KILLING GCH EVERYWHERE BY A COFINALITY-PRESERVING FORCING NOTION OVER A MODEL OF GCH

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Abstract. Starting from large cardinals we construct a pair (V_1, V_2) of models of ZFC with the same cardinals and cofinalities such that GCH holds in V_1 and fails everywhere in V_2.

1. Introduction

Easton's classical result showed that over any model of GCH, one can force any reasonable pattern of the power function $\lambda \mapsto 2^\lambda$ on the regular cardinals λ, preserving cardinals and cofinalities. Subsequently, much work has been done on the singular cardinal problem, whose aim is to characterise the patterns of the power function on all cardinals, including the singular ones. Typically in this work, large cardinals are used to obtain patterns of power function behaviour at singular cardinals after applying subtle forcings which change cofinalities or even collapse cardinals. This leads one to ask: Is it possible to obtain a failure of GCH everyhwere by forcing over a model of GCH without changing cofinalities? If so, can one have a fixed finite gap in the resulting model, meaning that $2^\lambda = \lambda^{+n}$ for some finite $n > 1$ for all λ?

In this paper we prove the following theorem.

Theorem 1.1. *Assume GCH+there exists a $(\kappa+4)-$strong cardinal κ. Then there is a pair (V_1, V_2) of models of ZFC such that:*

(a) V_1 and V_2 have the same cardinals and cofinalities,

(b) GCH holds in V_1,

(c) $V_2 \models$ "$\forall\lambda, 2^\lambda = \lambda^{+3}$".

Remark 1.2. *In fact it suffices to have a Mitchell increasing sequence of extenders of length κ^+, each of them $(\kappa+3)-$strong. Thus the exact strength that we need for fixed gap of 3 is a measurable cardinal κ with $o(\kappa) = \kappa^{+3} + \kappa^+$. It is also easy to extend our result to* *an arbitrary finite gap n instead of 3. Then what we need is a measurable cardinal κ with $o(\kappa) = \kappa^{+n} + \kappa^{+}$. We focus on the case $n = 3$ as it is typical of all cases $n \geq 3$ (the case $n = 2$ is easier).*

The rest of this paper is devoted to the proof of this theorem. The proof is based on the extender-based Radin forcing developed by C. Merimovich in the papers [3], [4]. We try to make the proof self-contained, thus we start with some preliminaries and facts from these papers.

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2. Extender Sequences

Suppose $j : V^* \to M^* \supseteq V^*_\lambda, crit(j) = \kappa$. Define an extender sequence (with projections)

$$E(0) = \langle \langle E_\alpha(0) : \alpha \in A \rangle, \langle \pi_{\beta,\alpha} : \beta, \alpha \in A, \beta \geq_j \alpha \rangle \rangle$$

on κ by:

- $A = [\kappa, \lambda)$,
- $\forall \alpha \in A, E_\alpha(0)$ is the $\kappa-$complete ultrafilter on κ defined by

  $$X \in E_\alpha(0) \Leftrightarrow \alpha \in j(X)$$

  We write $E_\alpha(0)$ as U_α.
- $\forall \alpha, \beta \in A$

  $$\beta \geq_j \alpha \Leftrightarrow \beta \geq \alpha \text{ and for some } f \in {}^\kappa\kappa,\ j(f)(\beta) = \alpha$$
- $\beta \geq_j \alpha \Rightarrow \pi_{\beta,\alpha} : \kappa \to \kappa$ is such that $j(\pi_{\beta,\alpha})(\beta) = \alpha$

Let's recall the main properties of this sequence (see [2])

(1) $\langle \lambda, \leq_j \rangle$ is a κ^+-directed partial order,

(2) $\forall \alpha, \kappa \leq_j \alpha$,

(3) U_κ is a normal measure on κ,

(4) $\forall \alpha, U_\alpha$ is a $P-$point ultrafilter over κ, i.e for any $f : \kappa \to \kappa$ there is $X \in U_\alpha$ such that $\forall \nu < \kappa, |X \cap f^{-1''}(\nu)| < \kappa$,

(5) $\pi^{-1''}_{\beta,\alpha}(X) \in U_\beta \Leftrightarrow X \in U_\alpha$,

(6) $\forall \alpha, \pi_{\alpha,\alpha} = id$,

(7) $\forall \gamma \geq_j \beta \geq_j \alpha$ there is $X \in U_\gamma$ such that $\forall \nu \in X, \pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$,

(8) $\forall \gamma \geq_j \alpha, \beta$ where $\alpha \neq \beta$ there is $X \in U_\gamma$ such that $\forall \nu \in X, \pi_{\gamma,\alpha}(\nu) \neq \pi_{\gamma,\beta}(\nu)$,

(9) $\forall \beta \geq_j \alpha, \forall \nu < \kappa, \pi_{\beta,\kappa}(\nu) = \pi_{\alpha,\kappa}(\pi_{\beta,\alpha}(\nu))$,

(10) $\forall \alpha, \beta, \forall \nu < \kappa, \pi_{\alpha,\kappa}(\nu) = \pi_{\beta,\kappa}(\nu)$; we denote the latter by ν^0.

Now suppose that we have defined the sequence $\langle E(\tau') : \tau' < \tau \rangle$. If $\langle E(\tau') : \tau' < \tau \rangle \notin M^*$ we stop the construction and set

$$\forall \alpha \in A, \bar{E}_\alpha = \langle \alpha, E(0), ..., E(\tau'), ... : \tau' < \tau \rangle$$

and call $\bar{E}_\alpha$ an extender sequence of length τ ($\mathrm{l}(\bar{E}_\alpha) = \tau$).

If $\langle E(\tau') : \tau' < \tau \rangle \in M^*$ then we define an extender sequence (with projections)

$$E(\tau) = \langle \langle E_{\langle \alpha, E(\tau'):\tau'<\tau \rangle}(\tau) : \alpha \in A \rangle, \langle \pi_{\langle \beta, E(\tau'):\tau'<\tau \rangle, \langle \alpha, E(\tau'):\tau'<\tau \rangle} : \beta, \alpha \in A, \beta \geq_j \alpha \rangle \rangle$$

on V_κ by:

- $X \in E_{\langle \alpha, E(\tau'):\tau'<\tau \rangle}(\tau) \Leftrightarrow \langle \alpha, E(\tau') : \tau' < \tau \rangle \in j(X)$,
- for $\beta \geq_j \alpha$ in $A, \pi_{\langle \beta, E(\tau'):\tau'<\tau \rangle, \langle \alpha, E(\tau'):\tau'<\tau \rangle}(\langle \nu, d \rangle) = \langle \pi_{\beta,\alpha}(\nu), d \rangle$

Note that $E_{\langle \alpha, E(\tau'):\tau'<\tau \rangle}(\tau)$ concentrates on pairs of the form $\langle \nu, d \rangle$ where $\nu < \kappa$ and d is an extender sequence. This makes the above definition well-defined.

We let the construction run until it stops due to the extender sequence not being in M^*.

Definition 2.1. (1) *$\bar{\mu}$ is an extender sequence if there are $j : V^* \to M^*$ and $\bar{\nu}$ such that $\bar{\nu}$ is an extender sequence derived from j as above (i.e $\bar{\nu} = \bar{E}_\alpha$ for some α) and $\bar{\mu} = \bar{\nu} \restriction \tau$ for some $\tau \leq \mathrm{l}(\bar{\nu})$,*

(2) *$\kappa(\bar{\mu})$ is the ordinal of the beginning of the sequence (i.e $\kappa(\bar{E}_\alpha) = \alpha$),*

(3) *$\kappa^0(\bar{\mu}) = (\kappa(\bar{\mu}))^0$ (i.e $\kappa^0(\bar{E}_\alpha) = \kappa$)),*

(4) *The sequence $\langle \bar{\mu_1}, ..., \bar{\mu_n} \rangle$ of extender sequences is ^{0}−increasing if $\kappa^0(\bar{\mu_1}) < ... < \kappa^0(\bar{\mu_n})$,*

(5) *The extender sequence $\bar{\mu}$ is permitted to a ^{0}−increasing sequence $\langle \bar{\mu_1}, ..., \bar{\mu_n} \rangle$ of extender sequences if $\kappa^0(\bar{\mu_n}) < \kappa^0(\bar{\mu})$,*

(6) *$X \in \bar{E}_\alpha \Leftrightarrow \forall \xi < \mathrm{l}(\bar{E}_\alpha), X \in E_\alpha(\xi)$,*

(7) $\bar{E} = \langle \bar{E}_\alpha : \alpha \in A\rangle$ *is an extender sequence system if there is* $j : V^* \to M^*$ *such that each* $\bar{E}_\alpha$ *is derived from* j *as above and* $\forall \alpha, \beta \in \mathrm{A}, \mathrm{l}(\bar{E}_\alpha) = \mathrm{l}(\bar{E}_\beta)$. *Call this common length, the length of* $\bar{E}, \mathrm{l}(\bar{E})$,

(8) *For an extender sequence* $\bar{\mu}$, *we use* $\bar{E}(\bar{\mu})$ *for the extender sequence system containing* $\bar{\mu}$ *(i.e* $\bar{E}(\bar{E}_\alpha) = \bar{E}$*),*

(9) $\mathrm{dom}(\bar{E}) = A$,

(10) $\bar{E}_\beta \geq_{\bar{E}} \bar{E}_\alpha \Leftrightarrow \beta \geq_j \alpha$.

3. Finding generic filters

Start with GCH and construct an extender sequence system $\bar{E} = \langle \bar{E}_\alpha : \alpha \in \mathrm{dom}\,\bar{E}\rangle$ where $\mathrm{dom}\,\bar{E} = [\kappa, \kappa^{+3})$ and $\mathrm{l}(\bar{E}) = \kappa^+$ such that $j_{\bar{E}} : V^* \to M^*_{\bar{E}} \supseteq V^*_{\kappa+3}$. We may suppose that $\bar{E}$ is derived from an elementary embedding $j : V^* \to M^*$. Consider the following elementary embeddings $\forall \tau' < \tau < \mathrm{l}(\bar{E})$

$$j_\tau : V^* \to M^*_\tau \simeq \mathrm{Ult}(V^*, E(\tau)),$$

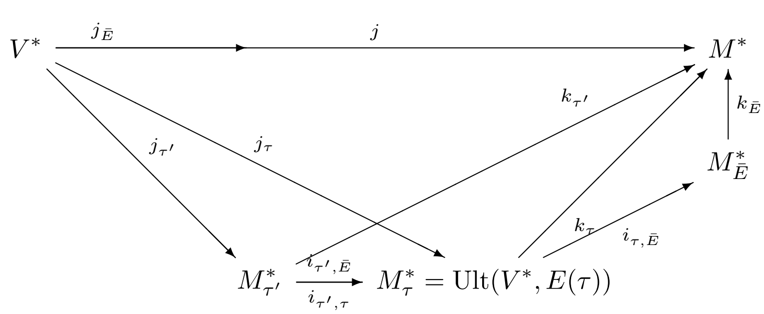
$$k_\tau(j_\tau(f)(\bar{E}_\alpha \restriction \tau)) = j(f)(\bar{E}_\alpha \restriction \tau),$$

$$i_{\tau',\tau}(j_{\tau'}(f)(\bar{E}_\alpha \restriction \tau')) = j_\tau(f)(\bar{E}_\alpha \restriction \tau'),$$

$$\langle M^*_{\bar{E}}, i_{\tau,\bar{E}}\rangle = \lim \mathrm{dir}\langle\langle M^*_\tau \mid \tau < \mathrm{l}(\bar{E})\rangle, \langle i_{\tau',\tau} \mid \tau' \leq \tau < \mathrm{l}(\bar{E})\rangle\rangle.$$

We restrict $\mathrm{l}(\bar{E})$ by demanding $\forall \tau < \mathrm{l}(\bar{E})\ \bar{E} \restriction \tau \in M^*_\tau$.

Thus we get the following commutative diagram.



Note that

- the critical point of elementary embeddings originating in V^* is κ,
- the critical point of elementary embeddings originating in other models is κ^{+4} as computed in that model.

Thus we get

$$\begin{aligned}
&\operatorname{crit} i_{\tau',\tau} = \operatorname{crit} k_{\tau'} = \operatorname{crit} i_{\tau',\bar{E}} = (\kappa^{+4})_{M^*_{\tau'}},\\
&\operatorname{crit} k_{\tau} = \operatorname{crit}(i_{\tau,\bar{E}}) = (\kappa^{+4})_{M^*_{\tau}},\\
&\operatorname{crit} k_{\bar{E}} = (\kappa^{+4})_{M^*_{\bar{E}}}.
\end{aligned}$$

Each of these models catches $V^{M^*}_{\kappa^{+3}} = V^*_{\kappa^{+3}}$ hence compute κ^{+3} to be the same ordinal in all models. The larger τ is the more resemblence there is between M^*_τ and M^*, and hence with V^* towards $V^*_{\kappa^{+4}}$. This can be observed by noting that

$$\kappa^{+4}_{M^*_{\tau'}} < j_{\tau'}(\kappa) < \kappa^{+4}_{M^*_\tau} < j_\tau(\kappa) < \kappa^{+4}_{M_{\bar{E}^*}} \leq \kappa^{+4}_{M^*} \leq \kappa^{+4}.$$

We also factor through the normal ultrafilter to get the following commutative diagram

$$\begin{tikzcd}
V^* \arrow[rr, "j_{\bar{E}}"] \arrow[d, "i_U"'] \arrow[drr, "j_\tau" near start] & & M^*_{\bar{E}} \\
N^* \simeq \operatorname{Ult}(V^*, U) \arrow[rr, "i_{U,\tau}"'] \arrow[urr, "i_{U,\bar{E}}"' near start] & & M^*_\tau \arrow[u, "i_{\tau,\bar{E}}"']
\end{tikzcd}
\qquad
\begin{aligned}
&U = E_\kappa(0),\\
&i_U : V^* \to N^* \simeq \operatorname{Ult}(V^*, U),\\
&i_{U,\tau}(i_U(f)(\kappa)) = j_\tau(f)(\kappa),\\
&i_{U,\bar{E}}(i_U(f)(\kappa)) = j_{\bar{E}}(f)(\kappa).
\end{aligned}$$

N^* catches V^* only up to $V^*_{\kappa+1}$ and we have

$$\kappa^+ < \operatorname{crit} i_{U,\tau} = \operatorname{crit} i_{U,\bar{E}} = \kappa^{++}_{N^*} < i_U(\kappa) < \kappa^{++}.$$

Definition 3.1. *Let*

(1) $\mathbb{R}^{\mathrm{Col}}_U = \operatorname{Col}(\kappa^{+6}, i_U(\kappa))_{N^*}$,

(2) $\mathbb{R}^{\mathrm{Add},1}_U = \operatorname{Add}(\kappa^{+}, \kappa^{+4})_{N^*}$,

(3) $\mathbb{R}^{\mathrm{Add},2}_U = \operatorname{Add}(\kappa^{++}, \kappa^{+5})_{N^*}$,

(4) $\mathbb{R}^{\mathrm{Add},3}_U = \operatorname{Add}(\kappa^{+3}, \kappa^{+6})_{N^*}$,

(5) $\mathbb{R}^{\mathrm{Add},4}_U = (\operatorname{Add}(\kappa^{+4}, i_U(\kappa)^{+}) \times \operatorname{Add}(\kappa^{+5}, i_U(\kappa)^{++}) \times \operatorname{Add}(\kappa^{+6}, i_U(\kappa)^{+3}))_{N^*}$,

(6) $\mathbb{R}^{\mathrm{Add}}_U = \mathbb{R}^{\mathrm{Add},1}_U \times \mathbb{R}^{\mathrm{Add},2}_U \times \mathbb{R}^{\mathrm{Add},3}_U \times \mathbb{R}^{\mathrm{Add},4}_U$,

(7) $\mathbb{R}_U = \mathbb{R}^{\mathrm{Add}}_U \times \mathbb{R}^{\mathrm{Col}}_U$.

Definition 3.2. *Let*

(1) $\mathbb{R}_\tau^{\mathrm{Col}} = \mathrm{Col}(\kappa^{+6}, j_\tau(\kappa))_{M_\tau^*}$,

(2) $\mathbb{R}_\tau^{\mathrm{Add},1} = \mathrm{Add}(\kappa^{+}, \kappa^{+4})_{M_\tau^*}$,

(3) $\mathbb{R}_\tau^{\mathrm{Add},2} = \mathrm{Add}(\kappa^{++}, \kappa^{+5})_{M_\tau^*}$,

(4) $\mathbb{R}_\tau^{\mathrm{Add},3} = \mathrm{Add}(\kappa^{+3}, \kappa^{+6})_{M_\tau^*}$,

(5) $\mathbb{R}_\tau^{\mathrm{Add},4} = (\mathrm{Add}(\kappa^{+4}, j_\tau(\kappa)^{+}) \times \mathrm{Add}(\kappa^{+5}, j_\tau(\kappa)^{++}) \times \mathrm{Add}(\kappa^{+6}, j_\tau(\kappa)^{+3}))_{M_\tau^*}$,

(6) $\mathbb{R}_\tau^{\mathrm{Add}} = \mathbb{R}_\tau^{\mathrm{Add},1} \times \mathbb{R}_\tau^{\mathrm{Add},2} \times \mathbb{R}_\tau^{\mathrm{Add},3} \times \mathbb{R}_\tau^{\mathrm{Add},4}$,

(7) $\mathbb{R}_\tau = \mathbb{R}_\tau^{\mathrm{Add}} \times \mathbb{R}_\tau^{\mathrm{Col}}$.

Definition 3.3. *Let*

(1) $\mathbb{R}_{\bar{E}}^{\mathrm{Col}} = \mathrm{Col}(\kappa^{+6}, j_{\bar{E}}(\kappa))_{M_{\bar{E}}^*}$,

(2) $\mathbb{R}_{\bar{E}}^{\mathrm{Add},1} = \mathrm{Add}(\kappa^{+}, \kappa^{+4})_{M_{\bar{E}}^*}$,

(3) $\mathbb{R}_{\bar{E}}^{\mathrm{Add},2} = \mathrm{Add}(\kappa^{++}, \kappa^{+5})_{M_{\bar{E}}^*}$,

(4) $\mathbb{R}_{\bar{E}}^{\mathrm{Add},3} = \mathrm{Add}(\kappa^{+3}, \kappa^{+6})_{M_{\bar{E}}^*}$,

(5) $\mathbb{R}_{\bar{E}}^{\mathrm{Add},4} = (\mathrm{Add}(\kappa^{+4}, j_{\bar{E}}(\kappa)^{+}) \times \mathrm{Add}(\kappa^{+5}, j_{\bar{E}}(\kappa)^{++}) \times \mathrm{Add}(\kappa^{+6}, j_{\bar{E}}(\kappa)^{+3}))_{M_{\bar{E}}^*}$,

(6) $\mathbb{R}_{\bar{E}}^{\mathrm{Add}} = \mathbb{R}_{\bar{E}}^{\mathrm{Add},1} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},2} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},3} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},4}$,

(7) $\mathbb{R}_{\bar{E}} = \mathbb{R}_{\bar{E}}^{\mathrm{Add}} \times \mathbb{R}_{\bar{E}}^{\mathrm{Col}}$.

Also define the forcing notion $\mathbb{P}$ as follows

$$\mathbb{P} = \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_3 = \mathrm{Add}(\kappa^{+}, \kappa^{+4}) \times \mathrm{Add}(\kappa^{++}, \kappa^{+5}) \times \mathrm{Add}(\kappa^{+3}, \kappa^{+6}).$$

and let $G = G_1 \times G_2 \times G_3$ be $\mathbb{P}-$generic over V^*. It is clear that $V^*[G]$ is a cofinality-preserving generic extension of V^* and that GCH holds in $V^*[G]$ below and at κ.

Remark 3.4. . *We require that $G_2 \times G_3$ contains some special element, that we will specify later.*

Lemma 3.5. *(a) $G_U = \langle i_U^{''} G_1 \rangle \times \langle i_U^{''} G_2 \rangle \times \langle i_U^{''} G_3 \rangle$ is $\mathbb{P}_U = i_U(\mathbb{P})-$generic over N^*,*

(b) $G_\tau = \langle j_\tau^{''} G_1 \rangle \times \langle j_\tau^{''} G_2 \rangle \times \langle j_\tau^{''} G_3 \rangle$ is $\mathbb{P}_\tau = j_\tau(\mathbb{P})-$generic over M_τ^*,

(c) $G_{\bar{E}} = \langle \bigcup_{\tau < \mathrm{l}(\bar{E})} i_{\tau,\bar{E}}^{''} G_\tau \rangle$ is $\mathbb{P}_{\bar{E}} = j_{\bar{E}}(\mathbb{P})-$generic over $M_{\bar{E}}^*$,.

Proof. (a) Suppose $D \in N^*$ is dense open in $\mathbb{P}_U$. Let $D = i_U(f)(\kappa)$ for some function $f \in V$ on κ. Then

$$D^* = \{\alpha < \kappa : f(\alpha) \text{ is dense open in } \mathbb{P}\} \in U.$$

Since $\mathbb{P}$ is κ^+-closed, $\bigcap_{\alpha\in D^*} f(\alpha)$ is dense open in $\mathbb{P}$. Let $p \in G \cap \bigcap_{\alpha\in D^*} f(\alpha)$. Then $i_U(p) \in G_U \cap D$.

(b) Suppose $D \in M^*_\tau$ is dense open in $\mathbb{P}_\tau$. Let $D = j_\tau(f)(\bar{E}_\alpha \restriction \tau)$ for some function $f \in V$ on V_κ. Then

$$D^* = \{\bar{\nu} \in V_\kappa : f(\bar{\nu}) \text{ is dense open in } \mathbb{P}\} \in E_\alpha(\tau).$$

Since $\mathbb{P}$ is κ^+-closed, $\bigcap_{\bar{\nu}\in D^*} f(\bar{\nu})$ is dense open in $\mathbb{P}$. Let $p \in G \cap \bigcap_{\bar{\nu}\in D^*} f(\bar{\nu})$. Then $j_\tau(p) \in G_\tau \cap D$.

(c) Suppose $D \in M^*_{\bar{E}}$ is dense open in $\mathbb{P}_{\bar{E}}$. Let $\tau < \mathrm{l}(\bar{E})$ and $D_\tau \in M^*_\tau$ be such that $D = i_{\tau,\bar{E}}(D_\tau)$. By elementarity D_τ is dense open in $\mathbb{P}_\tau$. Let $p \in G_\tau \cap D_\tau$. Then $i_{\tau,\bar{E}}(p) \in G_{\bar{E}} \cap D$. $\square$

The following lemma is now trivial.

Lemma 3.6. *The generic filters above are such that*

(a) $i''_U[G] \subseteq G_U$,

(b) $j''_\tau[G] \subseteq G_\tau$,

(c) $j''_{\bar{E}}[G] \subseteq G_{\bar{E}}$,

(d) $i''_{U,\tau'}[G_U] \subseteq G_\tau$,

(e) $i''_{\tau',\tau}[G_{\tau'}] \subseteq G_\tau$,

(f) $i''_{\tau,\bar{E}}[G_\tau] \subseteq G_{\bar{E}}$.

It then follows that we have the following lifting diagram.

$$\begin{tikzcd}
V = V^*[G] \arrow[rr, "j_{\bar{E}}", "j_\tau"'] \arrow[d, "i_U"'] \arrow[dr, "j_{\tau'}"'] & & M_{\bar{E}} = M^*_{\bar{E}}[G_{\bar{E}}] \\
N = N^*[G_U] \arrow[r, "i_{U,\tau'}"'] & M_{\tau'} = M^*_{\tau'}[G_{\tau'}] \arrow[r, "i_{\tau',\tau}"'] \arrow[ur, "i_{\tau',\bar{E}}"] & M_\tau = M^*_\tau[G_\tau] \arrow[u, "i_{\tau,\bar{E}}"']
\end{tikzcd}$$

Lemma 3.7. *In $V^*[G]$ there are I_U, I_τ and $I_{\bar{E}}$ such that*

(a) *I_U is R_U-generic over $N^*[G_U]$,*

(b) *I_τ is $R_\tau-$generic over $M^*_\tau[G_\tau]$,*

(c) *$I_{\bar{E}}$ is $R_{\bar{E}}-$generic over $M^*_{\bar{E}}[G_{\bar{E}}]$,*

(d) *The generics are so that we have the following lifting diagram*

$$\begin{tikzcd} & & M_{\bar{E}}[I_{\bar{E}}] \\ N[I_U] \arrow[r, "i^*_{U,\tau'}"'] & M_{\tau'}[I_{\tau'}] \arrow[ur, "i^*_{\tau',\bar{E}}"] \arrow[r, "i^*_{\tau',\tau}"'] & M_\tau[I_\tau] \arrow[u, "i^*_{\tau,\bar{E}}"'] \end{tikzcd}$$

Proof. We will prove the lemma in a sequence of claims.

Claim 3.8. $I^{\mathrm{Add},1}_{\bar{E}} = G_1 \cap \mathbb{R}^{\mathrm{Add},1}_{\bar{E}}$ *is* $\mathbb{R}^{\mathrm{Add},1}_{\bar{E}}$*−generic over* $M^*_{\bar{E}}$.

Proof. Suppose A is a maximal antichain of $\mathbb{R}^{\mathrm{Add},1}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $X = \bigcup\{\mathrm{dom}(p) : p \in A\}$. As $|A| \leq \kappa^+$, we have $|X| \leq \kappa^+$, and of course A is a maximal antichain of $\mathrm{Add}(\kappa^+, X)_{M^*_{\bar{E}}}$. For simlpicity let us assume $|X| = \kappa^+$. Now we have $\mathrm{Add}(\kappa^+, X)_{M^*_{\bar{E}}} = \mathrm{Add}(\kappa^+, X)$ and hence A is a maximal antichain of $\mathrm{Add}(\kappa^+, X)$. It then follows that A is a maximal antichain of $\mathrm{Add}(\kappa^+, \kappa^{+4})$. Let $p \in G_1 \cap A$. Then $p \in I^{\mathrm{Add},1}_{\bar{E}} \cap A$. □

Claim 3.9. $I^{\mathrm{Add},2}_{\bar{E}} = G_2 \cap \mathbb{R}^{\mathrm{Add},2}_{\bar{E}}$ *is* $\mathbb{R}^{\mathrm{Add},2}_{\bar{E}}$*−generic over* $M^*_{\bar{E}}$.

Proof. Suppose A is a maximal antichain of $\mathbb{R}^{\mathrm{Add},2}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $X = \bigcup\{\mathrm{dom}(p) : p \in A\}$. As $|A| \leq \kappa^{++}$, we have $|X| \leq \kappa^{++}$, and of course A is a maximal antichain of $\mathrm{Add}(\kappa^{++}, X)_{M^*_{\bar{E}}}$. For simlpicity let us assume $|X| = \kappa^{++}$. Now we have $\mathrm{Add}(\kappa^{++}, X)_{M^*_{\bar{E}}} = \mathrm{Add}(\kappa^{++}, X)$ and hence A is a maximal antichain of $\mathrm{Add}(\kappa^{++}, X)$. It then follows that A is a maximal antichain of $\mathrm{Add}(\kappa^{++}, \kappa^{+5})$. Let $p \in G_2 \cap A$. Then $p \in I^{\mathrm{Add},2}_{\bar{E}} \cap A$. □

Claim 3.10. $I^{\mathrm{Add},3}_{\bar{E}} = G_3 \cap \mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$ *is* $\mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$*−generic over* $M^*_{\bar{E}}$.

Proof. Suppose A is a maximal antichain of $\mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $X = \bigcup\{\mathrm{dom}(p) : p \in A\}$. As $|A| \leq \kappa^{+3}$, we have $|X| \leq \kappa^{+3}$, and of course A is a maximal antichain of $\mathrm{Add}(\kappa^{+3}, X)_{M^*_{\bar{E}}}$. For simlpicity let us assume $|X| = \kappa^{+3}$. Now we have $\mathrm{Add}(\kappa^{+3}, X)_{M^*_{\bar{E}}} = \mathrm{Add}(\kappa^{+3}, X)$ and hence A is a maximal antichain of $\mathrm{Add}(\kappa^{+3}, X)$. It then follows that A is a maximal antichain of $\mathrm{Add}(\kappa^{+3}, \kappa^{+6})$. Let $p \in G_3 \cap A$. Then $p \in I^{\mathrm{Add},3}_{\bar{E}} \cap A$. □

It follows from the above claims that

Claim 3.11. $I^{\mathrm{Add},1}_{\bar{E}} \times I^{\mathrm{Add},2}_{\bar{E}} \times I^{\mathrm{Add},3}_{\bar{E}}$ *is* $\mathbb{R}^{\mathrm{Add},1}_{\bar{E}} \times \mathbb{R}^{\mathrm{Add},2}_{\bar{E}} \times \mathbb{R}^{\mathrm{Add},3}_{\bar{E}}$*−generic over* $M^*_{\bar{E}}$.

Claim 3.12. $I^{\mathrm{Add},1}_U = \langle i^{-1\prime\prime}_{U,\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle$ *is* $\mathbb{R}^{\mathrm{Add},1}_U$*−generic over* N^*.

Proof. Let A be a maximal antichain of $\mathbb{R}_U^{\mathrm{Add},1}$ in N^*. Then $i_{U,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},1}$ in $M_{\bar{E}}^*$. Since $|A| \leq \kappa^+$, and $crit(i_{U,\bar{E}}) = \kappa_{N^*}^{++} > \kappa^+$, we have $i_{U,\bar{E}}(A) = i''_{U,\bar{E}}(A)$. Then $I_{\bar{E}}^{\mathrm{Add},1} \cap i''_{U,\bar{E}}(A) \neq \emptyset$, which implies $I_U^{\mathrm{Add},1} \cap A \neq \emptyset$. $\square$

Now consider the forcing notion $\mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4} \times \mathbb{R}_U^{\mathrm{Col}}$. Working in $M_{\bar{E}}^*$, this forcing notion is κ^+-closed and there are only κ^+-many maximal antichains of it which are in N^*. Thus we can define a descending sequence $\langle\langle p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle}, p_{\langle\alpha,\mathrm{Add},4\rangle}, p_{\langle\alpha,\mathrm{Col}\rangle}\rangle : \alpha < \kappa^+\rangle$ of conditions such that $I_U^{\mathrm{Add},2} \times I_U^{\mathrm{Add},3} \times I_U^{\mathrm{Add},4} \times I_U^{\mathrm{Col}} = \{p \in \mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4} \times \mathbb{R}_U^{\mathrm{Col}} : \exists\alpha < \kappa^+, \langle p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle}, p_{\langle\alpha,\mathrm{Add},4\rangle}, p_{\langle\alpha,\mathrm{Col}\rangle}\rangle \leq p\}$ is $\mathbb{R}_U^{\mathrm{Add},2} \times \mathbb{R}_U^{\mathrm{Add},3} \times \mathbb{R}_U^{\mathrm{Add},4} \times \mathbb{R}_U^{\mathrm{Col}}$-generic over N^*. Also note that this generic filter is in $M_{\bar{E}}^*$.

We may also note that $\{i_{U,\bar{E}}(p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle}) : \alpha < \kappa^+\} \subseteq \mathbb{R}_{\bar{E}}^{\mathrm{Add},2} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},3}$, and since this forcing is $\kappa^{++}-$closed, there is $\langle p_{\langle\mathrm{Add},2\rangle}, p_{\langle\mathrm{Add},3\rangle}\rangle \in \mathbb{R}_{\bar{E}}^{\mathrm{Add},2} \times \mathbb{R}_{\bar{E}}^{\mathrm{Add},3}$ such that $\forall\alpha < \kappa^+, \langle p_{\langle\mathrm{Add},2\rangle}, p_{\langle\mathrm{Add},3\rangle}\rangle \leq i_{U,\bar{E}}(p_{\langle\alpha,\mathrm{Add},2\rangle}, p_{\langle\alpha,\mathrm{Add},3\rangle})$. We may suppose that $\langle p_{\langle\mathrm{Add},2\rangle}, p_{\langle\mathrm{Add},3\rangle}\rangle \in G_2 \times G_3$ (see Remark 3.4).

Let $I_U = I_U^{\mathrm{Add},1} \times I_U^{\mathrm{Add},2} \times I_U^{\mathrm{Add},3} \times I_U^{\mathrm{Add},4} \times I_U^{\mathrm{Col}}$. It follows from the above results that I_U is $\mathbb{R}_U-$generic over N^*.

Claim 3.13. (a) *$I_\tau^{\mathrm{Add},1} = \langle i_{\tau,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},1})\rangle$ is $\mathbb{R}_\tau^{\mathrm{Add},1}-$generic over M_τ^*,*

(b) *$I_\tau^{\mathrm{Add},2} = \langle i_{\tau,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},2})\rangle$ is $\mathbb{R}_\tau^{\mathrm{Add},2}-$generic over M_τ^*,*

(c) *$I_\tau^{\mathrm{Add},3} = \langle i_{\tau,\bar{E}}^{-1''}(I_{\bar{E}}^{\mathrm{Add},3})\rangle$ is $\mathbb{R}_\tau^{\mathrm{Add},3}-$generic over M_τ^*.*

Proof. (a) Let A be a maximal antichain of $\mathbb{R}_\tau^{\mathrm{Add},1}$ in M_τ^*. Then $i_{\tau,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},1}$ in $M_{\bar{E}}^*$. Since $|A| \leq \kappa^+$, and $crit(i_{\tau,\bar{E}}) = \kappa_{M_\tau^*}^{+4} > \kappa^+$, we have $i_{\tau,\bar{E}}(A) = i''_{\tau,\bar{E}}(A)$. Then $I_{\bar{E}}^{\mathrm{Add},1} \cap i''_{\tau,\bar{E}}(A) \neq \emptyset$, which implies $I_\tau^{\mathrm{Add},1} \cap A \neq \emptyset$.

(b) Let A be a maximal antichain of $\mathbb{R}_\tau^{\mathrm{Add},2}$ in M_τ^*. Then $i_{\tau,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},2}$ in $M_{\bar{E}}^*$. Since $|A| \leq \kappa^{++}$, and $crit(i_{\tau,\bar{E}}) = \kappa_{M_\tau^*}^{+4} > \kappa^{++}$, we have $i_{\tau,\bar{E}}(A) = i''_{\tau,\bar{E}}(A)$. Then $I_{\bar{E}}^{\mathrm{Add},2} \cap i''_{\tau,\bar{E}}(A) \neq \emptyset$, which implies $I_\tau^{\mathrm{Add},2} \cap A \neq \emptyset$.

(c) Let A be a maximal antichain of $\mathbb{R}_\tau^{\mathrm{Add},3}$ in M_τ^*. Then $i_{\tau,\bar{E}}(A)$ is a maximal antichain of $\mathbb{R}_{\bar{E}}^{\mathrm{Add},3}$ in $M_{\bar{E}}^*$. Since $|A| \leq \kappa^{+3}$, and $crit(i_{\tau,\bar{E}}) = \kappa_{M_\tau^*}^{+4} > \kappa^{+3}$, we have $i_{\tau,\bar{E}}(A) = i''_{\tau,\bar{E}}(A)$. Then $I_{\bar{E}}^{\mathrm{Add},3} \cap i''_{\tau,\bar{E}}(A) \neq \emptyset$, which implies $I_\tau^{\mathrm{Add},1} \cap A \neq \emptyset$. $\square$

As $U \in M^*_{\bar{E}}$ and $\forall \tau < \mathrm{l}(\bar{E})$ $E(\tau) \in M^*_{\bar{E}}$ we have the following diagram

$$\begin{tikzcd} M^*_{\bar{E}} \arrow[d, "i^{\bar{E}}_U"'] \arrow[dr, "j^{\bar{E}}_\tau"] & \\ N^{*\bar{E}} \arrow[r, "i^{\bar{E}}_{U,\tau}"'] & M^{*\bar{E}}_\tau \end{tikzcd}$$

$$U = E_\kappa(0),$$

$$i^{\bar{E}}_U : M^*_{\bar{E}} \to N^{*\bar{E}} \simeq \mathrm{Ult}(M^*_{\bar{E}}, U),$$

$$j^{\bar{E}}_\tau : M^*_{\bar{E}} \to M^{*\bar{E}}_\tau \simeq \mathrm{Ult}(M^*_{\bar{E}}, E(\tau)),$$

$$i^{\bar{E}}_{U,\tau}(i^{\bar{E}}_U(f)(\kappa)) = j^{\bar{E}}_\tau(f)(\kappa).$$

Recall that we have $I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U \in M^*_{\bar{E}}$ which is $\mathbb{R}^{\mathrm{Add},4}_U \times \mathbb{R}^{\mathrm{Col}}_U$-generic over $N^{*\bar{E}}$.

Claim 3.14. *There is $I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau \in M^*_{\bar{E}}$ which is $\mathbb{R}^{\mathrm{Add},4}_\tau \times \mathbb{R}^{\mathrm{Col}}_\tau-$generic over M^*_τ.*

Proof. We follow the idea from [3]. For this set

(1) $\mathbb{R}^{\bar{E},\mathrm{Col}}_\tau = \mathrm{Col}(\kappa^{+6}, j^{\bar{E}}_\tau(\kappa))_{M^{*\bar{E}}_\tau}$,

(2) $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau = (\mathrm{Add}(\kappa^{+4}, j^{\bar{E}}_\tau(\kappa)^{+}) \times \mathrm{Add}(\kappa^{+5}, j^{\bar{E}}_\tau(\kappa)^{++}) \times \mathrm{Add}(\kappa^{+6}, j^{\bar{E}}_\tau(\kappa)^{+3}))_{M^{*\bar{E}}_\tau}$,

$\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau \times \mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$ and $\mathbb{R}^{\mathrm{Add},4}_\tau \times \mathbb{R}^{\mathrm{Col}}_\tau$ are coded in $V^{M^*_\tau}_{j_\tau(\kappa)+3}$, $V^{M^{*\bar{E}}_\tau}_{j^{\bar{E}}_\tau(\kappa)+3}$ respectively. $V^{M^*_\tau}_{j_\tau(\kappa)+3}$, $V^{M^{*\bar{E}}_\tau}_{j^{\bar{E}}_\tau(\kappa)+3}$ are determined by $V^{V^*}_{\kappa+3}$, $V^{M^*_{\bar{E}}}_{\kappa+3}$ (and $E(\tau)$, of course). As $E(\tau) \in M^*_{\bar{E}}$ and $V^{V^*}_{\kappa+3} = V^{M^*_{\bar{E}}}_{\kappa+3}$ we get that $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau \times \mathbb{R}^{\bar{E},\mathrm{Col}}_\tau = \mathbb{R}^{\mathrm{Add},4}_\tau \times \mathbb{R}^{\mathrm{Col}}_\tau$.

By the same reasoning, each antichain of $\mathbb{R}^{\mathrm{Add},4}_\tau \times \mathbb{R}^{\mathrm{Col}}_\tau$ appearing in M^*_τ is also an antichain of $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau \times \mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$ appearing in $M^{*\bar{E}}_\tau$. Hence, if $I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau \in M^*_{\bar{E}}$ is $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau \times \mathbb{R}^{\bar{E},\mathrm{Col}}_\tau-$generic filter over $M^{*\bar{E}}_\tau$ then it is also $\mathbb{R}^{\mathrm{Add},4}_\tau \times \mathbb{R}^{\mathrm{Col}}_\tau-$generic over M^*_τ.

Let $I^{\mathrm{Add},4}_\tau \times I^{\mathrm{Col}}_\tau = \langle i^{\bar{E}\prime\prime}_{U,\tau}(I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U) \rangle$. We show that it is as required. So let $D \in M^{*\bar{E}}_\tau$ be dense open in $\mathbb{R}^{\bar{E},\mathrm{Add},4}_\tau \times \mathbb{R}^{\bar{E},\mathrm{Col}}_\tau$. Then $D = j^{\bar{E}}_\tau(f)(\bar{E}_\alpha \restriction \tau)$ for some function $f \in M^*_{\bar{E}}$ on V_κ. It then follows that in $M^*_{\bar{E}}$

$$D^* = \{\bar{\nu} \in V_\kappa : f(\bar{\nu}) \text{ is dense open in } \mathbb{R}^{\mathrm{Add},4}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}\} \in E_\alpha(\tau).$$

It is easily seen that

$$B = \{\mu : |\{\bar{\nu} \in D^* : \kappa^0(\bar{\nu}) = \mu\}| \leq \mu^{+3}\} \in E_\kappa(0).$$

Thus for each $\mu \in B$ we can find $f^*(\mu)$ such that for all $\bar{\nu} \in D^*$ with $\kappa^0(\bar{\nu}) = \mu$ we have $f^*(\mu) \subseteq f(\bar{\nu})$ is dense open (in $M^*_{\bar{E}}$). Hence

$$N^{*\bar{E}} \models \text{“} i^{\bar{E}}_U(f^*)(\kappa) \text{ is dense open ”}.$$

and

$$M^{*\bar{E}}_{\tau} \models \text{“}j^{\bar{E}}_{\tau}(f^*)(\kappa) \subseteq j^{\bar{E}}_{\tau}(f)(\bar{E}_\alpha \upharpoonright \tau)\text{”}.$$

So there is $g \in M^*_{\bar{E}}$ such that $i^{\bar{E}}_U(g)(\kappa) \in (I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U) \cap i^{\bar{E}}_U(f^*)(\kappa)$. It then follows that $j^{\bar{E}}_{\tau}(g)(\kappa) \in i^{\bar{E}''}_{U,\tau}(I^{\mathrm{Add},4}_U \times I^{\mathrm{Col}}_U) \cap j^{\bar{E}}_{\tau}(f^*)(\kappa)$. This means that $(I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}) \cap j^{\bar{E}}_{\tau}(f)(\bar{E}_\alpha \upharpoonright \tau) \neq \emptyset$. The result follows.

□

Now let $I_\tau = I^{\mathrm{Add},1}_{\tau} \times I^{\mathrm{Add},2}_{\tau} \times I^{\mathrm{Add},3}_{\tau} \times I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}$. It follows that I_τ is $\mathbb{R}_\tau-$generic over M^*_τ.

Claim 3.15. *$I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}} = \langle \bigcup_{\tau < \mathrm{l}(\bar{E})} i''_{\tau,\bar{E}}(I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}) \rangle$ is $\mathbb{R}^{\mathrm{Add},4}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}-$generic over $M^*_{\bar{E}}$.*

Proof. Let D be a dense open subset of $\mathbb{R}^{\mathrm{Add},4}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$ in $M^*_{\bar{E}}$. Let $\tau < l(\bar{E})$ and $D_\tau \in M^*_\tau$ be such that $D = i_{\tau,\bar{E}}(D_\tau)$. By elementarity D_τ is dense open in $\mathbb{R}^{\mathrm{Add},4}_{\tau} \times \mathbb{R}^{\mathrm{Col}}_{\tau}$. Let $\langle p, q\rangle \in (I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}) \cap D_\tau$. Then $\langle i_{\tau,\bar{E}}(p), i_{\tau,\bar{E}}(q)\rangle \in (I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}) \cap D$. □

Let $I_{\bar{E}} = I^{\mathrm{Add},1}_{\bar{E}} \times I^{\mathrm{Add},2}_{\bar{E}} \times I^{\mathrm{Add},3}_{\bar{E}} \times I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}$. It follows that $I_{\bar{E}}$ is $\mathbb{R}_{\bar{E}}-$generic over $M^*_{\bar{E}}$.

To summarize, so far we have shown the following

- I_U is $\mathbb{R}_U-$generic over N^*,
- I_τ is $\mathbb{R}_\tau-$generic over M^*_τ,
- $I_{\bar{E}}$ is $\mathbb{R}_{\bar{E}}-$generic over $M^*_{\bar{E}}$.

Before continuing we recall Easton's lemma.

Lemma 3.16. *(Easton's Lemma). Let λ be regular uncountable, and suppose that $\mathbb{P}$ satisfies the $\lambda-c.c.$ and $\mathbb{Q}$ is $\lambda-$closed. Then*

(*a*) $\Vdash_{\mathbb{P}\times\mathbb{Q}}$“*$\lambda$ is a regular uncountable cardinal*”,

(*b*) $\Vdash_{\mathbb{Q}}$“*$\mathbb{P}$ satisfies the $\lambda - c.c.$*”,

(*c*) $\Vdash_{\mathbb{P}}$“*$\mathbb{Q}$ is $\lambda-$distributive*”.

Claim 3.17. *I_U is $\mathbb{R}_U-$generic over $N^*[G_U]$.*

Proof. First note that in N^* the forcing notions $\mathbb{R}_U$ and $\mathbb{P}_U$ are $i_U(\kappa)^+-c.c$ and $i_U(\kappa)^+-$closed respectively. Now let A be a maximal antichain of $\mathbb{R}_U$ in $N^*[G]$. By Easton's Lemma

$|A| \leq i_U(\kappa)$, hence again by Easton's Lemma $A \in N^*$. It follows that $I_U \cap A \neq \emptyset$, as I_U is $\mathbb{R}_U-$generic over N^*. The result follows. $\square$

By similar arguments

Claim 3.18. *I_τ is $\mathbb{R}_\tau-$generic over $M^*_\tau[G_\tau]$.*

Claim 3.19. *$I_{\bar{E}}$ is $\mathbb{R}_{\bar{E}}-$generic over $M^*_{\bar{E}}[G_{\bar{E}}]$.*

It remains to prove part (d) of lemma 3.7. Before going into details let's recall a simple observation.

Claim 3.20. (a) $V^{N^{*\bar{E}}}_{i^{\bar{E}}_U(\kappa)+3} = V^{N^*}_{i_U(\kappa)+3}$,

(b) $i^{\bar{E}}_{U,\tau} \restriction V^{N^{*\bar{E}}}_{i^{\bar{E}}_U(\kappa)+3} = i_{U,\tau} \restriction V^{N^*}_{i_U(\kappa)+3}$,

(c) $i^{\bar{E}}_{\tau',\tau} \restriction V^{N^{*\bar{E}}}_{i^{\bar{E}}_U(\kappa)+3} = i_{\tau',\tau} \restriction V^{N^*}_{i_U(\kappa)+3}$.

Claim 3.21. $i^{''}_{U,\tau'}(I_U) \subseteq I_{\tau'}$.

Proof. We have

(1) $i^{''}_{U,\tau'}(I^{\text{Add},1}_U) \subseteq I^{\text{Add},1}_{\tau'}$: This is because $i^{''}_{U,\tau'}(I^{\text{Add},1}_U) = i^{''}_{U,\tau'}(\langle i^{-1''}_{U,\bar{E}}(I^{\text{Add},1}_{\bar{E}})\rangle) \subseteq \langle i^{-1''}_{\tau',\bar{E}}(I^{\text{Add},1}_{\bar{E}})\rangle = I^{\text{Add},1}_{\tau'}$,

(2) $i^{''}_{U,\tau'}(I^{\text{Add},2}_U) \subseteq I^{\text{Add},2}_{\tau'}$: It suffices to show that $\forall \alpha < \kappa^+, i_{U,\tau'}(p_{\langle \alpha, \text{Add},2\rangle}) \in I^{\text{Add},2}_{\tau'}$. But we have $p_{\langle \text{Add},2\rangle} \in I^{\text{Add},2}_{\bar{E}}$ and $\forall \alpha < \kappa^+, p_{\langle \text{Add},2\rangle} \leq i_{U,\bar{E}}(p_{\langle \alpha, \text{Add},2\rangle})$. It then follows that $\forall \alpha < \kappa^+, i_{U,\tau'}(p_{\langle \alpha, \text{Add},2\rangle}) = i^{-1}_{\tau',\bar{E}}(i_{U,\bar{E}}(p_{\langle \alpha, \text{Add},2\rangle})) \geq i^{-1}_{\tau',\bar{E}}(p_{\langle \text{Add},2\rangle})$. But now note that by our definition $i^{-1}_{\tau',\bar{E}}(p_{\langle \text{Add},2\rangle}) \in I^{\text{Add},2}_{\tau'}$. It then follows that $i_{U,\tau'}(p_{\langle \alpha, \text{Add},2\rangle}) \in I^{\text{Add},2}_{\tau'}$,

(3) $i^{''}_{U,\tau'}(I^{\text{Add},3}_U) \subseteq I^{\text{Add},3}_{\tau'}$: By the same argument as in (2) using the fact that $p_{\langle \text{Add},3\rangle} \in I^{\text{Add},3}_{\bar{E}}$,

(4) $i^{''}_{U,\tau'}(I^{\text{Add},4}_U \times I^{\text{Col}}_U) \subseteq I^{\text{Add},4}_{\tau'} \times I^{\text{Col}}_{\tau'}$: Trivial by the definition of $I^{\text{Add},4}_{\tau'} \times I^{\text{Col}}_{\tau'}$ and the previous Claim.

The result follows. $\square$

Claim 3.22. $i^{''}_{\tau',\tau}(I_{\tau'}) \subseteq I_\tau$.

Proof. We have

(1) $i''_{\tau',\tau}(I^{\mathrm{Add},1}_{\tau'}) \subseteq I^{\mathrm{Add},1}_{\tau}$: Because $i''_{\tau',\tau}(I^{\mathrm{Add},1}_{\tau'}) = i''_{\tau',\tau}(\langle\langle i^{-1''}_{\tau',\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle\rangle) \subseteq \langle i^{-1''}_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\bar{E}})\rangle = I^{\mathrm{Add},1}_{\tau}$,

(2) $i''_{\tau',\tau}(I^{\mathrm{Add},2}_{\tau'}) \subseteq I^{\mathrm{Add},2}_{\tau}$: By the same argument as in (1),

(3) $i''_{\tau',\tau}(I^{\mathrm{Add},3}_{\tau'}) \subseteq I^{\mathrm{Add},3}_{\tau}$: By the same argument as in (1),

(4) $i''_{\tau',\tau}(I^{\mathrm{Add},4}_{\tau'} \times I^{\mathrm{Col}}_{\tau'}) \subseteq I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}$: Because $i''_{\tau',\tau}(I^{\mathrm{Add},4}_{\tau'} \times I^{\mathrm{Col}}_{\tau'}) = i''_{\tau',\tau}(\langle\langle i^{\bar{E}''}_{U,\tau'}(I^{\mathrm{Add},4}_{U} \times I^{\mathrm{Col}}_{U})\rangle\rangle) \subseteq \langle i^{\bar{E}''}_{U,\tau}(I^{\mathrm{Add},4}_{U} \times I^{\mathrm{Col}}_{U})\rangle = I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}$.

The result follows.

□

Claim 3.23. $i''_{\tau,\bar{E}}(I_\tau) \subseteq I_{\bar{E}}$.

Proof. We have

(1) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\tau}) \subseteq I^{\mathrm{Add},1}_{\bar{E}}$: Trivial as $i''_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\tau}) = i''_{\tau,\bar{E}}(\langle\langle i^{-1''}_{\tau,\bar{E}}(I^{\mathrm{Add},1}_{\tau})\rangle\rangle) \subseteq I^{\mathrm{Add},1}_{\bar{E}}$,

(2) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},2}_{\tau}) \subseteq I^{\mathrm{Add},2}_{\bar{E}}$: As in (1),

(3) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},3}_{\tau}) \subseteq I^{\mathrm{Add},3}_{\bar{E}}$: As in (1),

(4) $i''_{\tau,\bar{E}}(I^{\mathrm{Add},4}_{\tau} \times I^{\mathrm{Col}}_{\tau}) \subseteq I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}$: Trivial by the definition of $I^{\mathrm{Add},4}_{\bar{E}} \times I^{\mathrm{Col}}_{\bar{E}}$.

The result follows.

□

This completes the proof of lemma 3.7. □

We iterate $j_{\bar{E}}$ and consider the following diagram

$$\begin{tikzcd}
V \arrow[rr, "j_{\bar{E}}=j^{0,1}_{\bar{E}}"] \arrow[d, "i_U"'] \arrow[dr, "j_{\tau_1}"] & & M_{\bar{E}} \arrow[rr, "j^{1,2}_{\bar{E}}"] \arrow[d, "i^2_U"'] \arrow[dr, "j^2_{\tau_2}"] & & M^2_{\bar{E}} \arrow[rr, "j^{2,3}_{\bar{E}}"] \arrow[d, "i^3_U"'] \arrow[dr, "j^3_{\tau_3}"] & & M^3_{\bar{E}} \arrow[r, dotted] & {} \\
N \arrow[r, "i_{U,\bar{E}}", "i_{U,\tau_1}"'] & M_{\tau_1} \arrow[ur, "i_{\tau_1,\bar{E}}"'] & N^2 \arrow[r, "i^2_{U,\bar{E}}", "i^2_{U,\tau_2}"'] & M^2_{\tau_2} \arrow[ur, "i^2_{\tau_2,\bar{E}}"'] & N^3 \arrow[r, "i^3_{U,\bar{E}}", "i^3_{U,\tau_3}"'] & M^3_{\tau_3} \arrow[ur, "i^3_{\tau_3,\bar{E}}"'] & &
\end{tikzcd}$$

where

$$j^0_{\bar{E}} = \mathrm{id},$$

$$j^n_{\bar{E}} = j^{0,n}_{\bar{E}},$$

$$j^{m,n}_{\bar{E}} = j^{n-1,n}_{\bar{E}} \circ \cdots \circ j^{m+1,m+2}_{\bar{E}} \circ j^{m,m+1}_{\bar{E}}.$$

Let $R(-,-) = R^{\mathrm{Add}}(-,-) \times R^{\mathrm{Col}}(-,-)$ be a function such that

$$i^2_U(R^{\mathrm{Add}})(\kappa, i_U(\kappa)) = \mathbb{R}^{\mathrm{Add}}_U,$$

$$i^2_U(R^{\mathrm{Col}})(\kappa, i_U(\kappa)) = \mathbb{R}^{\mathrm{Col}}_U,$$

where i^2_U is the iterate of i_U. Then we will have

$$i^2_U(R)(\kappa, i_U(\kappa)) = \mathbb{R}_U.$$

The following is trivial.

Lemma 3.24. *(a)* $j^2_{\bar{E}}(R^{\mathrm{Add}})(\kappa, j_{\bar{E}}(\kappa)) = \mathbb{R}^{\mathrm{Add}}_{\bar{E}}$,

(b) $j^2_{\bar{E}}(R^{\mathrm{Col}})(\kappa, j_{\bar{E}}(\kappa)) = \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$,

(c) $j^2_{\bar{E}}(R)(\kappa, j_{\bar{E}}(\kappa)) = \mathbb{R}_{\bar{E}}$.

Cardinal structure in $N^*[I_U]$. The following lemma gives us everything that we need about the model $N^*[I_U]$.

Lemma 3.25. *(a) In $N^*[I_U]$ there are no cardinals in $[\kappa^{+7}, i_U(\kappa)]$ and all other N^*-cardinals are preserved,*

(b) The power function differs from the power function of N^* at the following points: $2^{\kappa^+} = \kappa^{+4}, 2^{\kappa^{++}} = \kappa^{+5}, 2^{\kappa^{+3}} = \kappa^{+6}, 2^{\kappa^{+4}} = i_U(\kappa)^+, 2^{\kappa^{+5}} = i_U(\kappa)^{++}, 2^{\kappa^{+6}} = i_U(\kappa)^{+3}$.

Cardinal structure in $M^*_\tau[I_\tau]$ and $M^*_{\bar{E}}[I_{\bar{E}}]$. The following lifting says everything which we can possibly say.

$$\begin{tikzcd}
 & & M^*_{\bar{E}}[I_{\bar{E}}] \\
N^*[I_U] \arrow[r, "i^*_{U,\tau'}"'] & M^*_{\tau'}[I_{\tau'}] \arrow[ur, "i^*_{\tau',\bar{E}}"] \arrow[r, "i^*_{\tau',\tau}"'] & M^*_\tau[I_\tau] \arrow[u, "i^*_{\tau,\bar{E}}"']
\end{tikzcd}$$

The forcing notion $\mathbb{P}_{\bar{E}}$,due to Merimovich, which we define later, adds a club to κ. For each ν_1, ν_2 successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{+3}]$ of the generic extension is the same as the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$ of $M^*_{\bar{E}}[I_{\bar{E}}]$.

Cardinal structure in $N^*[I^{\mathrm{Col}}_U]$. The following lemma gives us anything that we need about the model $N^*[I^{\mathrm{Col}}_U]$.

Lemma 3.26. *(a) In $N^*[I^{\mathrm{Col}}_U]$ there are no cardinals in $[\kappa^{+7}, i_U(\kappa)]$ and all other N^*-cardinals are preserved,* *(b) GCH holds in* $N^*[I_U^{\mathrm{Col}}]$.

Cardinal structure in $M^*_\tau[I_\tau^{\mathrm{Col}}]$ **and** $M^*_{\bar{E}}[I_{\bar{E}}^{\mathrm{Col}}]$**.** The following lifting says everything which we can possibly say.

$$\begin{tikzcd}
 & & M^*_{\bar{E}}[I^{Col}_{\bar{E}}] \\
N^*[I^{Col}_U] \arrow[r, "i^*_{U,\tau'}"'] & M^*_{\tau'}[I^{Col}_{\tau'}] \arrow[r, "i^*_{\tau',\tau}"'] \arrow[ur, "i^*_{\tau',\bar{E}}"] & M^*_\tau[I^{Col}_\tau] \arrow[u, "i^*_{\tau,\bar{E}}"']
\end{tikzcd}$$

The forcing notion $\mathbb{R}_{\bar{E}_\kappa}$ which we define later, adds a club to κ. For each ν_1, ν_2 successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{+3}]$ of the generic extension is the same as the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{+3}]$ of $M^*_{\bar{E}}[I^{Col}_{\bar{E}}]$.

4. Redefining extender Sequences

We define a new extender sequence system $\bar{F} = \langle \bar{F}_\alpha : \alpha \in \mathrm{dom}(\bar{F}) \rangle$ by:

- $\mathrm{dom}(\bar{F}) = \mathrm{dom}(\bar{E})$,
- $\mathrm{l}(\bar{F}) = \mathrm{l}(\bar{E})$
- $\leq_{\bar{F}} = \leq_{\bar{E}}$,
- $F(0) = E(0)$,
- $I(\tau) = I_\tau$,
- $\forall 0 < \tau < \mathrm{l}(\bar{F}), F(\tau) = \langle \langle F_\alpha(\tau) : \alpha \in \mathrm{dom}(\bar{F}) \rangle, \langle \pi_{\beta,\alpha} : \beta, \alpha \in \mathrm{dom}(\bar{F}), \beta \geq_{\bar{F}} \alpha \rangle \rangle$ is such that

  $$X \in F_\alpha(\tau) \Leftrightarrow \langle \alpha, F(0), I(0), ..., F(\tau^{'}), I(\tau^{'}), ... : \tau^{'} < \tau \rangle \in j_{\bar{E}}(X),$$

  and

  $$\pi_{\beta,\alpha}(\langle \xi, d \rangle) = \langle \pi_{\beta,\alpha}(\xi), d \rangle,$$

- $\forall \alpha \in \mathrm{dom}(\bar{F}), \bar{F}_\alpha = \langle \alpha, F(\tau), I(\tau) : \tau < \mathrm{l}(\bar{F}) \rangle$.

Also let $I(\bar{F})$ be the filter generated by $\bigcup_{\tau < \mathrm{l}(\bar{F})} i^{''}_{\tau,\bar{E}} I(\tau)$. Then $I(\bar{F})$ is $\mathbb{R}_{\bar{E}}-$generic over $M_{\bar{E}}$. Let us write $I(\bar{F}) = I^{\mathrm{Add}}(\bar{F}) \times I^{\mathrm{Col}}(\bar{F})$ corresponding to $\mathbb{R}_{\bar{E}} = \mathbb{R}^{\mathrm{Add}}_{\bar{E}} \times \mathbb{R}^{\mathrm{Col}}_{\bar{E}}$.

From now on we work with this new definition of extender sequence system and use $\bar{E}$ to denote it.

Definition 4.1. (1) $T \in \bar{E}_\alpha \Leftrightarrow \forall \xi < \mathrm{l}(\bar{E}_\alpha), T \in E_\alpha(\xi)$,

(2) $T \backslash \bar{\nu} = T \backslash V^*_{\kappa^0(\bar{\nu})}$,

(3) $T \restriction \bar{\nu} = T \cap V^*_{\kappa^0(\bar{\nu})}$.

We now define two forcing notions $\mathbb{P}_{\bar{E}}$ and $\mathbb{R}_{\bar{E}_\kappa}$.

5. Definition of the forcing notion $\mathbb{P}_{\bar{E}}$

This forcing notion is essentially the forcing notion of [4]. We give it in detail for completeness and later use. First we define a forcing notion $\mathbb{P}^*_{\bar{E}}$.

Definition 5.1. *A condition p in $\mathbb{P}^*_{\bar{E}}$ is of the form*

$$p = \{\langle \bar{\gamma}, p^{\bar{\gamma}} \rangle : \bar{\gamma} \in s\} \cup \{\langle \bar{E}_\alpha, T, f, F \rangle\}$$

where

(1) $s \in [\bar{E}]^{\leq \kappa}, \min \bar{E} = \bar{E}_\kappa \in s$,

(2) $p^{\bar{E}_\kappa} \in V^*_{\kappa^0(\bar{E})}$ *is an extender sequence such that* $\kappa(p^{\bar{E}_\kappa})$ *is inaccessible (we allow* $p^{\bar{E}_\kappa} = \emptyset$*). Write* p^0 *for* $p^{\bar{E}_\kappa}$.

(3) $\forall \bar{\gamma} \in s \backslash \{\min(s)\}, p^{\bar{\gamma}} \in [V^*_{\kappa^0(\bar{E})}]^{<\omega}$ *is a* ^{0}*-increasing sequence of extender sequences and* $\max \kappa(p^{\bar{\gamma}})$ *is inaccessible,*

(4) $\forall \bar{\gamma} \in s, \kappa(p^0) \leq \max \kappa(p^{\bar{\gamma}})$,

(5) $\forall \bar{\gamma} \in s, \bar{E}_\alpha \geq \bar{\gamma}$,

(6) $T \in \bar{E}_\alpha$,

(7) $\forall \bar{\nu} \in T, | \{\bar{\gamma} \in s : \bar{\nu}$ *is permitted to* $p^{\bar{\gamma}}\} | \leq \kappa^0(\bar{\nu})$,

(8) $\forall \bar{\beta}, \bar{\gamma} \in s, \forall \bar{\nu} \in T$, *if* $\bar{\beta} \neq \bar{\gamma}$ *and* $\bar{\nu}$ *is permitted to* $p^{\bar{\beta}}, p^{\bar{\gamma}}$, *then* $\pi_{\bar{E}_\alpha, \bar{\beta}}(\bar{\nu}) \neq \pi_{\bar{E}_\alpha, \bar{\gamma}}(\bar{\nu})$,

(9) f *is a function such that*

  (9.1) $\mathrm{dom}(f) = \{\bar{\nu} \in T : \mathrm{l}(\bar{\nu}) = 0\}$,

  (9.2) $f(\nu_1) \in R(\kappa(p^0), \nu_1^0)$. *If* $p^0 = \emptyset$, *then* $f(\nu_1) = \emptyset$,

(10) F *is a function such that*

  (10.1) $\mathrm{dom}(F) = \{\langle \bar{\nu_1}, \bar{\nu_2} \rangle \in T^2 : \mathrm{l}(\bar{\nu_1}) = \mathrm{l}(\bar{\nu_2}) = \emptyset\}$,

  (10.2) $F(\nu_1, \nu_2) \in R(\nu_1^0, \nu_2^0)$,

  (10.3) $j^2_{\bar{E}}(\alpha, j_{\bar{E}}(\alpha)) \in I(\bar{E})$.

We write $\mathrm{mc}(p), \mathrm{supp}(p), T^p, f^p$ and F^p for $\bar{E}_\alpha, s, T, f$ and F respectively.

Definition 5.2. *For $p, q \in \mathbb{P}^*_{\bar{E}}$, we say p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) iff*

(1) $\operatorname{supp}(p) \supseteq \operatorname{supp}(q)$,

(2) $\forall \bar{\gamma} \in \operatorname{supp}(q), p^{\bar{\gamma}} = q^{\bar{\gamma}}$,

(3) $\operatorname{mc}(p) \geq_{\bar{E}} \operatorname{mc}(q)$,

(4) $\operatorname{mc}(p) >_{\bar{E}} \operatorname{mc}(q) \Rightarrow \operatorname{mc}(q) \in \operatorname{supp}(p)$,

(5) $\forall \bar{\gamma} \in \operatorname{supp}(p) \setminus \operatorname{supp}(q), \max \kappa^0(p^{\bar{\gamma}}) > \bigcup \bigcup j_{\bar{E}}(f^q)(\kappa(\operatorname{mc}(q)))$,

(6) $T^p \leq \pi^{-1''}_{\operatorname{mc}(p), \operatorname{mc}(q)} T^q$,

(7) $\forall \bar{\gamma} \in \operatorname{supp}(q), \forall \bar{\nu} \in T^p$, *if $\bar{\nu}$ is permitted to $p^{\bar{\gamma}}$, then*

$$\pi_{\operatorname{mc}(p), \bar{\gamma}}(\bar{\nu}) = \pi_{\operatorname{mc}(q), \bar{\gamma}}(\pi_{\operatorname{mc}(p), \operatorname{mc}(q)}(\bar{\nu})),$$

(8) $\forall \nu_1 \in \operatorname{dom}(f^p), f^p(\nu_1) \leq f^q o \pi_{\operatorname{mc}(p), \operatorname{mc}(q)}(\nu_1)$,

(9) $\forall \langle \nu_1, \nu_2 \rangle \in \operatorname{dom}(F^p), F^p(\nu_1, \nu_2) \leq F^q o \pi_{\operatorname{mc}(p), \operatorname{mc}(q)}(\nu_1, \nu_2)$.

We are now ready to define the forcing notion $\mathbb{P}_{\bar{E}}$.

Definition 5.3. *A condition p in $\mathbb{P}_{\bar{E}}$ is of the form*

$$p = p_n^\frown ... ^\frown p_0$$

where

- $p_0 \in \mathbb{P}^*_{\bar{E}}, \kappa^0(p^0_0) \geq \kappa^0(\bar{\mu}_1)$,
- $p_1 \in \mathbb{P}^*_{\bar{\mu}_1}, \kappa^0(p^0_1) \geq \kappa^0(\bar{\mu}_2)$,

  $\vdots$
- $p_n \in \mathbb{P}^*_{\bar{\mu}_n}$.

and $\langle \bar{\mu}_n, ..., \bar{\mu}_1, \bar{E} \rangle$ is a $^0-$inceasing sequence of extender sequence systems, that is $\kappa^0(\bar{\mu}_n) < ... < \kappa^0(\bar{\mu}_1) < \kappa^0(\bar{E})$.

Definition 5.4. *For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) iff*

$$p = p_n^\frown ... ^\frown p_0$$

$$q = q_n^\frown ... ^\frown q_0$$

where

- $p_0, q_0 \in \mathbb{P}^*_{\bar{E}}, p_0 \leq^* q_0$,
- $p_1, q_1 \in \mathbb{P}^*_{\bar{\mu}_1}, p_1 \leq^* q_1$,

  $\vdots$

- $p_n, q_n \in \mathbb{P}^*_{\bar{\mu}_n}, p_n \leq^* q_n$.

Now let $p \in \mathbb{P}_{\bar{E}}$ and $\bar{\nu} \in T^p$. We define $p_{\langle \bar{\nu} \rangle}$ a one element extension of p by $\bar{\nu}$.

Definition 5.5. *Let $p \in \mathbb{P}_{\bar{E}}, \bar{\nu} \in T^p$ and $\kappa^0(\bar{\nu}) > \bigcup\bigcup j_{\bar{E}}(f^{p,\mathrm{Col}})(\kappa(\mathrm{mc}(p)))$, where $f^{p,\mathrm{Col}}$ is the collapsing part of f^p. Then $p_{\langle \bar{\nu} \rangle} = p_1 ^\frown p_0$ where*

(1) $\mathrm{supp}(p_0) = \mathrm{supp}(p)$,

(2) $\forall \bar{\gamma} \in \mathrm{supp}(p_0)$, $p_0^{\bar{\gamma}} = \begin{cases} \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) & \text{if } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}} \text{ and } \mathrm{l}(\bar{\nu}) > 0, \\ \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) & \text{if } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}}, \mathrm{l}(\bar{\nu}) = 0 \text{ and } \bar{\gamma} = \bar{E}_\kappa, \\ p^{\bar{\gamma}} {}^\frown \langle \pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) \rangle & \text{if } \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}}, \mathrm{l}(\bar{\nu}) = 0 \text{ and } \bar{\gamma} \neq \bar{E}_\kappa, \\ p^{\bar{\gamma}} & \text{otherwise}. \end{cases}$

(3) $\mathrm{mc}(p_0) = \mathrm{mc}(p)$,

(4) $T^{p_0} = T^p \backslash \bar{\nu}$,

(5) $\forall \nu_1 \in T^{p_0}, f^{p_0}(\nu_1) = F^p(\kappa(\bar{\nu}), \nu_1)$,

(6) $F^{p_0} = F^p$,

(7) *if* $\mathrm{l}(\bar{\nu}) > 0$ *then*

 (7.1) $\mathrm{supp}(p_1) = \{\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu}) : \bar{\gamma} \in \mathrm{supp}(p)$ *and* $\bar{\nu}$ *is permitted to* $p^{\bar{\gamma}}\}$,

 (7.2) $p_1^{\pi_{\mathrm{mc}(p),\bar{\gamma}}(\bar{\nu})} = p^{\bar{\gamma}}$,

 (7.3) $\mathrm{mc}(p_1) = \bar{\nu}$,

 (7.4) $T^{p_1} = T^p \restriction \bar{\nu}$,

 (7.5) $f^{p_1} = f^p \restriction \bar{\nu}$,

 (7.6) $F^{p_1} = F^p \restriction \bar{\nu}$,

(8) *if* $\mathrm{l}(\bar{\nu}) = 0$ *then*

 (8.1) $\mathrm{supp}\, p_1 = \{\pi_{\mathrm{mc}(p),0}(\bar{\nu})\}$,

 (8.2) $p_1^{\pi_{\mathrm{mc}(p),0}(\bar{\nu})} = p^{\bar{E}_\kappa}$,

 (8.3) $\mathrm{mc}(p_1) = \bar{\nu}^0$,

 (8.4) $T^{p_1} = \emptyset$,

 (8.5) $f^{p_1} = f^p(\kappa(\bar{\nu}))$,

 (8.6) $F^{p_1} = \emptyset$.

We use $(p_{\langle \bar{\nu} \rangle})_0$ and $(p_{\langle \bar{\nu} \rangle})_1$ for p_0 and p_1 respectively. We also let $p_{\langle \bar{\nu_1}, \bar{\nu_2} \rangle} = (p_{\langle \bar{\nu_1} \rangle})_1^\frown (p_{\langle \bar{\nu_1} \rangle})_{0 \langle \bar{\nu_2} \rangle}$ and so on.

The above definition is the key step in the definition of the forcing relation $\leq$.

Definition 5.6. *For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is a $1-$point extension of q ($p \leq^1 q$) iff*

$$p = p_{n+1}^\frown ...^\frown p_0$$

$$q = q_n^\frown ...^\frown q_0$$

and there is $0 \leq k \leq n$ such that

- $\forall i < k, p_i, q_i \in \mathbb{P}^*_{\bar{\mu}_i}, p_i \leq^* q_i,$
- $\exists \bar{\nu} \in T^{q_k}, (p_{k+1})^\frown p_k \leq^* (q_k)_{\langle \bar{\nu} \rangle}$
- $\forall i > k, p_{i+1}, q_i \in \mathbb{P}^*_{\bar{\mu}_i}, p_{i+1} \leq^* q_i,$

where $\bar{\mu}_0 = \bar{E}$.

Definition 5.7. *For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is an $n-$point extension of q ($p \leq^n q$) iff there are $p^n, ..., p^0$ such that*

$$p = p^n \leq^1 ... \leq^1 p^0 = q.$$

Definition 5.8. *For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is an extension of q ($p \leq q$) iff there is some n such that $p \leq^n q$.*

Suppose that H is $\mathbb{P}_{\bar{E}}-$generic over V. For $\alpha \in \mathrm{dom}(\bar{E})$ set

$$C^\alpha_H = \{\max \kappa(p_0^{\bar{E}_\alpha}) : p \in H\}.$$

Theorem 5.9. (a) *$V[H]$ and V have the same cardinals $\geq \kappa$,*

(b) *κ remains strongly inaccessible in $V[H]$*

(c) *C^α_H is unbounded in κ,*

(d) *C^κ_H is a club in κ,*

(e) *$\alpha \neq \beta \Rightarrow C^\alpha_H \neq C^\beta_H$,*

(f) *Let $\lambda = \min(C^\kappa_H)$, and let K be $\mathrm{Col}(\omega, \lambda^+)_{V[H]}-$generic over $V[H]$. Then*

$$CARD^{V[H][K]} \cap \kappa = (\lim(C^\kappa_H) \cup \{\mu^+, ..., \mu^{+6} : \mu \in C^\kappa_H\} \backslash \lambda^{++}) \cup \{\omega\},$$

(g) *$V[H][K] \models$ "$\forall \lambda \leq \kappa, 2^\lambda = \lambda^{+3}$".*

Proof. Essentially the same as in [4]. $\square$

6. Definition of the forcing notion $\mathbb{R}_{\bar{E}_\kappa}$

We now define another forcing notion $\mathbb{R}_{\bar{E}_\kappa}$. It is essentially the Radin forcing corresponding to $\bar{E}_\kappa$ with interleaving collapses (see also [3]).

Definition 6.1. *A condition in* $\mathbb{R}_{\bar{E}_\kappa}$ *is of the form*

$$p = \langle\langle \bar{\mu}_n, s^n, S^n, f^n, F^n\rangle, ..., \langle \bar{\mu}_0, s^0, S^0, f^0, F^0\rangle\rangle$$

where

(1) $\bar{\mu}_n, ..., \bar{\mu}_0$ *are minimal extender sequences,*

(2) $\bar{\mu}_0 = \bar{E}_\kappa$,

(3) $\forall i \leq n-1, \kappa(\bar{\mu}_{i+1}) < \kappa^0(\bar{\mu}_i)$,

(4) $\forall i \leq n, S^i \in \bar{\mu}_i$,

(5) $\forall i \leq n, s^i \in V_{\kappa^0(\bar{\mu}_i)}$ *is an extender sequence such that* $\kappa(s^i)$ *is inaccessible,*

(6) $\forall i \leq n, f^i$ *is a function such that*

(6.1) $\mathrm{dom}(f^i) = \{\bar{\nu} \in S^i : \mathrm{l}(\bar{\nu}) = 0\}$,

(6.2) $f^i(\nu_1) \in R^{\mathrm{Col}}(\kappa(s^i), \nu_1^0)$,

(7) $\forall i \leq n, F^i$ *is a function such that*

(7.1) $\mathrm{dom}(F^i) = \{\langle \bar{\nu_1}, \bar{\nu_2}\rangle \in (S^i)^2 : \mathrm{l}(\bar{\nu}_1) = \mathrm{l}(\bar{\nu}_2) = 0\}$,

(7.2) $F^i(\langle \nu_1, \nu_2\rangle) \in R^{\mathrm{Col}}(\nu_1^0, \nu_2^0)$,

(7.3) $j^2_{\bar{E}}(F^i)(\kappa(\bar{\mu}_i), j_{\bar{E}}(\kappa(\bar{\mu}_i)) \in I^{\mathrm{Col}}(\bar{E})$.

Definition 6.2. *For* $p, q \in \mathbb{R}_{\bar{E}_\kappa}$ *we say* p *is a Prikry extension of* q *(*$p \leq^* q$ *or* $p \leq^1 q$*) iff* p *and* q *are of the form*

$$p = \langle\langle \bar{\mu}_n, s^n, S^n, f^n, F^n\rangle, ..., \langle \bar{\mu}_0, s^0, S^0, f^0, F^0\rangle\rangle$$

$$q = \langle\langle \bar{\mu}_n, t^n, T^n, g^n, G^n\rangle, ..., \langle \bar{\mu}_0, t^0, T^0, g^0, G^0\rangle\rangle$$

where $\forall i \leq n$

(1) $s^i = t^i$,

(2) $S^i \subseteq T^i$,

(3) $f^i \leq g^i$,

(4) $F^i \leq G^i$.

Definition 6.3. *Let* $p = \langle\langle\bar{\mu}_n, s^n, S^n, f^n, F^n\rangle, ..., \langle\bar{\mu}_0, s^0, S^0, f^0, F^0\rangle\rangle \in \mathbb{R}_{\bar{E}_\kappa}$, *and let* $\langle\bar{\nu}\rangle \in S^i, \kappa^0(\bar{\nu}) > \bigcup\bigcup j_{\bar{E}}(f^i)(\kappa(\bar{\mu}_i))$. *We define* $p_{\langle\bar{\nu}\rangle}$ *as follows*

- *if* $\mathrm{l}(\bar{\nu}) > 0$, *then*

$$\begin{aligned} p_{\langle\bar{\nu}\rangle} = \langle\langle\bar{\mu}_n, s^n, S^n, f^n, F^n\rangle, ..., \\ \langle\bar{\mu}_{i+1}, s^{i+1}, S^{i+1}, f^{i+1}, F^{i+1}\rangle, \\ \langle\bar{\nu}, s^i, S^i \restriction \bar{\nu}, f^i \restriction \bar{\nu}, F^i \restriction \bar{\nu}\rangle, \\ \langle\bar{\mu}_i, \bar{\nu}, S^i\backslash\bar{\nu}, F^i(\kappa(\bar{\nu}, -)), F^i\rangle, \\ \langle\bar{\mu}_{i-1}, s^{i-1}, S^{i-1}, f^{i-1}, F^{i-1}\rangle, ..., \\ \langle\bar{\mu}_0, s^0, S^0, f^0, F^0\rangle\rangle \end{aligned}$$

- *if* $\mathrm{l}(\bar{\nu}) = 0$, *then*

$$\begin{aligned} p_{\langle\bar{\nu}\rangle} = \langle\langle\bar{\mu}_n, s^n, S^n, f^n, F^n\rangle, ..., \\ \langle\bar{\mu}_{i+1}, s^{i+1}, S^{i+1}, f^{i+1}, F^{i+1}\rangle, \\ \langle\bar{\nu}^0, s^i, \emptyset, f^i(\kappa(\bar{\nu})), \emptyset\rangle, \\ \langle\bar{\mu}_i, \bar{\nu}, S^i\backslash\bar{\nu}, F^i(\kappa(\bar{\nu}, -)), F^i\rangle, \\ \langle\bar{\mu}_{i-1}, s^{i-1}, S^{i-1}, f^{i-1}, F^{i-1}\rangle, ..., \\ \langle\bar{\mu}_0, s^0, S^0, f^0, F^0\rangle\rangle \end{aligned}$$

Definition 6.4. *Let* $p, q \in \mathbb{R}_{\bar{E}_\kappa}$, *where* $q = \langle\langle\bar{\mu}_n, s^n, S^n, f^n, F^n\rangle, ..., \langle\bar{\mu}_0, s^0, S^0, f^0, F^0\rangle\rangle$. *We say* p *is a* $1-$*point extension of* q $(p \leq^1 q)$ *iff there are* i *and* $\langle\bar{\nu}\rangle \in S^i$ *such that* $p \leq^* q_{\langle\bar{\nu}\rangle}$.

Definition 6.5. *Let* $p, q \in \mathbb{R}_{\bar{E}_\kappa}$. *We say* p *is an* $n-$*point extension of* q $(p \leq^n q)$ *iff there are* $p^n, ..., p^0$ *such that*

$$p = p^n \leq^1 ... \leq^1 p^0 = q.$$

Definition 6.6. *Let* $p, q \in \mathbb{R}_{\bar{E}_\kappa}$. *We say* p *is an extension of* q $(p \leq q)$ *iff there is* n *such that* $p \leq^n q$.

Suppose G is $\mathbb{R}_{\bar{E}_\kappa}-$generic over V. Set

$$C = \{\kappa(s^0) : s^0 \text{ appears in in some } p \in G\}.$$

Theorem 6.7. (*a*) $V[G]$ *and* V *have the same cardinals* $\geq \kappa$,

(*b*) κ *remains strongly inaccessible in* $V[G]$, (*c*) C *is a club in* κ,

(*d*) *Let* $\lambda = \min(C)$ *and let* K *be* $\mathrm{Col}(\omega, \lambda^+)_{V[G]}-$*generic over* $V[G]$. *Then*

$$\mathrm{C}ARD^{V[G][K]} \cap \kappa = (\lim(C) \cup \{\mu^+, ..., \mu^{+6} : \mu \in C\} \backslash \lambda^{++}) \cup \{\omega\},$$

(*e*) $V[G][K] \models$ *"GCH".*

Proof. Essentially the same as in [3] and [4]. □

7. Projection of $\mathbb{P}_{\bar{E}}$ into $\mathbb{R}_{\bar{E}_\kappa}$

We now define a projection

$$\pi : \mathbb{P}_{\bar{E}} \to \mathbb{R}_{\bar{E}_\kappa}$$

as follows

Suppose $p = p_n^\frown ...^\frown p_0$ where

- $p_0 \in \mathbb{P}^*_{\bar{E}}, \kappa^0(p^0_0) \geq \kappa^0(\bar{\mu}_1)$,
- $p_1 \in \mathbb{P}^*_{\bar{\mu}_1}, \kappa^0(p^0_1) \geq \kappa^0(\bar{\mu}_2)$,

  $\vdots$
- $p_n \in \mathbb{P}^*_{\bar{\mu}_n}$.

and $\langle \bar{\mu}_n, ..., \bar{\mu}_1, \bar{E} \rangle$ is a ${}^0-$inceasing sequence of extender sequence systems. For each $i \leq n$ set $f^{p_i} = f^{p_i,\mathrm{Add}} \times f^{p_i,\mathrm{Col}}$ and $F^{p_i} = F^{p_i,\mathrm{Add}} \times F^{p_i,\mathrm{Col}}$ which correspond to $R = R^{\mathrm{Add}} \times R^{\mathrm{Col}}$. Let

$$\pi(p) = \langle \langle \min \bar{\mu}_n, p^0_n, \pi_{\mathrm{mc}(p_n),0} T^{p_n}, f^{p_n,\mathrm{Col}} \circ \pi^{-1}_{\mathrm{mc}(p_n),0}, F^{p_n,\mathrm{Col}} \circ \pi^{-1}_{\mathrm{mc}(p_n),0} \rangle, ...,$$
$$\langle \bar{E}_\kappa, p^0_0, \pi_{\mathrm{mc}(p_0),0} T^{p_0}, f^{p_0,\mathrm{Col}} \circ \pi^{-1}_{\mathrm{mc}(p_0),0}, F^{p_0,\mathrm{Col}} \circ \pi^{-1}_{\mathrm{mc}(p_0),0} \rangle \rangle.$$

Let us note that $\pi(p) \in \mathbb{R}_{\bar{E}_\kappa}$ and π is well-defined.

Lemma 7.1. *π is a projection, i.e*

(*a*) $\pi(1_{\mathbb{P}_{\bar{E}}}) = 1_{\mathbb{R}_{\bar{E}_\kappa}}$,

(*b*) π *is order preserving,*

(*c*) *if* $p \in \mathbb{P}_{\bar{E}}, q \in \mathbb{R}_{\bar{E}_\kappa}$ *and* $q \leq \pi(p)$ *then there is* $r \leq p$ *in* $\mathbb{P}_{\bar{E}}$ *such that* $\pi(r) \leq q$.

8. Completing the proof

Finally in this section we complete the proof of Theorem 1.1. Let H be $\mathbb{P}_{\bar{E}}-$generic over V and let $G = \langle \pi'' H \rangle$, the filter generated by $\pi'' H$. Then G is $\mathbb{R}_{\bar{E}_\kappa}-$generic over V.

Consider the clubs $C = \{\kappa(s^0) : s^0$ appears in in some $p \in G\}$ and $C^{\kappa}_H = \{\kappa(p^0_0) : p \in H\}$. It is easily seen that $C = C^{\kappa}_H$. Let $\lambda = \min(C)$. Note that the forcing notions $\mathbb{P}_{\bar{E}}$ and $\mathbb{R}_{\bar{E}_\kappa}$ add no new bounded subsets to λ^+, hence $\mathrm{Col}(\omega, \lambda^+)_{V[G]} = \mathrm{Col}(\omega, \lambda^+)_{V[H]}$, and hence if K is $\mathrm{Col}(\omega, \lambda^+)_{V[H]}-$generic over $V[H]$ then K is $\mathrm{Col}(\omega, \lambda^+)_{V[G]}-$generic over $V[G]$. Let

$$V_1 = V_\kappa^{V[G][K]}$$

$$V_2 = V_\kappa^{V[H][K]}$$

It follows that V_1 and V_2 are models of ZFC. We show that the pair (V_1, V_2) satisfies the requirements of the theorem.

(a) V_1 and V_2 have the same cardinals: This is trivial, since

$$\begin{aligned} CARD^{V_1} &= (\lim(C) \cup \{\mu^+, ..., \mu^{+6} : \mu \in C\} \backslash \lambda^{++}) \cup \{\omega\} \\ &= (\lim(C^{\kappa}_H) \cup \{\mu^+, ..., \mu^{+6} : \mu \in C^{\kappa}_H\} \backslash \lambda^{++}) \cup \{\omega\}. \\ &= CARD^{V_2}. \end{aligned}$$

(b) V_1 and V_2 have the same cofinalities: This is again trivial, since changing the cofinalities depends on the length of the extender sequence system used and not on its size.

(c) $V_1 \models$ "GCH": by Theorem 6.7(e).

(d) $V_2 \models$ "$\forall \lambda, 2^\lambda = \lambda^{+3}$": by Theorem 5.9$(g)$.

Theorem 1.1 follows.

Open question. Is it possible to kill GCH everywhere, preserving cofinalities, adding just a single real? (Allowing cofinalities to change, this was accomplished in [1].)

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