}$, where σ is any $\mathbb{P}$-name in V. $\mathbb{P}^{j(\kappa)+1}$ is $\varepsilon_{j(\kappa)+2}$-closed in $V[G_{j(\kappa)+1}]$, and $\varepsilon_{j(\kappa)+2} > \lambda$. Hence, $V[G]_{j(\kappa)+1} = V[G_{j(\kappa)+1}]_{j(\kappa)+1}$. $V[G_{j(\kappa)+1}]_{j(\kappa)+1} = V_\lambda[G_{j(\kappa)+1}]$, since λ is inaccessible in V, and since $\mathbb{P}^{j(\kappa)+1}$ is $\varepsilon_{j(\kappa)+2}$-closed. Likewise, $M[G^*]_{j(\kappa)+1} = M_\lambda[G^*_{j(\kappa)+1}]$. Since $V_\lambda[G_{j(\kappa)+1}] = M_\lambda[G^*_{j(\kappa)+1}]$ and $\mathbb{P}_{j(\kappa)+1}$ and $\mathbb{P}^*_{j(\kappa)+1}$ both collapse λ to $j(\kappa)^+$, $V[G]_{j(\kappa)+1} = M[G^*]_{j(\kappa)+1}$. Therefore, κ is hyperstrong in $V[G]$. $\dashv$

Phase 2: *Construction of* $\mathbb{R}$. The second stage of the forcing takes care of the cases which were untouched by $\mathbb{P}$. In Phase 1, we could not let $\mathbb{Q}_\alpha$ be a collapsing forcing when α was the successor of an inaccessible limit of ω_1-Erdős cardinals, because it was precisely at those points that we needed enough closure to lift the embedding. In Phase 2 we correct what was left undone in Phase 1 in order that all successors of regulars were ω_1-Erdős cardinals in V. We must be careful that what we obtained in regard to co-stationaity of the ground model after Phase 1 remains valid after Phase 2.

Recall that $\langle \delta_\alpha : \alpha \in \mathrm{Ord}\rangle$ enumerates all the cardinals in V which are both inaccessible and limits of ω_1-Erdős cardinals in V. The stages $\delta_\alpha + 1$ of the Phase 1 forcing $\mathbb{P}$ were where we did not collapse anything; $\mathbb{Q}_{\delta_\alpha+1}$ was the trivial forcing. These are the stages where we will now do a Lévy collapse.

For each $\alpha \in \text{Ord}$, recall that δ_α is fixed by $\mathbb{P}$, and in $V[G]$, $\mu_{\delta_\alpha+1} = (\delta_\alpha)^+$ and $\mu_{\delta_\alpha+2} = \varepsilon_{\delta_\alpha+2}$. For each ordinal α, we take $\mathbb{S}_\alpha = \text{Col}(\mu_{\delta_\alpha+1}, < \mu_{\delta_\alpha+2})$. Let $\mathbb{R}_{\alpha+1} = \mathbb{R}_\alpha * \mathbb{S}_\alpha$, where $\mathbb{R}_0$ is the trivial forcing. Let $\mathbb{R}$ be the reverse Easton iteration of the $\mathbb{R}_\alpha$, $\alpha \in \text{Ord}$. $\mathbb{R}$ is going to collapse the $\mu_{\delta_\alpha+2}$'s down so that $\mu_{\delta_\alpha+2}$ becomes $(\delta_\alpha)^{++}$, which is the same as $\aleph_{\alpha+2}$ in the final extension by $\mathbb{P} * \mathbb{R}$. Let $V' = V[G]$. Let H be V'-generic for $\mathbb{R}$, and let $W = V'[H]$.

Claim 2. κ remains superstrong in W.

Proof. Let $k : V' \to N$ be an elementary embedding witnessing the superstrength of κ in V'. Let $\mathbb{R}^*$ denote the version of $\mathbb{R}$ in N. Again, our goal is to create a generic H^* for $\mathbb{R}^*$ over N such that $k[H] \subseteq H^*$ and $W_{k(\kappa)} = N[H^*]_{k(\kappa)}$. $\mathbb{R}^*_{k(\kappa)} = \mathbb{R}_{k(\kappa)}$, since $V'_{k(\kappa)} = N_{k(\kappa)}$, so let $H^*_{k(\kappa)} = H_{k(\kappa)}$.

Next, construct the generic for $\mathbb{R}^{*,k(\kappa)}$ over $N[H^*_{k(\kappa)}]$. Let k^*_κ denote the lifting of k to $V'[H_\kappa]$, so that $k^*_\kappa : V'[H_\kappa] \to N[H_{k(\kappa)}]$ by $k^*_\kappa(\sigma^{H_\kappa}) = (k(\sigma))^{H_{k(\kappa)}}$, where σ is an $\mathbb{R}_\kappa$-name in V'. We assume that k is given by an ultrapower embedding, i.e., $N = \{k(g)(b) : g \in V', g : V'_\kappa \to V', \text{and } b \in V'_{k(\kappa)}\}$. Let D be a set-sized maximal antichain of $\mathbb{R}^{*,k(\kappa)}$ in $N[H^*_{k(\kappa)}]$. Let τ be an $\mathbb{R}^*_{k(\kappa)}$-name in N such that $D = \tau^{H^*_{k(\kappa)}}$, and let $g \in V'$ and $b \in V'_{k(\kappa)}$ such that $\tau = k(g)(b)$. In $V'[H_\kappa]$, $\mu_{\kappa+1} = \kappa^+$, so $\mathbb{S}_\kappa = \text{Col}(\kappa^+, < \varepsilon_{\kappa+2})$. Hence, $\mathbb{R}^\kappa$ is κ^+-closed. So in $V'[H_\kappa]$, there is a $p \in H^\kappa$ which extends an element of $(g(\bar{b}))^{H_\kappa}$ whenever $\bar{b} \in V'_\kappa$ and $(g(\bar{b}))^{H_\kappa}$ is predense in $\mathbb{R}^\kappa$. It follows that $N[H^*_{k(\kappa)}] \models (\exists r \in D \text{ such that } k^*_\kappa(p) \leq r)$. By elementarity, $k^*_\kappa[H^\kappa]$ is pairwise compatible. Let $H^{*,k(\kappa)} = \{q \in \mathbb{R}^{*,j(\kappa)} : \exists r \in G^\kappa\, (k^*_\kappa(r) \leq q)\}$, the upward closure in $\mathbb{R}^{*,j(\kappa)}$ of $k^*_\kappa[H^\kappa]$.

Thus, $k[H] \subseteq H^*$. Hence, k lifts to an elementary embedding $k^* : W \to N[H^*]$ with $k^*(\kappa) = k(\kappa)$. It remains to show that $W_{k(\kappa)} = N[H^*]_{k(\kappa)}$. $\mathbb{R}^{k(\kappa)} = \text{Col}(k(\kappa)^+, < \mu_{k(\kappa)+2}) * \mathbb{R}^{k(\kappa)+1}$, hence is $k(\kappa)^+$-closed. So $W_{k(\kappa)} = V'[H_{k(\kappa)}]_{k(\kappa)}$. $V'[H_{k(\kappa)}]_{k(\kappa)} = V'_{k(\kappa)}[H_{k(\kappa)}]$, since $k(\kappa)$ is inaccessible. Likewise, $N[H^*]_{k(\kappa)} = N_{k(\kappa)}[H^*_{k(\kappa)}]$. $H_{k(\kappa)} = H^*_{k(\kappa)}$, and $N_{k(\kappa)} = V'_{k(\kappa)}$. So we have $W_{k(\kappa)} = N[H^*]_{k(\kappa)}$. Thus, κ is superstrong in W. ⊣

As the last step, we check that every $\aleph_2$-c.c. forcing in W which adds a new subset of $\aleph_1$ achieves global co-stationarity of W.

Subclaim A. Let $\alpha \in \text{Ord}$. If μ_α is regular in V' and $\alpha \neq \delta_\gamma + 1$ for any $\gamma \in \text{Ord}$, then $*(V', \mu_\alpha)$ holds.

Proof. By Lemma 2.7, $*(V[G_{\alpha+1}], \mu_\alpha)$ holds. Since $\mathbb{P}^{\alpha+1}$ is $(\mu_\alpha)^+$-closed, $*(V', \mu_\alpha)$ holds, by Lemma 2.10. ⊣

Subclaim B. For each regular ρ in W, $*(W, \rho)$ holds.

Proof. *Case* 1. $\alpha \in \text{Ord}$, and in W, $(\delta_\alpha)^+ < \rho \leq \delta_{\alpha+1}$. Then $*(V', \rho)$ holds, by Subclaim A. $\mathbb{R}_{\alpha+1} = \mathbb{R}_\alpha * \text{Col}((\delta_\alpha)^+, < \varepsilon_{\delta_\alpha+2})$, so $\mathbb{R}_{\alpha+1}$ is $\varepsilon_{\delta_\alpha+2}$-c.c. $\rho \geq \varepsilon_{\delta_\alpha+2}$ in V', so $*(V'[H_{\alpha+1}], \rho)$ holds, by Lemma 2.9. $\mathbb{R}^{\alpha+1} = \text{Col}((\delta_{\alpha+1})^+, < \varepsilon_{\delta_{\alpha+1}+2}) * \mathbb{R}^{\alpha+2}$ is $(\delta_{\alpha+1})^+$-closed. So by Lemma 2.10, $*(W, \rho)$ holds.

Case 2. $\alpha \in \text{Ord}$ and $\rho = (\delta_\alpha)^+$. $\mathbb{S}_\alpha = \text{Col}((\delta_\alpha)^+, < \varepsilon_{\delta_\alpha+2})$, so $*(V'[H_{\alpha+1}], \rho)$ holds, by Lemma 2.7. $\mathbb{R}^{\alpha+1}$ is $(\delta_{\alpha+1})^+$-closed in $V'[H_{\alpha+1}]$, so $*(W, \rho)$ holds, by Lemma 2.10.

Case 3. α is a limit ordinal and $\rho = \delta_\alpha$. $*(V', \rho)$ holds, by Subclaim A, and δ_α is still strongly inaccessible in V'. If $\alpha < \rho$, then $|\mathbb{R}_\alpha| < \rho$. If $\alpha = \rho$, then $\mathbb{R}_\alpha$ is ρ-c.c. Either way, $*(V'[H_\alpha], \rho)$ holds, by Lemma 2.9. $\mathbb{R}^\alpha = \text{Col}(\rho^+, < \varepsilon_{\alpha+2}) * \mathbb{R}^{\alpha+1}$ is ρ^+-closed; so by Lemma 2.10, $*(W, \rho)$ holds. ⊣

These three cases cover all regular cardinals $\rho \in W$, so $*(W, \rho)$ holds for all regular ρ in W. By Theorem 2.8, we finally obtain global co-stationarity of W for any partial ordering which adds a new subset of $\aleph_1$ and is $\aleph_2$-c.c. (or adds a new subset of $\aleph_1$ and satisfies the $(\rho^+, \rho^+, < \rho)$-d.l. if ρ is less than the least regular limit cardinal, and the θ-c.c. if θ is the least regular limit cardinal). ⊣

THEOREM 3.2. *Assume V is a model of* ZFC *with an ω_1-Erdős hyperstrong cardinal κ and a measurable cardinal above $j(\kappa)$, where j is an elementary embedding witnessing the ω_1-Erdős hyperstrength of κ. Then there is an inner model $W \subseteq V$ with a proper class of Woodin cardinals such that every $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ achieves global co-stationarity of W. If moreover κ is an ω_1-Erdős hyperstrong cardinal which is a limit of ω_1-Erdős hyperstrong cardinals, then W has a proper class of superstrong cardinals.*

PROOF. Let φ be the sentence "There exists an ω_1-Erdős hyperstrong cardinal κ, a witnessing embedding $j : V \to N$, and a measurable cardinal above $j(\kappa)$". Let V be model of ZFC satisfying φ, and let θ be a regular cardinal large enough that $H_\theta \models \varphi$. Let T'' be a countable elementary submodel of H_θ and T' be the transitive collapse of T''. Since T' is countable we can force CH over T' by forcing with the collection of functions from ordinals $\alpha < \omega_1$ into 2^ω. Let T denote the extension model. Then T is still countable and $T \models \varphi$.

In T, let κ denote an ω_1-Erdős hyperstrong cardinal, $j : V \to N$ be a witnessing embedding, and μ denote a measurable cardinal above $j(\kappa)$. Define $\mathbb{P}_{j(\kappa)+1}$ in T as in Phase 1 in the proof of Theorem 3.1. Let $G \in V$ be $\mathbb{P}_{j(\kappa)+1}$-generic over T. Define $\mathbb{R}_{j(\kappa)}$ as in Phase 2 in the proof of Theorem 3.1. Let $H \in V$ be $\mathbb{R}_{j(\kappa)}$-generic over $T[G]$. Let $M = T[G][H]$. $M \in V$. By the same arguments, κ remains superstrong in M. Moreover, $*(M, \rho)$ holds for all regular cardinals $\rho \in M$ with $\aleph_2 \leq \rho \leq j(\kappa)$.

Since the forcing $\mathbb{P}_{j(\kappa)+1} * \mathbb{R}_{j(\kappa)}$ has cardinality in T much smaller than μ, μ is still measurable in M and carries a measure in M which is iterable in V. Iterate out μ to obtain an inner model $W' \subseteq V$ which agrees with M up to sets of rank less than μ. That is, letting θ be a regular cardinal in V above μ, form (in V) the ultrapower of $H(\theta)$ with respect to a measure on μ. Then there is an elementary embedding $j_1 : H(\theta) \to \text{Ult}_1$, where Ult_1 denotes $\text{Ult}(H(\theta), \mathscr{U}_0)$, where $\mathscr{U}_0$ is a measure on μ. $j_1(\mu) > \mu$ and $j_1(\mu)$ is measurable with measure $j_i(\mathscr{U}_0)$ in Ult_1; so there is an elementary embedding $j_2 : \text{Ult}_1 \to \text{Ult}_2$, where $\text{Ult}_2 = \text{Ult}(\text{Ult}_1, \mathscr{U}_1)$, where $\mathscr{U}_1 = j_1(\mathscr{U}_0)$ is a measure on $j_1(\mu)$. In this way, form the directed limit W' of ultrapowers iterating through all the ordinals in V. This gives us an inner model W' of V such that W' and V are the same everywhere below rank μ. κ is still superstrong in W', so let $\mathscr{U}$ be a measure on κ in W'. Let H' denote the H_{κ^+} of W'. As before, iterate $\mathscr{U}$ out to obtain an inner model W satisfying $*(W, \rho)$ for each regular $\rho \in W$ with $\rho \geq \aleph_2$, and as κ is superstrong in W', W has a proper class of Woodin cardinals. That is, starting with the model $H(\kappa)$ in W', use a measure on κ and form the directed limit (over all ordinals in W') of iterations of ultrapowers. This gives an inner model W of W' which is elementarily equivalent to $H(\kappa)$.

If in addition, V has an ω_1-Erdős hyperstrong cardinal which is also a limit of ω_1-Erdős hyperstrong cardinals, then by the proof of Theorem 3.1, κ is a limit of superstrong cardinals in W'. It follows that there is a proper class of superstrong cardinals in W. ⊣

We conclude with the following open problems.

PROBLEM 3.3. What is the internal consistency strength of global co-stationarity for $\aleph_2$-c.c. forcings which add a new subset of $\aleph_1$? Does the existence of an inner model for this property follow from the existence of an inner model with a large cardinal property weaker than superstrength?

PROBLEM 3.4. What is the consistency strength of global co-stationarity for $\aleph_2$-c.c. forcings which add a new subset of $\aleph_1$ together with the existence of a measurable cardinal?

Our final open problem refers to [3], where we showed that "$\mathscr{P}_{\aleph_3}(\aleph_{\omega_2}) \setminus V$ is stationary in $V^{\mathbb{C}}$ where $\mathbb{C}$ is $\aleph_2$-Cohen forcing" is equiconsistent with $\aleph_2$ many measurable cardinals.

PROBLEM 3.5. Is global co-stationarity for $\aleph_2$-Cohen forcing consistent?

REFERENCES

[1] JAMES E. BAUMGARTNER, *On the size of closed unbounded sets*, ***Annals of Pure and Applied Logic***, vol. 54 (1991), pp. 195–227.

[2] A. BELLER, R. JENSEN, and P. WELCH, ***Coding the Universe***, Cambridge University Press, 1982.

[3] NATASHA DOBRINEN and SY-DAVID FRIEDMAN, *Co-stationarity of the ground model*, this JOURNAL, vol. 71 (2006), no. 3, pp. 1029–1043.

[4] SY-DAVID FRIEDMAN, *Internal consistency and the inner model hypothesis*, ***The Bulletin of Symbolic Logic***, vol. 12 (2006), no. 4, pp. 591–600.

[5] SABINE KOPPELBERG, ***Handbook of Boolean Algebra***, vol. 1, North-Holland, 1989.

[6] DAVID W. KUEKER, *Löwenheim-Skolem and interpolation theorems in infinitary languages*, ***Bulletin of the American Mathematical Society***, vol. 78 (1972), pp. 211–215.

[7] TELIS K. MENAS, *On strong compactness and supercompactness*, ***Annals of Mathematical Logic***, vol. 7 (1974/75), pp. 327–359.

UNIVERSITÄT WIEN, KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
WÄHRINGER STRAßE 25
1090 WIEN, AUSTRIA
Current address: University of Denver, Department of Mathematics, 2360 S Gaylord St., Denver, CO 80208, USA
E-mail: ndobrine@du.edu
URL: http://www.math.du.edu/dobrinen/

UNIVERSITÄT WIEN, KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
WÄHRINGER STRAßE 25
1090 WIEN, AUSTRIA
E-mail: sdf@logic.univie.ac.at
URL: http://www.logic.univie.ac.at/∼sdf/