The non-absoluteness of model existence in uncountable cardinals for $L_{\omega_1,\omega}$

Sy-David Friedman (KGRC, Vienna)*

Tapani Hyttinen (University of Helsinki) †

Martin Koerwien (CRM, Barcelona) ‡

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Abstract

For sentences ϕ of $L_{\omega_1,\omega}$, we investigate the question of absoluteness of ϕ having models in uncountable cardinalities. We first observe that having a model in $\aleph_1$ is an absolute property, but having a model in $\aleph_2$ is not as it may depend on the validity of the Continuum Hypothesis. We then consider the GCH context and provide sentences for any $\alpha \in \omega_1 \setminus \{0, 1, \omega\}$ for which the existence of a model in $\aleph_\alpha$ is non-absolute (relative to large cardinal hypotheses). Finally, we present a *complete* sentence for which model existence in $\aleph_3$ is non-absolute.

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Throughout, we assume ϕ is an $L_{\omega_1,\omega}$ sentence which has infinite models. By the downward Löwenheim-Skolem-Theorem, ϕ must have a countable model, so the property "having a countable model" is an absolute property of such sentences in the sense that its validity does not depend on the properties of the set-theoretic universe we work in. More precisely, if $V \subseteq W$ are transitive models of ZFC with the same ordinals and $\phi \in V$, $V \models$"ϕ is an $L_{\omega_1,\omega}$-sentence" (with a natural set-theoretic coding of such sentences), then $V \models$"ϕ has a countable model" if and only if $W \models$"ϕ has a countable model". The purpose of this paper is to investigate the question of how far we can replace "countable" by higher cardinalities.

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A main tool for absoluteness considerations is Shoenfields absoluteness Theorem (Theorem 25.20 in [8]). It states that any property expressed by either a $\mathbf{\Sigma}^1_2$ or a $\mathbf{\Pi}^1_2$ formula is absolute between transitive models of ZFC with the same ordinals. As John Baldwin observed in [2], it follows from results of [7] that the property of ϕ having arbitrarily large models is absolute (it can be expressed in form of the existence of an infinite indiscernible sequence, which by Shoenfield is absolute). Since the Hanf number of the logic $L_{\omega_1,\omega}$ equals $\beth_{\omega_1}$, is follows that the existence of models in cardinalities above that number is absolute. Therefore the context we are interested in is where ϕ (absolutely) does not have a model of size $\beth_{\omega_1}$.

1 The case $\aleph_1$

For *complete* sentences ϕ (meaning that any model of ϕ satisfies the same $L_{\omega_1,\omega}$ sentences), having a model in $\aleph_1$ is an absolute notion. We have the following characterization (see also [2]) of ϕ having a model of size $\aleph_1$ (which is a $\mathbf{\Sigma}^1_1$ property and therefore absolute by Shoenfield's absoluteness Theorem):

> $(*)$ There exist two countable models M, N of ϕ such that M is a proper elementary (in the fragment of ϕ) substructure of N.

To see that this is a characterization, note first that if ϕ has an uncountable model, $(*)$ holds by Löwenheim-Skolem. For the converse, we use the completeness of ϕ which implies that any two countable models of ϕ are isomorphic (by Scott's isomorphism Theorem, since ϕ must imply Scott sentences of countable models). Then, as $N \cong M$, we can find a proper countable $L_{\omega_1,\omega}$-elementary extension of N as well and continue this procedure ω_1 many times (taking unions at limit stages). The union of this elementary chain will then be a model of ϕ of size $\aleph_1$.

If the sentence is not complete, there might be examples of ϕ having an uncountable model, where $(*)$ fails (Gregory claimed the existence of such an example in [6]). However having a model of size $\aleph_1$ turns out to be absolute in general[1]. We have to provide a slightly more subtle criterion to deal with possibly incomplete ϕ. To state it, we have to regard the sentence ϕ as a set-theoretic object using standard coding of formulas of $L_{\omega_1,\omega}$. ϕ can thus be regarded as a hereditarily countable set.

The following property which (again by Shoenfield) is absolute, characterizes ϕ having a model of size $\aleph_1$:

> $(**)$ There is a countable transitive model U of ZFC$^-$ (ZFC without the power set axiom) containing ϕ with $U \models$ "ω_1 exists, ϕ is hereditarily countable, and there

[1] This has also been observed recently by Paul Larson. His argument uses iterated generic ultrapowers. Rami Grossberg points out, he knew of this fact already in the 1980's but did not publish it, and that others like Shelah, Barwise and Keisler most likely knew of it even earlier.

is a model of ϕ with universe ω_1".

First, suppose ϕ has a model M of size $\aleph_1$, say one with universe ω_1. As both ϕ and M are elements of H_{ω_2} (the collection of sets hereditarily of size at most $\aleph_1$), we have $H_{\omega_2} \models \mathrm{ZFC}^- +$ "there is a model of ϕ with universe ω_1". Now it suffices to take a countable (first order) elementary substructure $U \prec H_{\omega_2}$ containing ϕ, and U will have the properties of $(**)$.

Conversely, assuming $(**)$ holds for some countable U, we can take an elementary extension U' of U where all (in the sense of U) hereditarily countable sets are unchanged and all (in U) uncountable ones become sets of size $\aleph_1$ (using Corollary A of Theorem 36 in [10]). In particular this is true for the ω_1 of U' on which we know a model M of ϕ lives (note that $U' \models (M \models \phi)$ implies that $M \models \phi$ in the real universe; to see this, use that M contains the fragment of ϕ and satisfaction for formulas in this fragment is absolute between M and the real universe"). So we get a model of ϕ of size $\aleph_1$.

There is another absolute criteria characterizing ϕ having an uncountable model, but it requires going beyond the logic $L_{\omega_1,\omega}$. Let us consider the extension $L_{\omega_1,\omega}(Q)$ of $L_{\omega_1,\omega}$ obtained by adding an extra quantifier Q with the semantics "there exist uncountably many". As is shown in [3], $L_{\omega_1,\omega}(Q)$ admits a completeness theorem which actually has a very natural (absolute) deduction calculus. Now the statement

$(***)$ There is a proof of $\neg Qx(x = x)$ starting from ϕ

characterizes ϕ having only countable models. Thus the negation of $(***)$ is an (absolute) property characterizing ϕ having an uncountable model. Note that this argument shows that model existence in $\aleph_1$ is absolute even for $L_{\omega_1,\omega}(Q)$ sentences.

2 Going beyond $\aleph_1$

It is not generally true that the existence of a model of size $\aleph_2$ is an absolute property.

A very simple way to see this is to take any sentence ϕ that has models exactly up to size continuum. We easily find even *complete* sentences with this property. Then clearly, ϕ has a model of size $\aleph_2$ if and only if the continuum hypothesis fails.

More generally, such a sentence has a model of size $\aleph_\alpha$ if and only if $2^{\aleph_0} \geq \aleph_\alpha$. So for any $\alpha > 1$, the existence of a model of size $\aleph_\alpha$ is non-absolute.

There are many examples of complete $L_{\omega_1,\omega}$-sentences in the literature having models exactly up to size continuum, but they are mostly more complicated than necessary for our purposes, because their authors have been interested in additional properties. Therefore we provide here a very simple such example:

Let the language L consist of countably many binary relation symbols E_n ($n < \omega$), and let $\sigma \in L_{\omega_1,\omega}$ be the conjunction of

- All E_n are equivalence relations such that E_0 has two classes and each E_n-class is the union of exactly two E_{n+1}-classes.
- $\forall x, y((\bigwedge_{n<\omega} E_n(x,y)) \to x = y)$

It is an easy back-and-forth argument to show that any two countable models of σ are isomorphic, so σ is complete. Every model represents a set of branches through a full binary tree, so there cannot be models greater than the continuum. On the other hand, the Cantor space 2^ω together with the relations "$E_n(x, y)$ if and only if x and y coincide on the $n + 1$ first components" is a model of σ of size continuum.

3 Going beyond $\aleph_1$ under the assumption of GCH

As we have seen, playing with the cardinal exponential function provides trivial examples for the non-absoluteness of the existence of models of cardinality greater than $\aleph_1$. A next natural question is if this is the only non-absoluteness phenomenon there is. That is, under the additional assumption of GCH, does the existence of models in cardinalities greater than $\aleph_1$ become an absolute notion? We will provide different incomplete sentences and later on even a complete one that show the answer is negative.

3.1 A reminder about two-cardinal properties

As we will see later, there is an interesting connection between classical first-order two-cardinal properties and model existence for $L_{\omega_1,\omega}$-sentences. We recall the following definition:

Definition 1. *Let T be a first-order theory in a signature containing a unary predicate P. Given two infinite cardinals κ, λ, we say that T* admits (κ, λ) *if there is a model of T of size κ such that $P^M = \{a \in M | M \models P(a)\}$ has cardinality λ.*

As is already exposed in Chang-Keisler's classical textbook [4] in chapter 7.2, admitting certain pairs (κ, λ) is a non-absolute property for certain theories. There, examples are given where admitting (κ^+, κ) is equivalent to the existence of a special κ^+-Aronszajn tree or where admitting (κ^{++}, κ) is equivalent to the existence of a κ^+ Kurepa tree (or equivalently a κ^+ Kurepa family).

3.2 Some set theory

We now recall the two classical concepts of Kurepa families and special Aronszajn trees. The first-order examples in [4] showing non-absoluteness of the existence of certain two-cardinal models and our later exposed examples of $L_{\omega_1,\omega}$-sentences showing non-absoluteness of model existence in certain cardinalities code those objects in their models. The coding is such that the existence of a certain two-cardinal model or the existence of a model in a certain cardinality is equivalent to the existence of such an object (which is independent from ZFC+GCH as we will see in the following).

Definition 2. *Let κ be any infinite cardinal. A κ^+* Kurepa family *is a family $\mathcal{F}$ of subsets of some set A with $|A| = \kappa^+$, such that $|\mathcal{F}| > \kappa^+$ and for any subset $B \subset A$ with $|B| = \kappa$, $|\{X \cap B | X \in \mathcal{F}\}| \leq \kappa$.*

Let KH_{κ^+} be the statement that there exists a κ^+ Kurepa family.

It is folklore that the existence of Kurepa families in different $\aleph_\alpha$ ($\alpha < \omega_1$) is independent from one another. We will now describe the formal arguments for the cases we need (essentially the same arguments would work more generally for "switching on and off" independently the existence of Kurepa families in different $\aleph_\alpha$). In the constructible universe, KH_{κ^+} is true for all cardinals κ (this follows from the fact that $\diamondsuit^+$ holds at successor cardinals in L, see [9]). On the other hand we have:

Theorem 3. *The consistency of "ZFC+there are uncountably many inaccessible cardinals" implies the consistency of "ZFC+GCH+$\forall\alpha < \omega_1 \neg \mathrm{KH}_{\aleph_{\alpha+1}}$"*

Proof. This is a slight generalisation of Silver's argument that if κ is inaccessible then after forcing with $\mathrm{Coll}(\omega_1, < \kappa)$, the forcing to convert κ into $\aleph_2$ with countable conditions, $\mathrm{KH}_{\aleph_1}$ fails (see [8]).

Assume GCH, let κ_0 be $\aleph_1$ and define $(\kappa_\beta)_{0<\beta<\omega_1}$ inductively: set $\kappa_{\beta+1}$ the least inaccessible cardinal greater than κ_β and for $\beta < \omega_1$ a limit ordinal set $\kappa_\beta = \sup\{\kappa_\gamma | \gamma < \beta\}$. Let P be the fully supported product of the forcings $\mathrm{Coll}(\kappa_\beta, < \kappa_{\beta+1})$ for $\beta < \omega_1$. Then in the extension, κ_β equals $\aleph_{\beta+1}$. We claim that $\mathrm{KH}_{\kappa_\beta}$ fails for each $\beta < \omega_1$.

Indeed, the forcing P can be factored as $P(< \beta) \times P(\geq \beta)$ where $P(< \beta)$ refers only to the collapses $\mathrm{Coll}(\kappa_\gamma, < \kappa_{\gamma+1})$ for $\gamma < \beta$ and $P(\geq \beta)$ refers only to the the collapses $\mathrm{Coll}(\kappa_\gamma, < \kappa_{\gamma+1})$ for $\gamma \geq \beta$. Similarly, $V[G]$ factors as $V[G(< \beta)][G(\geq \beta)]$. In the model $V[G(< \beta)]$, $\kappa_{\beta+1}$ is still inaccessible, so we can apply Silver's argument to conclude that $\mathrm{KH}_{\kappa_\beta}$ fails in $V[G(< \beta)][G(\geq \beta)] = V[G]$, using the closure of the forcing $P(\geq \beta)$ under sequences of length less than κ_β. $\square$

Definition 4. *A* tree *is a partially ordered set $(T, <)$ such that for any element $t \in T$, the set $\{x | x < t\}$ is well ordered by $<$. The* rank $\mathrm{rk}(t)$ *of t is the order type of $\{x | x < t\}$. For any ordinal α, let $T_\alpha = \{t \in T | \mathrm{rk}(t) = \alpha\}$.*

For any cardinal κ, a κ^+-tree is a tree T such that $T_{\kappa^+} = \emptyset$ and for all $\alpha < \kappa^+$, $0 < |T_\alpha| < \kappa^+$. T is normal*, if*

- $|T_0| = 1$
- *every element has at least two immediate successors*
- *for any $t \in T$ and α with $\mathrm{rk}(t) < \alpha < \kappa^+$, there is some $t' > t$ with $\mathrm{rk}(t') = \alpha$.*

A normal κ^+-tree T is a special κ^+-Aronszajn tree*, if there is some set A of size κ and a function $f : T \to A$ such that for all $t, t' \in T$, $t < t'$ implies $f(t) \neq f(t')$.*

It is a consequence of GCH that special κ-Aronszajn trees exist for all successor cardinals κ that are not successors of singular cardinals. Moreover, in the constructible universe, special Aronszajn trees exist even in successors of limit cardinals (this is a consequence of $\square_\kappa$, see [9]).

On the other hand, the consistency of "ZFC+$\exists\kappa$(κ supercompact)" implies the consistency of "ZFC+GCH+there are no special $\aleph_\alpha$-Aronszajn trees for all countable limit successors α":

We start with a model of GCH with a supercompact cardinal κ and force with $\mathrm{Coll}(\omega_1, <\kappa)$. As is argued in [5], this forcing preserves a stationary reflection property sufficient to ensure that Weak Square fails at $\aleph_\lambda$ for λ a limit ordinal of countable cofinality. By a result of Jensen in [9], Weak Square at a cardinal κ is equivalent to the existence of a special Aronszajn tree on κ^+.

3.3 Connecting first-order two-cardinal properties with $L_{\omega_1,\omega}$-model existence

We will describe how a first-order theory T can be turned into an $L_{\omega_1,\omega}$-sentence σ in such a way that T admitting certain (κ, λ) is equivalent to the existence of a model of σ of size κ.

We start with an ad-hoc definition of an $L_{\omega_1,\omega}$-sentence σ_0^α characterizing $\aleph_\alpha$ (for $\alpha < \omega_1$), which means that it (absolutely) has a model of size $\aleph_\alpha$, but no bigger model.

Let $L_0^\alpha = \{Q_\beta, a_n, <, F\}_{\beta \leq \alpha;\ n<\omega}$, where the Q_β are unary predicates, the a_n are constant symbols, $<$ is a binary and F a ternary relation symbol.

Let $\sigma_0^\alpha \in (L_0^\alpha)_{\omega_1,\omega}$ be the conjunction of the following sentences:

- The universe is the union of all Q_β
- $Q_0 = \{a_n | n < \omega\}$ where all a_n designate distinct elements.
- For any $\beta < \alpha$, $Q_{\beta+1}$ is disjoint from any Q_γ for all $\gamma \leq \beta$.

- For any limit ordinal $\beta \le \alpha$, $Q_\beta = \bigcup_{\gamma<\beta} Q_\gamma$
- $<$ linearly orders $Q_{\beta+1}$ for every $\beta < \alpha$ and $x < y$ implies that for some $\beta < \alpha$, both x and y belong to $Q_{\beta+1}$.
- $F(a, b, c)$ implies that for some $\beta < \alpha$, $a \in Q_{\beta+1}$, $b < a$ and $c \in Q_\beta$.
- For every $\beta < \alpha$ and every $a \in Q_{\beta+1}$, $F(a, \cdot, \cdot)$ defines a total injective function from $\{x|x < a\}$ into Q_β.

Note that for β a limit ordinal or zero, Q_β is not ordered by $<$ and if $\alpha = 0$, both $<$ and F are empty relations.

Clearly, if $M \models \sigma_0^\alpha$, then in M the ordering of $Q_{\beta+1}$ must be $|Q_\beta|$-like (i.e. any proper initial segment has cardinality at most $|Q_\beta|$). This implies that $|Q_{\beta+1}|$ is at most $|Q_\beta|^+$ and since Q_0 is countable by definition, we see inductively that the cardinality of each Q_β is bounded by $\aleph_\beta$ (and there exist models with $|Q_\beta| = \aleph_\beta$ for all $\beta \le \alpha$).

Now suppose we have a first-order theory T in a language containing a unary predicate P. For $\beta < \alpha < \omega_1$, we define the $L_{\omega_1,\omega}$-sentence $\sigma_T^{\alpha,\beta}$ as the conjunction of

- T
- σ_0^α
- $P = Q_\beta$

Proposition 5. *Let $\beta < \omega_1$ and $0 < n < \omega$. T admits $(\aleph_{\beta+n}, \aleph_\beta)$ if and only if $\sigma_T^{\beta+n,\beta}$ has a model of cardinality $\aleph_{\beta+n}$.*

Proof. If $M \models \sigma_T^{\beta+n,\beta}$ has cardinality $\aleph_{\beta+n}$, we must have $|Q_\beta| = \aleph_\beta$ in that model (here we use that n is finite!). Now the reduct of M to the language of T is a model of size $\aleph_{\beta+n}$ where P has size $\aleph_\beta$.

Conversely, given a model of T of size $\aleph_{\beta+n}$ where P has size $\aleph_\beta$, it is easy to expand this model to be a model of $\sigma_T^{\beta+n,\beta}$. □

Note that this Proposition becomes false if n is allowed to be infinite.

3.4 Examples of incomplete sentences: successor cardinals

We quote Chang-Keisler's results 7.2.11 and 7.2.13 from [4] (adapting the notation slightly):

- There is a sentence ϕ_1 in a finite language L such that for all infinite cardinals λ, ϕ_1 admits (λ^+, λ) if and only if there exists a special λ^+-Aronszajn tree.

- There is a sentence ϕ_2 in a suitable language such that for all infinite cardinals λ, ϕ_2 admits (λ^{++}, λ) if and only if a λ^+ Kurepa family exists.

From the preceding section we get thus infinitary sentences $\sigma_{\phi_1}^{\alpha+1,\alpha}$ and $\sigma_{\phi_2}^{\alpha+2,\alpha}$ such that

- $\sigma_{\phi_1}^{\alpha+1,\alpha}$ has a model of cardinality $\aleph_{\alpha+1}$ if and only if a special $\aleph_{\alpha+1}$-Aronszajn tree exists.
- $\sigma_{\phi_2}^{\alpha+2,\alpha}$ has a model of cardinality $\aleph_{\alpha+2}$ if and only if a $\aleph_{\alpha+1}$ Kurepa family exists.

Now recalling the set-theoretic facts from section 3.2, we get the following results:

Theorem 6. *Let $\alpha < \omega_1$. Assuming the existence of uncountably many inaccessible cardinals, model-existence in $\aleph_{\alpha+2}$ is a non-absolute notion modulo ZGC+GCH for $L_{\omega_1,\omega}$-sentences.*

Theorem 7. *Let $\alpha < \omega_1$ be the successor of a limit ordinal. Assuming the existence of a supercompact cardinal, model-existence in $\aleph_{\alpha+1}$ is a non-absolute notion modulo ZGC+GCH for $L_{\omega_1,\omega}$-sentences.*

At this point, we have covered all cases of successor cardinals $\aleph_\alpha$ for $1 < \alpha < \aleph_{\omega_1}$.

3.5 Examples of incomplete sentences: limit cardinals

We would also like to find examples of (incomplete) sentences where model existence in $\aleph_\alpha$ is non-absolute modulo ZFC+GCH for countable limit ordinals α. With a slight variation of our examples involving special Aronszajn trees, we can deal with limits that are greater than ω.

Since the construction is rather straightforward, we will only give an informal description of it.

The sentence ϕ_2 used to prove Theorem 7 which is given explicitly in [4] involves essentially a binary relation T coding a tree and a unary predicate U and has the property that whenever $M \models \phi_2$ and $|M| = |U^M|^+$, then T has a subtree which is a special $|M|$-Aronszajn-tree.

Now, fixing some $\alpha < \omega_1$ greater than ω, we start with the sentence σ_0^α (see section 3.3) and for all $\beta < \alpha$, we add the theory ϕ_2 relativised to $\bigcup_{\gamma \leq \beta+1} Q_\gamma$ (i.e. the set $\bigcup_{\gamma \leq \beta+1} Q_\gamma$ with the induced structure in the language of ϕ_2 is a model of ϕ_2) with Q_β taking the role of U. I.e. we are coding special Aronszajn trees at *every* level $Q_{\beta+1}$ where $|Q_{\beta+1}| = |Q_\beta|^+$.

The result is a sentence σ_1^α for which (assuming consistency of supercompact cardinals) the existence of a model of size $\aleph_\alpha$ is non-absolute modulo ZFC+GCH. The reason is that if no special $\aleph_{\omega+1}$-Aronszajn tree exists, the maximum cardinality of a model of σ_1^α is $\aleph_\omega$ since whenever for some $\gamma < \alpha$, $|Q_{\gamma+1}| = |Q_\gamma|^+ = \aleph_{\omega+1}$, a special $\aleph_{\omega+1}$-Aronszajn tree will be coded in the model.

4 A complete sentence

Both the first-order examples from [4] and our $L_{\omega_1,\omega}$-examples from the preceding section are highly incomplete (i.e. many first-order or $L_{\omega_1,\omega}$-statements are undecided) and it seems a very non-trivial task to turn them into complete theories while conserving the properties that matter to us.

We will now introduce a method of completing incomplete $L_{\omega_1,\omega}$-sentences that has the benefits of providing fairly explicit axiomatizations as well as some means of constructing models of the resulting complete sentence with certain properties. This method will then be applied to an incomplete sentence coding $\aleph_2$ Kurepa trees (similar to the examples from the preceding section).

Definition 8. *Let $\sigma \in L_{\omega_1,\omega}$.*

- *A* σ-chain *is a family $(M_\alpha)_{\alpha<\lambda}$ of models of σ such that whenever $\alpha < \beta < \lambda$, we have $M_\alpha \subset M_\beta$.*
- *σ is* preserved under chains *if for any σ-chain $(M_\alpha)_{\alpha<\lambda}$, $M = \bigcup_{\alpha<\lambda} M_\alpha$ is a model of σ.*

As in the classical first-order case, it is still true that any Π_2-sentence is preserved under chains, i.e. any sentence of the form $\forall \bar{x} \exists \bar{y} \psi(\bar{x}, \bar{y})$. where ψ is quantifier-free (but possibly infinitary). We have to be a little careful with the definition of Π_2 as for example infinite disjunctions of universal formulas might not be preserved under chains. A simple example is given by the sentence

$$\sigma = \bigvee_{S \subset \omega \text{ finite}} \forall x (U(x) \leftrightarrow \bigvee_{i \in S} x = a_i)$$

in the language of countably many constants a_i and a unary predicate U. This sentence expresses that U is finite.

Definition 9. *Let $\sigma \in L_{\omega_1,\omega}$.*

- *Set $S_{\mathrm{qf}}(\sigma) = \{\mathrm{tp}_{\mathrm{qf}}(\bar{a}) | \exists M \models \sigma \ (\bar{a} \in M)\}$ (where $\mathrm{tp}_{\mathrm{qf}}(\bar{a})$ is the quantifier-free type of $\bar{a}$).*
- *σ is* qf-small *if $S_{\mathrm{qf}}(\sigma)$ is countable.*

Note that by the downward Löwenheim-Skolem-Theorem, we can define $S_{\mathrm{qf}}(\sigma)$ by referring only to countable models of σ.

Definition 10. *Suppose σ is qf-small.*

- *For any pair $p(\bar{x}), q(\bar{x}\bar{y}) \in S_{\mathrm{qf}}(\sigma)$ define the sentence $\sigma_{p,q} = \forall \bar{x}(p(\bar{x}) \to \exists \bar{y}\; q(\bar{x}\bar{y}))$.*
- *Set $\sigma^* = \sigma \wedge \bigwedge_{p,q \in S_{\mathrm{qf}}(\sigma); p \subset q} \sigma_{p,q}$*

If σ is preserved under chains, then σ^* is as well. However, there are consistent σ for which σ^* is inconsistent. An example would be the sentence $\sigma = \forall a, b, c, d(R(a,b) \wedge R(c,d) \to a = c \wedge b = d)$ which expresses that exactly two points are R-related.

Proposition 11. *For any σ, if σ^* is consistent, it is complete.*

Proof. We show $\aleph_0$-categoricity. Let $M, N \models \sigma^*$ be countable and suppose f is a finite partial isomorphism mapping a tuple $\bar{a} \in M$ to a tuple $\bar{b} \in N$. Now let $c \in M$ be any point and set $p = \mathrm{tp}_{\mathrm{qf}}(\bar{a})$ $(= \mathrm{tp}_{\mathrm{qf}}(\bar{b}))$ and $q = \mathrm{tp}_{\mathrm{qf}}(\bar{a}c)$. Since $N \models \sigma_{p,q}$, we find a $d \in N$ with $\bar{b}d \models q$, so we can extend f by mapping c to d. Now after enumerating both M and M we can construct a total isomorphism as the union of finite partial isomorphisms by adding every point of M to the domain and every point of N to the range eventually. □

Definition 12. *A sentence $\sigma \in L_{\omega_1,\omega}$ has the* extension property for countable models (EPC)*, if for any countable $M \models \sigma$ and $p(\bar{x}) \subset q(\bar{x}\bar{y})$ in $S_{\mathrm{qf}}(\sigma)$, whenever some $\bar{a} \in M$ realizes p, there is a countable $N \models \sigma$ with $M \subset N$ containing some $\bar{b}$ with $\bar{a}\bar{b} \models q$.*

Theorem 13. *Suppose $\sigma \in L_{\omega_1,\omega}$ is preserved under chains, is qf-small and has the EPC. Then*

1. *σ^* is consistent.*
2. *any countable model of σ has an extension that is a model of σ^*.*
3. *σ^* is the only completion of σ with property 2 that is still preserved under chains.*

Proof. Let $M \models \sigma$ be countable. Enumerate all possible pairs $(\bar{a}, q)$ where $\bar{a} \in M$ and $\mathrm{tp}_{\mathrm{qf}}(\bar{a}) \subset q \in S_{\mathrm{qf}}(\sigma)$ as $((\bar{a}_n, q_n))_{n<\omega}$. Construct a $\subset$-chain $(M_n)_{n<\omega}$ of models of σ such that in M_n we add a tuple $\bar{b}_n$ with the property that $\bar{a}_n\bar{b}_n \models q_n$. Let $M^1 = \bigcup_{n<\omega} M_n$. Do the same procedure for M^1 in place of M to get some M^2. Repeat this ω many more times and set $N = \bigcup_{k<\omega} M^k$. Since σ is preserved under chains we still have $N \models \sigma$, and we just added all necessary witnesses in the chains to satisfy all $\sigma_{p,q}$ as well, so we have constructed a model of σ^* that contains the model M we started with.

The uniqueness of σ^* follows from the fact that if some τ has the same properties, including being preserved under chains, we can form a $\subset$-chain $(M_n)_{n<\omega}$ with $M_{2n} \models \sigma^*$ and $M_{2n+1} \models \tau$ for all n. Then by preservation under chains, the union must be a model of both σ^* and τ and we conclude by completeness of both sentences. □

Now we turn to the definition of an incomplete sentence coding $\aleph_2$ Kurepa families, which we will then complete by the described technique.

Our language will be $\mathcal{L} = \{S, L, U, V, E_n, <, R, F, G, H, \}_{n<\omega}$, where S and L are unary predicates, all E_n as well as $U, V, <, R$ are binary relations and F, G and H are ternary relations.

Before we give the formal definition of our sentence, we describe informally what a model of it looks like:

- $(L, <)$ is a linear order.
- The elements of S code subsets of L via the relation R such that any two of them coincide on an initial segment of L with a maximum element and are disjoint above that initial segment.
- F defines a binary function mapping two elements of S to the point of L where they become disjoint.
- For every $a \in L$, U and V define sets $U_a = \{x|U(a,x)\}$, $V_a = \{x|V(a,x)\}$ and all those sets are pairwise disjoint.
- The E_n are such that every set U_a and V_a with the restrictions of the E_n satisfies the theory of binarily splitting equivalence relations, given in section 2. In particular, all these sets have size at most $2^{\aleph_0} = \aleph_1$.
- G codes bijections between every initial segment $\{x|x < a\}$ and the set U_a. This makes $(L, <)$ $\aleph_2$-like.
- H codes intersections of sets coded by elements of S with initial segments $\{x|x < a\}$ as elements of V_a. Consequently, on each initial segment, there are at most $\aleph_1$ many possibilities for the sets coded by elements of S.

Let σ be the conjunction of the following statements:

(A1) Both $U(x, y)$ or $V(x, y)$ imply $x \in L$. Writing $U_x = \{y|U(x, y)\}$ and $V_x = \{y|V(x, y)\}$, the sets L, S, U_x, V_x (for all $x \in L$) are pairwise disjoint and their union is everything.

(A2) All E_n define equivalence relations on every set U_x and V_x where on every U_x or V_x, E_0 has exactly two classes and every E_n-class is the union of exactly two E_{n+1}-classes. In addition, $\bigwedge_{n<\omega} xE_ny$ implies $x = y$.

(A3) $<$ is a linear ordering of L. For $x \in L$ we write $L_{<x} = \{y \in L|y < x\}$ and $L_{\leq x} = L_{<x} \cup \{x\}$.

(A4) $F(s, t, x)$ implies $s, t \in S$ and $x \in L$. F defines a symmetric function from $S \times S$ to L.

(A5) $R \subset S \times L$. For $s \in S$ we write $R_s = \{x \in L|R(s, x)\}$. For any two distinct $s, t \in S$, R_s and R_t are identical on $L_{\leq F(s,t)}$ and disjoint on $L \setminus L_{\leq F(s,t)}$.

(A6) $G(x, y, z)$ implies $x \in L$, $y < x$ and $z \in U_x$. For every $x \in L$, $G(x, \cdot, \cdot)$ defines a bijective function $G_x : L_{<x} \to U_x$ by $G_x(y) = z$ if and only if $G(x, y, z)$.

(A7) $H(x, y, z)$ implies $x \in L$, $y \in S$ and $z \in V_x$. For every $x \in L$, $H(x, \cdot, \cdot)$ defines a surjective function $H_x : S \to V_x$ by $H_x(y) = z$ if and only if $H(x, y, z)$. H_x has the property that $H_x(s) = H_x(t)$ if and only if $F(s, t) \geq x$.

It is easy to construct a model of σ, but σ is not a complete sentence. We verify that it satisfies the hypotheses of Theorem 13. The axioms are all at most Π_2-statements, so we have preservation under chains. Also, since the equivalence relations E_n are refining and $\mathcal{L} \setminus \{E_n\}_{n<\omega}$ is finite, $S_{\mathrm{qf}}(\sigma)$ is countable.

Towards showing EPC, let $M \models \sigma$ be countable, $\bar{a} = (a_1, \ldots, a_n) \in M$ and let $p(\bar{x}), q(\bar{x}, y) \in S_{\mathrm{qf}}(\sigma)$ with $\bar{a} \models p$ and $p \subset q$ (note that it suffices to consider a single variable y instead of an arbitrary tuple). We want to find some $N \supset M$ and $b \in N$ such that $\bar{a}b \models q$. There are several cases:

- Suppose $S(y) \in q(\bar{x}, y)$. We have to add a new set R_y to M respecting the requirements of q and the axioms of σ. The requirements can be $R(y, x_i)$, $\neg R(y, x_i)$ as well as $F(y, x_j) = x_i$, $F(y, x_j) \neq x_i$ and $H(x_i, y) = x_j$, $H(x_i, y) \neq x_j$, $H(x_i, y) E_n x_j$, $\neg H(x_i, y) E_n x_j$ for components x_i, x_j in $\bar{x}$ and $n < \omega$ (G does not matter here since it does not involve elements from S). We define the set R_y as follows: Take the maximal element $z \in L$ occurring in $\bar{x}$ such that either
  - $q \vdash F(y, x_i) = z$ for some x_i in $\bar{x}$ or
  - $q \vdash R(y, z)$ and there is some $s \in S$ with $M \models R(s, z)$.

  Writing $A = \{a \in L | q \vdash R(y, a)\}$, we set $R_y = A \cup (R_{x_i} \cap L_{\leq z})$ in the first case and $R_y = A \cup (R_s \cap L_{\leq z})$ in the second case (choose any such s arbitrarily). If we are in the second case and q implies $F(y, s) \neq z$, we add a new element w to L which is greater than z and declare $R(s, w)$, $R(y, w)$, $F(s, y) = w$. To turn M with the additional y (and possibly w) into a model of σ, we have to set the F- and H-relations which can be done straightforwardly (respecting possible requirements from q for H; we may have to add new points to sets V_a for $a > z$). In case we added the point w, we also have to add new sets U_w, V_w as well as a new point to each U_a for $a > w$, and extend G accordingly.
- Now suppose $L(y) \in q(\bar{x}, y)$. Add a new element z to L for y in an arbitrary cut that complies with the conditions $x_i < y$ or $x_i > y$ contained in q. Add $R(x_i, z)$ whenever demanded by q and for any other $s \in S$ add $R(s, z)$ if and only if $R(t, z)$ and $F(s, t) > z$ for some element $t \in S$. Finally, we have to add new sets U_z and V_z as well as a new point a to each U_w with $w > z$ and declare $G(w, z, a)$. We may have to add a new point to sets V_w for $w > z$ too.
- Should $U_{x_i}(y)$ or $V_{x_i}(y)$ belong to q, it is easy to see that there must already be some $b \in M$ with $\bar{a}b \models q$.

Now we apply Theorem 13 to σ. Immediately we see that σ^* implies:

- The ordering on L is dense without endpoints.
- Every set R_s is dense (and thus unbounded) and co-dense in L.
- $s \neq t$ implies $R_s \neq R_t$ ("R is extensional").

But we know more about the properties of σ^*. The countable model of σ^* is extendible, so there is an uncountable model. In addition, we have seen in the verification of EPC that we have a lot of freedom in adding new elements to countable models of σ, and thus to models of σ^*, so that we can conclude the existence of models of σ^* with

- $(L, <)$ isomorphic to a proper initial segment of $\eta_1 \cdot \omega_2$, where η_1 is the saturated dense linear order without endpoints of size $\aleph_1$ (we assume GCH).
- all $(U_x, E_n)_{n<\omega}$ and $(V_x, E_n)_{n<\omega}$ isomorphic to $(2^\omega, F_n)$ where we define $\xi F_n \rho$ if and only if $\xi(k) = \rho(k)$ for all $k \leq n$.

Now we consider the class $\mathbb{P}$ of all such models with the additional properties

- S is a subset of ω_3 of size $\aleph_1$ (so all models in $\mathbb{P}$ will have size $\aleph_1$).
- The sets U_x and V_x $(x \in L)$ equal $2^\omega \times \{(x, 0)\}$ and $2^\omega \times \{(x, 1)\}$ respectively and the E_n defined on them are the natural ones (compare with F_n above).

We order the elements of $\mathbb{P}$ by the superstructure relation $\supset$. Since σ^* is preserved under unions, the poset $(\mathbb{P}, \supset)$ is ω_2-closed (meaning every sequence of length less than ω_2 of elements of $\mathbb{P}$ has a lower $\supset$-bound; clearly the union of the chain of models will do).

Now we show that $(\mathbb{P}, \supset)$ has the ω_3-cc. Take any $X \subset \mathbb{P}$ of size $\aleph_3$. We shall find two elements of X which have a common extension. By the pigeonhole-principle and the delta-system-lemma, we may assume that

- the domains of the elements of X form a delta-system.
- the L-part of all elements of X is identical.
- the structure of all pairs $M, N \in X$ agree on the root of the delta-system.
- the collection of sets R_s $(s \in S)$ is identical for all elements of X.

Two models $M, N \in X$ may only differ on their S-part. We would like to make the union $M \cup N$ into a model of σ. The problem is that if the models are not already identical, there will be $x \in S^M$, $y \in S^N$ outside the root such that $R_x = R_y$, so $F(x, y)$ cannot be defined in a way that axiom (A5) holds. The solution is to end-extend L in order to make R_x and R_y disjoint on a final segment.

Suppose that in $\eta_1 \cdot \omega_2$, L is an initial segment contained in $\{x | x < a\}$. Enumerate the elements of $S^M \setminus S^N$ as $(s_\alpha)_{\alpha<\mu}$ (for some $\mu \leq \aleph_1$). Now inductively do the following: given $\alpha < \omega_1$ there is a unique $t \in S^N \setminus S^M$ such that $R_{s_\alpha} = R_t$. Set $R(s_\alpha, a)$, $R(t, a)$, $F(s_\alpha, t) = a$ and $R(s_\alpha, a_\alpha)$ (but *not* $R(t, a_\alpha)$), where $a_\alpha \in \eta_1 \cdot \omega_2$ is greater than a and any already chosen a_β ($\beta < \alpha$). Now we have to add sets U_{a_α} and V_{a_α} and extend G and H to get a model M' of σ containing both M and N. Note that we do not have to add any point to the U_x, V_x for $x \in L$ which is fortunate since that would be impossible. Finally, we observe that the proof of Theorem 13 can be adapted (using an induction of ordertype ω_1 instead of ω and intelligent enough bookkeeping) in such a way to not only obtain an extension of M' which is a model of σ^*, but even one that is an element of $\mathbb{P}$.

Let G be a $\mathbb{P}$-generic filter over V. $\bigcup G$ will be a model of σ of size $\aleph_3^V$. But since the forcing is ω_2-closed and has ω_3-cc, all cardinals are preserved and in particular $\aleph_3^{V[G]} = \aleph_3^V$. I.e. we get a model of σ of size $\aleph_3$ in a generic extension. In addition, the forcing preserves GCH.

On the other hand, any such model codes an $\aleph_2$ Kurepa family which means that it is consistent with ZFC+GCH (assuming the existence of an inaccessible cardinal) that σ has no model of size $\aleph_3$.

5 Final observations

The question of absoluteness of model-existence (under ZFC+GCH) in $\aleph_\omega$ remains open. On the other hand, the technique of finding complete examples described in section 4 should be applicable more widely to obtain complete examples of non-absoluteness of model existence (under ZFC+GCH) in cardinals greater than $\aleph_3$. Interestingly, however, this method seems to be problematic for finding examples for model existence in $\aleph_2$, at least with the approach of trying to code Kurepa families. The reason is that it seems difficult to code an ω_1-like ordering without making many elements definable over others (or even getting infinite definable closures over finite tuples), which destroys any chance to have EPC.

As a last remark, our use of the concept of Kurepa families has the slight flaw that in order to find set-theoretic universes which do not contain such families, we have to assume the existence of inaccessible cardinals. For the special Aronszajn technique, we even have to assume the consistency of supercompact cardinals. It would be nice to find $L_{\omega_1,\omega}$ sentences for which under GCH the existence of models of certain cardinalities is not absolute, without assuming the existence of large cardinals.

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higher cardinalities to achieve non-absoluteness of model existence in cardinalities above $\aleph_2$. He also was the first to recognize a problem in dealing with $\aleph_\omega$ by this construction.

Sy-David Friedman
Kurt Gödel Research Center for Mathematical Logic
Währinger Strasse 25, 1090 Wien, Austria
sdf@logic.univie.ac.at

Tapani Hyttinen
Department of Mathematics University of Helsinki
P.O. Box 68 (Gustaf Hällströmin katu 2b), FI-00014 Helsinki
tapani.hyttinen@helsinki.fi

Martin Koerwien
Kurt Gödel Research Center for Mathematical Logic
Währinger Strasse 25, 1090 Wien, Austria
koerwien@math.uic.edu

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