PROJECTIVE MAD FAMILIES

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ABSTRACT. Using almost disjoint coding we prove the consistency of the existence of a Π^1_2 definable ω-mad family of infinite subsets of ω (resp. functions from ω to ω) together with $\mathfrak{b} = 2^\omega = \omega_2$.

1. INTRODUCTION

A classical result of Mathias [7] states that there exists no Σ^1_1 definable mad family of infinite subsets of ω. One of the two main results of [4] states that there is no Σ^1_1 definable ω-mad family of functions from ω to ω. It is the purpose of this paper to analyse how low in the projective hierarchy one can consistently find a mad subfamily of $[\omega]^\omega$ or ω^ω.

Recall that $a, b \in [\omega]^\omega$ are called *almost disjoint*, if $a \cap b$ is finite. An infinite set A is said to be an *almost disjoint family* of infinite subsets of ω (or an almost disjoint subfamily of $[\omega]^\omega$) if $A \subset [\omega]^\omega$ and any two elements of A are almost disjoint. A is called a *mad family* of infinite subsets of ω (abbreviated from "maximal almost disjoint"), if it is maximal with respect to inclusion among almost disjoint families of infinite subsets of ω. Given an almost disjoint family $A \subset [\omega]^\omega$, we denote by $\mathcal{L}(A)$ the set $\{b \in [\omega]^\omega : b$ is not covered by finitely many $a \in A\}$. Following [6] we define a mad subfamily A of $[\omega]^\omega$ to be *ω-mad*, if for every $B \in [\mathcal{L}(A)]^\omega$ there exists $a \in A$ such that $|a \cap b| = \omega$ for all $b \in B$.

Two functions $a, b \in \omega^\omega$ are called *almost disjoint*, if they are almost disjoint as subsets of $\omega \times \omega$, i.e. $a(k) \neq b(k)$ for all but finitely many $k \in \omega$. A set A is said to be an *almost disjoint family* of functions (or an almost disjoint subfamily of ω^ω) if $A \subset \omega^\omega$ and any two elements of A are almost disjoint. A is called a *mad family* of functions, if it is maximal with respect to inclusion among almost disjoint families of functions. Given an almost disjoint family $A \subset \omega^\omega$, we denote by $\mathcal{L}(A)$ the set $\{b \in \omega^\omega : b$ is not covered by finitely many $a \in A\}$. A mad subfamily A of ω^ω is *ω-mad*[1], if for every $B \in [\mathcal{L}(A)]^\omega$ there exists $a \in A$ such that $|a \cap b| = \omega$ for all $b \in B$.

The following theorems are the main results of this paper.

Theorem 1. *It is consistent that $2^\omega = \mathfrak{b} = \omega_2$ and there exists a Π^1_2 definable ω-mad family of infinite subsets of ω.*

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[1]Such families of functions are called *strongly maximal* in [4, 9]. We call them ω-mad just to keep the analogy with the case of subsets of $[\omega]^\omega$.

Theorem 2. *It is consistent that $2^\omega = \mathfrak{b} = \omega_2$ and there exists a Π^1_2 definable ω-mad family of functions.*

By [8, Theorem 8.23], in L there exists a mad subfamily of $[\omega]^\omega$ which is Π^1_1 definable. Moreover, $V = L$ implies the existence of a Π^1_1 definable ω-mad subfamily A of ω^ω, see [4, § 3]. It is easy to check that $A \cup \{\{n\} \times \omega : n \in \omega\}$ is actually an ω-mad family of subsets of $\omega \times \omega$ for every ω-mad subfamily A of ω^ω, and hence Π^1_1 definable ω-mad subfamilies of $[\omega]^\omega$ exist under $V = L$ as well.

Regarding the models of $\neg$CH, it is known that ω-mad subfamilies of $[\omega]^\omega$ remain so after adding any number of Cohen subsets, see [5] and references therein. Combining Corollary 53 and Theorem 65 from [9], we conclude that the ground model ω-mad families of functions remain so in forcing extensions by countable support iterations of a wide family of posets including Sacks and Miller forcings. If $A \in V$ is a Π^1_1 definable almost disjoint family whose Π^1_1 definition is provided by formula $\varphi(x)$, then $\varphi(x)$ defines an almost disjoint family in any extension V' of V (this is a straightforward consequence of the Shoenfield's Absoluteness Theorem). Thus if a ground model Π^1_1 definable mad family *remains mad* in a forcing extension, it remains Π^1_1 definable by means of the same formula. From the above it follows that the Π^1_1 definable ω-mad family in L of functions constructed in [4, § 3] remains Π^1_1 definable and ω-mad in $L[G]$, where G is a generic over L for the countable support iteration of Miller forcing of length ω_2. Thus the essence of Theorems 1 and 2 is the existence of projective ω-mad families combined with the inequality $\mathfrak{b} > \omega_1$, which rules out all mad families of size ω_1.

It is not known whether in ZFC one can prove the existence of Σ^1_1 mad families of functions or of ω-mad families of functions; see [9].

2. Preliminaries

In this section we introduce some notions and notation needed for the proofs of Theorems 1 and 2, and collect some basic facts about T-proper posets, see [2] for more details.

Proposition 3. (1) *There exists an almost disjoint family $R = \{r_{\langle\zeta,\xi\rangle} : \zeta \in \omega \cdot 2, \xi \in \omega_1^L\} \in L$ of infinite subsets of ω such that $R \cap M = \{r_{\langle\zeta,\xi\rangle} : \zeta \in \omega \cdot 2, \xi \in (\omega_1^L)^M\}$ for every transitive model M of* ZF$^-$.

(2) *There exists an almost disjoint family $\mathcal{F} = \{f_{\langle\zeta,\xi\rangle} : \zeta \in \omega \cdot 2, \xi \in \omega_1^L\} \in L$ of functions such that $\mathcal{F} \cap M = \{f_{\langle\zeta,\xi\rangle} : \zeta \in \omega \cdot 2, \xi \in (\omega_1^L)^M\}$ for every transitive model M of* ZF$^-$.

Proof sketch. Let $r^*_{\zeta,\xi}$ be the L-least real coding the ordinal $(\omega^2 \cdot \xi) + \zeta$ and let $r_{\zeta,\xi}$ be the set of numbers coding a finite initial segment of $r^*_{\zeta,\xi}$. Similarly for functions. □

One of the main building blocks of the required ω-mad family will be suitable sequences of stationary in L subsets of ω_1 given by the following proposition which may be proved in the same way as [1, Lemma 14].

Say that a transitive ZF$^-$ model M is *suitable* iff $M \vDash$"ω_2 exists and $\omega_2 = \omega_2^L$".

Proposition 4. *There exists a* Σ_1 *definable over* L_{ω_2} *tuple* $\langle T_0, T_1, T_2\rangle$ *of mutually disjoint* L*-stationary subsets of* ω_1 *and* Σ_1 *definable over* L_{ω_2} *sequences* $\bar S = \langle S_\alpha : \alpha < \omega_2\rangle$, $\bar S' = \langle S'_\alpha : \alpha < \omega_2\rangle$ *of pairwise almost disjoint* L*-stationary subsets of* ω_1 *such that*

- $S_\alpha \subset T_2$ *and* $S'_\alpha \subset T_1$ *for all* $\alpha \in \omega_2$*;*
- *Whenever* M, N *are suitable models of ZF*$^-$ *such that* $\omega_1^M = \omega_1^N$, $\bar S^M$ *agrees with* $\bar S^N$ *on* $\omega_2^M \cap \omega_2^N$*. Similarly for* $\bar S'$*.*

The following standard fact gives an absolute way to code an ordinal $\alpha < \omega_2$ by a subset of ω_2.

Fact 5. *There exists a formula* $\phi(x, y)$ *and for every* $\alpha < \omega_2^L$ *a set* $X_\alpha \in ([\omega_1]^{\omega_1})^L$ *such that*

(1) *For every suitable model* M *containing* $X_\alpha \cap \omega_1^M$, $\phi(x, X_\alpha \cap \omega_1^M)$ *has a unique solution in* M*, and this solution equals* α *provided* $\omega_1^M = \omega_1^L$*;*
(2) *For arbitrary suitable models* M, N *with* $\omega_1^M = \omega_1^N$ *and* $X_\alpha \cap \omega_1^M \in M \cap N$*, the solutions of* $\phi(x, X_\alpha \cap \omega_1^M)$ *in* M *and* N *coincide*[2]*.*

Let γ be a limit ordinal and $r : \gamma \to 2$. We denote by $\mathrm{Even}(r)$ the set $\{\alpha < \gamma : r(2\alpha) = 1\}$. For ordinals $\alpha < \beta$ we shall denote by $\beta - \alpha$ the ordinal γ such that $\alpha + \gamma = \beta$. If B is a set of ordinals above α, then $B - \alpha$ stands for $\{\beta - \alpha : \beta \in B\}$. Observe that if ζ is an indecomposable ordinal (e.g., ω_1^M for some countable suitable model of ZF$^-$), then $((\alpha + B) \cap \zeta) - \alpha = B \cap \zeta$ for all B and $\alpha < \zeta$. This will be often used for $B = X_\alpha$.

For $x, y \in \omega^\omega$ we say that y *dominates* x and write $x \le^* y$ if $x(n) \le y(n)$ for all but finitely many $n \in \omega$. The minimal size of a subset B of ω^ω such that there is no $y \in \omega^\omega$ dominating all elements of B is denoted by $\mathfrak{b}$. It is easy to see that $\omega < \mathfrak{b} \le 2^\omega$. We say that a forcing notion $\mathbb{P}$ *adds a dominating real* if there exists $y \in \omega^\omega \cap V^{\mathbb{P}}$ dominating all elements of $\omega^\omega \cap V$.

Definition 6. Let $T \subset \omega_1$ be a stationary set. A poset $\mathbb{P}$ is T*-proper*, if for every countable elementary submodel $\mathcal{M}$ of H_θ, where θ is a sufficiently large cardinal, such that $\mathcal{M} \cap \omega_1 \in T$, every condition $p \in \mathbb{P} \cap \mathcal{M}$ has an $(\mathcal{M}, \mathbb{P})$-generic extension q.

The following theorem includes some basic properties of T-proper posets.

Theorem 7. *Let* T *be a stationary subset of* ω_1*.*

(1) *Every* T*-proper poset* $\mathbb{P}$ *preserves* ω_1*. Moreover,* $\mathbb{P}$ *preserves the stationarity of every stationary set* $S \subset T$*.*
(2) *Let* $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\zeta : \xi \le \delta, \zeta < \delta\rangle$ *be a countable support iteration of* T*-proper posets. Then* $\mathbb{P}_\delta$ *is* T*-proper. If, in addition, CH holds in* V*,* $\delta \le \omega_2$*, and the* $\dot{\mathbb{Q}}_\zeta$*'s are forced to have size at most* ω_1*, then* $\mathbb{P}_\delta$ *is* ω_2*-c.c. If, moreover,* $\delta < \omega_2$*, then CH holds in* $V^{\mathbb{P}_\delta}$*.*

[2] In what follows the phrase "X codes an ordinal β in a suitable ZF$^-$ model M" means that there exists $\alpha < \omega_2^L$ such that $X = \omega_1^M \cap X_\alpha \in M$ and $\phi(\beta, X)$ holds in M.

3. Proof of Theorem 1

We start with the ground model $V = L$. Recursively, we shall define a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$. The desired family A is constructed along the iteration: for cofinally many α's the poset $\mathbb{Q}_\alpha$ takes care of some countable family B of infinite subsets of ω which might appear in $\mathcal{L}(A)$ in the final model, and adds to A some $a_\alpha \in [\omega]^\omega$ almost disjoint from all elements of A_α such that $|a \cap b| = \omega$ for all $b \in B$ (here A_α stands for the set of all elements of A constructed up to stage α). Our forcing construction will have some freedom allowing for further applications.

We proceed with the definition of $\mathbb{P}_{\omega_2}$. For successor α let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for some proper forcing of size ω_1 adding a dominating real. For a subset s of ω and $l \in |s|$ ($= \mathrm{card}(s) \leq \omega$) we denote by $s(l)$ the l'th element of s. In what follows we shall denote by $E(s)$ and $O(s)$ the sets $\{s(2i) : 2i \in |s|\}$ and $\{s(2i+1) : 2i+1 \in |s|\}$, respectively. Let us consider some limit α and a $\mathbb{P}_\alpha$-generic filter G_α. Suppose also that

$$(*) \qquad \forall B \in [A_\alpha]^{<\omega}\, \forall r \in R\, (|E(r) \setminus \cup B| = |O(r) \setminus \cup B| = \omega)$$

Observe that equation $(*)$ yields $|E(r) \setminus \cup B| = |O(r) \setminus \cup B| = \omega$ for every $B \in [R \cup A_\alpha]^{<\omega}$ and $r \in R \setminus B$. Let us fix some function $F : Lim \cap \omega_2 \to L_{\omega_2}$ such that $F^{-1}(x)$ is unbounded in ω_2 for every $x \in L_{\omega_2}$. Unless the following holds, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Suppose that $F(\alpha)$ is a sequence $\langle \dot{b}_i : i \in \omega \rangle$ of $\mathbb{P}_\alpha$-names such that $b_i = \dot{b}_i^{G_\alpha} \in [\omega]^\omega$ and none of the b_i's is covered by a finite subfamily of A_α. In this case $\mathbb{Q}_\alpha := \dot{\mathbb{Q}}_\alpha^{G_\alpha}$ is the two-step iteration $\mathbb{K}^0_\alpha * \dot{\mathbb{K}}^1_\alpha$ defined as follows.

In $V[G_\alpha]$, $\mathbb{K}^0_\alpha$ is some $T_0 \cup T_2$-proper poset of size ω_1. Our proof will not really depend on $\mathbb{K}^0_\alpha$. $\mathbb{K}^0_\alpha$ is reserved for some future applications, see section 5.

Let us fix some $\mathbb{K}^0_\alpha$-generic filter h_α over $V[G_\alpha]$ and find a limit ordinal $\eta_\alpha \in \omega_1$ such that there are no finite subsets J, E of $(\omega \cdot 2) \times (\omega_1 \setminus \eta_\alpha)$, A_α, respectively, and $i \in \omega$, such that $b_i \subset \bigcup_{\langle \zeta, \xi \rangle \in J} r_{\langle \zeta, \xi \rangle} \cup \bigcup E$. (The almost disjointness of the $r_{\langle \zeta, \xi \rangle}$'s imply that if $b_i \subset \bigcup R' \cup \bigcup A'$ for some $R' \in [R]^{<\omega}$ and $A' \in [A_\alpha]^{<\omega}$, then $b_i \setminus \bigcup A'$ has finite intersection with all elements of $R \setminus R'$. Together with equation $(*)$ this easily yields the existence of such an η_α.) Let z_α be an infinite subset of ω coding a surjection from ω onto η_α. For a subset s of ω we denote by $\bar{s}$ the set $\{2k+1 : k \in s\} \cup \{2k : k \in (\sup s \setminus s)\}$. In $V[G_\alpha * h_\alpha]$, $\mathbb{K}^1_\alpha$ consists of sequences $\langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$[3] satisfying the following conditions:

- (i) c_k is a closed, bounded subset of $\omega_1 \setminus \eta_\alpha$ such that $S_{\alpha+k} \cap c_k = \emptyset$ for all $k \in \omega$;
- (ii) $y_k : |y_k| \to 2$, $|y_k| > \eta_\alpha$, $y_k \upharpoonright \eta_\alpha = 0$, and $\mathrm{Even}(y_k) = (\{\eta_\alpha\} \cup (\eta_\alpha + X_\alpha)) \cap |y_k|$;
- (iii) $s \in [\omega]^{<\omega}$, $s^* \in \big[\{r_{\langle m, \xi \rangle} : m \in \bar{s}, \xi \in c_m\} \cup \{r_{\langle \omega+m, \xi \rangle} : m \in \bar{s}, y_m(\xi) = 1\} \cup A_\alpha\big]^{<\omega}$. In addition, for every $2n \in |s \cap r_{\langle 0,0 \rangle}|$, $n \in z_\alpha$ if and only if there exists $m \in \omega$ such that $(s \cap r_{\langle 0,0 \rangle})(2n) = r_{\langle 0,0 \rangle}(2m)$; and

[3]The tuples $\langle s, s^* \rangle$ and $\langle c_k, y_k : k \in \omega \rangle$ will be referred to as the *finite part* and the *infinite part* of the condition $\langle \langle s, s^* \rangle, \langle c_k, y_k : k \in \omega \rangle \rangle$, respectively.

(*iv*) For all $k \in \bar{s} \cup (\omega \setminus (\max\ \bar{s}))$, limit ordinals $\xi \in \omega_1$ such that $\eta_\alpha < \xi \leq |y_k|$, and suitable ZF$^-$ models M containing $y_k \restriction \xi$ and $c_k \cap \xi$ with $\omega_1^M = \xi$, ξ is a limit point of c_k, and the following holds in M: $(\mathrm{Even}(y_k) - \min \mathrm{Even}(y_k)) \cap \xi$ codes a limit ordinal $\bar{\alpha}$ such that $S^M_{\bar{\alpha}+k}$ is non-stationary.

For conditions $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $\vec{q} = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ in $\mathbb{K}^1_\alpha$, we let $\vec{q} \leq \vec{p}$ (by this we mean that $\vec{q}$ is stronger than $\vec{p}$) if and only if

(*v*) (t, t^*) extends (s, s^*) in the almost disjoint coding, i.e. t is an end-extension of s and $t \setminus s$ has empty intersection with all elements of s^*;

(*vi*) If $m \in \bar{t} \cup (\omega \setminus (\max\ \bar{t}))$, then d_m is an end-extension of c_m and $y_m \subset z_m$.

This finishes our definition of $\mathbb{P}_{\omega_2}$. Before proving that the statement of our theorem holds in $V^{\mathbb{P}_{\omega_2}}$ we shall establish some basic properties of $\mathbb{K}^1_\alpha$. In Claims 8, 9, 10, 11, and Corollary 12 below we work in $L[G_\alpha * h_\alpha]$.

Claim 8. (Fischer, Friedman [1, Lemma 1].) *For every condition* $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$ *and every* $\gamma \in \omega_1$ *there exists a sequence* $\langle d_k, z_k : k \in \omega\rangle$ *such that* $\langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$, $\langle\langle s, s^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle \leq \vec{p}$, *and* $|z_k|, \max d_k \geq \gamma$ *for all* $k \in \omega$.

Claim 9. *For every* $\vec{p} \in \mathbb{K}^1_\alpha$ *and open dense* $D \subset \mathbb{K}^1_\alpha$ *there exists* $\vec{q} \leq \vec{p}$ *with the same finite part as* $\vec{p}$ *such that whenever* $\vec{p}_1$ *is an extension of* $\vec{q}$ *meeting* D *with finite part* $\langle r_1, r_1^*\rangle$, *then already some condition* $\vec{p}_2$ *with the same infinite part as* $\vec{q}$ *and finite part* $\langle r_1, r_2^*\rangle$ *for some* r_2^* *meets* D.

Proof. Let $\vec{p} = \langle\langle t_0, t_0^*\rangle, \langle d_k^0, z_k^0 : k \in \omega\rangle\rangle$ and let $\mathcal{M}$ be a countable elementary submodel of H_θ containing $\mathbb{K}^1_\alpha$, $\vec{p}$, X_α, and D, and such that $j := \mathcal{M} \cap \omega_1 \notin \bigcup_{k \in \bar{t_0} \cup (\omega \setminus (\max \bar{t_0}))} S_{\alpha+k}$.

Let $\{\langle \vec{r}_n, s_n\rangle : n \in \omega\}$ be a sequence in which every pair $\langle \vec{r}, s\rangle \in (\mathbb{K}^1_\alpha \cap \mathcal{M}) \times [\omega]^{<\omega}$ with $\vec{p} \geq \vec{r}$ appears infinitely often. Let $\langle j_n : n \in \omega\rangle$ be increasing and cofinal in j. Using Claim 8, by induction on n construct sequences $\langle d_k^n, z_k^n : k \in \omega\rangle \in \mathcal{M}$ as follows:

If there exists $\vec{r}_{1,n} \in D \cap \mathcal{M}$ below both $\vec{r}_n$ and $\langle\langle t_0, t_0^*\rangle, \langle d_k^n, z_k^n : k \in \omega\rangle\rangle$ and with finite part of the form $\langle s_n, s_n^*\rangle$ for some s_n^*, then let $\langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle$ be the infinite part of $\vec{r}_{1,n}$, extended further in such a way that $\langle\langle t_0, t_0^*\rangle; \langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$ and $|z_k^{n+1}|, \max d_k^{n+1} \geq j_n$ for all $n \in \omega$ and $k \in \bar{t_0} \cup (\omega \setminus (\max \bar{t_0}))$. If there is no such $\vec{r}_{1,n}$, then let d_k^{n+1} be an arbitrary end-extension of d_k^n and z_k^{n+1} be an extension of z_k^n such that $|z_k^{n+1}|, \max d_k^{n+1} \geq j_n$ for all $n \in \omega$ and $k \in \bar{t_0} \cup (\omega \setminus (\max \bar{t_0}))$, and $\langle\langle t_0, t_0^*\rangle; \langle d_k^{n+1}, z_k^{n+1} : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$.

Set $d_k = \bigcup_{n\in\omega} d_k^n \cup \{j\}$ and $z_k = \bigcup_{n\in\omega} z_k^n$ for all $k \in \omega \setminus F$, $d_k = z_k = \emptyset$ for $k \in F$, and $\vec{q} = \langle\langle t_0, t_0^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. We claim that $\vec{q}$ is as required.

Let us show first that $\vec{q} \in \mathbb{K}^1_\alpha$. Only item (*iv*) of the definition of $\mathbb{K}^1_\alpha$ for $k \in \bar{t_0} \cup (\omega \setminus (\max \bar{t_0}))$ and $\xi = j$ must be verified. Fix such a k and suitable ZF$^-$ model M containing z_k and d_k with $\omega_1^M = j$. Let $\bar{M}$ be the Mostowski collapse of $\mathcal{M}$ and $\pi : \mathcal{M} \to \bar{M}$ be the corresponding isomorphism. Let us note that $j = \omega_1^M = \omega_1^{\bar{M}}$. Since $X_\alpha \in \mathcal{M}$, and $\mathcal{M}$ is elementary submodel of H_θ, α is the unique solution of $\phi(x, X_\alpha)$ in $\mathcal{M}$, and hence $\bar{\alpha} := \pi(\alpha)$ is the unique solution of $\phi(x, X_\alpha \cap j = \pi(X_\alpha))$ in $\bar{M}$. In addition, $S^{\bar{M}}_{\bar{\alpha}+k} = \pi(S_{\alpha+k}) = S_{\alpha+k} \cap j$ for all $k \in \omega$. Applying Fact 5(2) and Proposition 4, we conclude that $\phi(\bar{\alpha}, X_\alpha \cap j)^M$ holds and $S^M_{\bar{\alpha}+k} = S^{\bar{M}}_{\bar{\alpha}+k} = S_{\alpha+k} \cap j$. Since $d_k \in M$, $d_k \cap S_{\alpha+k} = \emptyset$, and $d_k \setminus \{j\}$ is unbounded in $j = \omega_1^M$ by the construction of d_k, we conclude that $S^M_{\bar{\alpha}+k}$ is not stationary in M. This proves that $\vec{q} \in \mathbb{K}^1_\alpha$.

Now suppose that $\vec{p}_1 = \langle\langle r_1, r_1^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle \leq \vec{q}$ and $\vec{p}_1 \in D$. Since r_1, r_1^* are finite, there exists $m \in \omega$ such that $\vec{r} := \langle\langle r_1, r_1^* \cap \mathcal{M}\rangle, \langle d^m_k, z^m_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha \cap \mathcal{M}$. Let $n \geq m$ be such that $\vec{r}_n = \vec{r}$ and $s_n = r_1$. Since $\vec{p}_1$ is obviously a lower bound of $\vec{r}_n$ and $\langle\langle t_0, t_0^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle$ with finite part $\langle s_n, r_1^*\rangle$, there exists $\vec{p}_2 \in \mathcal{M} \cap D$ below both $\vec{r}_n$ and $\langle\langle t_0, t_0^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle$ with finite part $\langle s_n, r_2^*\rangle$ for some suitable $r_2^* \in \mathcal{M}$. Thus the first (nontrivial) alternative of the construction of d^{n+1}_k, z^{n+1}_k's took place. Without loss of generality, $\vec{r}_{1,n} = \vec{p}_2$. A direct verification shows that $\vec{p}_2 = \langle\langle s_n = r_1, r_2^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ is as required. □

Claim 10. *Let $\mathcal{M}$ be a countable elementary submodel of H_θ for sufficiently large θ containing all relevant objects with $i = \mathcal{M} \cap \omega_1$ and $\vec{p} \in \mathcal{M} \cap \mathbb{K}^1_\alpha$. If $i \notin \bigcup_{n \in \bar{s} \cup (\omega \setminus (\max \bar{s}))} S_{\alpha+n}$, then there exists an $(\mathcal{M}, \mathbb{K}^1_\alpha)$-generic condition $\vec{q} \leq \vec{p}$ with the same finite part as $\vec{p}$.*

Proof. Let $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $\langle D_n : n \in \omega\rangle$ be the collection of all open dense subsets of $\mathbb{K}^1_\alpha$ which are elements of $\mathcal{M}$, and $\langle i_n : n \in \omega\rangle$ be an increasing sequence of ordinals converging to i. Using Claims 8 and 9, inductively construct a sequence $\langle \vec{q}_n : n \in \omega\rangle \subset \mathcal{M} \cap \mathbb{K}^1_\alpha$, where $\vec{q}_n = \langle\langle s, s^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle$ and $\vec{q}_0 = \vec{p}$, such that

- (i) d^{n+1}_k is an end-extension of d^n_k and z^{n+1}_k is an extension of z^n_k for all $n \in \omega$ and $k \in \bar{s} \cup (\omega \setminus (\max \bar{s}))$;
- (ii) $|z^n_k|, \max d^n_k \geq i_n$ for all $n \geq 1$ and $k \in \bar{s} \cup \omega \setminus (\max \bar{s})$; and
- (iii) For every $n \geq 1$ and $\vec{r} = \langle\langle r, r^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle \leq \vec{q}_n$, $\vec{r} \in D_n$, there exists r_2^* such that $\vec{r}_2 := \langle\langle r_1, r_2^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle \in D_n$ and $\vec{r}_2 \leq \vec{q}_n$.

Set $d_k = \bigcup_{n \in \omega} d^n_k \cup \{i\}$ and $z_k = \bigcup_{n \in \omega} z^n_k$ for all $k \in \bar{s} \cup (\omega \setminus (\max \bar{s}))$, $d_k = z_k = \emptyset$ for all other $k \in \omega$, and $\vec{q} = \langle\langle t_0, t_0^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$. We claim that $\vec{q}$ is as required, i.e., $\vec{q} \in \mathbb{K}^1_\alpha$ and $D_n \cap \mathcal{M}$ is pre-dense below q for every $n \in \omega$. The fact that $\vec{q} \in \mathbb{K}^1_\alpha$ can be shown in the same way as in the proof of Claim 9.

Let us fix $n \in \omega$ and $\vec{r}_1 = \langle\langle t_1, t_1^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle \leq \vec{q}$. Without loss of generality, $\vec{r}_1 \in D_n$. Since $\vec{r}_1 \leq \vec{q}_n$, (iii) yields the existence of t_2^* such that $\vec{r}_2 := \langle\langle t_1, t_2^*\rangle, \langle d^n_k, z^n_k : k \in \omega\rangle\rangle \leq \vec{q}_n$ and $\vec{r}_2 \in D_n$. It is clear that $\vec{r}_2 \in \mathcal{M}$. We claim that $\vec{r}_2$ and $\vec{r}_1$ are compatible. Indeed, set $\vec{r}_3 = \langle\langle t_1, t_2^* \cup t_1^*\rangle, \langle d'_k, z'_k : k \in \omega\rangle\rangle$ and note that $\vec{r}_3 \leq \vec{r}_1, \vec{r}_2$. □

Let H_α be a $\mathbb{K}^1_\alpha$-generic filter over $L[G_\alpha * h_\alpha]$. Set $Y^\alpha_k = \bigcup_{\vec{p} \in H_\alpha} y_k$, $C^\alpha_k = \bigcup_{\vec{p} \in H_\alpha} c_k$, $a_\alpha = \bigcup_{\vec{p} \in H_\alpha} s$, $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, and $S^* = \bigcup_{\vec{p} \in H_\alpha} s^*$, where

$\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$. The following statement is a consequence of the definition of $\mathbb{K}^1_\alpha$ and the genericity of H_α.

Claim 11. *(1)* $S^* = \{r_{\langle m,\xi\rangle} : m \in \overline{a_\alpha}, \xi \in C^\alpha_m\} \cup \{r_{\langle \omega+m,\xi\rangle} : m \in \overline{a_\alpha}, Y^\alpha_m(\xi) = 1\} \cup A_\alpha$;

(2) $a_\alpha \in [\omega]^\omega$;

(3) *If* $m \in \overline{a_\alpha}$, *then* $\mathrm{dom}(Y^\alpha_m) = \omega_1$ *and* C^α_m *is a club in* ω_1 *disjoint from* $S_{\alpha+m}$;

(4) a_α *is almost disjoint from all elements of* A_α;

(5) *If* $m \in \overline{a_\alpha}$, *then* $|a_\alpha \cap r_{\langle m,\xi\rangle}| < \omega$ *if and only if* $\xi \in C^\alpha_m$;

(6) *If* $m \in \overline{a_\alpha}$, *then* $|a_\alpha \cap r_{\langle \omega+m,\xi\rangle}| < \omega$ *if and only if* $Y^\alpha_m(\xi) = 1$;

(7) $|a_\alpha \cap b_i| = \omega$ *for all* $i \in \omega$;

(8) *For every* $n \in \omega$, $n \in z_\alpha$ *if and only if there exists* $m \in \omega$ *such that* $(a_\alpha \cap r_{\langle 0,0\rangle})(2n) = r_{\langle 0,0\rangle}(2m)$; *and*

(9) *Equation* $(*)$ *holds for* $\alpha + 1$, *i.e. for every* $r \in R$ *and a finite subfamily* B *of* $A_{\alpha+1}$, B *covers neither a cofinite part of* $E(r)$ *nor of* $O(r)$.

Proof. Items (1), (2), (4), and (9) are straightforward. Items (2), (5), (6), and (8) follow from the inductive assumption $(*)$. Item (3) is a consequence of Claim 8.

We are left with the task to prove (7). Let us fix $l, i \in \omega$ and denote by $D_{l,i}$ the set of conditions $\langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$ such that $(s \setminus l) \cap b_i \neq \emptyset$. It suffices to show that $D_{l,i}$ is dense in $\mathbb{K}^1_\alpha$. Fix a condition $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$ and set $x = b_i \setminus \cup s^*$. Note that $x \in [\omega]^\omega$ by our choice of η_α and items $(i), (ii)$ of the definition of $\mathbb{K}^1_\alpha$. Two cases are possible.

1. $|x \setminus r_{\langle 0,0\rangle}| = \omega$. Then

$$\vec{q} := \langle\langle s \cup \{\min(x \setminus (r_{\langle 0,0\rangle} \cup l \cup \max s))\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$$

is an element of $D_{l,i}$ and is stronger than $\vec{p}$.

2. $x \subset^* r_{\langle 0,0\rangle}$. Without loss of generality, $x \setminus r_{\langle 0,0\rangle} \subset l$. Suppose that $|s \cap r_{\langle 0,0\rangle}| = 2j - 1$ for some $j \in \omega$ (the case of even $|s \cap r_{\langle 0,0\rangle}|$ is analogous and simpler). Let $y = r_{\langle 0,0\rangle} \setminus \cup s^*$ and note that $x \subset^* y$. By $(*)$, $|y \cap E(r_{\langle 0,0\rangle})| = |y \cap O(r_{\langle 0,0\rangle})| = \omega$. Denote by m_e and m_o the minima of the sets $(y \cap E(r_{\langle 0,0\rangle}) \setminus (l \cup (\max s + 1))$ and $(y \cap O(r_{\langle 0,0\rangle}) \setminus (l \cup (\max s + 1))$, respectively. Set

$$\vec{r} := \langle\langle s \cup \{m_e\} \cup \{\min(x \setminus (m_e + 1))\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$$

if $j \in z_\alpha$ and

$$\vec{r} := \langle\langle s \cup \{m_o\} \cup \{\min(x \setminus (m_o + 1))\}, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$$

otherwise. A direct verification shows that $\vec{r} \in D_{l,i}$ and $\vec{r} \leq \vec{p}$. □

Corollary 12. $\dot{\mathbb{Q}}_\alpha$ *is* T_0*-proper. Consequently,* $\mathbb{P}_{\omega_2}$ *is* T_0*-proper and hence preserves cardinals.*

More precisely, for every condition $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle \in \mathbb{K}^1_\alpha$ *the poset* $\{\vec{r} \in \mathbb{K}^1_\alpha : \vec{r} \leq \vec{p}\}$ *is* $\omega_1 \setminus \bigcup_{n \in \bar{s} \cup (\omega \setminus (\max \bar{s}))} S_{\alpha+n}$*-proper. Consequently,* $S_{\alpha+n}$ *remains stationary in* $V^{\mathbb{P}_{\omega_2}}$ *for all* $n \in \omega \setminus \overline{a_\alpha}$.

Let G be a $\mathbb{P}_{\omega_2}$-generic filter over L. The following lemma shows that A is a Π^1_2 definable subset of $[\omega]^\omega$ in $L[G]$ and thus finishes the proof of Theorem 1.

Lemma 13. *In $L[G]$ the following conditions are equivalent:*

(1) *$a \in A$;*

(2) *For every countable suitable model M of* ZF$^-$ *containing a as an element there exists $\bar\alpha < \omega_2^M$ such that $S^M_{\bar\alpha+k}$ is nonstationary in M for all $k \in \overline{a}$.*

Proof. (1) $\to$ (2). Fix $a \in A$ and find $\alpha < \omega_2$ such that $a = a_\alpha$. Fix also a countable suitable model M of ZF$^-$ containing a_α as an element. By Claim 11(5, 6, 8), $z_\alpha \in M$ and $C^\alpha_k \cap \omega_1^M, Y^\alpha_k \restriction \omega_1^M \in M$ for all $k \in \overline{a_\alpha}$. Therefore $\eta_\alpha < \omega_1^M$. Since $\langle\langle\emptyset,\emptyset\rangle, \langle C^\alpha_k \cap (\omega_1^M+1), Y^\alpha_k \restriction \omega_1^M : k \in \omega\rangle\rangle$ is a condition in $\mathbb{K}^1_\alpha$, item (iv) of the definition of $\mathbb{K}^1_\alpha$ ensures that for every $k \in \overline{a_\alpha}$, $\mathrm{Even}(Y^\alpha_k \restriction \omega_1^M) - \min \mathrm{Even}(Y^\alpha_k \restriction \omega_1^M)$ codes a limit ordinal $\bar\alpha_k \in \omega_2^M$ such that $S^M_{\bar\alpha_k+k}$ is nonstationary in M. By item (ii) of the definition of $\mathbb{K}^1_\alpha$,

$$\mathrm{Even}(Y^\alpha_k \restriction \omega_1^M) - \min \mathrm{Even}(Y^\alpha_k \restriction \omega_1^M) = X_\alpha \cap \omega_1^M$$

for every $k \in I$, and hence $\bar\alpha_k$'s do not depend on k.

(2) $\to$ (1). Let us fix a fulfilling (2) and observe that by Löwenheim-Skolem, (2) holds for arbitrary (not necessarily countable) suitable model of ZF$^-$ containing a. In particular, it holds in $M = L_{\omega_8}[G]$. Observe that $\omega_2^M = \omega_2^{L[G]} = \omega_2^L$, $\vec{S}^M = \vec{S}$, and the notions of stationarity of subsets of ω_1 coincide in M and $L[G]$. Thus there exists $\alpha < \omega_2$ such that $S_{\alpha+k}$ is nonstationary for all $k \in \overline{a}$. Since the stationarity of some $S_{\alpha+k}$'s has been destroyed, Corollary 12 together with the T_2-properness of $\mathbb{K}^0_\xi$'s implies that $\dot{\mathbb{Q}}_\alpha$ is not trivial. Now, the last assertion of Corollary 12 easily imples that $a = a_\alpha$. □

4. Proof of Theorem 2

The proof is completely analogous to that of Theorem 1. Therefore we just define the corresponding poset $\mathbb{P}_{\omega_2}$, the use of the poset $\mathbb{M}^1_\alpha$ defined below instead of $\mathbb{K}^1_\alpha$ at the α's stage of iteration being the only significant change. We leave it to the reader to verify that the proof of Theorem 1 can be carried over.

For successor α let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for some proper forcing of size ω_1 adding a dominating real. Let us consider some limit α and a $\mathbb{P}_\alpha$-generic filter G_α. Suppose also that we have already constructed an almost disjoint family $A_\alpha \subset \omega^\omega$ such that

$$(**) \qquad \forall E \in [A_\alpha]^{<\omega}\, \forall f \in F\, (|f \restriction (2\omega) \setminus \cup E| = |f \restriction (2\omega+1) \setminus \cup E| = \omega)$$

Equation $(**)$ yields

$$\forall E \in [\mathcal{F} \cup A_\alpha]^{<\omega}\, \forall f \in \mathcal{F} \setminus E\, (|f \restriction (2\omega) \setminus \cup E| = |f \restriction (2\omega+1) \setminus \cup E| = \omega).$$

Let $F : Lim \cap \omega_2 \to L_{\omega_2}$ be the same as in the proof of Theorem 1. Unless the following holds, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial poset. Suppose that $F(\alpha)$ is a sequence $\langle \dot b_i : i \in \omega\rangle$ of $\mathbb{P}_\alpha$-names such that $b_i = \dot b_i^{G_\alpha} \in \omega^\omega$

and none of the b_i's is covered by a finite subfamily of A_α. In this case $\mathbb{Q}_\alpha := \dot{\mathbb{Q}}_\alpha^{G_\alpha}$ is the two-step iteration $\mathbb{K}^0_\alpha * \dot{\mathbb{M}}^1_\alpha$ defined as follows.

In $V^{\mathbb{P}_\alpha}$, $\mathbb{K}^0_\alpha$ is some $T_0 \cup T_2$-proper poset of size ω_1.

Let us fix a recursive bijection $\psi : \omega \times \omega \to \omega$ and $s \in \omega^{<\omega}$. Set $\mathrm{sq}(s) = \mathrm{dom}(s) \times (\mathrm{dom(s)} + \mathrm{ran}(s))$ and

$$\bar{\bar{s}} = \{2k+1 : k \in \psi(s)\} \cup \{2k : k \in \psi(\mathrm{sq}(s) \setminus s)\}.$$

In $V^{\mathbb{P}_\alpha * \mathbb{K}^0_\alpha}$ find an ordinal $\eta_\alpha \in \omega_1$ such that there are no finite subsets J, E of $(\omega \cdot 2) \times (\omega_1 \setminus \eta_\alpha)$, A_α, respectively, and $i \in \omega$, such that $b_i \subset \bigcup_{\langle\zeta,\xi\rangle \in J} f_{\langle\zeta,\xi\rangle} \cup \bigcup E$. $\mathbb{M}^1_\alpha$ consists of sequences $\langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ satisfying the following conditions:

- $(i)_f$ Conditions (i)-(ii) from the definition of $\mathbb{K}^1_\alpha$ in the proof of Theorem 1 hold;
- $(ii)_f$ $s \in \omega^{<\omega}$, $s^* \in \big[\{f_{\langle m,\xi\rangle} : m \in \bar{\bar{s}}, \xi \in c_m\} \cup \{f_{\langle\omega+m,\xi\rangle} : m \in \bar{\bar{s}}, y_m(\xi) = 1\} \cup A_\alpha\big]^{<\omega}$. In addition, for every $2n \in |s \cap f_{\langle 0,0\rangle}|$, $n \in z_\alpha$ if and only if there exists $m \in \omega$ such that $s(j) = f_{\langle 0,0\rangle}(2m)$, where j is the $2n$'th element of the domain of $s \cap f_{\langle 0,0\rangle}$; and
- $(iii)_f$ For all $m \in \bar{\bar{s}} \cup \{2k, 2k+1 : k \in \psi((\omega \setminus \mathrm{dom}(s)) \times \omega)\}$, limit ordinals $\xi \in \omega_1$ such that $\eta_\alpha < \xi \leq |y_m|$, and suitable ZF^- models M containing $y_m \restriction \xi$ and $c_m \cap \xi$ with $\omega_1^M = \xi$, ξ is a limit point of c_m, and the following holds in M: $(\mathrm{Even}(y_m) - \min \mathrm{Even}(y_m)) \cap \xi$ codes a limit ordinal $\bar{\alpha}$ such that $S^M_{\bar{\alpha}+m}$ is non-stationary.

For conditions $\vec{p} = \langle\langle s, s^*\rangle, \langle c_k, y_k : k \in \omega\rangle\rangle$ and $\vec{q} = \langle\langle t, t^*\rangle, \langle d_k, z_k : k \in \omega\rangle\rangle$ in $\mathbb{M}^1_\alpha$, $\vec{q} \leq \vec{p}$ if and only if

- $(iv)_f$ $s \subset t$, $s^* \subset t^*$, and $t \setminus s$ has empty intersection with all elements of s^*;
- $(v)_f$ If $m \in \bar{\bar{s}} \cup \{2k, 2k+1 : k \in \psi((\omega \setminus \mathrm{dom}(s)) \times \omega)\}$, then d_m is an end-extension of c_m and $y_m \subset z_m$.

5. Final remarks

The fact that $S'_\alpha \cap S_\beta = \emptyset$ for all $\alpha, \beta < \omega_2$ together with the freedom to choose $\mathbb{K}^0_\alpha$ to be an arbitrary $T_0 \cup T_2$-proper forcing of size ω_1 allow for combining the proofs of Theorems 1, 2 and [1, Theorem 1]. In addition, we could take $\dot{\mathbb{K}}^0_\alpha$ to be a name for a two-step iteration with second component equal to the poset used in the proof of [1, Theorem 1] at stage α, and first component equal to a name of a c.c.c. poset of size ω_1 (Theorem 7(2) allows us to arrange a suitable bookkeeping of such names). This gives us the following statements.

Theorem 14. *It is consistent with Martin's Axiom that there exists a Δ^1_3 definable wellorder of the reals and a Π^1_2 definable ω-mad family of infinite subsets of ω.*

Theorem 15. *It is consistent with Martin's Axiom that there exists a Δ^1_3 definable wellorder of the reals and a Π^1_2 definable ω-mad family of functions.*

The following questions remain open. In all questions we are interested in families of infinite subsets of ω as well as in families of functions from ω to ω.

Question 16. Is it consistent to have $\mathfrak{b} > \omega_1$ with a Σ^1_2 definable (ω-)mad family?

Question 17. Is it consistent to have $\omega_1 < \mathfrak{b} < 2^\omega$ with a Π^1_2 definable (ω-)mad family?

In the proofs of Theorems 1 and 2 we ruled out all mad families of size ω_1 by making $\mathfrak{b}$ big. Alternatively, one could use the methods developed in [1] and prove the consistency of $\omega_1 = \mathfrak{b} < \mathfrak{a} = \omega_2$ together with a Δ^1_3 definable ω-mad family. This suggests the following

Question 18. Is it consistent to have $\mathfrak{b} < \mathfrak{a}$ and a Π^1_2 definable (ω-)mad family?

Question 19. Is a projective (ω-)mad family consistent with $\mathfrak{b} \geq \omega_3$?

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