ANALYTIC EQUIVALENCE RELATIONS AND BI-EMBEDDABILITY

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ABSTRACT. Louveau and Rosendal [5] have shown that the relation of bi-embeddability for countable graphs as well as for many other natural classes of countable structures is complete under Borel reducibility for analytic equivalence relations. This is in strong contrast to the case of the isomorphism relation, which as an equivalence relation on graphs (or on any class of countable structures consisting of the models of a sentence of $\mathcal{L}_{\omega_1\omega}$) is far from complete (see [5, 2]).

In this article we strengthen the results of [5] by showing that not only does bi-embeddability give rise to analytic equivalence relations which are complete under Borel reducibility, but in fact *any* analytic equivalence relation is *Borel equivalent* to such a relation. This result and the techniques introduced answer questions raised in [5] about the comparison between isomorphism and bi-embeddability. Finally, as in [5] our results apply not only to classes of countable structures defined by sentences of $\mathcal{L}_{\omega_1\omega}$, but also to discrete metric or ultrametric Polish spaces, compact metrizable topological spaces and separable Banach spaces, with various notions of embeddability appropriate for these classes, as well as to actions of Polish monoids.

1. Introduction

Given two n-ary analytic (i.e. $\mathbf{\Sigma}^1_1$) relations R, R' defined on standard Borel spaces X, X', we say that R is *Borel reducible* to R' ($R \leq_B R'$ in symbols) if there is a Borel measurable function $f \colon X \to X'$ such that

$$\forall x_1, \dots, x_n \in X (R(x_1, \dots, x_n) \iff R'(f(x_1), \dots, f(x_n)).$$

We say that R and R' are *Borel equivalent* ($R \sim_B R'$ in symbols) if both $R \leq_B R'$ and $R' \leq_B R$, and that R is *complete* if $R' \leq_B R$ for any n-ary analytic relation R'.

The quasi-order $\leq_B$ can be used to measure the relative complexity of the relations R and R', and has been used (explicitly or implicitly) in many areas of mathematics to solve various classification problems — see [3] for a brief but informative discussion on this topic. For this reason, the relation $\leq_B$ has been extensively studied in the case $n = 1$, and in the case $n = 2$ when restricting either to quasi-orders (reflexive and transitive relations) or to equivalence relations (symmetric quasi-orders). Logic provides a nice example of analytic equivalence relation, namely the isomorphism relations (denoted by $\cong$) on the class of countable models

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of some $\mathcal{L}_{\omega_1\omega}$-sentence, $\mathcal{L}$ a countable language: unfortunately, it turns out that these relations are in a sense too special, as there are other relations (such as E_1) which are not even Borel reducible to $\cong$. This means the the relation of isomorphism is very far from being able to produce (up to Borel equivalence) all analytic equivalence relations. More generally, one can see that the latter gives rise to an extremely complex structure under Borel reducibility, in which many "pathologies" (such as e.g. infinite descending chains, or infinite antichains) appear, even in very simple subclasses.

Regarding completeness, there are of course complete analytic equivalence relations, but, to our knowledge, the unique technique to produce examples which are "concrete" from the mathematical point of view is to first find examples of complete analytic quasi-orders R and then pass to the associated equivalence relation $E_R = R \cap R^{-1}$, a method developed by Louveau and Rosendal in their [5]. Among other things, in that paper they proved quite surprisingly that when considering the natural counterpart of isomorphism, namely embeddability, on countable graphs, one gets a complete analytic quasi-order (denoted by $\sqsubseteq$): this in particular implies that the analytic equivalence relation of bi-embeddability (denoted[1] by $\equiv$) on countable graphs is complete. Building on the work of Louveau and Rosendal, we show in this paper that the embeddability relation has the further stronger "universal" property that *any* analytic quasi-order is indeed Borel equivalent to $\sqsubseteq$ restricted to the class of countable graphs satisfying a corresponding $\mathcal{L}_{\omega_1\omega}$-sentence, whence the similar result regarding analytic equivalence relations and bi-embeddability. As we will see in Remark 3.4, there is an *effective* version of this result. Moreover, following [5], the same result can be extended to the context of analysis and monoid actions.

We finally want to stress that the Borel equivalences obtained here are of a very special kind, and give rise to a stronger relationship (which was already considered as an interestening notion in [2]) between arbitrary analytic equivalence relations and bi-embeddability. In fact, the existence of a Borel reduction f between analytic equivalence relations E and F on standard Borel spaces X and Y implies that its lifting $\hat{f} \colon X/_E \to Y/_F \colon [x]_E \mapsto [f(x)]_F$ is well-defined and injective: therefore $E \leq_B F$ if and only if there is a Borel $f \colon X \to Y$ such that $\hat{f}$ is a well-defined embedding of $X/_E$ into $Y/_F$, and, consequently, $E \sim_B F$ if and only if there are Borel functions $f \colon X \to Y$ and $g \colon Y \to X$ such that $\hat{f}$ and $\hat{g}$ are well-defined embeddings of $X/_E$ into $Y/_F$ and of $Y/_F$ into $X/_E$, respectively. Note, however, that with this definition the embeddings $\hat{f}$ and $\hat{g}$ may have very little in common. A stronger requirement (already proposed in [2]) would be to ask for reductions f and g such that $\hat{g} = \hat{f}^{-1}$, so that $\hat{f}$ is indeed an isomorphism between $X/_E$ and $Y/_F$: in this case E and F are said to be *Borel isomorphic*[2]. As it can be easily checked, our construction will give that for every analytic equivalence relation E on a standard Borel space X there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that E is not only Borel equivalent, but Borel isomorphic to the bi-embeddability relation on the class Mod_φ

[1]We would like to warn the reader that in this paper, as in [5], the symbol $\equiv$ denotes bi-embeddability and *not* elementary equivalence.

[2]Note that to get that E and F are Borel isomorphic it is sufficient to just produce an *injective* Borel reduction $f \colon X \to Y$ of E into F such that $\hat{f}$ is surjective (hence a bijection), as in this case the function g defined by $g(y) = x \iff \exists z \in Y(zFy \wedge f(x) = z)$ turns out to be Borel and such that $\hat{g} = \hat{f}^{-1}$.

of countable models of φ (and the witness of this fact will also witness at the same time that equality on X is Borel isomorphic to isomorphism on Mod_φ). The same strong property will be provided also for the cases regarding analysis and monoid actions. However, at least to our knowledge, whether Borel isomorphism (or any of the other notions of equivalence between analytic equivalence relations that can be naturally formulated, see [2] for some examples) is a *strictly* finer relation than Borel equivalence is still an open problem.

2. Notation and preliminaries

Let ω denote the set of natural numbers. Given a nonempty set A, we will denote by ${}^{<\omega}A$ the collection of all finite sequences s of elements of A, and denote by $|s|$ its length. For each $i < |s|$, $s(i)$ will denote the $(i+1)$-st element of s, so that $s = \langle s(i) \mid i < |s| \rangle$. Moreover, we will write $s^\frown t$ for the concatenation of s and t (but for simplicity of notation we will simply write e.g. $n^\frown t$ rather than $\langle n \rangle^\frown t$). If $A = A_0 \times \ldots \times A_k$, we will identify each element $s \in {}^{<\omega}A$ with a sequence $(s_0, \ldots, s_k)$ of sequences of same length such that $s_i \in {}^{<\omega}A_i$ for each $i \leq k$. A *tree* T on A is simply a subset of ${}^{<\omega}A$ closed under initial subsequences. If T is a tree on A we call the *body* of T the set $[T]$ of all ω-sequences x of elements of A such that $x \restriction n \in T$ for each $n \in \omega$ (where $x \restriction n$ denotes the restriction of x to its first n digits). Moreover, if T is a tree on $A \times \omega$, the *projection* $p[T]$ of T is the collection of ω-sequences x of elements of A such that for some ω-sequence y of natural numbers $(x \restriction n, y \restriction n) \in T$ for each $n \in \omega$.

A special role in the paper will be played by the Cantor space ${}^{\omega}2$ of all binary sequences of length ω: in fact, since any two uncountable Polish spaces are Borel-isomorphic and all the notions considered here depend only on the Borel-structure of the spaces involved, we can restrict our attention without loss of generality to quasi-orders and equivalence relations defined on ${}^{\omega}2$.

Finally, we need to briefly recall some notation, definitions, and basic results from [5]. Given two sequences $s, t \in {}^{<\omega}\omega$ of the same length, we put $s \leq t$ if $s(i) \leq t(i)$ for every $i < |s| = |t|$, and define $s + t \in {}^{<\omega}\omega$ by setting $(s+t)(i) = s(i) + t(i)$. Given a tree T on $X \times \omega$, we say that T is *normal* if given $u \in {}^{<\omega}X$ and $s \in {}^{<\omega}\omega$, $(u, s) \in T$ implies $(u, t) \in T$ for every $s \leq t$. It is well-known that any analytic subset of ${}^{\omega}2 \times {}^{\omega}2$ (hence, in particular, any analytic quasi-order R on ${}^{\omega}2$) is the projection of a tree on $2 \times 2 \times \omega$: in [5, Theorem 2.4], Louveau and Rosendal proved the following stronger result, in which R is viewed as the projection of a normal tree S on $2 \times 2 \times \omega$ such that the reflexivity and transitivity properties of R are "mirrored" by corresponding local properties of S.

Proposition 2.1 (Louveau-Rosendal)**.** *Let R be any analytic quasi-order on ${}^{\omega}2$. Then there is a normal tree S on $2 \times 2 \times \omega$ such that:*

i) $R = p[S]$*;*
ii) *for every $u \in {}^{<\omega}2$ and $s \in {}^{<\omega}\omega$ of the same length, $(u, u, s) \in S$;*
iii) *for every $u, v, w \in {}^{<\omega}2$ and $s, t \in {}^{<\omega}\omega$ of the same length, if $(u, v, s) \in S$ and $(v, w, t) \in S$ then $(u, w, s + t) \in S$.*

From this one can easily get the following refinement, which will be needed for our result.

Corollary 2.2. *Let R be any analytic quasi-order on ${}^{\omega}2$. Then there is a normal tree S on $2 \times 2 \times \omega$ which satisfies i)–iii) of Proposition 2.1 and* *iv) for every $u, v \in {}^{<\omega}2$ of the same length, $(u, v, 0^{|u|}) \in S$ implies $u = v$.*

Proof. For $u \neq v \in {}^{<\omega}2$ of the same length, discard each element of the form $(u, v, 0^{|s|})$ from the tree S given by Proposition 2.1, and check that the new tree is normal and still satisfies i)–iii). □

A function $f \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ is said to be *Lipschitz* if $s \subseteq t \Rightarrow f(s) \subseteq f(t)$ and $|s| = |f(s)|$ for each $s, t \in {}^{<\omega}\omega$. Consider now the space $\mathcal{T}$ of all normal trees on $2 \times \omega$, and for $S, T \in \mathcal{T}$ put $S \leq_{max} T$ if and only if there exists a Lipschitz function $f \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $(u, s) \in S \Rightarrow (u, f(s)) \in T$ for every $u \in {}^{<\omega}2$ and $s \in {}^{<\omega}\omega$ of the same length (this in particular implies $p[S] \subseteq p[T]$).

Theorem 2.3 (Louveau-Rosendal). *The quasi-order $\leq_{max}$ is complete for analytic quasi-orders.*

In their proof, Louveau and Rosendal considered an arbitrary analytic quasi-order R on ${}^{\omega}2$ together with the tree S given by Proposition 2.1: then they defined for every $x \in {}^{\omega}2$ the normal tree $S^x = \{(u, s) \mid (u, x \restriction |u|, s) \in S\}$ and showed that xRy if and only if $S^x \leq_{max} S^y$ (for $x, y \in {}^{\omega}2$). If we now use Corollary 2.2 rather than Proposition 2.1 to carry out the same proof, we get the additional property that the continuous map which sends x to S^x is now injective (and moreover becomes also open in its range): if $x \neq y$ and s is such that $s \subseteq x$ but $s \nsubseteq y$ then $(s, 0^{|s|}) \in S^x$ but $(s, 0^{|s|}) \notin S^y$. Therefore, given a quasi-order R on ${}^{\omega}2$ and $x \in {}^{\omega}2$, from now on we will denote by S^x the normal tree defined as above *but using the tree S given by Corollary 2.2.*

The next ingredient for our proof is the collection OCT of the *ordered (countable) combinatorial trees*, i.e. the collection of those $G = \langle U_G, G, \leq_G \rangle$ such that $\langle U_G, G \rangle$ is a *combinatorial tree* (that is a connected and acyclic graph) and $\leq_G$ is any partial transitive relation on U_G which will be called the *order*[3] of G. Note that OCT "includes" the collection CT of all combinatorial trees considered in [5], as each of them can be canonically identified with the corresponding ordered combinatorial tree obtained by adjoining the empty set as order (and this identification clearly preserves the relations of embeddability and isomorphism). By this identification, one immediately gets from [5, Theorem 3.1] that $\sqsubseteq_{OCT}$, the relation of embeddability on OCT, is a complete analytic quasi-order. However, we will need a slightly different proof which will be given in Theorem 2.4. From the construction given in our new argument one could also restrict attention to *rooted OCT* (ordered combinatorial trees with a special distinguished element called root), or put many restrictions on $\leq_G$, requiring it to be reflexive and antisymmetric (i.e. an order in the common sense), connected (for each pair of elements $t, t' \in U_G$ there is $t'' \in U_G$ such that $t'' \leq_G t, t'$), linear (for each pair $t, t' \in U_G$ either $t \leq_G t'$ or $t' \leq_G t$), well-founded (there is no $\leq_G$-descending chain) and so on. One obviously could also enlarge the class of structures under consideration and get that the embeddability relation on any class of structures between OCT and the collection of all $\mathcal{L}$-structures, where $\mathcal{L} = \{P, Q\}$ is a language with just two binary relations, is a complete analytic quasi-order.

[3]This evident abuse of terminology is justified by the fact that in all constructions below, $\leq_G$ will always be either a linear well-founded order (in the usual sense) or its strict part.

Theorem 2.4. *The relation $\sqsubseteq_{OCT}$ of embeddability on OCT is complete for analytic quasi-orders.*

Proof. The proof is almost identical to the one of [5, Theorem 3.1]. To each normal tree $T \in \mathcal{T}$ on $2 \times \omega$ associate a combinatorial tree G_T defined as in [5]:

i) fix an enumeration $\theta\colon {}^{<\omega}2 \to \omega$ such that $|s| \leq |t|$ implies $\theta(s) \leq \theta(t)$;
ii) "double" the set ${}^{<\omega}\omega$, that is adjoin a new vertex s^* for each $s \in {}^{<\omega}\omega \setminus \{\emptyset\}$, and put an edge between s^* and s, and between s^* and the predecessor s^- of s as a sequence (this combinatorial tree, which does not depend on T, is denoted by G_0);
iii) for each pair $(u, s) \in T$ add vertices (u, s, x), where $x = 0^{(k)}$ or $x = 0^{(2\theta(u)+2)}{}^\frown 1^\frown 0^{(k)}$, and then link $(u, s, \emptyset)$ to s and (u, s, x) to (u, s, x^-) (where x^- is again the predecessor of x as a sequence).

Now define the order $\leq_T = \leq_{G_T}$ on U_{G_T} in the following way: for $s, t \in {}^{<\omega}\omega$ put $s \preceq t$ if and only if $|s| < |t|$ or $|s| = |t|$ and $s \leq_{lex} t$ (the symbol $\prec$ will denote the strict part of $\preceq$). Now for $g, g' \in U_{G_T}$, $s, t \in {}^{<\omega}\omega$, and $u, v, x, y \in {}^{<\omega}2$ put $g \leq_T g'$ in each of (and only) the following cases:

- $g = s$ and either $g' = t^*$ or $g' = (v, t, y)$, or $g = s^*$ and $g' = (v, t, y)$;
- $g = s$, $g' = t$, and $s \preceq t$;
- $g = s^*$, $g' = t^*$, and $s \preceq t$;
- $g = (u, s, x)$, $g' = (v, t, y)$ and

$$(s \prec t) \vee (s = t \wedge u \prec v) \vee (s = t \wedge u = v \wedge x \preceq y).$$

(Note in particular that $\leq_T$ is a well-founded linear order of length $\leq \omega^4$ but one could also define a suitable linear order of type ω as well, although this would make it considerably more difficult to check that the functions defined below are really embeddings, i.e. that they preserve the orders.)

Now we will show that the map $T \mapsto G_T$ is a reduction of $\leq_{max}$ to $\sqsubseteq_{OCT}$. Assume first that S, T are normal trees on $2 \times \omega$ such that $S \leq_{max} T$: as observed in Lemma 2.8 of [6], this can be witnessed by a Lipschitz $\leq_{lex}$-preserving function $f\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$, that is by an f such that $s \preceq s' \iff f(s) \preceq f(s')$ for every $s, s' \in {}^{<\omega}\omega$ (in particular f is injective). Now embed G_S into G_T sending s to $f(s)$, s^* to $f(s)^*$, and (u, s, x) to $(u, f(s), x)$, and check that both the graph and the order relations are preserved.

For the other direction, if $G_S \sqsubseteq G_T$ than G_S embeds in G_T as a combinatorial tree (disregarding the orders) and so $S \leq_{max} T$ by the second part of the proof of [5, Theorem 3.1]. $\square$

Remark 2.5. Let us note here that if the coding of G_T as a structure on ω is chosen in a careful way, e.g. as in the proof of Theorem 3.9, then the map which sends an arbitrary normal tree T into (the code of) G_T, which is clearly Borel, has very low topological complexity: in fact, it is continuous and open in its image.

3. The main results

We now want to prove our main results, namely that there are various natural quasi-orders arising in model theory, analysis and descriptive set theory such that each analytic quasi-order is indeed Borel equivalent to that specific quasi-order (on a suitable class of objects). This gives also several characterizations of both analytic quasi-orders and analytic equivalence relations, and shows that the notions of embeddability, homomorphism, and weak-homomorphism among countable structures (for model theory), the notions of isometric embeddability among discrete metric or ultrametric Polish spaces, of continuous embeddability among compact metrizable topological spaces, and of linear isometric embeddability among separable Banach spaces (for analysis), and the notion of closed or Borel action of Polish monoids (for descriptive set theory) are able to capture the great complexity of the whole structure of analytic quasi-orders and analytic equivalence relations (up to Borel equivalence).

3.1. **Morphisms in Model Theory.** The advantage of having used OCT in the previous section (rather than CT as in [5]) emanates from the following two lemmas.

Lemma 3.1. *Let S, T be normal trees, and G_S and G_T be defined as in the proof of Theorem 2.4. If $S \neq T$ then $G_S \not\cong G_T$.*

Proof. Suppose i is an isomorphism between G_S and G_T. Since the orders $\leq_S$ and $\leq_T$ coincide on ${}^{<\omega}\omega$ we have that $i \restriction {}^{<\omega}\omega$ must be the identity. Suppose now $(u,s) \in S$: as in the proof of Theorem 3.1 in [5], the point $(u,s,0^{2\theta(u)+2})$ must be sent in a point of the form $(u,i(s),0^{2\theta(u)+2}) = (u,s,0^{2\theta(u)+2})$, and the existence of such a point witnesses $(u,s) \in T$. Hence $S \subseteq T$. Exchanging the role of S and T and using i^{-1} instead of i one gets $T \subseteq S$, and therefore $S = T$. $\square$

As already noted in [5], the domain of each ordered combinatorial tree of the form G_T is formally different from ω, but nevertheless one can easily code (Borel-in-T) such a structure in another structure $\hat{G}_T$ with domain ω: for simplicity of presentation, as in the following lemma, we will often confuse the two structures G_T and $\hat{G}_T$. Let S_∞ be the Polish group of permutations on ω, $\mathcal{L} = \{P, Q\}$ be the relational language with just two binary symbols, and $j_{\mathcal{L}} \colon S_\infty \times Mod_{\mathcal{L}} \to Mod_{\mathcal{L}}$ be the usual (continuous) action of S_∞ on $Mod_{\mathcal{L}}$, the collection of all countable $\mathcal{L}$-structures. For every normal tree S on $2 \times \omega$ and $p \in S_\infty$, put $G_{S,p} = j_{\mathcal{L}}(p, G_S)$, where G_S is the ordered combinatorial tree obtained from S as in the proof of Theorem 2.4.

Lemma 3.2. *For every distinct $p, q \in S_\infty$ and every normal tree S on $2 \times \omega$, we have $G_{S,p} \neq G_{S,q}$.*

Proof. Let $\leq_{S,p}$ and $\leq_{S,q}$ be the linear orders on $G_{S,p}$ and $G_{S,q}$, respectively. Let g be the $\leq_S$-minimal element of G_S such that $p(g) \neq q(g)$. We claim that $p(g) \leq_{S,p} q(g)$ but $p(g) \not\leq_{S,q} q(g)$ (this implies that the two structures $G_{S,p}$ and $G_{S,q}$ are different). Assume toward a contradiction that $p^{-1}(q(g)) <_S g$: then $q(p^{-1}(q(g))) <_{S,q} q(g)$. But as $p(p^{-1}(q(g))) = q(g)$, the previous inequality shows that $p(p^{-1}(q(g))) \neq q(p^{-1}(q(g)))$, contradicting the $\leq_S$-minimality of g. Therefore $g \leq_S p^{-1}(q(g))$, which implies $p(g) \leq_{S,p} q(g)$.

Assume now towards a contradiction that $p(g) \leq_{S,q} q(g)$. Since $p(g) \neq q(g)$ (by hypothesis) we get $p(g) <_{S,q} q(g)$, which implies $q^{-1}(p(g)) <_S g$. Arguing as before (with p and q exchanged), we get a contradiction with the $\leq_S$-minimality of g. Therefore $p(g) \not\leq_{S,q} q(g)$, as required. $\square$

Now we are ready to prove our first main theorem.

Theorem 3.3. *If R is an analytic quasi-order on a standard Borel space X, then there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel equivalent to embeddability on $Mod_\varphi = \{x \in Mod_{\mathcal{L}} \mid x \vDash \varphi\}$.*

Proof. As already noted, we can assume $X = {}^\omega 2$. For $x \in {}^\omega 2$ let S^x be defined as in the previous section, so that the map which sends x to S^x is Borel and injective. Let R' be the quasi-order on $X \times S_\infty$ defined by $(x,p)R'(y,q) \iff xRy$. It is clear that R and R' are Borel equivalent (as witnessed by the maps $x \mapsto (x, id)$ and $(x,p) \mapsto x$), hence it is enough to prove the theorem for R'. Our plan is to find a Borel and invariant subset Z of $Mod_{\mathcal{L}}$ and a reduction of R' to the embeddability relation $\sqsubseteq$ with range Z, and then use the well-known fact (due to Lopez-Escobar, see e.g. [4, Theorem 16.8]) that such a Z must coincide with Mod_φ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ.

Using the same notation of the previous lemmas, consider the map f which sends (x,p) to $G_{S^x,p}$, which is clearly a Borel (in fact, continuous) map. First note that f reduces R' to the embedding relation $\sqsubseteq$, as

$$(x,p)R'(y,q) \iff xRy \iff S^x \leq_{max} S^y \iff G_{S^x} \sqsubseteq G_{S^y} \iff G_{S^x,p} \sqsubseteq G_{S^y,q}.$$

We now claim that f is injective. Assume $(x,p) \neq (y,q)$: if $x \neq y$ then $S^x \neq S^y$, and therefore by Lemma 3.1 we get that $G_{S^x} \ncong G_{S^y}$, which in turn implies that $G_{S^x,p} \ncong G_{S^y,q}$ as well (so that, in particular, $G_{S^x,p}$ and $G_{S^y,q}$ must be different). If instead $x = y$ but $p \neq q$, then by Lemma 3.2 we get $G_{S^x,p} \neq G_{S^x,q} = G_{S^y,q}$ and hence we are done.

Since $X \times S_\infty$ is a Borel set and f is Borel and injective, we get that $f(X \times S_\infty) \subseteq Mod_{\mathcal{L}}$ is a Borel set and that f^{-1} is Borel as well. But $f(X \times S_\infty)$ is clearly invariant under isomorphism, so $f(X \times S_\infty) = Mod_\varphi$ for some $\mathcal{L}_{\omega_1\omega}$-sentence φ. Since f and f^{-1} witness the Borel equivalence between R' and embeddability on Mod_φ, this concludes the proof. □

Remark 3.4. There is an *effective* version of Theorem 3.3 (as well as of Theorem 3.5 and the corollaries below). Using the fact that any Σ^1_1 subset A of the Cantor space ${}^\omega 2$ is the projection of a recursive (not necessarily pruned) tree on $2 \times \omega$, one can check that the proof of Corollary 2.2 gives that a Σ^1_1 quasi-order R on ${}^\omega 2$ is also the projection of a *recursive* normal tree with all the requested properties, and this can in turn be used to check that, once we have chosen a suitable coding for the target ordered combinatorial tree (e.g. a coding similar to the one which will be explicitly given in Theorem 3.9), the function f from ${}^\omega 2 \times S_\infty$ to $Mod_{\mathcal{L}}$ constructed in the previous proof is Δ^1_1-recursive (in fact, Σ^0_1-recursive). As f is injective and has a Δ^1_1 domain, we get from the effective version of the properties of Borel injective functions (see e.g. [8, Exercise 4D.7]) that range$(f) \in \Delta^1_1$ and f^{-1} is Δ^1_1-recursive. By [12, Theorem 3.14], any invariant Δ^1_1-subset of $Mod_{\mathcal{L}}$ is the class of models of some *computable* infinitary formula, that is of a formula in the effective version of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ where countable conjunction and disjunction are allowed only on effectively enumerable sets of formulas. Therefore we have the following: for every Σ^1_1 quasi-order R on ${}^\omega 2$ there is a computable infinitary formula φ such that R is Δ^1_1-equivalent (in fact, Δ^1_1-isomorphic) to embeddability on Mod_φ, where Δ^1_1-equivalence is simply the effectivization of $\sim_B$. Such result can then be naturally extended to all Σ^1_1 quasi-orders defined on spaces which are Δ^1_1-isomorphic to ${}^\omega 2$, that is to Σ^1_1 quasi-orders defined on *recursively presented* Polish spaces.

Now we will concentrate on other model-theoretic notions of morphism, namely homomorphism and weak-homomorphism. For simplicity of notation, the definitions are given just for the language $\mathcal{L}$ under consideration in this section, but can clearly be generalized in a straightforward way to arbitrary languages.

Definition 1. If G, G' are two $\mathcal{L}$-structures (where $\mathcal{L}$ is again the language containing just the two binary relational symbols P and Q), we say that G is *homomorphic* to G' if there is a function h such that for every g_0, g_1 in the domain of G, $g_0 P^G g_1 \iff h(g_0) P^{G'} h(g_1)$ and $g_0 Q^G g_1 \iff h(g_0) Q^{G'} h(g_1)$ (such an h will be called *homomorphism* between G and G').

Moreover, we say that G is *weakly-homomorphic* to G' just in case there is a function h (called *weak-homomorphism*) such that for every g_0, g_1 in the domain of G, $g_0 P^G g_1 \Rightarrow h(g_0) P^{G'} h(g_1)$ and $g_0 Q^G g_1 \Rightarrow h(g_0) Q^{G'} h(g_1)$.

The relevance of the notion of homomorphism between graphs is briefly described in [5]. One should also note that embeddings are just *injective* homomorphisms.

Theorem 3.5. *The relation of homomorphism (resp. weak-homomorphism) on OCT is a complete analytic quasi-order. Moreover, if R is an analytic quasi-order on a standard Borel space X then there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel equivalent to the relation of homomorphism (resp. weak-homomorphism) on Mod_φ.*

Proof. For the relation of homomorphism, one should simply note that since the order of an (isomorphic copy of an) ordered combinatorial tree of the form G_T is reflexive and antisymmetric, each homomorphism between $G_{S,p}$ and $G_{T,q}$ must be injective, i.e. homomorphisms and embeddings coincide on Mod_φ (where $G_{S,p}$, $G_{T,q}$ and Mod_φ are defined as in the proof of Theorem 3.3).

For the relation of weak-homomorphism, first replace the order $\leq_T$ of G_T with its strict part $<_T$ (denote such a structure by $\bar{G}_T$), where G_T is defined as above. By linearity of $\leq_T$, this implies that any weak-homomorphism h between $\bar{G}_S$ and $\bar{G}_T$ must be injective. But since the graph relation on any structure of the form $\bar{G}_T$ is connected, h must also be a homomorphism (hence an embedding): for each pair of distinct elements $g, g' \in \bar{G}_S$ either $h(g) \not<_T h(g')$ or $h(g') \not<_T h(g)$, so if $h(g) <_T h(g')$ (which in particular implies $h(g) \neq h(g')$, and hence also $g \neq g'$) then $g <_S g'$ (otherwise by linearity of $\leq_S$ we would have $g' <_S g$ and hence $h(g') <_T h(g)$, a contradiction!). Moreover, let $g, g' \in \bar{G}_S$ be such that $h(g)$ and $h(g')$ are linked by the graph relation of $\bar{G}_T$, and let $g = g_0, g_1, \dots, g_n = g'$ be the (unique) path in $\bar{G}_S$ which goes from g to g': since h is a weak-homomorphism and is injective, $h(g_0), \dots, h(g_1)$ must be a path from $h(g)$ to $h(g')$ and if $n > 1$ this would form a cycle because by hypothesis $h(g_0) = h(g)$ is linked to $h(g_n) = h(g')$ by the graph relation of $\bar{G}_T$, a contradiction with the absence of loops in $\bar{G}_T$! Hence $n = 1$ and $g = g_0$ must be linked to $g' = g_1$ by the graph relation of $\bar{G}_S$. Therefore, on (isomorphic copies of) structures of the form $\bar{G}_S$, the notions of weak-homomorphism, homomorphism and embedding coincide, and we can get the result by systematically replacing G_T with $\bar{G}_T$ in Theorem 2.4, Lemma 3.1, Lemma 3.2 and Theorem 3.3. □

Corollary 3.6. *Given a standard Borel space X, a binary relation R on X is an analytic quasi-order if and only if there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel equivalent to the relation of embeddability (resp. homomorphism, weak-homomorphism) on Mod_φ.*

Proof. One direction is given by Theorems 3.3 and 3.5, while for the other direction just note that "being an analytic quasi-order" is downward closed with respect to Borel reducibility. □

Corollary 3.7. *A binary relation E on a standard Borel space X is an analytic equivalence relation if and only if it is Borel equivalent to a bi-embeddability (resp. bi-homomorphism, bi-weak-homomorphism) relation.*

Proof. For the nontrivial direction, apply Corollary 3.6 to the quasi-order E. □

As observed in the introduction, what we have really shown is that given an analytic equivalence relation E on a standard Borel space X there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ and a *Borel* function $f \colon X \to Mod_\varphi$ which witnesses that E is *Borel isomorphic* to $\equiv$ on Mod_φ and that $=$ on X is *Borel isomorphic* to $\cong$ on Mod_φ. Moreover, if X is homeomorphic to ${}^\omega 2$, by Remark 2.5 and the observation following Theorem 2.3 we get that f has the further nice property of being a *homeomorphism* on its range (and hence is a topologically embedding, a very simple function).

Remark 3.8.
(1) Note that in Corollary 3.7 we get (by the symmetry of E) that there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that $E \sim_B \sqsubseteq \restriction Mod_\varphi$, but with the further property that embeddability and bi-embeddability coincide on Mod_φ, that is if $x, y \in Mod_\varphi$ are such that $x \sqsubseteq y$ then automatically $y \sqsubseteq x$ as well.
(2) Suppose that E in Corollary 3.7 is $\cong$ on Mod_ψ, ψ an $\mathcal{L}_{\omega_1\omega}$-sentence of some countable language $\mathcal{L}$. It is worth nothing that in general the sentence φ such that $(Mod_\psi, \cong) \sim_B (Mod_\varphi, \equiv)$ obtained by Corollary 3.6 has in general very little in common with ψ, even if they share the same language.

3.2. **Embeddings in Analysis.** The following version of Theorem 3.3 gives some applications in analysis, but it is also interesting *per se* as it shows that one can replace the language $\mathcal{L}$ with two different binary symbols with the graph language.

Theorem 3.9. *If R is an analytic quasi-order on a standard Borel space X then there is a sentence ψ of $\mathcal{L}'_{\omega_1\omega}$ (where $\mathcal{L}'$ is the graph language with just one binary relational symbol) such that R is Borel equivalent to embeddability on Mod_ψ (in particular, Mod_ψ is the collection of "ordinary" graphs which satisfy ψ).*

Proof. The proof is a modification of the argument used in Theorem 3.3. Given a normal tree T on $2 \times \omega$, we will define a new combinatorial tree[4] G'_T (without any order), prove that [5, Theorem 3.1] still holds when replacing G_T with G'_T, and then slightly modify the argument used in Theorem 3.3 to get the new result. We can assume again that $X = {}^\omega 2$. First define G'_T: given a normal tree $T \in \mathcal{T}$, let G_T be defined as in the proof of [5, Theorem 3.1], that is as in the proof of Theorem 3.3 but without the order relation. Let $\# \colon {}^{<\omega}\omega \to \omega$ be any bijection. Now for every $s \in {}^{<\omega}\omega$, adjoin vertices s^+, s^{++} and (s^{++}, i^k) for $i \leq \#s + 2$ and $0 \neq k \in \omega$, and link s^+ to both s and s^{++}, (s^{++}, i) to s^{++}, and (s^{++}, i^k) to (s^{++}, i^{k+1}). This concludes the definition of G'_T.

Now it is easy to see how to reprove [5, Theorem 3.1]. For one direction, if f is an injective witness of $S \leq_{max} T$ such that $\#s \leq \#f(s)$ for every $s \in {}^{<\omega}\omega$ (the construction of such a witness from an arbitrary one is easy and is left to the

[4]Clearly in this proof we could also use *rooted* combinatorial trees instead of combinatorial trees: this will be used in Corollary 3.14.

reader), then we can define the embedding g from G'_S to G'_T by sending s to $f(s)$, s^* to $f(s)^*$, s^+ to $f(s)^+$, s^{++} to $f(s)^{++}$, (s^{++}, i^k) to $(f(s)^{++}, i^k)$, and (u, s, x) to $(u, f(s), x)$. For the other direction, note that all the points in G'_S have valence ≤ 2 except for those of the form $s \in {}^{<\omega}\omega$ (which have valence ω), s^{++} (which have valence $\#s + 4$), and $(u, s, 0^{2\theta(u)+2})$ (which have valence 3). Moreover, the distance from s to s^{++} is always 2 and vertices of the form $(u, s, 0^{2\theta(u)+2})$ are the unique vertices which have valence ≥ 3 and distance $2\theta(u) + 3$ from s. Using all these facts, together with those about distances and valences in G_0, we reach the conclusion that if g is an embedding of G'_S in G'_T, then $f = g \restriction {}^{<\omega}\omega$ is such that $\mathrm{range}(f) \subseteq {}^{<\omega}\omega$, $f(\emptyset) = \emptyset$, and f witnesses $S \leq_{max} T$ (the proof being exactly the same as in [5, Theorem 3.1]).

As before, each structure of the kind G'_T needs to be Borel-in-T coded into a structure $\hat{G}'_T$ with domain ω to fit the official definition of countable $\mathcal{L}'$-structure: this can be done in several ways, but in our case we need to specify a particular coding, at least for those G'_T coming from an infinite $T \in \mathcal{T}$. Let $\langle \cdot, \cdot \rangle$ be any bijection from $\omega \times \omega$ in ω. Then code vertices of the form s, s^*, s^+ and s^{++} by $\langle 0, \#s \rangle$, $\langle 1, \#s \rangle$, $\langle 2, \#s \rangle$ and $\langle 3, \#s \rangle$, respectively. Now let η_T be an enumeration of T such that $\eta_T((u, s)) \leq \eta_T((v, t))$ if and only if either $\#s < \#t$, or else $\#s = \#t$ and $u \leq_{lex} v$ (η_T is well-defined as in the latter case $s = t$, and $(u, s), (v, t) \in T$ implies $|u| = |s| = |t| = |v|$), and code each vertex of the form $(u, s, x) \in G'_T$, where $x \subseteq 0^{2\theta(u)+2}$, in an element of the form $\langle 4, n \rangle$ in such a way that for two such vertices (u, s, x) and (v, t, y) and corresponding codings $\langle 4, n \rangle$ and $\langle 4, m \rangle$ one has $n \leq m$ if and only if either $\eta_T((u, s)) < \eta_T((v, t))$ or $\eta_T((u, s)) = \eta_T((v, t))$ and $x \subseteq y$. Finally, let η_0 be an enumeration of $\langle (s, i) \mid s \in {}^{<\omega}\omega \wedge i \leq \#s + 2 \rangle$ such that $\eta_0((s, i)) \leq \eta_0((t, j))$ if and only if either $\#s < \#t$, or else $\#s = \#t$ and $i \leq j$, and define $\pi_0 \colon \omega \to {}^{<\omega}\omega$ and $\pi_1 \colon \omega \to \omega$ in such a way that $\eta_0(\pi_0(n), \pi_1(n)) = n$. Now for $k \in \omega$ code vertices of G'_T of the form (s^{++}, i^{k+1}), $(u, s, 0^{2\theta(u)+2 \frown} 0^\frown 0^k)$ and $(u, s, 0^{2\theta(u)+2 \frown} 1^\frown 0^k)$ into $\langle 3\eta_0((s, i)) + 5, k \rangle$, $\langle 3\eta_T((u, s)) + 6, k \rangle$ and $\langle 3\eta_T((u, s)) + 7, k \rangle$, respectively. This finishes the coding, and we will always identify G'_T with its coded version $\hat{G}'_T$.

Now consider the closed subgroup $H \subseteq S_\infty$ given by those bijections p such that $p(\langle n, k \rangle) = \langle m, k \rangle$, where m depends only on n (that is, if $p(\langle n, 0 \rangle) = \langle m, k \rangle$ then $p(\langle n, k \rangle) = \langle m, k \rangle$ for every $k \in \omega$) and the following conditions hold:

- $n = m$ if $n \leq 4$
- $m = 3\eta_0((\pi_0(j), l)) + 5$ if $n = 3j + 5$
- $m = n$ or $m = n + 1$ if $n = 3j + 6$
- $m = n$ or $m = n - 1$ if $n = 3j + 7$.

Notice that H consists exactly of all automorphisms of $\hat{G}'_T = G'_T$ for some/every infinite $T \in \mathcal{T}$. By a theorem of Burgess (see e.g. [4, Theorem 12.17]), there is a Borel selector for the equivalence relation on S_∞ whose classes are the (left) cosets of H. Let Y be the corresponding Borel transversal, and consider the quasi-order R' defined on $X \times Y$ by letting $(x, p) R' (y, q) \iff x R y$. Let $f \colon X \times Y \to CT$ be the Borel (in fact, continuous) function which sends (x, p) to $G'_{S^x, p} = j_{\mathcal{L}'}(p, G'_{S^x})$, where S^x is defined as in Section 2: it is immediate to check as before that f is a reduction of R' to $\sqsubseteq_{CT}$, so it is enough to show that f is injective and that its range is invariant under isomorphism (this allows to conclude our proof as in Theorem 3.3).

The first claim (injectivity of f) follows from the fact that each isomorphism i between combinatorial trees of the form G'_S and G'_T (for $S, T \in \mathcal{T}$ infinite, as is the case if they are of the form S^x because of the reflexivity of R) must belong to H: granting this, one should simply note that $f((x,p)) = f((y,q))$ implies that $q^{-1} \circ p \in H$, so that $p = q$ (as they belongs to the same left coset of H and both are in Y) and hence $G_{S^x} = G_{S^y}$, which in turn implies $x = y$ by injectivity of the map $x \mapsto S^x$. To prove the above statement, first note that since i must preserve both distances and valences, $i(s^{++}) = s^{++}$ because s^{++} is the unique vertex (both in G'_S and G'_T) with valence $\#s + 4$, and therefore $i(s) = s$ because s is the unique vertex of valence ω with distance 2 from s^{++}. As in the proof of Lemma 3.1, this implies that $S = T$ (as G_T and G'_T share the same vertices of valence 3), and hence $G'_S = G'_T$: but this means $i \in H$ as required.

Finally, for the second claim (invariance of range(f)) it suffices to show that range(f) is the saturation under isomorphism of the set $\{G'_{S^x} \mid x \in X\}$, so consider a structure of the form $j_{\mathcal{L}'}(p, G'_{S^x})$ for $x \in X$ and $p \in S_\infty$ (the other inclusion is obvious). Let $q \in Y$ be in the same (left) coset of p with respect to H, so that $q = p \circ h$ for some $h \in H$: then

$$j_{\mathcal{L}'}(p, G'_{S^x}) = j_{\mathcal{L}'}(q \circ h^{-1}, G'_{S^x}) = j_{\mathcal{L}'}(q, j_{\mathcal{L}'}(h^{-1}, G'_{S^x})) = j_{\mathcal{L}'}(q, G'_{S^x}) = f((x,q)),$$

since $h^{-1} \in H$ is necessarily an automorphism of G'_{s^x}. □

Clearly Theorem 3.9 implies Theorem 3.3 as each combinatorial tree can be identified with the ordered combinatorial tree with same graph relation and empty order, and this identification preserves (closure under) isomorphisms and embeddings. However, it seems that Theorem 3.9 is really much stronger than 3.3 — see the discussion in Section 4.

The construction above allows us also to show that the relation of homomorphism on graphs is a complete analytic quasi-order (a fact already noted in [5, Theorem 3.5]), and that for each analytic quasi-order R on X there is an $\mathcal{L}'_{\omega_1\omega}$-sentence ψ such that R is Borel equivalent to the relation of homomorphism on Mod_ψ. This follows from the next proposition and the fact the none of the combinatorial trees involved in the proof of Theorem 3.9 have vertices of valence 1.

Proposition 3.10. *Assume that G is a combinatorial tree such that in G there is no pair of vertices of valence 1 with distance 2 from each other, and G' is an arbitrary graph. Then every homomorphism from G to G' is injective (hence an embedding).*

Proof. It suffices to prove that if $h \colon G \to G'$ is a homomorphism and g_0, g_1 are distinct vertices of G such that $h(g_0) = h(g_1)$ then these vertices have both valence 1 and distance 2 from each other. This is an easy consequence of the next claim.

Claim 3.10.1. Let $g_0, \dots, g_n$ be a chain of vertices in G with $n \geq 3$. Then $h(g_i) \neq h(g_j)$ for distinct $i, j \leq n$.

Proof of the claim. By induction on $n \geq 3$. If $n = 3$ first we have that $h(g_i) \neq h(g_{i+1})$ ($i \leq 2$) because g_i is linked to g_{i+1} in G. Then $h(g_0) \neq h(g_2)$ because otherwise $h(g_3)$ would be linked to $h(g_0)$ and therefore g_3 would be linked to g_0, a contradiction with the acyclicity of G. The same argument (using g_1 instead of g_0 in the second case) shows that $h(g_1) \neq h(g_3)$ and $h(g_0) \neq h(g_3)$.

For the inductive step, consider a chain $g_0, \dots, g_{n+1}$: since the claim must hold for both the chains $g_0, \dots, g_n$ and $g_1, \dots, g_{n+1}$, we need only to check that $h(g_0) \neq$

$h(g_{n+1})$. But $h(g_0) = h(g_{n+1})$ would contradict the acyclicity of G again (since it implies that g_1 is linked to g_{n+1}), hence we are done. $\square$ *Claim*

$\square$

Remark 3.11.
1. Despite the previous result, we should note that the construction given in Theorem 3.9 cannot be used to prove the analogous statement about the relation of weak-homomorphism on combinatorial trees, as any two such trees are always bi-weak-homomorphic. To see this it is enough to show that any combinatorial tree G is indeed bi-weak-homomorphic to the combinatorial tree $\bar{G}$ on ω in which $n < m$ are linked just in case $m = n + 1$. In fact, choose any vertex g_0 of G: the map which sends an arbitrary vertex of G to its distance from g_0 is a weak-homomorphism of G into $\bar{G}$. Conversely, choose vertices g_0, g_1 in G such that there is an edge between them: the map which sends $2k + i$ to g_i (for $i = 0, 1$ and $k \in \omega$) is a weak-homomorphism of $\bar{G}$ into G.
2. A different proof of Theorem 3.9 can be given using an argument similar to the one of Theorem 3.3: in fact it is possible to define for each normal tree T on $2 \times \omega$ a combinatorial tree G^+_T such that $S \neq T \Rightarrow G^+_S \ncong G^+_T$ and each G^+_T has no nontrivial automorphism, and then prove the desired results as in the proof of Theorem 3.3 but using these last properties instead of Lemmas 3.1 and 3.2. However the combinatorial trees G^+_T would not be pruned (i.e. they would have vertices with valence 1), while we will often need pruned combinatorial trees to get the applications to analysis (see the corollaries below). On the other hand, this alternative proof allows one to get an effective version of Theorem 3.9 analogous to the effective version of Theorem 3.3 provided in Remark 3.4.
3. Combining the variant suggested above with [5, Theorem 3.3], one gets that the combinatorial trees used in Theorem 3.9 can be substituted by countable partial orders or countable lattices (viewed as partial orders, or even viewed as countable lattices, that is as structures in the language $\mathcal{L}''$ containing two binary function symbols and satisfying the axioms of lattices). In fact, in [5, Theorem 3.3] a map $G \mapsto \leq_G$ from combinatorial trees on ω to countable partial orders (or to lattices) is constructed, and inspecting its proof one gets that each nontrivial (that is, different from the identity) isomorphism between $\leq_G$ and $\leq_H$ can be turned into a nontrivial isomorphism between G and H: therefore we get that the map $T \mapsto \leq_{G^+_T}$ is such that $S \neq T \Rightarrow \leq_{G^+_S} \ncong \leq_{G^+_T}$ and $\leq_{G^+_T}$ has only trivial automorphisms, hence we can conlude the proof as in Theorem 3.3 again.

The main advantage of using Theorem 3.9 is that we can get several applications in analysis as corollaries (we could not have used Theorem 3.3 because of the orderings). First consider the class $\mathscr{D}$ of *discrete Polish metric spaces* $(\mathcal{X}, d)$ (i.e. discrete separable complete metric spaces): since any discrete separable topological space is countable, we can identify each of them as a space on ω, i.e. we can put $\mathcal{X} = \omega$. Granting this identification we have the following result (recall that an isometric embedding is simply an injection between metric spaces which preserves distances, while an isometry is just a surjective isometric embedding).

Corollary 3.12. *If R is an analytic quasi-order on a standard Borel space X then there is a Borel class $\mathcal{C} \subseteq \mathscr{D}$ closed under isometry such that R is Borel equivalent to the relation of isometric embeddability on $\mathcal{C}$.*

Proof. With the notation established in the proof of Theorem 3.9, consider each graph of the form $G'_{S,p}$ as a discrete Polish space, where the distance is the geodesic distance on $G'_{S,p}$. Since one can recover such a distance from the graph structure and, conversely, the graph structure from the distance, it is clear that embeddings must correspond exactly to isometric embeddings and isomorphisms to isometries, therefore the result follows immediately from Theorem 3.9. □

If one wants to deal with uncountable Polish spaces, the standard procedure is to identify (up to isometry) each such space with a closed subset of the Polish Urysohn space $\mathbb{U}$ (where the class $F(\mathbb{U})$ of closed subsets of $\mathbb{U}$ is endowed with the Effros-Borel topology), and then consider the analytic relation of isometric embeddability on $F(\mathbb{U})$. If we now look at the Borel class $\mathscr{U} \subseteq F(\mathbb{U})$ of ultrametric Polish spaces (i.e. metric Polish spaces whose distance d is such that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$) we get the following:

Corollary 3.13. *For every analytic quasi-order R on a standard Borel space X there is a Borel class $\mathcal{C} \subseteq \mathscr{U}$ of pairwise non-isometric ultrametric Polish spaces such that R is Borel equivalent to the relation of isometric embeddability on $\mathcal{C}$.*

Proof. Identify each element of the form G'_T (fot $T \in \mathcal{T}$) with the set U_T of all maximal paths α starting from $\emptyset$, equipped with the distance d_T defined by $d_T(\alpha_0, \alpha_1) = 0$ if $\alpha_0 = \alpha_1$, and $d_T(\alpha_0, \alpha_1) = 2^{-n}$ if the set of vertices belonging to both α_0 and α_1 has cardinality n. It is clear that $U_T = (U_T, d_T)$ is an ultrametric Polish space, and that $G'_S \cong G'_T$ if and only if U_S is isometric to U_T: this is because any isomorphism (resp. embedding) between G'_S and G'_T can be canonically converted into an isometry (resp. isometric embedding) between U_S and U_T, and *vice-versa.* Now consider the Borel map g which sends $x \in X$ to the isometric copy of U_{S^x} in $\mathbb{U}$ (which is an element of $\mathscr{U}$): it is clearly injective, and by Theorem 3.9 has the further property that $x \neq y$ implies that $g(x)$ and $g(y)$ are not isometric; therefore it is enough to put $\mathcal{C} = \operatorname{range}(g)$. □

Now we turn our attention to continuous embeddability. Each compact metrizable space can be identified, up to homeomorphism, with an element of the space $K(I)$, the space of all compact subspaces of the Hilbert cube $I = [0, 1]^\omega$ (with its Hausdorff topology). If we consider the relations of, respectively, continuous embeddability (given by injective continuous maps) and homeomorphism between elements of $K(I)$, we get, respectively, an analytic quasi-order and an analytic equivalence relation: also in this case, Theorem 3.9 leads to the following corollary[5].

[5]We have proved this result for the class $K([0, 1]^2)$ of compact subsets of $[0, 1]^2$, but since any such space can be naturally identified, up to homeomorphism, with an element of $K(I)$ the corollary holds with $K([0, 1]^2)$ replaced by $K(I)$ as well. Nevertheless, we cannot replace $K([0, 1]^2)$ by $K([0, 1])$ because, as already noted in [5], the notion of continuous embeddability on $K([0, 1])$ gives just a pre-well-ordering of type $\omega_1 + 2$.

Corollary 3.14. *For every analytic quasi-order R on a standard Borel space X there is a Borel class $\mathcal{C} \subseteq K([0,1]^2)$ of pairwise non-homeomorphic compact metrizable spaces such that R is Borel equivalent to the relation of continuous embeddability on $\mathcal{C}$.*

Proof. Consider the construction given in [5, Theorem 4.5] which defines a Borel map $S \mapsto K_S$ from rooted combinatorial trees to $K(I)$. Analyzing that proof, it is clear that for distinct rooted combinatorial trees S, T one gets $S \cong T$ if and only if K_S and K_T are homeomorphic. Therefore the Borel map g which sends $x \in X$ to $K_{G'_{Sx}}$ is such that $g(x)$ is non-homeomorphic to $g(y)$ for distinct $x, y \in X$. Taking $\mathcal{C} = \mathrm{range}(g)$ we get the result. $\square$

Finally, we look at separable Banach spaces. Any such space is linearly isometric to a closed subspace of $C([0,1])$ with the sup norm, so the class of separable Banach spaces can be identified with the Borel subset $\mathscr{B} \subseteq F(C([0,1]))$ of all closed linear subspaces of $C([0,1])$ (which is a standard Borel space). A function between two separable Banach spaces B and B' is said to be a *linear isometric embedding* (resp. *linear isometry*) if it is linear and norm-preserving (resp. linear, norm-preserving and onto): the corresponding relations of linear isometric embeddability and linear isometry on $\mathscr{B}$ are, respectively, an analytic quasi-order and an analytic equivalence relation. As noted in [5], recent results by Godefroy and Kalton show that on $\mathscr{B}$ these two relations coincide with isometric embeddability and isometry, respectively.

Corollary 3.15. *For every analytic quasi-order R on a standard Borel space X there is a Borel class $\mathcal{C} \subseteq \mathscr{B}$ of pairwise non-linear isometric (resp. non-isometric) separable Banach spaces isomorphic to c_0 such that R is Borel equivalent to the relation of linear isometric embeddability (resp. isometric embeddability) on $\mathcal{C}$.*

Proof. Given a combinatorial tree G, consider the construction of the space $(c_0, \| \|_G)$ given in the proof of [5, Theorem 4.6]. Since that proof shows that $G_0 \cong G_1$ if and only if $(c_0, \| \|_{G_0})$ and $(c_0, \| \|_{G_1})$ are linear isometric, the map g which sends $x \in X$ to $(c_0, \| \|_{G'_{Sx}})$ is Borel and strongly injective (in the sense that if $x \neq y \in X$ then $(c_0, \| \|_{G_{Sx}})$ and $(c_0, \| \|_{G_{Sy}})$ are not linear isometric) by the proof of Theorem 3.9 again. Therefore it is again enough to put $\mathcal{C} = \mathrm{range}(g)$. $\square$

Obviously, all the previous corollaries can be naturally translated into the context of analytic equivalence relations.

Corollary 3.16. *Let E be a binary relation on a standard Borel space X. Then the following are equivalent:*

(1) *E is an analytic equivalence relation on X;*
(2) *there is a Borel class $\mathcal{C} \subseteq \mathscr{D}$ of discrete Polish metric spaces closed under isometry such that E is Borel equivalent to the relation of isometric bi-embeddabbility on $\mathcal{C}$;*
(3) *there is a Borel class $\mathcal{C} \subseteq \mathscr{U}$ of pairwise non-isometric ultrametric Polish spaces such that E is Borel equivalent to the relation of isometric bi-embeddability on $\mathcal{C}$;*
(4) *there is a Borel class $\mathcal{C} \subseteq K(I)$ (or even just $\mathcal{C} \subseteq K([0,1]^2)$) of pairwise non-homeomorphic compact metrizable spaces such that E is Borel equivalent to the relation of continuous bi-embeddability on $\mathcal{C}$;*

(5) *there is a Borel class $\mathcal{C} \subseteq \mathscr{B}$ of pairwise non-linear isometric (resp. non-isometric) separable Banach spaces isomorphic to c_0 such that E is Borel equivalent to the relation of linear isometric bi-embeddability (resp. isometric bi-embeddability) on $\mathcal{C}$.*

3.3. **Actions of groups and monoids.** A special kind of analytic equivalence relations is one which is induced by the continuous (resp. Borel) action of a group: given a Polish space X, a Polish group G (that is, a group equipped with a Polish topology such that the map $(g, h) \mapsto gh^{-1}$ is continuous), and an action a of G on X (that is a function $a\colon G \times X \to X$ such that $a(e, x) = x$ and $a(g,(a(h, x))) = a(gh, x)$ for every $g, h \in G$ and $x \in X$), for each $x, y \in X$ we put $xE^G_X y \iff \exists g \in G(a(g, x) = y)$, and it is easy to check that if a is a continuous (or just Borel) function that E^G_X is an analytic equivalence relation on X. Note however that such an analytic equivalence relation cannot be complete, as there are analytic equivalence relations, such as E_1, that are not Borel reducible to it (see e.g. [3, Theorem 8.2]).

The natural counterpart of Polish groups in the quasi-order context are Polish monoids, i.e. semigroups with identity which are equipped with a Polish topology such that the monoid operation is a continuous function. However, it is not clear what should be the right generalization of the notion of *action*: in [5], Louveau and Rosendal chose to define the action of the monoid G on the Polish space X exactly as an action of a group, that is as a function $a\colon G \times X \to X$ such that $a(e, x) = x$ and $a(g,(a(h, x))) = a(gh, x)$ for every $g, h \in G$ and $x \in X$, and then proved in [5, Theorem 5.1] that there is a Polish monoid G acting continuously on a Polish space X, such that E^G_X is complete for analytic quasi-orders. There is however a difficulty with this notion of monoid action: if one looks at some natural quasi-order R induced by some class of morphisms acting on X, then the monoid G consisting of such morphisms together with the composition operation and the usual identity (and topologized in a natural way) should have a "natural" action on X inducing R, and this will not in general be true using the notion of action defined above. For example, consider the relation of embeddability on the space X_G of (codes for) countable graphs: in this case the collection of morphisms is simply the Polish monoid S^-_∞, the set of all injective (not necessarily onto) functions from ω into itself, topologized as a subspace of the Baire space. But there is no natural function assigning to each pair $(p, x) \in S^-_\infty \times X_G$ a *unique* $y \in X_G$ such that $x \sqsubseteq y$, as there are too many y in which x can be embedded. One option is to change the notion of monoid in such a way that each of its elements determines in a unique way the target graph, not only how x is embedded into it. We have taken this option in the next section in the special case of embeddings on graphs; however, this approach can be used just for some specific cases. This suggests that the "right" definition of an action of a monoid on a Polish space should be that of a relation (or, equivalently, of a multi-valued function) rather than that of a function: here is our proposal.

Definition 2. Let X be a Polish space and G a Polish monoid. An action A of G on X is a relation $A \subseteq G \times X \times X$ such that for every $x, y, z \in X$ and $g, h \in G$ the following holds:

i) $(e, x, x) \in A$;
ii) if $(h, x, y) \in A$ and $(g, y, z) \in A$ then $(gh, x, z) \in A$.

The action A is said to be *closed* (resp. *Borel*, *analytic*) if it is closed (resp. Borel, analytic) as a subspace of the Polish space $G \times X \times X$.

Note that the usual group action can be identified (passing from functions to their graph) with those monoid actions which turn out to be functions from $G \times X$ into X, that is with those $A \subseteq G \times X \times X$ such that for every $(g, x) \in G \times X$ there is a unique $y \in X$ such that $(g, x, y) \in A$. Being closed (resp. Borel) for monoid actions is the analogous of being continuous (resp. Borel) for group actions, although the analogy is not exact in the first case as there are functions with closed graphs which are not continuous[6]. In the case of group actions the distinction between Borel and analytic simply disappears, because a function is Borel if and only its graph is Borel if and only if its graph is analytic if and only if it is analytic-measurable.

For A an action of G on X we put $x E_X^{G,A} y \iff \exists g \in G((g, x, y) \in A)$, and it is easy to check that $E_X^{G,A}$ is a quasi-order (by the properties of an action), and that if A is analytic then $E_X^{G,A}$ is analytic: we will call $E_X^{G,A}$ the analytic quasi-order induced by the action A of G on X. Note that if X is just a standard Borel space (rather than Polish), then we can speak of closed actions on X just in case we fix in advance some Polish topology on X compatible with its Borel structure, but we can unambiguously speak of Borel (or analytic) actions, as these notions only depends on the Borel structure of X.

Now consider again the example of embeddings on the space of graphs X_G: with our new definition, the monoid S_∞^- has a natural closed action on X_G, namely

$$A = \{(p, x, y) \in A \mid \forall n, m(n R^{\mathcal{A}_x} m \iff p(n) R^{\mathcal{A}_y} p(m))\}$$

(where $\mathcal{A}_x$ and $\mathcal{A}_y$ are the graphs coded by x and y, respectively), and it is clear that the quasi-order $E_{X_G}^{S_\infty^-, A}$ is just the relation of embeddability on X_G.

Theorem 3.17. *Each analytic quasi-order R on a standard Borel space X is Borel equivalent to a quasi-order on ${}^\omega 2$ induced by a closed action of S_∞^-. Moreover, R itself is induced by a Borel action of S_∞^-.*

Proof. As any standard Borel space X is Borel-isomorphic to ${}^\omega 2$, we just need to prove that any quasi-order R' on ${}^\omega 2$ is induced by a closed action A' of S_∞^- on ${}^\omega 2$. Let A be the natural closed action A of S_∞^- on the space of (codes for) graphs X_G, and let $f \colon {}^\omega 2 \times Y \to X_G$ be the reduction defined in the proof of Theorem 3.9, where Y is the Borel transversal defined in that proof. Clearly we can assume $id \in Y$. Since f is a continuous function, so is $h \colon {}^\omega 2 \to X_G$ defined by $x \mapsto f((x, id))$: therefore the action $A' \subseteq S_\infty^- \times {}^\omega 2 \times {}^\omega 2$ given by $(p, x, y) \in A' \iff (p, h(x), h(y)) \in A$ is closed and clearly induces R because h is a reduction of R' to $\sqsubseteq$ (which is $E_{X_G}^{S_\infty^-, A}$). $\square$

4. Applications of the main results and techniques

We would like to explore in this section some applications of the methods and results obtained above. In [5], Louveau and Rosendal suggested that one analyze the possible relationships between bi-embeddability $\equiv_{\mathcal{C}}$ and isomorphism $\cong_{\mathcal{C}}$ on some $\mathcal{L}_{\omega_1 \omega}$-elementary class $\mathcal{C}$ of countable structures (that is on the class of countable models of some $\mathcal{L}_{\omega_1 \omega}$-sentence, $\mathcal{L}$ some countable language). As they already noted,

[6]As an example of this fact, consider the function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ if $x = 0$ and $f(x) = x^{-1}$ otherwise.

these relations can behave very differently. A trivial example is the class $\mathcal{C}$ of countable well-founded linear order of length bounded by some fixed $\alpha < \omega_1$: in this case $\equiv_\mathcal{C}$ and $\cong_\mathcal{C}$ coincide. In contrast, some deep results show that $\equiv_\mathcal{C}$ and $\cong_\mathcal{C}$ can be extremely far apart.

Example 1. Let $\mathcal{C}$ be the collection of countable linear orders: then $\equiv_\mathcal{C}$ has only $\aleph_1$ classes (since by Laver's proof of Fraïssé conjecture $\sqsubseteq_\mathcal{C}$ is a bqo), whereas $\cong_\mathcal{C}$ is S_∞-complete (that is as complicated as it can be) by [2].

Example 2. If $\mathcal{C}$ is the collection of countable graphs (or of combinatorial trees, partial orders, lattices, and so on) then $\equiv_\mathcal{C}$ is a complete analytic equivalence relation by [5], while $\cong_\mathcal{C}$ is just S_∞-complete.

Louveau and Rosendal raised the question of whether one can increase these gaps, namely:

Question 1. *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ with $\cong_\mathcal{C}$ S_∞-complete but $\equiv_\mathcal{C}$ with countably many classes?*

Question 2. *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ with $\equiv_\mathcal{C}$ complete analytic but $\cong_\mathcal{C}$ not S_∞-complete?*

In the same vein, a third natural question concerning some possible limitations to the method developed in [5] was asked as well:

Question 3. *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ with $\equiv_\mathcal{C}$ complete analytic but $\sqsubseteq_\mathcal{C}$ not a complete analytic quasi-order?*

The results obtained in the previous section can be used to answer Questions 2 and 3: in fact, as already observed, Theorem 3.3 and Theorem 3.9 both show (for different languages $\mathcal{L}$) that for every analytic quasi-order R there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $R \sim_B \sqsubseteq_\mathcal{C}$ (hence $E_R = R \cap R^{-1} \sim_B \equiv_\mathcal{C}$) and $= \sim_B \cong_\mathcal{C}$. This result can be used to find e.g. $\mathcal{L}_{\omega_1\omega}$-elementary classes $\mathcal{C}$ such that $\equiv_\mathcal{C} = \cong_\mathcal{C}$ (take R to be the equality relation on ${}^\omega 2$) or such that $\equiv_\mathcal{C} \neq \cong_\mathcal{C}$ but still $\equiv_\mathcal{C} \sim_B \cong_\mathcal{C}$ (take R to be any equivalence relation on ${}^\omega 2$ Borel equivalent to the equality relation but such that at least one equivalence class has more than one element: then if $\mathcal{C}$ is the class resulting by the application of our result, $\equiv_\mathcal{C}$ will be strictly coarser than $\cong_\mathcal{C}$ but $\cong_\mathcal{C} \sim_B = \sim_B R = E_R \sim_B \equiv_\mathcal{C}$).

Another trivial application of the same result allows one to answer Question 2: let R be any complete analytic quasi-order, and let $\mathcal{C}$ be the class corresponding to R with respect to the result quoted above. Then $\sqsubseteq_\mathcal{C}$ (and hence also $\equiv_\mathcal{C}$) must be complete analytic (in the corresponding classes of relations), but $\cong_\mathcal{C}$ is just smooth (in fact, Borel equivalent to the equality relation). The same kind of argument shows that we can realize any possible relationship between $\cong_\mathcal{C}$ and $\equiv_\mathcal{C}$ (such as $\equiv_\mathcal{C} <_B \cong_\mathcal{C}$, or $\equiv_\mathcal{C}$ and $\cong_\mathcal{C}$ Borel incomparable, and so on) as long as we are content to have $\cong_\mathcal{C} \sim_B =$. An interesting related problem would be to determine which are the possible pairs of degrees $(\cong_\mathcal{C}, \equiv_\mathcal{C})$ for $\mathcal{C}$ an $\mathcal{L}_{\omega_1\omega}$-elementary class, but apart from the previous results and some obvious limitations this seems to be difficult.

Question 3 can be answered in the same vein as Question 2: let R be a complete analytic equivalence relation (so that, in particular, $E_R = R$), and let $\mathcal{C}$ be the resulting class obtained with Theorem 3.3 or Theorem 3.9. Then $\equiv_\mathcal{C}$ is clearly complete analytic (as $\equiv_\mathcal{C} \sim_B E_R = R$), but $\sqsubseteq_\mathcal{C}$ can not be complete as a quasi-order since it is an equivalence relation (in fact, as already observed, in this case we get $\sqsubseteq_\mathcal{C} = \equiv_\mathcal{C}$).

Question 2 and 3 can be modified in the following way: given an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$, call the relation $\sqsubseteq_{\mathcal{C}}$ (resp. $\equiv_{\mathcal{C}}$) *universal* if for every analytic quasi-order R (resp. every analytic equivalence relation E) there is an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}' \subseteq \mathcal{C}$ such that $R \sim_B \sqsubseteq_{\mathcal{C}'}$ (resp. $E \sim_B \equiv_{\mathcal{C}'}$). Note that if one of $\sqsubseteq_{\mathcal{C}}$ or $\equiv_{\mathcal{C}}$ is universal then it must be also complete analytic in the corresponding class of analytic relations. Moreover, as Corollary 3.7 shows, if $\mathcal{C}$ is such that $\sqsubseteq_{\mathcal{C}}$ is universal then $\equiv_{\mathcal{C}}$ must be universal as well, and Theorems 3.3 and 3.9 can be rephrased as "there exists an $\mathcal{L}_{\omega_1\omega}$-elementary class such that $\sqsubseteq_{\mathcal{C}}$ (and hence also $\equiv_{\mathcal{C}}$) is universal". Here are the natural modifications of Question 2 and Question 3:

Question 4. *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ such that $\equiv_{\mathcal{C}}$ is universal but $\cong_{\mathcal{C}}$ not S_∞-complete?*

Question 5. *Is there an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ with $\equiv_{\mathcal{C}}$ universal but $\sqsubseteq_{\mathcal{C}}$ not universal?*

In both cases the answer is positive again, and it is even possible to have a single $\mathcal{C}$ such that $\equiv_{\mathcal{C}}$ is universal but $\cong_{\mathcal{C}}$ is not S_∞-complete and $\sqsubseteq_{\mathcal{C}}$ is not an complete analytic quasi-order (hence, in particular, not universal): in fact, it is enough to consider an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}'$ such that $\equiv_{\mathcal{C}'}$ is universal and then apply Theorem 3.3 or Theorem 3.9 to such a relation in order to get an $\mathcal{L}_{\omega_1\omega}$-elementary class $\mathcal{C}$ with the desired properties (one has just to check that $\equiv_{\mathcal{C}}$ is indeed universal because any Borel subset of $\mathcal{C}'$, hence in particular any $\mathcal{L}_{\omega_1\omega}$-elementary subclass of $\mathcal{C}'$, will be mapped to an $\mathcal{L}_{\omega_1\omega}$-elementary subclass of $\mathcal{C}$ by the construction given in the proofs of the quoted theorems).

The answer to Question 1 is of a different nature, and doesn't involve the results of this paper, but we put it here for the sake of completeness and because the following construction (due to H. Friedman and Stanley, see [2]) will be used below for different applications. Let $\langle \cdot, \cdot \rangle \colon \omega \times \omega \to \omega$ be any bijection: for $\emptyset \neq s \in {}^{<\omega}\omega$ define the *relevant pair of s* to be $rp(s) = (s(n), s(m))$, where n, m are such that s has length $\langle n, m \rangle + 1$. Given (a code for) any countable graph x, we will define a set-theoretical tree T_x in the following way: the domain of T_x is given by ${}^{<\omega}\omega \sqcup \omega$ (where $\sqcup$ means disjoint union) if x is not the empty graph and by ${}^{<\omega}\omega$ otherwise, and we order ${}^{<\omega}\omega$ with the inclusion relation. Finally, if x contains at least one edge we adjoin to each element $\emptyset \neq s \in {}^{<\omega}\omega$ such that $rp(s)$ is a pair of linked vertices in x a new terminal immediate successor taken from ω, in such a way that each natural number is the terminal successor of one (and only one) element of ${}^{<\omega}\omega$ (this can be done since for each $n, m \in \omega$ there are infinitely many $s \in {}^{<\omega}\omega$ for which $rp(s) = (n, m)$). As proved in [2], each isomorphism between graphs x and y can be naturally "lifted" to a permutation of ${}^{<\omega}\omega$, and then extended to an isomorphism of T_x and T_y. Conversely, an isomorphism j between T_x and T_y must send elements of ${}^{<\omega}\omega$ to elements of ${}^{<\omega}\omega$ of the same length: therefore we can reconstruct from j an isomorphism i between x and y using a back and forth argument. Now let $\mathcal{C}$ be the class of infinite set-theoretical countable trees of height ω such that each node is either terminal or has infinitely many immediate successor, and such that in the latter case at most one of those successors is terminal (we leave to the reader to show that there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ in the language of trees $\mathcal{L}$ such that $\mathcal{C} = Mod_\varphi$): then each tree of the form T_x belongs to $\mathcal{C}$, whence $\cong_{\mathcal{C}}$ is S_∞-complete, but each element of $\mathcal{C}$ can be embedded in any other element of $\mathcal{C}$ since any set-theoretical tree of height ω can be embedded in ${}^{<\omega}\omega$, and the last condition defining $\mathcal{C}$ easily imply that a tree in $\mathcal{C}$ must contain an isomorphic copy of ${}^{<\omega}\omega$ as subtree. This proves that $\mathcal{C}$ is such that $\cong_{\mathcal{C}}$ is S_∞-complete but $\equiv_{\mathcal{C}}$ has exactly 1 equivalence class (hence, in particular, provides another situation in which $\equiv_{\mathcal{C}} <_B \cong_{\mathcal{C}}$, but with the further property that $\cong_{\mathcal{C}}$ is as complicated as it can be).

The following different application of the methods developed in the previous section can in particular be intended as an explanation of why we heuristically feel that Theorem 3.9 is really a stronger result than Theorem 3.3: our explanation is based on some connections with the Vaught's conjecture, so let us first recall some definitions and results regarding that topic. Here is the statement of (a version of) Vaught's conjecture, which is still open.

Vaught's Conjecture. *For every countable language $\mathcal{L}$ and every $\mathcal{L}_{\omega_1\omega}$-sentence φ, there are either at most countably many or perfectly many non-isomorphic countable models of φ.*

Vaught's conjecture can be used to measure the "complexity" of various theories in the following way. Given an $\mathcal{L}_{\omega_1\omega}$-sentence φ, we say that φ (or, equivalently, the collection of all its countable models Mod_φ) *strongly satisfies Vaught's conjecture*, if for every $\mathcal{L}_{\omega_1\omega}$-sentence ψ such that $\psi \Rightarrow \varphi$ we have that either there are at most countably many non-isomorphic countable models of ψ, or else there are perfectly many non-isomorphic countable models of ψ. This can be equivalently rephrased by requiring that every Borel invariant (with respect to isomorphism) subset X of Mod_φ has either contably many or perfectly many $\cong$-classes. Note also that this is stronger than just requiring that φ has either countably many or perfectly many non-isomorphic countable models. In what follows we confuse (countable) $\mathcal{L}_{\omega_1\omega}$-theories T with the $\mathcal{L}_{\omega_1\omega}$-sentences given by the (infinite) conjunction of the sentences in T. Rubin [9, 10], Miller [7] and Steel [11] showed that the theories of, respectively, linear orders, unary operations (or even just graphs such that each connected component has only finitely many loops), and trees (partially ordered sets in which the set of predecessors of any element is linearly ordered) strongly satisfy Vaught's conjecture. The result of Miller, in particular, implies that also the theory of (rooted) combinatorial trees strongly satisfies Vaught's conjecture.

Theories which strongly satisfy Vaught's conjecture can be regarded in this context as "simple" theories. On the opposite side there are those theories which have the property that if they strongly satisfy Vaught's conjecture then Vaught's conjecture 4 holds: we call these theories *Vaught's conjecture-complete* (*VC-complete* for short), and can be regarded in the present context as the most complicated theories[7]. It is a folklore result that the theory of graphs is VC-complete. We will now show that also the theory of rooted ordered combinatorial trees $ROCT$ is VC-complete, and therefore (unless Vaught's conjecture is proved!) more complicated than the theory of (rooted) combinatorial trees: this shows that Theorem 3.9 is in a sense stronger than Theorem 3.3, because it proves that the "universality" property of $\sqsubseteq_{ROCT}$ expressed by Theorem 3.3 is shared also by the relation of embeddability on a "simpler" theory.

[7]There is an analogy between Vaught's conjecture and the problem $P = NP$ in theoretical computer science: theories correspond to decision problems, theories strongly satisfying Vaugh's conjecture correspond to decision problems belonging to P, and VC-complete theories correspond to NP-complete decision problems.

Theorem 4.1. *The theory of (rooted) ordered combinatorial trees (whose ordering is given by an equivalence relation) is VC-complete.*

Proof. The proof is a combination of the Friedman-Stanley's construction explained above with the technique developed in the proof of Theorem 3.9. Let X_G and X_{ROCT} be the set of (codes for) countable graphs and rooted ordered combinatorial trees[8], respectively. It is enough to show that for every Borel invariant $X \subseteq X_G$ there is a Borel invariant $Z \subseteq X_{ROCT}$ such that $(X, \cong) \sim_B (Z, \cong)$. Note that we can always assume that X doesn't contain the empty graph. To each $x \in X$, associate the rooted ordered combinatorial tree G_x defined on ${}^{<\omega}\omega \sqcup \omega$ (with $\emptyset$ as root) by stipulating that two elements of G_x are adiacent just in case one of them is an immediate successor of the other in the tree T_x defined above, and that two elements are in the order relation of G_x if and only if either they belong to ω or else they belong to ${}^{<\omega}\omega$ and share the same relevant pair (note that this is an equivalence relation which moreover is *independent* from the graph x). If we choose to link natural numbers to sequences in a careful way, e.g. respecting any fixed order of ${}^{<\omega}\omega$ of type ω, then the map which sends x to (the code of) G_x is Borel (in fact, continuous). By repeating the Friedman-Stanley's proof about the trees T_x sketched above, it is easy to check that this map reduces the isomorphism relation on any Borel invariant $X \subseteq X_G$ to the isomorphism relation on X_{ROCT} (it is enough to check that the "lifting" of any isomorphism between graphs x and y to a permutation of ${}^{<\omega}\omega$ preserves the order relations of G_x and G_y).

For simplicity of notation, identify S_∞ with the group of permutations of ${}^{<\omega}\omega \sqcup \omega$. Consider the following closed subgroups of S_∞:

$$\begin{aligned} H_1 = \{p \in S_\infty \mid & \forall s,t \in {}^{<\omega}\omega (rp(s) = rp(t) \iff rp(p(s)) = rp(p(t)) \wedge \\ & \wedge \forall n \in \omega (p(n) = n)\} \end{aligned}$$

and

$$H_2 = \{p \in S_\infty \mid \forall s \in {}^{<\omega}\omega (p(s) = s)\}.$$

By [4, Theorem 12.17] again, there is a Borel selector for the equivalence relation on S_∞ whose classes are the (left) cosets of H_1. Let Y be the corresponding Borel transversal, and consider the equivalence relation E on $X \times H_2 \times Y$ (X as before) defined by $(x, p_1, q_1) E (y, p_2, q_2) \iff x \cong y$. Clearly E is Borel equivalent to isomorphism on X, and the map $f \colon X \times H_2 \times Y \to X_{ROCT} \colon (x, p, q) \mapsto j_{\mathcal{L}}(q \circ p, G_x)$, $\mathcal{L}$ the language of rooted ordered combinatorial trees, is a continuous reduction of E into $\cong$ on X_{ROCT}. As in the proof of Theorem 3.9, it is therefore enough to show that f is injective and that its range Z is invariant under isomorphism.

Assume that $x, y \in X$, $p_1, p_2 \in H_2$ and $q_1, q_2 \in Y$ are such that $f((x, p_1, q_1)) = f((y, p_2, q_2))$. Then $p_2^{-1} \circ q_2^{-1} \circ q_1 \circ p_1$ is an isomorphism between G_x and G_y and since it must respect their order relations and both p_2^{-1} and p_1 are the identity on T_∞, we must conclude that $q_2^{-1} \circ q_1 \in H_1$: but then q_1 and q_2 are in the same left coset of H_1, which means $q_1 = q_2$ (since Y is a transversal). This implies $x = y$ because $p_2^{-1} \circ q_2^{-1} \circ q_1 \circ p_1 = p_2^{-1} \circ p_1 \in H_2$ is an isomorphism between G_x and G_y,

[8]As it will be easy to check, we can also replace rooted ordered combinatorial trees with ordered combinatorial trees: in the construction of the G_x's given below, instead of specifying $\emptyset$ as root, simply adjoin two new vertices r and r', and then link r to both $\emptyset$ and r'. It is easy to check that any embedding between two such trees must again send $\emptyset$ into itself, so the rest of the argument can be carried out exactly in the same way.

and hence we get also $p_2^{-1} \circ p_1 = id$, that is $p_1 = p_2$. This proves the injectivity of f.

Finally we will prove that range(f) is the closure under isomorphism of $\{G_x \mid x \in X\}$. First note that if $h \in H_1$ and $x \in X$, then there is $y \in X$ and $p \in H_2$ such that $j_{\mathcal{L}}(p, G_y) = j_{\mathcal{L}}(h, G_x)$ (this is because X is invariant under isomorphism). Now choose $r \in S_\infty$ and $x \in X$, and let $q \in Y$ be in the same left coset of H_1 as r, so that $r = q \circ h$ for some $h \in H_1$. Let y and p be as in the observation above. Then $j_{\mathcal{L}}(q \circ p, G_y) = j_{\mathcal{L}}(q \circ h, G_x) = j_{\mathcal{L}}(r, G_x)$, so that $f((y, p, q)) = j_{\mathcal{L}}(r, G_x)$ and we are done. □

Note that an obvious modification of the previous proof gives that also the theory of ordered trees (that is of set-theoretical trees with an extra transitive relation on their nodes) is VC-complete.

Besides showing that *ordered* (rooted) combinatorial trees have seemingly more "universal" properties than combinatorial trees (namely, the fact of being VC-complete), Theorem 4.1 should also be compared with the well-known argument used to show that the theory of graphs is VC-complete. In that case, one makes a crucial use of *infinitely many* loops: if one could find a similar argument which avoids loops (or uses only finitely many of them in each connected component), then applying Miller's result [7] one would get a proof of Vaught's conjecture. Theorem 4.1 shows that we can completely avoid loops, but (as far as we know) we have to compensate for this with the addition of a new transitive relation.

5. Open problems

We collect in this section some open problems related to some of the results presented in Section 3.

We say that a function f is an *epimorphism* between countable $\mathcal{L}$-structures H and G if it is a surjective homomorphism of H onto G, and that f is a *weak-epimorphism* if it is a surjective weak-homomorphism. Clearly the relations "$G \preceq H$ if and only if G is the epimorphic image of H" and "$G \preceq_w H$ if and only if G is the weak-epimorphic image of H" are analytic quasi-orders, and Camerlo [1] showed that $\preceq_w$ on countable graphs is complete for analytic quasi-orders. Therefore we have the following natural question.

Question 6. *Is it true that for every analytic quasi-order R on a standard Borel space X there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ ($\mathcal{L}$ some countable language) such that R is Borel equivalent to $\preceq$ (respectively, $\preceq_w$) when restricted to Mod_φ?*

Let us point out that combinatorial graphs are unlikely to play any role in answering this question: Proposition 3.10 shows that almost every epimorphism between combinatorial trees (in particular, every epimorphism between the combinatorial trees used in our constructions) is actually an isomorphism, so that $\preceq$ on combinatorial trees is probably more or less as complex as $\cong$ on combinatorial trees, while $\preceq_w$ on combinatorial trees seems to be a very simple relation, as it can be shown e.g. that any combinatorial tree with unbounded diameter is bi-weak-epimorphic to the combinatorial tree $\bar{G}$ defined in Remark 3.11. It could be that the combination of the construction given in [1] with the techniques developed in this paper would yield an answer to Question 6, but we leave this possibility for future research.

A similar questions can be asked about the relation of elementary embeddability between countable $\mathcal{L}$-structures, although (as far as we know) it is not even known if the corresponding quasi-order is complete analytic.

Question 7. *Is there any $\mathcal{L}_{\omega_1\omega}$-sentence φ ($\mathcal{L}$ some countable language) such that the the quasi-order induced by elementary embeddability on Mod_φ is complete analytic?*

Is it true that for every analytic quasi-order R on a standard Borel space X there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ such that R is Borel equivalent to elementary embeddability on Mod_φ?

Regarding Section 3.2. There is an evident qualitative difference between Corollary 3.12 on the one hand, and Corollaries 3.13, 3.14 and 3.15 on the other: in the first case we get a Borel class $\mathcal{C}$ which is *saturated* with respect to isometry, while in the second case we get classes $\mathcal{C}$ which are very far from being saturated with respect to the corresponding bijective morphisms (namely isometries, homeomorphisms and linear isometries, respectively).

Question 8. *Can one strenghten Corollaries 3.13, 3.14 and 3.15 by requiring that the Borel class $\mathcal{C}$ is closed under, respectively, isometries, homeomorphisms and linear isometries?*

What is missing to positively answer this question is a method which allows one to saturate "in a Borel way" the class $\mathcal{C}$, something which is possible if the class of bijective morphisms would form a natural Polish group. For example, if the class of isometries between arbitrary ultrametric Polish spaces (viewed as elements of $F(\mathbb{U})$) would form a Polish group acting on $F(\mathbb{U})$, then it would be very likely that we could repeat the argument used in Theorem 3.9 to prove that the saturation of $\mathcal{C}$ is still Borel. However, one should note that these isomorphisms are just *partial* isomorphisms of $\mathbb{U}$ into itself, and in general they cannot be extended to an automorphism of $\mathbb{U}$, so the fact that the automorphisms of $\mathbb{U}$ form a Polish group acting on $F(\mathbb{U})$ is of no use here. Moreover, we can not see any other natural way to view these partial isomorphisms as a Polish group acting on $F(\mathbb{U})$, therefore we suspect that an aswer to Question 8 would necessarily employ different techniques.

We end with a question regarding Section 3.3. Call an action of a Polish monoid *functional* if it is the graph of a function (so that functional actions coincide with the actions considered in [5]).

Question 9. *Is it true that any analytic quasi-order R on a standard Borel space X is Borel equivalent to a quasi-order induced by a continuous (or just Borel)* functional *action of a Polish monoid?*

The best result that we have in this direction is the following: the quasi-order of embeddability on countable graphs is induced by the continuous functional action of a Polish monoid (although neither the monoid nor the action are very "natural"). This strenghten a little bit [5, Theorem 5.1], since it gives a more canonical and natural example of an complete analytic quasi-order induced by a functional continuous action of a Polish monoid. Our example can also be easily extended to cover other important cases, such as embeddability on combinatorial trees and so on, but it seems that it doesn't work in the general context of Borel invariant subsets of $Mod_{\mathcal{L}}$ (such a generality would clearly answer Question 9 by Theorem 3.9). Here is the construction (we leave to the reader the proof of the fact that the proposed monoid is Polish and that the action is continuous): the monoid $M \subseteq S_\infty^- \times {}^\omega 2 \times X_G$ is defined by

$$M = \{(p,u,v) \mid \forall n \in \omega(u(i) = 1 \iff \exists m \in \omega(p(m) = n)) \wedge \\ \wedge \forall n,m \in \omega(u(n) = 1 \wedge u(m) = 1 \Rightarrow v(\langle n,m\rangle) = 0)\}.$$

The real u is used to determine whether or not an element is in the range of the injection p, and is instrumental in proving that the operation of the monoid defined below is continuous. The graph v defines the structure of the target graphs of the action of M *outside* the range of the embedding p — see the discussion in Section 3.3. Define now first the functional action of M on X_G as the function

$$a \colon M \times X_G \to X_G \colon (g,x) \mapsto y,$$

where $g = (p,u,v)$ and y is such that

$$y(\langle n,m\rangle) = \begin{cases} x(\langle p^{-1}(n), p^{-1}(m)\rangle) & \text{if } u(n) = u(m) = 1, \\ v(\langle n,m\rangle) & \text{otherwise.} \end{cases}$$

Finally, define the product operation $(h,g) \mapsto hg$ in the unique way which really turns a into an action, that is in such a way that $a(h,a(g,x)) = a(hg,x)$: if $g = (p_1,u_1,v_1)$ and $h = (p_2,u_2,v_2)$ then $hg = (q,t,w)$, where $q = p_2 \circ p_1$, $t(n) = 1 \iff \exists m(q(m) = n)$, and

$$w(\langle n,m\rangle) = \begin{cases} 0 & \text{if } t(n) = t(m) = 1, \\ v_2(\langle n,m\rangle) & \text{if } v_2(n) = 0 \text{ or } v_2(m) = 0, \\ v_1(\langle p_2^{-1}(n), p_2^{-1}(m)\rangle) & \text{otherwise.} \end{cases}$$

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