The number of normal measures

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Abstract

There have been numerous results showing that a measurable cardinal κ can carry exactly α normal measures in a model of GCH, where α is a cardinal at most κ^{++}. Starting with just one measurable cardinal, we have [9] (for $\alpha = 1$), [10] (for $\alpha = \kappa^{++}$, the maximum possible) and [1] (for $\alpha = \kappa^+$, after collapsing κ^{++}). In addition, under stronger large cardinal hypotheses, one can handle the remaining cases: [12] (starting with a measurable cardinal of Mitchell order α), [2] (as in [12], but where κ is the *least* measurable cardinal and α is less than κ, starting with a measurable of high Mitchell order) and [11] (as in [12], but where κ is the *least* measurable cardinal, starting with an assumption weaker than a measurable cardinal of Mitchell order 2). In this article we treat all cases by a uniform argument, starting with only one measurable cardinal and applying a cofinality-preserving forcing. The proof uses κ-Sacks forcing and the "tuning fork" technique of [8]. In addition, we explore the possibilities for the number of normal measures on a cardinal at which the GCH fails.

Theorem 1 *Assume GCH. Suppose that κ is measurable and let α be a cardinal at most κ^{++}. Then in a cofinality-preserving forcing extension, κ carries exactly α normal measures.*

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Proof. By [6], we may first force V to be of the form $L[U][R]$ for some real R, where U is a normal measure on κ. Using GCH in the ground model, this forcing is cofinality-preserving. So we assume that V is of this form, and as the real R will play no role in the proof, we further assume that V is simply $L[U]$ for some normal measure U, which by [9] is unique.

The case $\alpha = 0$ is easy, as adding one κ-Cohen set kills the measurability of κ. We focus now on the case $\alpha = 2$, which brings out the main ideas of the proof.

Let $j : V \to M$ be the ultrapower embedding given by the normal measure U; thus every element of M is of the form $j(f)(\kappa)$ for some function f with domain κ. Our plan is to define an iteration P (with specially-chosen support) of length $\kappa + 1$, with the two properties below. Let G be P-generic. For any elementary embedding $k : V[G] \to N$, the *measure* U_k *derived from* k is defined by: $A \in U_k$ iff $A \subseteq \kappa$ and $\kappa \in k(A)$.

1. In $V[G]$ there are exactly two $G^* \subseteq j(P)$ which are $j(P)$-generic over M and which contain $j[G]$ as a subset.
2. Suppose that U^* is any normal measure on κ in $V[G]$. Then U^* is the normal measure derived from an embedding $j^* : V[G] \to M^*$ where j^* extends j.

Now notice the following:

Lemma 2 *Suppose that* $j_0^* : V[G] \to M_0^*$ *and* $j_1^* : V[G] \to M_1^*$ *are elementary embeddings extending* j*. Then the following are equivalent:*
(i) j_0^* *equals* j_1^**.*
(ii) $j_0^*(G)$ *equals* $j_1^*(G)$*.*
(iii) The normal measure U_0^* *derived from* j_0^* *equals the normal measure* U_1^* *derived from* j_1^**.*

Proof. First note that M_i^* equals $M[j_i^*(G)]$ as j_i^* extends j. It follows that j_i^* is just an ultrapower embedding (given by the normal measure U_i^* derived from j_i^*), as every element of $M[j_i^*(G)]$ is of the form $j(f)(\kappa)^{j_i^*(G)} = j_i^*(f^*)(\kappa)$ where $f^* : \kappa \to V[G]$ is defined by $f^*(\alpha) = f(\alpha)^G$. Therefore (iii) implies (i). Also j_i^* is uniquely determined by $j_i^*(G)$, as $j_i^*(\sigma^G) = j(\sigma)^{j_i^*(G)}$ for each

P-name σ, hence (ii) implies (i). The implications (i) implies (ii) and (i) implies (iii) are trivial. $\Box$ (Lemma 2)

The theorem now follows: Property 1 and the lemma imply that there are exactly two normal measures on κ in $V[G]$ which are derived from elementary embeddings $j^* : V[G] \to M^*$ extending j. Property 2 implies that any normal measure on κ in $V[G]$ is indeed of this form. So there are exactly two normal measures on κ in $V[G]$.

We turn now to a description of the iteration P. Our notation for iterations is as follows: $P(\alpha)$ denotes stage α of the iteration, $P(< \alpha)$ denotes the iteration below α and for $\alpha < \beta$ we decompose $P(< \beta)$ naturally as $P(< \alpha) * P(\alpha) * P(\alpha, \beta)$. We must specify each $P(\alpha)$ and also the support to be used to define $P(< \alpha)$ for limit α.

First we specify each $P(\alpha)$. Our iteration has length $\kappa + 1$, so $P(\alpha)$ is defined only for $\alpha \leq \kappa$. We take $P(\alpha)$ to be trivial unless α is inaccessible, in which case $P(\alpha)$ is a two-step iteration $\mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$. The first factor, $\mathrm{Sacks}(\alpha)$ is α-Sacks forcing, whose conditions are perfect α-trees, i.e., subsets T of $2^{<\alpha}$ which are closed under initial segments, closed under increasing sequences of length $< \alpha$ and with the property that for some closed unbounded $C \subseteq \alpha$, both $s * 0$ and $s * 1$ belong to T whenever $s \in T$ has length in C (see [8]). To define $\mathrm{Code}(\alpha)$, we take advantage of the following lemma. Recall that we have assumed that V equals $L[U]$ where U is the (unique) normal measure on κ and that j denotes the embedding $j : V \to M$ resulting from the ultrapower via U.

Lemma 3 *There exists a sequence $\vec{X}^\kappa = \langle X_i^\kappa \mid i < \kappa^+ \rangle$ of pairwise disjoint stationary subsets of $\kappa^+ \cap Cof(\kappa)$ such that $\vec{X}^\kappa$ belongs to M.*

Proof. $V = L[U]$ satisfies Jensen's $\diamondsuit_{\kappa^+}$ Principle on cofinality κ: There is a sequence $\langle S_\beta \mid \beta < \kappa^+ \rangle$ such that $S_\beta \subseteq \beta$ for each $\beta < \kappa^+$ and for any $X \subseteq \kappa^+$, the set of $\beta < \kappa^+$ of cofinality κ such that $S_\beta = X \cap \beta$ is stationary (see [5]). In fact, using Jensen's hierarchy for $L[U]$ (described in [13]), which has better condensation properties than the usual $L[U]$-hierarchy, the sequence $\vec{S} = \langle S_\beta \mid \beta < \kappa^+ \rangle$ can be chosen to be definable over $H(\kappa^+)$ with parameter κ. As V and M (the ultrapower of V by the measure U) have the same $H(\kappa^+)$, it follows that $\vec{S}$ belongs to M. For $i < \kappa^+$ let X_i^κ be the set of $\beta < \kappa^+$ of cofinality κ such that $S_\beta = \{i\}$. Then $\langle X_i^\kappa \mid i < \kappa^+\rangle$ also belongs to M. $\square$ (Lemma 3)

Fix $\vec{X}^\kappa$ as in Lemma 3. Then as $\vec{X}^\kappa$ belongs to M, we may also fix a function $f : \kappa \to V$ such that for each inaccessible $\alpha < \kappa$, $f(\alpha)$ is an α^+-sequence of disjoint stationary subsets of $\alpha^+ \cap \mathrm{Cof}(\alpha)$ and $j(f)(\kappa) = \vec{X}^\kappa$. Write $f(\alpha)$ as $\vec{X}^\alpha = \langle X_i^\alpha \mid i < \alpha^+\rangle$.

Recall that we wish to define Code(α), where $P(\alpha) = \mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$. Let $S(\alpha)$ denote the α-Sacks generic added at the first stage of this two-step iteration. We view $S(\alpha)$ as a subset of α. A condition in Code(α) is a closed, bounded subset c of α^+. For conditions c, d in Code(α), we say that d *extends* c, written $d \leq c$, iff:

1. d end-extends c (i.e., d contains c and all elements of $d \setminus c$ are greater than $\max(c)$).
2. For $i < \alpha$: If i belongs to $S(\alpha)$ then $d \setminus c$ is disjoint from X^α_{1+2i}; if i does not belong to $S(\alpha)$ then $d \setminus c$ is disjoint from X^α_{1+2i+1}.
3. For $i \leq \max(c)$: If i belongs to c then $d \setminus c$ is disjoint from $X^\alpha_{\alpha+2i}$; if i does not belong to c then $d \setminus c$ is disjoint from $X^\alpha_{\alpha+2i+1}$.

Now we define the desired iteration P of length $\kappa + 1$:

$P(0)$ is trivial.
$P(\alpha)$ is trivial unless $\alpha \leq \kappa$ is inaccessible, in which case $P(\alpha) = \mathrm{Sacks}(\alpha) * \mathrm{Code}(\alpha)$.
$P(< \lambda)$ is the *nonstationary support* limit of the $P(< \alpha)$, $\alpha < \lambda$, for limit ordinals λ. I.e., p belongs to $P(< \lambda)$ iff p belongs to the inverse limit of the $P(< \alpha)$, $\alpha < \lambda$, and if λ is inaccessible then the set of $\alpha < \lambda$ such that $p(\alpha)$ is nontrivial is a nonstationary subset of λ.

The following fact will be used repeatedly in what follows.

Lemma 4 *Suppose that $\lambda \leq \kappa$ is inaccessible and $\langle \alpha_i \mid i < \lambda\rangle$ is the increasing enumeration of a closed unbounded subset of λ. Also suppose that $p_0 \geq p_1 \geq \cdots$ is a λ-sequence of conditions in $P(< \lambda)$ where p_{i+1} agrees with p_i up to and including α_i for each $i < \lambda$, and p_γ is the greatest lower bound of the p_i, $i < \gamma$, for limit $\gamma < \lambda$. Then there is a condition p in $P(< \lambda)$ which extends each p_i.*

Proof. Let $p(i)$ be $p_{\alpha_i}(i)$ for each $i < \lambda$. We need only verify that the p defined in this way is indeed a condition, as then it will clearly extend each p_i. If $i < \lambda$ is a limit and α_i belongs to the support of p, then α_i belongs to the support of p_j for some $j < i$; it follows that the support of p restricted to the α_i, i limit, is the diagonal union of the supports of the p_i, $i < \lambda$, and is therefore nonstationary. □

Lemma 5 *P preserves cofinalities.*

Proof. Suppose that α is an infinite regular cardinal; we show that any ordinal of cofinality greater than α in V also has cofinality greater than α in $V[G]$ for P-generic G. As P decomposes as $P(< \alpha) * \text{Sacks}(\alpha) * \text{Code}(\alpha) * P(\alpha, \kappa + 1)$ where $\text{Sacks}(\alpha)$ obeys α-fusion (see [8]) and $P(\alpha, \kappa + 1)$ is $< \alpha^+$-closed, it suffices to prove the result for $P(< \alpha)$ and for $\text{Code}(\alpha)$. We first consider $P(< \alpha)$. Suppose that $\dot{f}$ is a $P(< \alpha)$-name for a function from α into Ord. Let $\langle \alpha_i \mid i < \alpha \rangle$ enumerate the limit cardinals less than α in increasing order. If p is a condition in $P(< \alpha)$, then by Lemma 4 we may successively extend p to conditions p_i, $i \leq \alpha$, so that p_{i+1} agrees with p_i up to stage α_i and forces that there are at most α_i^+ possibilities for $\dot{f}(i)$, taking greatest lower bounds at limit stages. This is because $P(< \alpha_i^+)$ has a dense subset of size α_i^+ and $P(\alpha_i^+, \alpha)$ is $< \alpha_i^{++}$-closed. The resulting condition p_α forces that there are at most α_i^+ possibilities for $\dot{f}(i)$ for each $i < \alpha$, and therefore forces that $\dot{f}$ cannot be cofinal in an ordinal of V-cofinality greater than α.

To prove the result for $\text{Code}(\alpha)$, it suffices to show that $\text{Code}(\alpha)$ is $< \alpha^+$-distributive (i.e., the intersection of α-many open dense sets is dense). Notice that $\text{Code}(\alpha)$ is $< \alpha$-closed (i.e., descending β-sequences of conditions have lower bounds for $\beta < \alpha$) and when extending conditions, there is nothing to prohibit adding elements of X_0^α. So, given a sequence $\langle D_i \mid i < \alpha \rangle$ of open dense sets and a condition c, we extend c to $c = c_0 \geq c_1 \geq \cdots$ in $\alpha + 1$ steps, so that c_{i+1} meets D_i for each $i < \alpha$, c_λ is the greatest lower bound of the c_i, $i < \lambda$, for limit $\lambda < \alpha$ and the supremum of the $\max(c_i)$'s is an element of X_0^α. This is easily done, using the fact that X_0^α is a stationary subset of $\alpha^+ \cap \text{Cof}(\alpha)$. □ (Lemma 5)

A similar proof yields the following.

Lemma 6 *If G is P-generic, then any function $f^* : \kappa \to \kappa$ in $V[G]$ is dominated by a function $f : \kappa \to \kappa$ in V.*

Proof. P decomposes as $P(<\kappa) * \text{Sacks}(\kappa) * \text{Code}(\kappa)$ where $\text{Sacks}(\kappa)$ satisfies the desired domination property (see [8]) and $\text{Code}(\kappa)$ is $<\kappa^+$-distributive Thus it suffices to prove the result for $P(<\kappa)$. But the previous proof shows that if $\dot{f}$ is a $P(<\kappa)$-name for a function from κ to κ and p is a condition in $P(<\kappa)$, then some $q \leq p$ forces that $\dot{f}(i)$ is less than some $\kappa_i < \kappa$, for each $i < \kappa$; thus q forces $\dot{f}$ to be dominated by g where $g(i) = \kappa_i$. $\square$ (Lemma 6)

By yet another, similar argument we have:

Lemma 7 *For inaccessible $\alpha \leq \kappa$, $P(<\alpha) * \text{Sacks}(\alpha)$ preserves the stationarity of subsets of α^+.*

Proof. It suffices to show that $P(<\alpha)$ preserves the stationarity of subsets of α^+ and that $\text{Sacks}(\alpha)^{G(<\alpha)}$ has this property in $V[G(<\alpha)]$ for $P(<\alpha)$-generic $G(<\alpha)$.

For the first statement, suppose that X is a stationary subset of α^+ and $p \in P(<\alpha)$ forces $\dot{C}$ to be a closed unbounded subset of α^+; we must find $q \leq p$ which forces that some element of X belongs to $\dot{C}$. If $X \cap \text{Cof}(\bar{\alpha})$ is stationary for some $\bar{\alpha}$ less than α, then this is easy, as $P(<\alpha)$ factors as $P(<\bar{\alpha}^+) * P(\bar{\alpha}^+, \alpha)$, where the first factor has size less than α and the second factor is forced to be $<\bar{\alpha}^+$-closed. So assume that $X \cap \text{Cof}(\alpha)$ is stationary. Now much as in the previous proof, we can use Lemma 4 to build a sequence $p = p_0 \geq p_1 \geq \cdots$ of length $\alpha + 1$, taking greatest lower bounds at limit stages, together with a continuous, increasing sequence $\beta_0 < \beta_1 < \cdots$ of length $\alpha + 1$, such that each p_{i+1} forces $\dot{C}$ to intersect the interval (β_i, β_{i+1}) for each i; thus $q = p_\alpha$ forces $\dot{C}$ to contain β_α. Moreover, for some closed unbounded $D \subseteq \alpha^+$, each β in D of cofinality α is of the form β_α for some such choice of the p_i's and β_i's. By choosing β in $D \cap X$ we obtain $q \leq p$ forcing β to belong to $\dot{C}$ with β in X, as desired.

Now suppose that X is a stationary subset of α^+ in the model $V[G(<\alpha)]$, where $G(<\alpha)$ is $P(<\alpha)$-generic, and the condition $T \in \text{Sacks}(\alpha)^{G(<\alpha)}$ forces $\dot{C}$ to be a closed unbounded subset of α^+. We wish to find $T^* \leq T$ which forces some ordinal in X to belong to $\dot{C}$. If $X \cap \text{Cof}(\bar{\alpha})$ is stationary for some $\bar{\alpha} < \alpha$, then this is easy, as $\text{Sacks}(\alpha)$ is $<\alpha$-closed. So assume that $X \cap \text{Cof}(\alpha)$ is stationary. Now as in the previous argument, but using α-fusion, we can build a sequence $T = T_0 \geq T_1 \geq \cdots$ of length $\alpha + 1$, taking greatest lower bounds at limit stages, together with a continuous, increasing sequence $\beta_0 < \beta_1 < \cdots$ of length $\alpha + 1$, so that each T_{i+1} has the same i-th splitting level as T_i and forces $\dot{C}$ to intersect the interval (β_i, β_{i+1}); thus $T^* = T_\alpha$ forces $\dot{C}$ to contain β_α. Now the proof finishes exactly as in the previous argument. □ (Lemma 7)

It follows from Lemma 7 that the stationary sets X_i^α, $i < \alpha^+$, used to define Code(α) remain stationary after forcing with $P(< \alpha) * \text{Sacks}(\alpha)$.

Lemma 8 *Suppose that g is Code(α)-generic (over $V[G(< \alpha)][S(\alpha)]$) and let $C(\alpha)$ be the union of the conditions in g. Then in the generic extension we have:*
(1) For $i < \alpha$, i belongs $S(\alpha)$ iff X^α_{1+2i} is nonstationary and i does not belong to $S(\alpha)$ iff X^α_{1+2i+1} is nonstationary.
(2) For $i < \alpha^+$, i belongs to $C(\alpha)$ iff $X^\alpha_{\alpha+2i}$ is nonstationary and i does not belong to $C(\alpha)$ iff $X^\alpha_{\alpha+2i+1}$ is nonstationary.
(3) There is a unique Sacks(α) $*$ Code(α)-generic over $V[G(< \alpha)]$.

Proof. (1) By the definition of extension for $C(\alpha)$, it follows immediately that for $i < \alpha$, X^α_{1+2i} is nonstationary if i belongs to $S(\alpha)$ and X^α_{1+2i+1} is nonstationary if i does not belong to $S(\alpha)$. We must show that in the former case, X^α_{1+2i+1} remains stationary (the latter case is treated similarly). Suppose that $\dot{C}$ is a name for a CUB subset of α^+ and c is a condition. As in the proof of $< \alpha^+$-distributivity for Code(α), there are CUB-many $\beta < \alpha^+$ with the property that we can build an α-sequence $c = c_0 \geq c_1 \geq \cdots$ with c_{i+1} forcing some ordinal greater than $\max(c_i)$ into $\dot{C}$ and with β equal to the supremum of the $\max(c_i)$'s. As X^α_{1+2i+1} is stationary (in the ground model) we can choose such a β in X^α_{1+2i+1}, which proves that the latter set is indeed stationary in the generic extension.
(2) Just like (1).
(3) Using (1) and (2), another Sacks(α) $*$ Code(α)-generic would give rise to an inner model of $V[G(< \alpha)][S(\alpha)][C(\alpha)]$ in which some stationary set of $V[G(< \alpha)][S(\alpha)][C(\alpha)]$ is not stationary, a contradiction. □ (Lemma 8)

Our aim is to show that if G is P-generic over V then there are exactly two normal measures on κ in $V[G]$.

Lemma 9 *In $V[G]$ there are precisely two $G^* \subseteq j(P)$ which are $j(P)$-generic over M and which contain $j[G]$ as a subset.*

Proof. Write $G^*(j(\kappa))$ as $S^*(j(\kappa)) * C^*(j(\kappa))$. By the results of [8] (see Lemmas 4 and 6 of that paper), given a choice for $G^*(< j(\kappa))$, there are exactly two possibilities for $S^*(j(\kappa))$, each of which is generic. And given a choice for $G^*(< j(\kappa)) * S^*(j(\kappa))$, there is exactly one generic choice for $C^*(j(\kappa))$, as the forcing $\mathrm{Code}(\kappa)$ is $< \kappa^+$-distributive and therefore the image (under the extension to $V[G(<\kappa)*S(\kappa)]$ of j) of the generic $C(\kappa)$ for $\mathrm{Code}(\kappa)$ specified by $G(\kappa)$ generates a generic $C^*(j(\kappa))$. So it suffices to show that there is exactly one generic choice for $G^*(< j(\kappa))$.

Let P^* denote $j(P)$, and for any $p \in P^*(< j(\kappa))$, let $p(\leq \kappa)$ denote $p \upharpoonright \kappa+1$ and $p(> \kappa)$ denote $p \upharpoonright (\kappa, j(\kappa))$. Note that for any dense $D^* \subseteq P^*(< j(\kappa))$ in M, there is a condition $\bar{p} \in G(< \kappa)$ such that $j(\bar{p}) = p$ *reduces* D^* into $P^*(\leq \kappa) = P^*(< \kappa) * P^*(\kappa)$, in the sense that $\{q \in P^*(\leq \kappa) \mid q \cup p(> \kappa)$ meets $D^*\}$ is dense in $P^*(\leq \kappa)$ below $p(\leq \kappa)$: D^* is of the form $j(f)(\kappa)$ where $f : \kappa \to V$ and $f(i)$ is dense on $P(< \kappa)$ for each $i < \kappa$. For each inaccessible $i < \kappa$, any condition in $P(< \kappa)$ can be extended strictly above i to reduce $f(i)$ into $P(\leq i)$, as the forcing $P(\leq i)$ has a dense subset of size i^{++} and the forcing $P(> i)$ is i^{+++}-closed. It follows that the set of $\bar{p} \in P(< \kappa)$ such that $\bar{p}$ reduces $f(i)$ into $P(\leq i)$ for each inaccessible $i < \kappa$ is dense, as using Lemma 4 we can successively extend a given $p \in P(< \kappa)$ in κ steps, taking greatest lower bounds at limit stages and only extending strictly above i at each stage i to reduce $f(i)$ into $P(\leq i)$. Thus as $G(< \kappa)$ is $P(< \kappa)$-generic, there is a condition $\bar{p} \in G(< \kappa)$ which reduces each $f(i)$, i inaccessible and less than κ, into $P(\leq i)$. It follows that $p = j(\bar{p})$ reduces D^* into $P^*(\leq \kappa)$, as desired.

In particular, $G^*(< j(\kappa))$ is uniquely determined by $G^*(\leq \kappa)$, as it must contain each $j(\bar{p})$, $\bar{p} \in G(< \kappa)$ and these conditions, together with $G^*(\leq \kappa)$, decide which conditions belong to $G^*(< j(\kappa))$. So it only remains to show that there is only one generic choice for $G^*(\leq \kappa)$. Clearly there is only one choice for $G^*(< \kappa)$, as this must equal $G(< \kappa)$. And there is only one choice for $G^*(\kappa)$ by Lemma 8(3). $\square$ (Lemma 9)

To complete the proof of Theorem 1 in the case $\alpha = 2$ we show:

Lemma 10 *Suppose that U^* is any normal measure on κ in $V[G]$. Then U^* is the normal measure derived from an embedding $j^* : V[G] \to M^*$ where j^* extends j.*

Proof. Suppose that U^* is a normal measure on κ in $V[G]$. Let $k^* : V[G] \to N^*$ be the ultrapower of $V[G]$ via U^*. Then N^* is of the form $N[H^*]$ where $k = k^* \restriction V : V \to N$ and H^* is $k(P)$-generic over N, $k[G] \subseteq H^*$.

Now by [9], the embedding $k : V \to N$ is obtained by iterating the measure U. We claim that in fact, k equals j, which proves the lemma. Otherwise, $j(\kappa)$ is less than $k(\kappa)$ and therefore can be written as $k^*(f^*)(\kappa)$ for some function $f^* : \kappa \to \kappa$ in $V[G]$. By Lemma 6, choose $f : \kappa \to \kappa$ in V which dominates f^*. Thus $k^*(f^*)(\kappa) < k^*(f)(\kappa) = k(f)(\kappa)$ and therefore $k(f)(\kappa)$ is greater than $j(\kappa)$. But we can write write k as $k' \circ j$, where k' is obtained by iterating the measure $j(U)$ at $j(\kappa)$ and therefore has critical point $j(\kappa)$. As $j(f)(\kappa) < j(\kappa)$, it follows that $k(f)(\kappa) = k'(j(f)(\kappa)) = j(f)(\kappa)$, a contradiction. □ (Lemma 10)

We now consider the cases $\alpha > 2$. Let us say that a subtree T of $\kappa^{<\kappa}$ is *suitable* iff it is closed under initial segments and under unions of length less than κ.

If α is greater than 2 but still less than κ, then instead of using binary-splitting κ-trees, use fully α-splitting κ-trees, i.e., suitable subtrees T of $\alpha^{<\kappa}$ with the property that for some closed unbounded $C \subseteq \kappa$, $s * i$ belongs to T whenever $s \in T$ has length in C and i is less than α . Then everything works in the same way, except there are now exactly α-many choices for G^*, depending on the value at κ for the generic chosen by $G^*(j(\kappa))$ for α-splitting $j(\kappa)$-Sacks forcing.

If α equals κ then we use suitable subtrees T of $\kappa^{<\kappa}$ such that for some closed unbounded $C \subseteq \kappa$, $s * i$ belongs to T whenever the length of $s \in T$ belongs to C and i is less than the length of s. Now there are exactly κ-many possible values at κ for the (new kind of) $j(\kappa)$-Sacks generic chosen by $G^*(j(\kappa))$.

If α is κ^+ then we use suitable subtrees T of $\kappa^{<\kappa}$ with the property that for some closed unbounded $C \subseteq \kappa$, $s * i$ belongs to T whenever $|s|$ = the length of s belongs to C and i is less than $|s|^+$. Then there are exactly κ^+-many choices for the $j(\kappa)$-Sacks product generic chosen by $G^*(j(\kappa))$. Finally, if α equals κ^{++} then none of the above arguments are needed, as the result was already established in [10] (also see Lemma 6 of [4], which shows that one can simply add a single α-Cohen set to each inaccessible $\alpha \le \kappa$ with Easton support to obtain the desired result).

When GCH fails

We now consider the possible number of normal measures on a cardinal κ when GCH fails. Over a model of GCH, we can obviously force $2^{\kappa^{++}}$ to be any cardinal of cofinality greater than κ^{++} without adding normal measures; so the interesting questions concern the values of 2^κ and 2^{κ^+}. We shall focus on obtaining β normal measures on κ for cardinals $\beta \le 2^\kappa$, as for larger β one does not need the methods of this paper, but only the simpler methods of [10].

Regarding a failure of the GCH at κ^+, we have the following.

Theorem 11 *Suppose that $V = L[U]$ where U is a normal measure on κ, β is a cardinal at most κ^+ and γ is a cardinal of cofinality greater than κ^+. Then in a cofinality-preserving generic extension there are exactly β normal measures on κ and $2^{\kappa^+} = \gamma$.*

Proof. First perform the cofinality-preserving iteration P above to obtain a model where there are exactly β normal measures on κ and GCH holds. Then force with $\mathrm{Add}(\kappa^+, \gamma)$, the forcing that adds γ-many κ^+-Cohen sets via a κ-support product. As in Lemma 10, if G is generic for $P * \mathrm{Add}(\kappa^+, \gamma)$, then any measure ultrapower embedding $k^* : V[G] \to N^*$ is a lifting of j, the ultrapower of V by U. And as in Lemma 9, using the fact that the forcing $\mathrm{Add}(\kappa^+, \gamma)$ is $< \kappa^+$-closed ($< \kappa^+$-distributivity is enough) there are exactly β such liftings, giving rise to exactly β-many normal measures on κ. $\Box$ (Theorem 11)

Regarding a failure of the GCH at κ we begin with the following result.

Theorem 12 *Assume the consistency of a $P_2\kappa$-hypermeasurable, i.e., a cardinal κ for which there is an elementary embedding $j : V \to M$ with critical point κ and $V_{\kappa+2}$ contained in M. Then it is consistent that for some κ, $2^\kappa = \kappa^{++}$ and there is a unique normal measure on κ.*

Proof. We may assume that V is a "minimal extender model with a $P_2\kappa$-hypermeasurable", i.e., $V = K = L[E]$ where E is a coherent sequence of extenders and only the last extender F on the E-sequence witnesses that its critical point κ is $P_2\kappa$-hypermeasurable (see [13]). Moreover, V satisfies GCH and in any generic extension $V[G]$ of V, each $k : K \to K^*$ is obtained by normally iterating the extenders on the E-sequence. Let κ denote the critical point of F and let $j : V \to M$ denote the ultrapower of V via F. Thus $V_{\kappa+2}$, or equivalently $H(\kappa^{++})$, belongs to M.

We again describe an iteration P of length $\kappa + 1$ which is nontrivial only at inaccessible $\alpha \leq \kappa$.

In [8] the product with supports of size α of α^{++}-many copies of Sacks(α) was used at each inaccessible stage $\alpha \leq \kappa$ to preserve the measurability of κ and force $2^\kappa = \kappa^{++}$. This forcing however creates many new normal measures on κ and therefore we modify it as follows. Let Sacks$^*(\alpha)$ denote the forcing with *$*$-perfect α-trees*, i.e., subtrees T of $2^{<\alpha}$ which are closed under initial segments, closed under increasing sequences of length less than α and with the property that for some closed unbounded $C \subseteq \alpha$, all nodes of T of length a *singular* element of C are splitting nodes of T. Then like Sacks(α), Sacks$^*(\alpha)$ is $< \alpha$-closed and satisfies α-fusion. Its advantage for the present proof is that whereas use of Sacks(α) gives rise to at least two possible liftings $j^* : V[G] \to M[G^*]$ of j, corresponding to the two different values at κ of the generic specified by $G^*(j(\kappa))$ for Sacks($j(\kappa)$), use of Sacks$^*(\alpha)$ imposes a unique value at κ for the generic specified by $G^*(j(\kappa))$ for Sacks$^*(j(\kappa))$, and therefore helps to guarantee a unique lifting. We will however still need a version of the Code(α) forcing to guarantee a unique choice for $G^*(\kappa)$.

Thus for inaccessible $\alpha \leq \kappa$ we take $P(\alpha)$ to be Sacks$^*(\alpha, \alpha^{++}) *$ Code(α), where Sacks$^*(\alpha, \alpha^{++})$ is the product of α^{++}-many copies of Sacks$^*(\alpha)$ with support of size α, and Code(α), defined below, is the natural analogue of the Code(α) forcing used in the proof of Theorem 1. The forcing Sacks$^*(\alpha, \alpha^{++})$ is $< \alpha$-closed, obeys α-fusion, is α^{++}-cc and therefore preserves cofinalities (the proof is exactly as for Sacks(α, α^{++})). A generic for it corresponds to a sequence $\langle S(i) \mid i < \alpha^{++} \rangle$, where each $S(i) \subseteq \alpha$ is Sacks$^*(\alpha)$-generic over $V[G(<\alpha)]$.

As in Lemma 3, we may use the good condensation properties of the

$L[E]$-hierarchy (see [13]) to obtain a $\diamondsuit_{\kappa^{++}}$-sequence $\vec{S} = \langle S_\beta \mid \beta < \kappa^{++}\rangle$ on cofinality κ^+ which is definable over $H(\kappa^{++})$ from the parameter κ, and therefore belongs to M. Moreover $\vec{S}$ is of the form $j(f)(\kappa)$ where f has domain κ. Now for $i < \kappa^{++}$ let X_i^κ be the set of $\beta < \kappa^{++}$ of cofinality κ^+ such that $S_\beta = \{i\}$. Then the sequence $\langle X_i^\kappa \mid i < \kappa^{++}\rangle$ of disjoint stationary sets is of the form $j(g)(\kappa)$ where g has domain κ and $g(\alpha) = \langle X_i^\alpha \mid i < \alpha^{++}\rangle$ is a sequence of disjoint stationary subsets of $\alpha^{++} \cap \mathrm{Cof}(\alpha^+)$ for each inaccessible $\alpha < \kappa$. For inaccessible $\alpha \leq \kappa$ we use $\langle X_i^\alpha \mid i < \alpha^{++}\rangle$ to define $\mathrm{Code}(\alpha)$.

Let $S(\alpha, \alpha^{++}) = \langle S(i) \mid i < \alpha^{++}\rangle$ denote the generic added by the forcing $\mathrm{Sacks}^*(\alpha, \alpha^{++})$. A condition in $\mathrm{Code}(\alpha)$ is a closed, bounded subset c of α^{++}. For conditions c, d in $\mathrm{Code}(\alpha)$ we say that d *extends* c, written $d \leq c$, iff:

1. d end-extends c.
2. For $i, j < \alpha^{++}$, if j belongs to $S(i)$ then $d \setminus c$ is disjoint from $X^\alpha_{1+4\langle i,j\rangle}$; if j does not belong to $S(i)$ then $d \setminus c$ is disjoint from $X^\alpha_{1+4\langle i,j\rangle+1}$.
3. For $i < \max(c)$, if i belongs to c then $d \setminus c$ is disjoint from X^α_{1+4i+2}; if i does not belong to c then $d \setminus c$ is disjoint from X^α_{1+4i+3}.

In the previous definition, $\langle \cdot, \cdot \rangle$ denotes the Gödel pairing function on the ordinals. This completes the definition of $P(\alpha) = \mathrm{Sacks}^*(\alpha, \alpha^{++}) * \mathrm{Code}(\alpha)$ for inaccessible $\alpha \leq \kappa$. The iteration P is the length $\kappa + 1$ iteration of the $P(\alpha)$'s with nonstationary support. In analogy to the previous proof, we have the following lemmas:

Lemma 13 *P preserves cofinalities.*

Proof. As in the proof of Lemma 5, we show that for each infinite regular α, if an ordinal has cofinality greater than α then it still does after forcing with P. As P factors as $P(<\alpha) * \mathrm{Sacks}^*(\alpha, \alpha^{++}) * \mathrm{Code}(\alpha) * P(\alpha, \kappa+1)$ where $\mathrm{Code}(\alpha) * P(\alpha, \kappa + 1)$ is $< \alpha^+$-closed and $\mathrm{Sacks}^*(\alpha, \alpha^{++})$ obeys α-fusion, it suffices to prove the result for $P(< \alpha)$. This is done exactly as in the proof of Lemma 5. $\Box$ (Lemma 13)

Lemma 14 *If G is P-generic, then for any function $f^* : \kappa \to Ord$ in $V[G]$ there is a function $g : \kappa \to [Ord]^{<\kappa}$ in V such that for each $\alpha < \kappa$, $f^*(\alpha)$ is an element of $g(\alpha)$ and $g(\alpha)$ has cardinality α^{++}.*

Proof. Factor P as $P(<\kappa) * \text{Sacks}^*(\kappa, \kappa^{++}) * \text{Code}(\kappa)$ and correspondingly write $V[G]$ as $V[G(<\kappa)][S(\kappa, \kappa^{++})][C(\kappa)]$. As $\text{Code}(\kappa)$ is $<\kappa^+$-distributive, the given function f^* in fact belongs to $V[G(<\kappa)][S(\kappa, \kappa^{++})]$. Recall that $\text{Sacks}^*(\kappa, \kappa^{++})$ obeys κ-fusion. This implies that if $\dot{f}$ is a name for a function from κ into the ordinals and p is a condition in $\text{Sacks}^*(\kappa, \kappa^{++})$, then there is $q \leq p$ forcing that for each $\alpha < \kappa$, $\dot{f}(\alpha)$ takes one of at most card $((2^\alpha)^\alpha) = \alpha^+$-many values (corresponding to choices of nodes on the α-th splitting level of at most α-many trees). Thus there is a function g as in the statement of the lemma which belongs to $V[G(<\kappa)]$. And the argument given in the proof of Lemma 5 shows that if p belongs to $P(<\alpha)$ and $\dot{g}$ is a name for a function from κ into the $[\text{Ord}]^{<\kappa}$ then there is $q \leq p$ and a closed unbounded subset C of κ such that for α in C, q forces $\dot{g}(\alpha)$ to take one of at most α^{++}-many values (corresponding to choices for conditions in $P(<\alpha) * P(\alpha)$). We may further extend q at nonstationary-many places to force such a bound on the number of possible values for $\dot{g}(\alpha)$ for all $\alpha < \kappa$. Then putting these two approximation results together we obtain the lemma. □ (Lemma 14)

Lemma 15 *For inaccessible $\alpha \leq \kappa$, $P(<\alpha) * \text{Sacks}^*(\alpha, \alpha^{++})$ preserves the stationarity of subsets of α^{++}.*

Proof. This is clear, because $P(<\alpha)$ has a dense subset of size α^+ and $\text{Sacks}^*(\alpha, \alpha^{++})$ is (forced by $P(<\alpha)$ to be) α^{++}-cc. □ (Lemma 15)

Lemma 16 *Suppose that g is Code(α)-generic (over $V[G(<\alpha)][S(\alpha, \alpha^{++})]$, where $S(\alpha, \alpha^{++}) = \langle S(i) \mid i < \alpha^{++} \rangle$) and let $C(\alpha)$ be the union of the conditions in g. Then in the generic extension we have:*
(1) For $i, j < \alpha^{++}$, j belongs $S(i)$ iff $X^\alpha_{1+4\langle i,j\rangle}$ is nonstationary and j does not belong to $S(i)$ iff $X^\alpha_{1+4\langle i,j\rangle+1}$ is nonstationary.
(2) For $i < \alpha^{++}$, i belongs to $C(\alpha)$ iff X^α_{1+4i+2} is nonstationary and i does not belong to $C(\alpha)$ iff X^α_{1+4i+3} is nonstationary.
(3) There is a unique $\text{Sacks}^*(\alpha, \alpha^{++}) * \text{Code}(\alpha)$-generic over $V[G(<\alpha)]$.

Proof. Just as in the proof of Lemma 8. □ (Lemma 16)

Now we complete the proof of the Theorem 12 as follows.

Lemma 17 *In $V[G]$ there is precisely one $G^* \subseteq j(P)$ which is $j(P)$-generic over M and which contains $j[G]$ as a subset.*

Proof. We argue as in Lemma 9. $G^*(<\kappa)$ is unique as it must equal $G(<\kappa)$, which is indeed generic for $j(P)(<\kappa) = P(<\kappa)$. And $G^*(\kappa)$ is the unique generic for $j(P)(\kappa) = P(\kappa)$ by Lemma 16. Using nonstationary supports as in Lemma 9, it then follows that $G^*(< j(\kappa))$ is the unique generic for $P^*(< j(\kappa))$ containing $j[G(<\kappa)]$. Therefore we must only argue that there is a unique generic choice for $G^*(j(\kappa))$.

In [8], it is shown that if $j^* : V[G(<\kappa)] \to M[G^*(< j(\kappa))]$ is the canonical extension of $j : V \to M$ then the range of j^* on the Sacks(κ, κ^{++})-generic specified by $G(\kappa)$ determines a unique Sacks$(j(\kappa))$-generic $S_0^*(i)$ for $i < j(\kappa^{++})$ not in the range of j and exactly two Sacks$(j(\kappa))$-generics $S_0^*(i), S_1^*(i)$ for i in the range of j. (More precisely, the intersection of the i-th components of the $j^*(\vec{T})$, where $\vec{T}$ belongs to the Sacks(κ, κ^{++})-generic specified by $G(\kappa)$, is a single Sacks$(j(\kappa))^{M[G^*(<j(\kappa))]}$-generic for $i < \kappa^{++}$ not in the range of j, and is the union of two distinct Sacks$(j(\kappa))^{M[G^*(<j(\kappa))]}$-generics for $i < \kappa^{++}$ in the range of j.) Moreover, the sequence $\langle S_0^*(i) \mid i < j(\kappa^{++})\rangle$ is generic for Sacks$(j(\kappa), j(\kappa^{++}))^{M[G(<j(\kappa))]}$ over $M[G(< j(\kappa))]$. The argument used there applied to the present context gives the same result with Sacks replaced by Sacks*, with the only difference that now the range of j^* on the Sacks$^*(\kappa, \kappa^{++})$-generic specified by $G(\kappa)$ determines a unique Sacks$^*(j(\kappa))$-generic $S^*(i)$ for *each* $i < j(\kappa^{++})$, including those $i < j(\kappa^{++})$ in the range of j. The reason is that if i equals $j(\bar{i})$, then some T in in the $\bar{i}$-th component of the Sacks$^*(\kappa, \kappa^{++})$-generic specified by $G(\kappa)$ has the property that all of its splitting nodes are of singular length, and therefore $j^*(T)$, a $j(\kappa)$-tree associated to the i-th component of Sacks$^*(j(\kappa), j(\kappa^{++}))$, has no splitting at κ. Thus we arrive at the desired conclusion: The image under j^* of the Sacks$^*(\kappa, \kappa^{++})$-generic specified by $G(\kappa)$ generates a Sacks$^*(j(\kappa), j(\kappa^{++})$-generic over $M[G^*(< j(\kappa))]$, and therefore there is a unique generic choice for the Sacks$^*(j(\kappa), j(\kappa^{++})$-generic specified by $G^*(j(\kappa))$.

Finally, as Code(κ) is $< \kappa^+$-distributive, the Code$(j(\kappa))$-generic specified by $G^*(j(\kappa))$ is also uniquely determined, as it is generated by the image of the Code(κ)-generic specified by $G(\kappa)$ (under the canonical extension of j to $V[G(<\kappa)][S(\kappa, \kappa^{++})]$, where $S(\kappa, \kappa^{++})$ is the Sacks$^*(\kappa, \kappa^{++})$-generic specified by $G(\kappa)$). $\Box$

Lemma 18 *Suppose that G is P-generic and U^* is any normal measure on κ in $V[G]$. Then U^* is the normal measure derived from an embedding $j^* : V[G] \to M^*$ where j^* extends j.*

Proof. Suppose that $k^* : V[G] \to N^*$ is the ultrapower via the normal measure $U^* \in V[G]$. Then N^* is of the form $N[G^*]$ where $k : V \to N$ is the restriction of k^* to V and G^* is $k(P)$-generic over N. As $V = K$ it follows that k results from a normal iteration of $K = L[E]$ via the extenders on its canonical extender sequence E. We claim that this iteration is in fact just the single ultrapower of K via F, the unique extender on the E-sequence which witnesses the $P_2\kappa$-hypermeasurability of κ, and therefore k equals j.

Clearly the first extender applied in the iteration that produces k has critical point κ. Now if this extender F_0 were not the extender F witnessing the $P_2\kappa$-hypermeasurability of κ then k factors as $k_1 \circ k_0$ where $k_0 : K \to K_0$ is the ultrapower via F_0 and $k_0(\kappa)$ is a strongly inaccessible cardinal of K_0 (and therefore of N) less than κ^{++}. As $N[G^*]$ contains all subsets of κ that belong to $V[G]$ and $2^\kappa = \kappa^{++}$ in $V[G]$, it follows that $k_0(\kappa)$ is not strongly inaccessible in $N[G^*]$. But G^* is generic over N for the forcing $k(P)$, which preserves strong inaccessibility. This contradiction implies that F_0 is indeed equal to F.

Now we claim that the iteration ends in one step, i.e., no extender with critical point greater than κ is applied in the iteration. (Recall that in a normal iteration, critical points strictly increase over the stages of the iteration.) For, suppose that k factors as $h \circ j$, where $h : M \to N$ is an iteration map with critical point $\lambda > \kappa$. Then λ is of the form $k^*(f^*)(\kappa)$ for some $f^* : \kappa \to \mathrm{Ord}$ in $V[G]$, as $k^* : V[G] \to N[G^*]$ is given by a measure ultrapower. By Lemma 14 there is a function g in V with domain κ such that $f^*(\alpha) \in g(\alpha)$ and $g(\alpha)$ has cardinality α^{++}, for each $\alpha < \kappa$. Thus λ is an element of $k^*(g)(\kappa) = k(g)(\kappa) = h(j(g))(\kappa) = h(j(g))(h(\kappa)) = h(j(g)(\kappa))$, and $j(g)(\kappa)$ has M-cardinality κ^{++}. In particular, $j(g)(\kappa)$ has M-cardinality less than the critical point of h and therefore $h(j(g)(\kappa))$ is included in the range of h. But this is impossible, as λ belongs to $h(j(g)(\kappa))$ and λ is the critical point of h. $\square$ (Lemma 18)

This completes the proof of Theorem 12.

It is not difficult to derive other possibilities for the number of normal measures on a cardinal where the GCH fails.

Theorem 19 *Suppose that V is a minimal extender model with a $P_2\kappa$-hypermeasurable and β is a cardinal at most κ^{++}. Then there is a cofinality-preserving generic extension in which $2^\kappa = \kappa^{++}$ and there are exactly β normal measures on κ.*

Proof. The case $\alpha = 0$ is easy, and the case $\alpha = 1$ is Theorem 12, whose proof uses an iteration of the forcings $\text{Sacks}^*(\alpha, \alpha^{++}) * \text{Code}(\alpha)$ for inaccessible $\alpha \leq \kappa$. To obtain exactly β normal measures, we consider the forcing $\text{Sacks}^\beta(\alpha)$, defined as follows: If β is less than α then a condition is a subtree T of $\alpha^{<\alpha}$ which is suitable (i.e., closed under initial segments and under unions of increasing sequences of length less than α) and with the property that for some closed unbounded $C \subseteq \alpha$, $s * i$ belongs to T whenever $s \in T$ has length in C and i is less than β. If β lies in the interval $[\alpha, \kappa)$ then we take $\text{Sacks}^\beta(\alpha)$ to be the trivial forcing. If β equals κ, κ^+ or κ^{++} then we replace "i is less than β" with "i is less than $|s|$, $|s|^+$ or $|s|^{++}$", respectively (where $|s|$ denotes the length of s). Now instead of iterating $\text{Sacks}^*(\alpha, \alpha^{++}) * \text{Code}(\alpha)$ for inaccessible $\alpha \leq \kappa$, we iterate $\text{Sacks}^*(\alpha, \alpha^{++}) * \text{Sacks}^\beta(\alpha) * \text{Code}(\alpha)$, where $\text{Code}(\alpha)$ is defined as before, but now codes not only the $\text{Sacks}^*(\alpha, \alpha^{++})$-generic but also the $\text{Sacks}^\beta(\alpha)$-generic. The rest of the proof works as before, except now there are exactly β-many choices for the $\text{Sacks}^{j(\beta)}(j(\kappa))$-generic specified by $G^*(j(\kappa))$, one for each possible value of this generic at κ. Thus we get exactly β normal measures on κ in the generic extension. $\Box$ (Theorem 19)

Of course the above work concerning the number of normal measures on κ when $2^\kappa = \kappa^{++}$ generalises readily to many other values of 2^κ. Here is a sample result, whose proof involves no major new ideas.

Theorem 20 *Suppose that $V = L[E]$ is an extender model with a last extender F and let $j_F : V \to M_F$ be the ultrapower via F. Suppose that F has critical point κ, $f : \kappa \to \kappa$, $f(\alpha)$ is a cardinal of cofinality greater than α for each $\alpha < \kappa$, $H(j_F(f)(\kappa))$ is included in M_F (i.e., F witnesses that κ is "f-hypermeasurable") and F is the only extender on the E-sequence with this property. Also suppose that $\beta \leq j_F(f)(\kappa)$ is a cardinal of the form $j_F(g)(\kappa)$*

for some $g : \kappa \to \kappa$. Let γ denote $j_F(f)(\kappa)$. Then there is a cofinality-preserving generic extension in which $2^\kappa = \gamma$ and there are exactly β normal measures on κ.

For example, Theorem 20 can be used to obtain a model in which 2^κ is the least weakly Mahlo cardinal greater than κ and there are exactly λ normal measures on κ, where λ is the least weakly inaccessible greater than κ.

We end with some open questions.

Q1. Can the hypothesis of Theorem 12 be weakened to "$o(\kappa) = \kappa^{++}$"?
Q2. Is it consistent that the number of normal measures on some cardinal κ be a cardinal greater than 2^κ but of cofinality at most κ^+?
Q3. Do similar results hold for normal measures on $P_\kappa\lambda$ when λ is greater than κ?

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