LARGE CARDINALS AND Δ_1 DEFINABILITY OF THE NONSTATIONARY IDEAL

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ABSTRACT. We show that for a measurable cardinal κ, the restriction of the ideal of nonstationary subsets of κ to any given set of measure 0 can be made Δ_1 definable with parameters from $H(\kappa^+)$, preserving the measurability of κ. We also show that starting with one measurable cardinal, one can force NS_{ω_1} to be both precipitous and Δ_1-definable with parameters from $H(\omega_2)$.

1. INTRODUCTION

In this article we consider the definability of the ideal NS_κ of nonstationary subsets of an uncountable regular cardinal κ. It is easy to see that NS_κ is Σ_1 definable with parameter κ. In Gödel's L, NS_κ is not Δ_1 definable with parameters from $H(\kappa^+)$ (see [3]), but surprisingly, in the case $\kappa = \aleph_1$, NS_{ω_1} can be Δ_1 definable (with parameters from $H(\omega_2)$), as was shown in [10]. (Note that by reflection, a subset of $H(\kappa^+)$ is Δ_1 definable with parameters from $H(\kappa^+)$ iff it is Δ_1 definable over $H(\kappa^+)$ with parameters from $H(\kappa^+)$.) For larger κ, the entire NS_κ can be Δ_1 definable with parameters from $H(\kappa^+)$ for successor κ (see [4]) and for all uncountable regular κ it is possible for the restriction of NS_κ to a stationary set to be Δ_1 definable with parameters from $H(\kappa^+)$ (see [6]). In this paper we consider the Δ_1 definability of restrictions of NS_κ to a stationary set in the large cardinal context.

A first observation is the following: If κ is measurable or even just weak compact, then the full NS_κ cannot be Δ_1 with parameters from $H(\kappa^+)$. This is proved in Proposition 2.1 below.

We next show that if U is a normal measure on κ and the stationary set A is of measure zero (i.e. does not belong to U) then it is possible to force NS_κ restricted to A to be Δ_1, preserving the stationarity of A and the measurability of κ (witnessed by a normal measure extending U).

Finally, we show that starting with one measurable cardinal, one can force NS_{ω_1} to be both Δ_1 definable and precipitous.

Our notation is rather standard (cf [8]). The reader is assumed to be familiar with large cardinal and forcing arguments. In particular, the lifting argument via master conditions appears frequently in the proof. [1] contains the basic definitions and arguments used in this article.

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2. Δ_1-DEFINABILITY OF RESTRICTIONS OF THE NONSTATIONARY IDEAL AT A MEASURABLE CARDINAL

Proposition 2.1. *Suppose that κ is weak compact. Then NS_κ is not Δ_1 definable.*

Proof. It suffices to show that if $\varphi(A)$ is any Σ_1 formula with free variable A denoting a subset of κ then:

$$\varphi(A) \text{ iff } X = \{\alpha < \kappa \mid \varphi(A \cap \alpha)\} \text{ contains a club.}$$

Given this, we see that the Δ_1 definability of NS_κ would entail the Δ_1 definability of any Σ_1 definable collection of subsets of κ, which is impossible by diagonalisation.

Now we prove the above equivalence. First suppose that $\varphi(A)$ holds. Let $(M_i \mid i < \kappa)$ be a continuous κ-chain of elementary submodels of $H(\kappa^+)$ of size less than κ which are transitive below κ and contain the parameter A. Le κ_i denote $M_i \cap \kappa$. Then the κ_i's form a club C in κ. For each $i < \kappa$, $\varphi(A \cap \kappa_i)$ is true in the transitive collapse of M_i and therefore by persistence is true in V. This shows that X contains a club. So far we have only used the regularity of κ.

Now suppose that $\varphi(A)$ fails and therefore fails in $H(\kappa^+)$. For any club C there is some α in C such that $\varphi(A \cap \alpha)$ fails in $H(\alpha^+)$, using Π^1_1 reflection for the weak compact cardinal κ (note that Π^1_1 over $H(\kappa)$ is equivalent to Π_1 over $H(\kappa^+)$). This shows that X does not contain a club, finishing the proof. □

So for a measurable cardinal, which is also weak compact, the best we can hope for is the Δ_1 definability of the restriction of NS_κ to a costationary set.

Theorem 2.1. *Assume that GCH holds and κ is a measurable cardinal. Suppose U is a normal measure on κ and T is a stationary subset of κ of measure* 0*. Then in a cofinality-preserving forcing extension:*

1. *κ remains a measurable cardinal (witnessed by a normal measure extending U).*
2. *T remains stationary of measure* 0*.*
3. *$NS_\kappa \restriction T = \{S \mid S \in NS_\kappa$ and $S \subseteq T\}$ is Δ_1-definable with parameters from $H(\kappa^+)$.*

Proof. Let $j : V \to M$ be the ultrapower embedding given by U. Fix a function $f : \kappa + 1 \to P(\kappa)$ such that $f(\kappa) = T$, $j(f)(\kappa) = T$ and $f(\eta)$ is a stationary subset of η whenever $\eta < \kappa$ is regular. (Note that $f(\eta) = T \cap \eta$ for measure one many $\eta < \kappa$.) Fix a bijection $k : \kappa^+ \to \kappa^+ \times \kappa^+ \in M$ such that $\forall \alpha < \kappa^+$ $(k(\alpha))_0 \leq \alpha$ and let f_k with domain $\kappa + 1$ witness that $f_k(\kappa) = k$, $j(f_k)(\kappa) = k$ and $f_k(\eta) : \eta^+ \to \eta^+ \times \eta^+$ is a bijection such that $\forall \alpha < \eta^+$ $(f_k(\eta)(\alpha))_0 \leq \alpha$ for all regular $\eta < \kappa$. Fix a sequence $\vec{f} = \langle f_\beta \mid \beta \in [\kappa, \kappa^+) \rangle \in M$ of functions where $f_\beta : \kappa \to \beta$ is a bijection. Define the canonical functions $g_\beta : \gamma \mapsto \operatorname{ot} f_\beta[\gamma]$ for $\gamma < \kappa$. Fix a sequence $\vec{C} = \langle C_\beta \mid \beta \in [\kappa, \kappa^+) \rangle$ of club subsets of κ in M such that for all $\gamma_1 < \gamma_2 \in C_\beta$, $g_\beta(\gamma_1) < g_\beta(\gamma_2)$. Let $f_{\vec{f}}$ and $f_{\vec{C}}$ be such that $f_{\vec{f}}(\kappa) = \vec{f}$, $f_{\vec{C}}(\kappa) = \vec{C}$, $j(\{f_{\vec{f}}, f_{\vec{C}}\})(\kappa) = \{\vec{f}, \vec{C}\}$ and for all regular $\eta < \kappa$, $f_{\vec{f}}(\eta)$ and $f_{\vec{C}}(\eta)$ satisfy the properties of $\vec{f}$ and $\vec{C}$ with κ replaced by η. We write $f_{\vec{f}}(\eta) = (f^\eta_\beta \mid \beta \in [\eta, \eta^+))$ and $f_{\vec{C}}(\eta) = (C^\eta_\beta \mid \beta \in [\eta, \eta^+))$.

The forcing $\mathbb{P}$ is defined as a length $\kappa+1$ reverse Easton iteration $\langle \mathbb{P}_\eta, \dot{\mathbb{P}}^\eta \mid \eta \leq \kappa \rangle$ such that $\dot{\mathbb{P}}^\eta$ is trivial unless η is inaccessible.

Fixing any inaccessible cardinal $\eta \leq \kappa$, we define $\dot{\mathbb{P}}^\eta = \langle \mathbb{P}^\eta_\beta, \dot{\mathbb{Q}}^\eta_\beta \mid \beta < \eta^+ \rangle$ as a length η^+ iteration with supports of size less than η. $\dot{\mathbb{P}}^\eta$ is very similar to the forcing $\mathbb{P}$ defined in the proof of Theorem 49(4) of [3]. However, we slightly change the definition to fit our context.

In $V^{\mathbb{P}_\eta}$, $\dot{\mathbb{P}}^\eta$ is designed to force that $NS_\eta \restriction f(\eta)$ is Δ_1-definable over $H(\eta^+)$. It is sufficient to force that there is an $S \supset f(\eta)$ such that $NS_\eta \restriction S$ is Δ_1 over $H(\eta^+)$. In order to carry out the coding procedure, we also require that S is fat (i.e., its intersection with any club contains closed subsets of any size $< \eta$) and $\eta \setminus S$ is not reflecting to any $\alpha \in S$ (i.e., $S \cap \alpha$ is nonstationary in α for regular α in S). This can be achieved by forcing with

$\mathbb{P}^\eta_0 = \{p \in 2^{<\eta} \mid$ For all $\beta \in f(\eta) \cap \mathrm{dom}(p)$, $p(\beta) = 1$ and $\{\gamma < \beta \mid p(\gamma) = 0\}$ is not stationary in $\beta\}$,

ordered by end-extension. $\mathbb{P}^\eta_0$ is $(< \eta)$-strategically closed. We will prove a general form of this fact in Claim 2.1. If G is $\mathbb{P}^\eta_0$ generic over $V^{\mathbb{P}_\eta}$, then $S = (\bigcup G)^{-1}(1)$ satisfies the desired properties in the generic extension.

Let $\mathbb{Q}^\eta_1$ be $\mathrm{Add}(\eta, 1)^{V^{\mathbb{P}_\eta * \mathbb{P}^\eta_0}}$. $\mathbb{P}^\eta_0 * \mathbb{Q}^\eta_1$ is $(< \eta)$-distributive. Moreover, it preserves all stationary subsets of η. Let G be $\mathbb{Q}^\eta_1$ generic over $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_0}$. Define a sequence $\langle S_\beta \mid \beta \in [\eta, \eta^+)\rangle$ of subsets of η as follows: $\gamma \in S_\beta$ iff $(\bigcup G)(g_\beta(\gamma)) = 1 \wedge \gamma \in C_\beta \setminus S$. For all

$\beta \in [\eta, \eta^+)$, S_β is a stationary subset of $\eta \setminus S$. Also notice that modulo the nonstationary ideal, S_β does not depend on the choices of the f^η_β's and C^η_β's. For all $\beta \in [\eta, \eta^+)$, denote $(\eta \setminus S_\beta)$ by A_β.

For $\beta \in [2, \eta^+)$, we define $\mathbb{Q}^\eta_\beta$ by induction. From now on we work in $V^{\mathbb{P}_\eta}$. For a fat subset T of η, let $Sh(T)$ be the forcing poset for shooting a club through T. Assume that $\mathbb{P}^\eta_\beta$ has been defined. We define $\mathbb{Q}^\eta_\beta$ as follows: Let $\bar\beta$ and γ be such that $\beta = 2 + \eta * \bar\beta + \gamma$ where $\gamma < \eta$ (and $*$ denotes ordinal multiplication). Recall that $k_\eta = f_k(\eta) : \eta^+ \to \eta^+ \times \eta^+$ is a bijection such that for all $\beta < \eta^+$, $k_\eta(\beta) = \langle \beta_1, \beta_2\rangle$ is a pair of ordinals such that $\beta_1 \leq \beta$. Let $\dot T_{\bar\beta}$ be the $\bar\beta_2$-nd $\mathbb{P}^\eta_{\bar\beta_1}$-nice name for a subset of η. Since $\bar\beta_1 \leq \beta$, we can view $\dot T_{\bar\beta}$ as a $\mathbb{P}^\eta_\beta$ name. If $\beta \geq \omega$, let $\tilde\beta = \eta + \beta$, otherwise let $\tilde\beta = \eta + (\beta - 2)$. Now in $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\beta}$ we set

$$\mathbb{Q}^\eta_\beta = \begin{cases} Sh(A_{\tilde\beta}) & \text{if } \dot T_{\bar\beta} \notin NS_\eta \restriction S \wedge \exists\lambda((\gamma = 2 * \lambda \wedge \lambda \in \dot T_{\bar\beta}) \vee (\gamma = 2 * \lambda + 1 \wedge \lambda \notin \dot T_{\bar\beta})) \\ trivial & \text{otherwise.} \end{cases}$$

Claim 2.1. *For all $\beta \in [\eta, \eta^+)$:*

1. *In $V^{\mathbb{P}_\eta}$, $\mathbb{P}^\eta_\beta$ is η^+-c.c and $(< \eta)$-distributive.*
2. *In $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_0}$, for any $A \subseteq \eta$, both of the statements "$A \in NS_\eta \restriction S$" and "$A \notin NS_\eta \restriction S$" are preserved in $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\beta}$. In particular, $f(\eta)$ is stationary in $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\beta}$.*
3. *If $q \in \mathbb{P}^\eta_\beta$ and either $\gamma \geq \beta$ or q forces that $\dot{\mathbb{Q}}^\eta_\gamma$ is trivial, then q forces that $S_{\tilde\gamma}$ is stationary in $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\beta}$.*

Proof. (1) We will use the "flat condition" argument and prove this by induction. A $\mathbb{P}^\eta_\beta$ condition p is flat if

- $\forall\lambda \in \mathrm{spt}(p)$, $p \restriction \lambda$ decides "$\dot T_{\bar\lambda} \in NS_\eta \restriction S$".
- there is a unique $\gamma < \eta$ and a sequence $\langle p_i \mid i \in \mathrm{spt}(p)\rangle \in V^{\mathbb{P}_\eta}$ such that: $\forall\lambda \in \mathrm{spt}(p) \setminus 2$ $(p \restriction \lambda \Vdash p(\lambda) = p_\lambda \wedge \max(p_\lambda) = \gamma + 1)$, $p(0) = p_0$, $p(1) = p_1$, $\mathrm{dom}(p(0)) = \mathrm{dom}(p(1)) = \gamma + 1$ and $p(0)(\gamma) = 0$.

For a flat condition p, we denote the unique ordinal γ witnessing flatness by γ_p, the "height" of p. We will show by induction on β that for all $\xi \in \eta$, $D_\xi = \{p \in \mathbb{P}^\eta_\beta \mid p \text{ is flat} \wedge \gamma_p > \xi\}$ is dense. First assume that β is a limit ordinal with cofinality λ. By the induction hypothesis, the flat conditions with arbitrary height are dense in $\mathbb{P}^\eta_\delta$ for all $\delta < \beta$. Fix a condition $q \in \mathbb{P}^\eta_\beta$ and a sufficiently large regular cardinal θ. If $\lambda = \eta$, then $\mathrm{spt}(q)$ is bounded in β and it is easy to find the required stronger flat condition. Now assume $\lambda < \eta$ and $\langle \beta_i \mid i < \lambda\rangle$ is a sequence with supremum β. Fix a sequence $\vec M = \langle M_i \mid i < \lambda\rangle$ such that the following conditions hold:

(a) $\vec M$ is a continuous elementary chain of submodels of $H(\theta)$.
(b) For all $i < \lambda$, $|M_i| < \eta$, $\beta_i \in M_{i+1}$ and $M_i \cap \eta$ is transitive.
(c) $\{q, \mathbb{P}^\eta_\beta\} \subseteq M_0$. In particular, all the parameters which appear in the definition of $\mathbb{P}^\eta_\beta$ are in M_0.

(d) For all $i < \lambda$, $M_i^{<|M_i|} \in M_{i+1}$.

Let $\eta_i = M_i \cap \eta$. Then $C = \{\eta_i \mid i < \lambda\}$ is a club in $\eta_\lambda = \bigcup_{i<\lambda} \eta_i$. We construct a sequence of conditions $\vec{q} = \langle q_i \mid i < \lambda\rangle$ such that the following conditions hold:

(a) $q_0 < q$.
(b) For all $i_1 < i_2 < \lambda$, $q_{i_1} > q_{i_2}$ and $q_{i_1} \in M_{i_1+1}$.
(c) For each $i < \lambda$, $q_{i+1} \restriction \beta_i$ is flat and $\gamma_{q_{i+1}\restriction\beta_i} > \eta_i$. In particular, $\mathrm{dom}(q_{i+1}(0)) \supset \eta_i$, $q_{i+1}(0) \Vdash \mathrm{dom}(q_{i+1}(1)) \supset \eta_i$ and $q_{i+1}(0)$ decides the value of $q_{i+1}(1) \restriction \eta_i$. Denote this value by $A_i(1)$. Moreover, for all $\delta \in \mathrm{spt}(q_{i+1}) \cap [2, \beta_i)$, $q_{i+1} \restriction \delta$ decides "$\dot{T}_{\bar{\delta}} \in NS_\eta \restriction S$" and the value of $q_{i+1}(\delta) \cap \eta_i$. Denote this value by $A_i(\delta)$.
(d) For all $\delta \in \mathrm{spt}(q_{i+1}) \setminus 2$, $q_{i+1} \restriction \delta \Vdash \max(q_{i+1}(\delta)) > \eta_i$.
(e) $q_{i+1}(0)(\eta_i) = 1$.

We leave the details of the construction of $\vec{q}$ to the reader (or see [3]). We define the limit condition q_λ as follows:

$$q_\lambda(\delta) = \begin{cases} \bigcup q_i(\delta) \cup \{\langle \eta_\lambda, 1\rangle\} & \text{if } \delta = 0 \\ \bigcup A_i(1) \cup \{\langle \eta_\lambda, 1\rangle\} & \text{if } \delta = 1 \\ \bigcup_{\beta_i > \delta} A_i(\delta) \cup \{\eta_\lambda\} & \text{if } \delta \in \bigcup_{i<\lambda} \mathrm{spt}(q_i) \setminus 2 \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $q_\lambda(0) \Vdash \eta_\lambda \in S$, it is routine to check that q_λ is a flat condition stronger than q. Moreover, $\gamma_{q_\lambda} = \eta_\lambda$.

Now we turn to the case where $\beta = \epsilon + 1$ is a successor ordinal. Let q be a $\mathbb{P}^\eta_\beta$-condition. We construct a sequence of conditions $\langle q_n \mid n < \omega\rangle$ such that $q_0 < q$, $q_n \restriction \epsilon$ is flat, $q_n \restriction \epsilon \Vdash \sup q_n(\epsilon) > \gamma_{q_n}$ and $q_{n+1} \restriction \epsilon$ decides $q_n(\epsilon) \cap \gamma_{q_n}$. Define a supremum condition q_ω as in the last paragraph. It follows that q_ω is a flat condition stronger than q.

To prove that $\mathbb{P}^\eta_\beta$ is $(< \eta)$-distributive, it suffices to show that the suborder $\bar{\mathbb{P}}^\eta_\beta$ which consists of all flat conditions is η-strategically closed. Consider the game $G_\beta(\bar{\mathbb{P}}^\eta_\beta)$. A winning strategy of player II can be defined as follows. At even successor stage $\alpha < \eta$, player II chooses a condition p_α which is stronger than all previous plays such that $p_\alpha(0)(height(p_\alpha)) = 1$. At limit stage α, player II chooses q_α as the limit condition defined as follows:

$$q_\alpha(\delta) = \begin{cases} \bigcup q_i(\delta) \cup \{\langle \gamma_\alpha, 1\rangle\} & \text{if } \delta = 0 \text{ or } 1 \\ \bigcup q_i(\delta) \cup \{\gamma_\alpha\} & \text{if } \delta \in \bigcup_{i<\alpha} \mathrm{spt}(q_i) \setminus 2 \\ \emptyset & \text{otherwise.} \end{cases}$$

It is routine to verify that q_α is a condition stronger than all q_i's.

The proof of η^+-c.c for an iteration of shooting club forcings is well-known (see [1] for a proof using flat conditions and a Δ-system argument).

(2) Clearly if $A \in NS_\eta \restriction S$, then $\mathbb{P}_\eta * \mathbb{P}^\eta_\beta \Vdash A \in NS_\eta \restriction S$. Note that $\dot{\mathbb{Q}}^\eta_\beta$ is either trivial or the poset $Sh(A_{\bar{\beta}})$. It is known that iterating such forcings preserves the stationarity of all stationary subsets of S provided S satisfies that for all $\alpha \in S$, $\alpha \setminus S$ is nonstationary in α. (See [8] or the proof of (3)).

(3) First assume $\gamma < \beta$. Fix a name $\dot{C}$ for a club subset of η and any condition q' stronger than q. We need to find a condition $q_\omega < q'$ such that $q_\omega \Vdash \dot{C} \cap S_{\bar{\gamma}}$ is not empty. Fix a sufficiently large regular θ, fix $M \prec H(\theta)$ such that $|M| < \eta$, $M \cap \eta \in \eta \setminus S$ and $\{q', \gamma, \mathbb{P}^\eta_\beta\} \subseteq M$. Choose $\langle \eta_n \mid n < \omega\rangle$ cofinal in $\eta_\omega = M \cap \eta$. Construct a sequence of conditions $\langle q_n \mid n < \omega\rangle$ such that

(a) $q_0 < q'$ and for each $n < \omega$, $q_n \restriction 2 \Vdash q_n \restriction [2, \beta)$ is flat $\wedge\ \gamma_{q_n\restriction[2,\beta)} > \eta_n$.
(b) For each $n < \omega$, $\mathrm{dom}\, q_n(0) \supset \eta_n$.

(c) For each $n < \omega$, there is an $A_n \in V^{\mathbb{P}_\eta}$ such that $q_n(0) \Vdash q_n(1) = A_n$ and $\mathrm{dom}(A_n) \supset \eta_n$.

We omit the details of this construction. Let $B = \bigcup_{n<\omega} \mathrm{spt}(p_n)$. Note that the sequences $f_{\vec{f}}(\eta)$ and $f_{\vec{C}}(\eta)$ are in M. It follows that for all $\delta_1 < \delta_2 \in [\eta, \eta^+) \cap M$, $g_{\delta_1}(\eta_\omega) < g_{\delta_2}(\eta_\omega)$. Let ϵ be an ordinal greater than all $g_\delta(\eta_\omega)$'s for $\delta \in B$. Let $h : [\eta_\omega, \epsilon) \to 2$ be a function such that $h(g_\delta(\eta_\omega)) = 1 \leftrightarrow \delta \in B \vee \delta = \gamma$. The limit condition q_ω is defined as follows: $q_\omega(0) = \bigcup_{n<\omega} q_n(0) \cup \{\langle \eta, 0\rangle\}$, $q_\omega(1) = \bigcup_{n<\omega} A_n \cup h$, and $q_\omega(\delta) = \bigcup_{n<\omega} q_n(\delta) \cup \{\eta_\omega\}$ for $\delta \in B \setminus 2$. It is routine to verify that q_ω is a condition which forces that $\eta_\omega \in S_{\tilde{\gamma}} \cap \dot{C}$.

Finally suppose $\gamma \geq \beta$. The above argument still works. Instead of using $\gamma \in \bigcup_{n<\omega} \mathrm{spt}(p_n)$, we use the fact that γ is not in the domain of the iteration. □

We present the promised Δ_1 definition of $NS_\eta \restriction S$ in the following Claim:

Claim 2.2. *The following statements are true in $V^{\mathbb{P}_\eta * \dot{\mathbb{P}}^\eta}$:*

(1) *GCH holds and η remains inaccessible.*

(2) *$NS_\eta \restriction S$ is Δ_1 over $H(\eta^+)$. Therefore $NS_\eta \restriction f(\eta)$ is also Δ_1 over $H(\eta^+)$.*

Proof. (1) This follows from Claim 2.1 (1) and the fact that the cardinality of the set of flat conditions in $\mathbb{P}^\eta$ is η^+.

(2) We need only supply a Π_1 definition. It is sufficient to check that the following statement is true: $A \notin NS_\eta \restriction S$ if and only if

(∗): there are $\beta < \eta^+$, $\langle \{f_\gamma, C_\gamma, D_\gamma, B_\gamma\} \mid \gamma < \eta\rangle \in H(\eta^+)$ such that for all $\gamma < \eta$

- $0 < \beta$ can be divided by η.
- f_γ is a bijection from η to $\beta + \gamma$.
- C_γ is a club subset of η such that whenever $\gamma_1 < \gamma_2 \in C$, $g_{\beta+\gamma}(\gamma_1) < g_{\beta+\gamma}(\gamma_2) \in \eta$ (where $g_{\beta+\gamma}(\delta) = \mathrm{ot}\, f_\gamma[\delta]$).
- $\delta \in B_\gamma \leftrightarrow \bigcup G(g_{\beta+\gamma}(\delta)) = 1 \wedge \delta \in C_\gamma \setminus S$
- For all $\gamma < \eta$, D_γ is a club subset of η.
- $(\gamma \in A \to D_{2*\gamma} \cap B_{2*\gamma} = \emptyset) \wedge (\gamma \notin A \to D_{2*\gamma+1} \cap B_{2*\gamma+1} = \emptyset)$.

Suppose that $A \notin NS_\eta \restriction S$. By the η^+-c.c, there exists $\delta < \eta^+$ such that $A \in V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\delta}$. By Claim 2.1(2), in $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\delta}$ we have $A \notin NS_\eta \restriction S$. In $\mathbb{P}_\eta$ let $\dot{A}$ be a $\mathbb{P}^\eta_\delta$ name of A. By a bookkeeping argument and the definition of $\mathbb{P}^\eta$, there is an interval $[\eta * \bar{\beta}, \eta * (\bar{\beta}+1))$ such that $T_{\bar{\beta}} = \dot{A}$. Denote $\eta * \bar{\beta}$ by β. It follows that $\dot{\mathbb{Q}}^\eta_\delta$ is not trivial if $\delta \in [\beta, \beta+\eta)$ and $\exists \gamma(\gamma \in A \wedge \beta + 2 * \gamma = \delta) \vee (\gamma \notin A \wedge \beta + 2*\gamma + 1 = \delta)$. In $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_{\beta+\eta}}$, let $\langle D_\gamma \mid \gamma < \eta\rangle$ be a sequence of clubs such that $D_\gamma \cap S_{\beta+\gamma}$ is empty whenever $\dot{\mathbb{Q}}^\eta_{\beta+\gamma}$ is not trivial. Let $\langle \{f_\gamma, C_\gamma, B_\gamma\} \mid \gamma < \eta\rangle$ be such that $\{f_\gamma, C_\gamma, B_\gamma\} = \{f_{\vec{f}}(\eta)(\beta+\gamma), f_{\vec{C}}(\eta)(\beta+\gamma), S_{\beta+\gamma}\}$ for all $\gamma < \eta$. It is now routine to check that β and the sequence $\langle \{f_\gamma, C_\gamma, D_\gamma, B_\gamma\} \mid \gamma < \eta\rangle$ witness (∗).

For the converse, suppose $A \in NS_\eta \restriction S$. Assume there are β and $\langle \{f_\gamma, C_\gamma, D_\gamma, B_\gamma\} \mid \gamma < \eta\rangle$ witnessing (∗). Let $\bar{\beta}$ be such that $\eta * \bar{\beta} = \beta$. It follows from the discussion following the definition of the S_γ sequence that, $B_\gamma = S_{\beta+\gamma}$ modulo a nonstationary set. By (∗), there is a nonstationary $S_{\beta+\gamma}$. This implies that $\dot{\mathbb{Q}}^\eta_{\beta+\gamma}$ is not trivial and hence $T_{\bar{\beta}} \notin NS_\eta \restriction S$ in $V^{\mathbb{P}_\eta * \mathbb{P}^\eta_\beta}$. However, using Claim 2.1 (3), $T_{\bar{\beta}}$ is forced to be equal to A in $V^{\mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta}$. This is impossible as by Claim 2.1 (2), $T_{\bar{\beta}} \in NS_\eta \restriction S$ in $V^{\mathbb{P}_\eta * \dot{\mathbb{Q}}_\eta}$. □

The following is a summary of the properties of $\mathbb{P}$. Note that for every $\alpha < \kappa$, we can decompose $\mathbb{P}_\kappa$ as $\mathbb{P}_\alpha$ and $\mathbb{P}_{[\alpha,\kappa)}$ such that $\mathbb{P}_{[\alpha,\kappa)}$ is a $\mathbb{P}_\alpha$-name of an iteration of length $\kappa - \alpha$.

Fact 2.1. (1) *$\mathbb{P}$ preserves cofinalities and hence cardinalities. Moreover, $\mathbb{P} \Vdash$ GCH.*

(2) *If $\eta < \kappa$ is Mahlo, then $\mathbb{P}_\eta$ is η-c.c and $\mathbb{P}_\eta \Vdash \mathbb{P}_{[\eta+1,\kappa)}$ has an η^+-strategically closed dense subset, namely the set of conditions all of whose coordinates are forced to be flat.*
(3) $\mathbb{P} \Vdash NS_\kappa \restriction T$ *is Δ_1 definable over $H(\kappa^+)$.*

In the remaining part of the proof, we show how to lift the embedding j, which implies that κ is measurable in the generic extension. Let $H * G^\kappa$ be a $\mathbb{P} = \mathbb{P}_\kappa * \mathbb{P}^\kappa$ generic filter over V. We construct a lifting of j to $V[H * G^\kappa]$ in $V[H * G^\kappa]$. Firstly, we deal with $\mathbb{P}_\kappa$, for which H is the corresponding generic filter. Since $H(\kappa^+)^V = H(\kappa^+)^M$, $j(\mathbb{P})_{\kappa+1} = \mathbb{P}_{\kappa+1}$. Hence $H * G^\kappa$ is $j(\mathbb{P})_{\kappa+1}$ generic over M. Next, we need to construct a $j(\mathbb{P})_{[\kappa+1,j(\kappa))}$-generic filter over $M[H*G^\kappa]$ in $V[H*G^\kappa]$. It follows from Fact 2.1 (2) and the elementarity of j that $j(\mathbb{P})_{[\kappa+1,j(\kappa))}$ is $j(\kappa)$-c.c, of cardinality $j(\kappa)$ and has a $(\kappa^+)^M$-strategically closed dense subset in $M[H * G^\kappa]$. Thus in $M[H * G^\kappa]$ the size of the set of all maximal antichains of $j(\mathbb{P})_{[\kappa+1,j(\kappa))}$ is $j(\kappa)$. Since GCH holds in $V[H * G^\kappa]$, $V[H * G^\kappa] \models j(\mathbb{P})_{[\kappa+1,j(\kappa))}$ has a κ^+-strategically closed dense subset and κ^+ many maximal antichains in $M[H * G^\kappa]$. This means that we can construct the desired $j(\mathbb{P})_{[\kappa+1,j(\kappa))}$-generic filter h over $M[H * G^\kappa]$ in $V[H * G^\kappa]$. Hence we can lift j to $j' : V[H] \to M[H * G_{\kappa+1} * h]$.

We construct the final lifting using "partial master conditions", a technique first introduced by Magidor (see [1]). We need to construct a $j'(\mathbb{P}^\kappa)$ generic over $M[H*G^\kappa*h]$. By elementarity, in $M[H*G^\kappa*h]$, $j'(\mathbb{P}^\kappa)$ is a $j(\kappa^+)$-length iteration of cardinality $j(\kappa^+)$ with the $j(\kappa^+)$-c.c. Hence $j'(\mathbb{P}^\kappa)$ has $j(\kappa^+)$ many antichains in $M[H*G^\kappa*h]$. Since GCH holds in $V[H*G_{\kappa+1}]$, there are only κ^+ such antichains in $V[H * G_{\kappa+1}]$. Moreover, $V[H * G_{\kappa+1}] \models j'(\mathbb{P}^\kappa)$ has a κ^+-strategically closed dense subset. Thus in $V[H*G_{\kappa+1}]$, it is possible to construct a $j'(\mathbb{P}^\kappa)$-generic filter g^κ over $M[H*G_{\kappa+1}*h]$. However, in order to lift j', we need to ensure $j'[G^\kappa] \subseteq g^\kappa$.

For all $\alpha < \kappa^+$, $G^\kappa \restriction \alpha$ is $\mathbb{P}^\kappa_\alpha$ generic over $V[H]$. Let $p_\alpha : j(\alpha) \to V[H * G^\kappa]$ be defined as follows:

$$p_\alpha(\delta) = \begin{cases} \bigcup_{q\in G^\kappa\restriction\alpha} q(\delta) \cup \{\langle\kappa, 1\rangle\} & \text{if } \delta = 0. \\ \bigcup_{q\in G^\kappa\restriction\alpha} q(\delta)/G^\kappa_0 & \text{if } \delta = 1. \\ \bigcup_{q\in G^\kappa\restriction\alpha} q(\gamma)/G^\kappa_\gamma \cup \{\kappa\} & \text{if there is a } \gamma < \alpha \text{ such that } j(\gamma) = \delta. \\ \emptyset & \text{otherwise.} \end{cases}$$

It is routine to check that p_α is a j'-$\mathbb{P}^\kappa_\alpha$ master condition, namely all conditions in $j'[G^\kappa \restriction \alpha]$ are extended by p_α. By carefully selecting conditions, we can ensure that all p_α's are in g^α. The general idea of this selection process can be found in [1]. However, as we only assume that $j(\mathbb{P}^\kappa)$ has a κ^+-strategically closed dense subset, we present the construction here.

Let $\langle A_\alpha \mid \alpha < \kappa^+\rangle$ enumerate all the maximal antichains of $j(\mathbb{P}^\kappa)$ in $M[H * G_{\kappa+1} * h]$ and let $\langle \beta_\alpha \mid \alpha < \kappa^+\rangle$ be an increasing sequence of ordinals of cofinality κ cofinal in κ^+ such that $A_\alpha \cap j(\mathbb{P}^\kappa_{\beta_\alpha})$ is maximal in $j(\mathbb{P}^\kappa_{\beta_\alpha})$. We construct a sequence of conditions $\langle q_\alpha \mid \alpha < \kappa^+\rangle$ in $j(\mathbb{P}^\kappa)$ such that

(1) $q_\alpha \in j(\mathbb{P}^\kappa_{\beta_\alpha})$.
(2) $q_\alpha < p_{\beta_\alpha}$.
(3) q_α extends some condition in A_α.

The key fact we need here is that there is a uniform winning strategy for player II in all $G_{\kappa^+}(j(\mathbb{P}^\kappa))/p_\kappa$. To be precise, we have

Claim 2.3. *There is a winning strategy τ of II in $G_{\kappa^+}(j(\mathbb{P}^\kappa)/p_\kappa)$ such that for all $\beta < \kappa^+$, $\tau \restriction j(\mathbb{P}^\kappa_\beta)$ is a winning strategy for $G_{\kappa^+}(j(\mathbb{P}^\kappa_\beta)/p_\beta)$.*

The proof of this claim is implicit in the proof of Claim 2.1, as the strategy used there does not depend on the value of β and works for any $\mathbb{P}^\kappa_\beta$. As a corollary, if for some β, all the previous moves are in $j(\mathbb{P}^\kappa_\beta)$, then τ returns a move of II in $j(\mathbb{P}^\kappa_\beta)$.

Now we construct $\langle q_\alpha \mid \alpha < \kappa^+\rangle$ as a legal play where II uses the strategy τ. At any stage, after τ chooses q'_α for II, since $q'_\alpha < p_\alpha$, I can choose $q_\alpha < q'_\alpha$ such that (1)-(3) hold. Being a legal play, the construction never terminates at any stage $\alpha < \kappa^+$. Let g^κ be the filter generated by the q_α's. It follows that $j'[G^\kappa] \subseteq g^\kappa$. Hence we can lift j' to $j'' : V[H * G^\kappa] \to M[H * G^\kappa * h * g^\kappa]$ in $V[H * G^\kappa]$, as desired. □

Remark 2.1. (1) *By examining the proof of Theorem 2.1, it is clear that it can be generalized to all large cardinals defined as a critical point of an elementary embedding (e.g. strong cardinal, supercompact, huge). In fact, the proof will be simpler since the use of partial master conditions can be avoided. In addition, measurability can be replaced by weak compactness provided some Π^1_1 sentence (with a subset of κ as parameter) is true at κ but fails to reflect to any ordinal in T.*

(2) *In our final model, for all inaccessible $\eta < \kappa$, $NS_\eta \restriction f(\eta)$ is Δ_1 definable over $H(\eta^+)$.*

3. Precipitousness and the Δ_1-definability of NS_{ω_1}

In this section, we prove that assuming the existence of one measurable cardinal, it is consistent that NS_{ω_1} is both Δ_1 definable and precipitous.

We begin by recalling the definition of canary tree forcing.

Definition 3.1 ([10])**.** *T is a canary tree if $|T| = 2^\omega$, T has no uncountable branch, and in any extension of the universe in which no new reals are added and in which some stationary subset of ω_1 is destroyed, T has an uncountable branch.*

Equivalently, T is a canary tree if $|T| = 2^\omega$, T has no uncountable branch, and for all stationary, costationary sets S, there is an order-preserving function from $Sh(S)$, the tree of closed subsets of S ordered by end-extension, to T. As shown in [9], if CH holds the existence of canary tree is equivalent to the Δ_1-definability of NS_{ω_1}. In [10], assuming GCH, Mekler and Shelah construct a forcing $\mathbb{P}_{\omega_2}$ that preserves cardinals and GCH and forces the existence of a canary tree. However, there is a flaw in their proof which was repaired by Hyttinen and Rautila [6]. We now describe the poset $\mathbb{P}_{\omega_2}$, with a milder repair in the sense that the forcing is closer to Mekler-Shelah's original version.

Let $\mathbb{Q}_0$ be the set of functions f such that $\mathrm{dom}(f)$ is a countable subset of $\mathrm{Lim}(\omega_1)$ and $\forall\delta \in \mathrm{dom}(f)(f(\delta) \in \delta^\delta)$, ordered by reverse inclusion. Under CH, $\mathbb{Q}_0$ is equivalent to $\mathrm{Add}(\omega_1, 1)$, the forcing that adds a single ω_1-Cohen subset of ω_1. From a $\mathbb{Q}_0$-generic G_0, define a subtree $T(G_0)$ of $\omega_1^{<\omega_1}$ by $t \in T(G_0)$ iff for all limit $\delta \leq \mathrm{dom}(t)$, $t \restriction \delta \neq (\cup G_0)(\delta)$. In $V[G_0]$, $T(G_0)$ has cardinality 2^ω and no cofinal branch. For a fixed stationary, costationary set S, we say $t \in T(G_0)$ is an *S-node* if for every limit ordinal $\delta \leq \mathrm{dom}(t)$ not in S, $t \restriction \delta \notin \delta^\delta$. The partial order $\mathbb{P}(S, G_0)$ which adds an order-preserving function from $Sh(S)$ to $T(G_0)$ is defined as follows: A condition p in $\mathbb{P}(S, G_0)$ is a pair (g, X) such that the following conditions hold:

- (c1) g is a countable order-preserving partial mapping from $Sh(S)$ to the S-nodes of $T(G_0)$.
- (c2) X is a countable subset of $\omega_1^{<\omega_1}$ such that each element of X is of successor length.
- (c3) $\forall c \in \mathrm{dom}(g)\forall t \in X(t \not\subseteq g(c))$.
- (c4) $\forall\langle c_i \mid i \in \omega\rangle \in \mathrm{dom}(g)^\omega(\langle c_i \mid i \in \omega\rangle$ is increasing $\to \bigcup_{i\in\omega} g(c_i) \in T(G_0))$.
- (c5) $\mathrm{dom}(g)$ is closed under initial segments with respect to $Sh(S)$.
- (c6) $\forall\langle c_i \mid i \in \omega\rangle \in \mathrm{dom}(g)^\omega(\langle c_i \mid i \in \omega\rangle$ is increasing $\wedge \sup_{i\in\omega}(\max(c_i)) \in S \to \sup_{i\in\omega} \mathrm{dom}(g(c_i)) \in S)$.

For any condition p, denote the corresponding g and X by g_p and X_p. Let $o(p) = \sup\{\mathrm{dom}(t) \mid t \in \mathrm{ran}(g_p) \text{ or } t \in X_p\}$. Then we say a condition q extends p if $g_p \subseteq g_q$, $X_p \subseteq X_q$ and for all $c \in \mathrm{dom}(g_q) \setminus cl(\mathrm{dom}(g_p))$, $\mathrm{dom}(g_q(c)) > o(p)$, where $cl(\mathrm{dom}(g_p)) = \{u \in Sh(S) \mid \forall\alpha < \sup(u)\exists\beta > \alpha(u \restriction \beta \in \mathrm{dom}(g_p)\})$, i.e $cl(\mathrm{dom}(g_p))$ is the closure of $\mathrm{dom}(g_q)$ under the order of $Sh(S)$. The following claim replaces the Claim 3.11 of [6]).

Claim 3.1. *Suppose S is a stationary, costationary subset of ω_1 in V_1, where V_1 is generic over $V[G_0]$ without adding reals. If G is $\mathbb{P}(S, G_0)$-generic over V_1, then $\bigcup G$ is an order-preserving function from $Sh(S)$ to $T(G_0)$.*

Proof. By our assumption $T(G_0)^{V_1} = T(G_0)$. It suffices to prove that for every $t \in Sh(S)$, the set

$$D_t = \{p \in \mathbb{P}(S, G_0) \mid t \in \mathrm{dom}(g_p)\}$$

is dense in $\mathbb{P}(S, G_0)$. So let $p \in \mathbb{P}(S, G_0)$ be such that $t \notin \mathrm{dom}(g_p)$. Let $B = \{u \in \mathrm{dom}(g_p) \mid t <_{Sh(S)} u\}$, where $t <_{Sh(S)} u$ means that t is stronger than u in the $Sh(S)$-order, i.e., u is a proper initial segment of t. It is also convenient to view t as a subset of S instead of a partial function from ω_1 to 2.

Let $b = \bigcup B$, $b' = \bigcup_{u \in B} g_p(u)$. If $b \in B$, then obviously $b' \in T(G_0)$. If $b \notin B$, then by (c4), $b' \in T(G_0)$. As $t \in Sh(S)$ and $t <_{Sh(S)} b$, we have $\delta = \cup b \in S$. By (c6), $\delta' = \mathrm{dom}(b') \in S$. Since $\delta' \in S$ and $g_p(u)$ is an S-node for every $u \in B$, b' is also an S-node. Denote $S \cap (t \setminus (b \cup \{\delta\}))$ by S_t.

We are going to construct an S-node $c \in T(G_0)$ such that c extends b' with $\mathrm{ot}((\mathrm{dom}(c) \setminus o(p)) \cap S) = \mathrm{ot}(S_t)$, c does not extend any element of X_p and $\mathrm{dom}(c) - 1 \in S$. c can be constructed as follows: Let η be the $\mathrm{ot}(S_t)$-th element of $S \setminus o(p)$. Fix a $\gamma > \max\{\eta, \sup_{x \in X_p}(\max\{\mathrm{dom}(x), \sup(\mathrm{ran}(x))\})\}$. We then define c extending b' by setting $\mathrm{dom}(c) = \eta + 1$ and $c(\beta) = \gamma$ for any $\beta \in \mathrm{dom}(c) \setminus \mathrm{dom}(b')$. It is an S-node as b' is an S-node and for all $\beta \in \mathrm{dom}(c) \setminus \mathrm{dom}(b')$, $c \restriction \beta \notin \beta^\beta$. To see $c \notin T(G_0)$,it follows again that for all limit $\beta \in \mathrm{dom}(c) \setminus \mathrm{dom}(b')$, $c \restriction \beta \notin \beta^\beta$. Finally as b' does not extend any element of X_p, if c extends some $x \in X_p$ then $\mathrm{dom}(x) > \mathrm{dom}(b')$. But then for any $\beta > \mathrm{dom}(b')$ in $\mathrm{dom}(c)$, $c(\beta)$ is not in $\mathrm{ran}(x)$. Contradiction.

Let the mapping g'_p be defined for every $u >_{Sh(S)} t$, $u \notin B$ by setting

$$g'_p(b \cup \{\delta\}) = c \restriction \mathrm{dom}(b') + 1$$

and

$$g'_p(u) = \text{ the unique } v \subseteq c \text{ such that } \mathrm{dom}(v) \text{ is the } \gamma_u\text{-th element of } (\mathrm{dom}(c) \setminus o(p)) \cap S,$$

for $u <_{Sh(S)} b \cup \{\delta\}$ where $\gamma_u = \mathrm{ot}(S \cap (u \setminus (b \cup \{\delta\})))$. It follows from the fact that c is an S-node that $g'_p(u)$ is an S-node for every $u \geq_{Sh(S)} t$, $u \notin B$. Define $q = (g_q, X_p)$ by setting $g_q = g_p \cup g'_p$ and $X_q = X_p$. It is routine to check that q is a condition with $t \in \mathrm{dom}(g'_p) \subseteq \mathrm{dom}(q)$. It is also clear that t witnesses $q < p$, as $\mathrm{dom}(g_q(u)) = \mathrm{dom}(g'_p(u)) > o(p)$ for any $u \in \mathrm{dom}(g_q) \setminus cl(\mathrm{dom}(g_p)) = \mathrm{dom}(g'_p) \setminus \{b \cup \{\delta\}\}$. □

We will also need the fact that $\mathbb{P}(S, G_0)$ does not add new countable sets of ordinals. This can be viewed as a warm-up for a later, more sophisticated argument involving an iteration of forcings of this form.

Claim 3.2. *Suppose S is a stationary, costationary subset of ω_1 in V_1, where V_1 is generic over $V[G_0]$ without adding reals. Then $\mathbb{P}(S, G_0)$ is $< \omega_1$-distributive in V_1.*

Proof. Work in V_1. Fix any condition q' and sequence $\langle D_n \mid n < \omega\rangle$ of dense open sets. We need to find $q < q'$ in the intersection of all D_n. Let θ be a large regular cardinal. Let M be a countable elementary submodel of $H(\theta)$ containing all relevant parameters. Let $\delta = \omega_1 \cap M$. Let $t = r(\delta)$ for some condition r in G_0 with $\mathrm{dom}(r) > \delta$. Let $\langle E_n \mid n < \omega\rangle$ be an enumeration of the dense open subsets in M.

Case 1) There is a successor $\xi < \delta$ such that $t \restriction \xi \notin M$. Choose a descending sequence of conditions $\langle q_n \mid n < \omega\rangle$ such that $q_0 = q'$ and $q_{n+1} \in E_n \cap M$. Define q by setting $g_q = \bigcup g_{q_n}$ and $X_q = \bigcup X_{q_n}$. g_q is clearly a function from $Sh(S)$ to $T(G_0)$. We verify that q is a condition. We only check (c4) and (c6) as the other requirements trivially hold.

Claim 3.3. *For any increasing $\langle c_i \mid i \in \omega\rangle \in dom(g_q)^\omega$, either all c_i belong to $dom(g_{q_n})$ for some fixed q_n, or $\sup_{i<\omega} \max(c_i) = \delta$.*

Proof. We first assume that there is a fixed condition q_n such that $c_i \in cl(\mathrm{dom}(g_{q_n}))$ for all $i < \omega$. Choose any c_i; as $c_{i+1} \in cl(\mathrm{dom}(g_{q_n}))$ and c_{i+1} is strictly stronger than c_i, there is a $\beta > \max c_i$ such that $c_{i+1} \restriction \beta \in \mathrm{dom}(g_{q_n})$. But then by (c5), c_i is weaker then $c_{i+1} \restriction \beta \in \mathrm{dom}(g_{q_n})$ and must be in $\mathrm{dom}(g_{q_n})$. It follows that all c_i are in $\mathrm{dom}(g_{q_n})$.

Now we turn to the case that for all q_n, there is a c_i not in $cl(\mathrm{dom}(g_{q_n}))$. Note that by Claim 3.1, for any $\epsilon < \delta$ the set $D_\epsilon = \{p \in \mathbb{P}(S, G_0) \mid o(p) > \epsilon\}$ is a dense open set in M. Now for any $\epsilon < \delta$ let q_n, q_m be such that $q_m < q_n$, $o(q_n) > \epsilon$, $c_{i_\epsilon} \in \mathrm{dom}(g_{q_m})$ and $c_i \notin cl(\mathrm{dom}(g_{q_n}))$. It follows from the definition of compatibility that $\mathrm{dom}(g_{q_m}(c_i)) > o(q_n) > \epsilon$. Thus $\delta \geq \sup_{i<\omega} \mathrm{dom}(g_q(c_i)) \geq \sup_{i<\omega} \max(c_i) = \delta$. This proves Claim 3.3. □

For (c4), we must show that for all increasing $\langle c_i \mid i \in \omega\rangle \in \mathrm{dom}(g)^\omega$, $\bigcup_{i\in\omega} g_q(c_i) \in T(G_0)$. If there is a q_n such that all c_i are in $\mathrm{dom}\, g_{p_n}$ then there is nothing to prove. Otherwise, $\sup_{i<\omega} \mathrm{dom}(g_q(c_i)) = \sup_{i<\omega} \max(c_i) = \delta$. However, as $t \restriction \xi$ is not in M, $g_q(c_i)$ cannot extend $t \restriction \xi$. Thus $\bigcup_{i\in\omega} g_q(c_i)$ cannot extend $t \restriction \xi$ and is not equal to t. As all the initial segments of $\bigcup_{i\in\omega} g_q(c_i)$ is in $T(G_0)$ and $\bigcup_{i\in\omega} g_q(c_i) \neq t$, $\bigcup_{i\in\omega} g_q(c_i) \in T(G_0)$.

For (c6), fix again an increasing $\langle c_i \mid i \in \omega\rangle \in \mathrm{dom}(g)^\omega$. If there is a q_n such that all c_i are in $\mathrm{dom}\, g_{q_n}$ then again there is nothing to prove. Otherwise, $\sup_{i<\omega} \mathrm{dom}(g_q(c_i)) = \sup_{i<\omega} \max(c_i) = \delta$. And thus $\sup_{i<\omega} \mathrm{dom}(g_q(c_i))$ is in S iff $\sup_{i<\omega} \max(c_i)$ is in S.

It follows that $q < q'$ is a condition in the intersection of all E_n and thus in the intersection of all D_n.

Case 2) For all $\xi < \delta$, $t \restriction \xi \in M$. Let $\xi < \delta$ be such that $\xi > o(q')$. Define q'' by setting $g_{q''} = g_{q'}$ and $X_{q''} = X_{q'} \cup \{t \restriction \xi\}$. It is routine to check that q'' is a condition in M. Choose $\langle q_n \mid n < \omega\rangle$ as in Case 1) but with the new requirement that $q_1 < q''$. Define q by $g_q = \bigcup g_{q_n}$ and $X_q = \bigcup X_{q_n}$; we verify that q is a condition. Again we need to check (c4) and (c6). We omit the proof of (c6) as it is the same as in Case 1). Note that Claim 3.3 remains true.

For (c4), we must show that for all increasing $\langle c_i \mid i \in \omega\rangle \in \mathrm{dom}(g)^\omega$, $\bigcup_{i\in\omega} g_q(c_i) \in T(G_0)$. If there is a q_n such that all c_i are in $\mathrm{dom}\, g_{q_n}$ then there is nothing to prove. Otherwise, $\sup_{i<\omega} \mathrm{dom}(g_q(c_i)) = \sup_{i<\omega} \max(c_i) = \delta$. Let c_i, q_n be such that $c_i \in \mathrm{dom}(g_{q_n})$ with $n > 1$ and $\mathrm{dom}(g_{q_n}(c_i)) > \xi$. By our construction, $t \restriction \xi \in X_{q_n}$. Hence, $g_{q_n}(c_i)$ cannot extend ξ. As $\mathrm{dom}(g_{q_n}(c_i)) > \xi$, $g_{q_n}(c_i) \restriction \xi \neq t \restriction \xi$. Thus $\bigcup_{i\in\omega} g_q(c_i) \neq t$ and must be in $T(G_0)$. □

The standard canary tree forcing is a length ω_2, countable support iteration. At stage 0, we use $\mathbb{Q}_0$ to add the tree $T(G_0)$. At each step $\beta > 0$, we choose a stationary, costationary set S in $V^{\mathbb{P}_\beta}$ via a bookkeeping function and then force with $\mathbb{P}(S, G_0)$. The forcing has a countably-closed dense subset and tails of the iteration are proper. In the final model, $T(G_0)$ will be the desired canary tree. In our case, we will use a variant of the standard canary tree forcing and use the elementary embedding witnessing measurability to control the construction. Our treatment of precipitousness is close to the original paper [7] (or see [8]) and does not follow more recent expositions such as in section 17.2 of [1].

Theorem 3.1. *$Con(ZFC$ + there is a measurable cardinal$)$ $\leftrightarrow$ $Con(ZFC+NS_{\omega_1}$ is both Δ_1-definable with parameters from $H(\omega_2)$ and precipitous$)$.*

Proof. The direction from right to left is well-known (see [7]). For the other direction, we start from a model of $ZFC + GCH + \kappa$ is measurable, and then construct a forcing extension which satisfies our requirement. The iteration we use is similar to the iteration in [10].

Let $j : V \to M$ witness the measurability of κ and let U be the corresponding normal measure. We first force with $\mathbb{P} = \mathrm{Col}(\omega, <\kappa)$ and let G be $\mathbb{P}$-generic over V. Let $\mathbb{Q} = \mathrm{Col}(\omega, [\kappa, < j(\kappa)))$. Then $j(\mathbb{P}) \cong \mathbb{P} \times \mathbb{Q}$. κ becomes $\omega_1^{V[G]}$ in $V[G]$ and whenever $\bar{G}$ is $\mathbb{Q}$-generic over $V[G]$, j can be lifted to $j_G : V[G] \to M[G * \bar{G}]$ $(= M[G \times \bar{G}])$. In $V[G * \bar{G}]$, $M[G * \bar{G}]^\omega \subseteq M[G * \bar{G}]$. In $V[G]$, I_G is the derived precipitous ideal defined by: $A \in I_G$ if $\Vdash_{\mathbb{Q}} \kappa \notin j_G(A)$. Note it is equivalent to say $A \in I_G$ if there is a $\mathbb{P}$-name $\dot{A}$ for A and some $p \in G$ such that $p \Vdash_{j(\mathbb{P})} \kappa \notin j(\dot{A})$.

In $V[G]$, we will define a length ω_2 countable support iteration $\mathbb{P}_{\omega_2}$. During the construction we will also verify some properties of intermediate stages of the iteration which are needed for later parts of the construction. In particular, we will show that each $\mathbb{P}_\alpha$ is $< \omega_1$-distributive and has a size ω_1 dense subset using the "flat condition" argument. Throughout the proof, we always work in $V[G]$ unless otherwise specified. We view κ and κ^+ as ordinals and freely adopt the convention that $\kappa = \omega_1$ and $\kappa^+ = \omega_2$ (in $V[G]$).

Let $\mathbb{Q}_0$ be the first iterant of canary tree forcing. Let G_0 be $\mathbb{Q}_0$ generic over $V[G]$. As $\mathbb{Q}_0$ is of size ω_1, $\mathbb{Q}_0$ can be completely embedded into $B(\mathbb{Q})$, the Boolean completion of $\mathbb{Q}$. [1] Let $\bar{G}$ be $\mathbb{Q}$-generic over $V[G]$. For each $t \in \kappa^\kappa \cap V[G * \bar{G}]$, we define $m_0^t = (\bigcup G_0) \cup \langle \kappa, t \rangle$. It is clear that the m_0^t are $j_G(\mathbb{Q}_0)$ conditions.

Since for any $p \in G_0$, m_0^t extends $j(p) = p$, all m_0^t are j_G-$\mathbb{Q}_0$ master conditions. Hence, j_G can be lifted to $j_1 : V[G*G_0] \to M[G*\bar{G}*\bar{G}_0]$, where $\bar{G}_0$ is $j(\mathbb{Q}_0)$ generic over $M[G * \bar{G}]$ extending m_0^t. Nevertheless, we will also consider the general lifting without mentioning the master condition. Let I_1 be the ideal defined by: $A \in I_1$ if there are $p \in G$, $q \in G_0$ and $\mathbb{P} * \mathbb{Q}_0$-name $\dot{A}$ for A such that

$$p * q \Vdash_{j(\mathbb{P}*\mathbb{Q}_0)} \begin{array}{l} \text{For any } \bar{G} \text{ which is } \mathbb{Q}\text{-generic over } M[G], \text{and any} \\ \bar{G}_0 \text{ containing } G_0 \text{ which is } j(\mathbb{Q}_0) \text{ generic over} M[G * \bar{G}],\ \kappa \notin j(\dot{A}). \end{array}$$

Here note that $G_0 \subseteq \bar{G}_0$ implies that we can lift the embedding j_G to j_1. Basically, a set A is in the ideal if there is a condition $p * q$ in $G * G_0$ which forces that $j(\dot{A})$ has measure 0 in any available lifting of j_1 defined in $V[G * \bar{G} * \bar{G}_0]$.[2] Apparently, the definition is independent of the choice of the name $\dot{A}$ and condition $p * q$. A standard argument shows that I_1 is normal.

By induction on $\alpha < \kappa^+$, we define the following objects:

- A countable support iterated forcing $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa^+ \rangle$;
- A sequence $\langle I_\alpha \mid \alpha < \kappa^+ \rangle$ such that each I_α is a proper normal ideal in $V[G * G_\alpha]$.
- For appropriate G, $\bar{G}$ and G_α, the "preconditions" m_α^t for any t which is forced to be in $V[G]^{\mathbb{Q}} \cap \kappa^\kappa$. ($m_\alpha^t$ is only a "precondition" as it is not guaranteed to be a condition in $j_G(\mathbb{P}_\alpha)$.)

[1] This can be done in $M[G]$ using the absorption of Boolean algebras into the Levy Collapse. (See section 14 of [1])

[2] The definition of this ideal is the key difference between [7] and [1].

To complete the inductive definition of the $\mathbb{P}_\alpha, \mathbb{Q}_\alpha$ we need to know that each $\mathbb{P}_\alpha$ has a dense subset of size ω_1. The "preconditions" m^t_α are needed for this purpose.

We now give the definition of the forcing $\mathbb{P}_\alpha$ and the ideals I_α.

For $\alpha \in [1, \kappa^+)$, we define $\mathbb{Q}_\alpha$ as follows: Assume that I_α, $\mathbb{P}_\alpha$ have been defined and that $\mathbb{P}_\alpha$ is $< \omega_1$-distributive and has a size ω_1 dense subset F_α. Let G_α be $\mathbb{P}_\alpha$-generic over $V[G]$. Let X_α be a $\mathbb{P}_\alpha$-name for a subset of κ selected via bookkeeping. If in $V[G * G_\alpha]$, X_α is forced to be in I_α, then let $\mathbb{Q}_\alpha$ be $Sh(\kappa \setminus X_\alpha)$. If the complement of X_α is forced to be in I_α, then let $\mathbb{Q}_\alpha$ be the trivial forcing. Otherwise, let $\mathbb{Q}_\alpha$ be $\mathbb{P}(X_\alpha, G_0)$. We will show later that $\mathbb{P}_{\alpha+1}$ has a dense subset $F_{\alpha+1}$ of size $\omega_1^{V[G]}$. Hence we can completely embed $F_{\alpha+1}$ into $B(\mathbb{Q})$. Let $G_{\alpha+1}$ be $\mathbb{P}_\alpha * \mathbb{Q}_\alpha$ generic over $V[G]$ extending G_α. We can then assume $G_{\alpha+1} \cap F_{\alpha+1}$ is in $M[G * \bar{G}]$. We define $I_{\alpha+1}$ as before, i.e. $A \in I_{\alpha+1}$ if there are $p \in G$, $q \in G_\alpha$ and $\mathbb{P} * \mathbb{P}_\alpha$-name $\dot{A}$ for A such that

$$p * q \Vdash_{j(\mathbb{P}*\mathbb{P}_\alpha)} \begin{array}{l}\text{For any } \bar{G} \text{ which is } \mathbb{Q}\text{-generic over } M[G] \text{ and any } \bar{G}_\alpha \supseteq j[G_\alpha] \\ \text{which is } j(\mathbb{P}_\alpha)\text{-generic over } M[G * \bar{G}],\ \kappa \notin j(\dot{A}),\end{array}$$

and let $j_{\alpha+1} : V[G * G_\alpha] \to M[G * \bar{G} * \bar{G_\alpha}]$ denote the associated lifting (which depends on the choices of $\bar{G}$, $\bar{G_\alpha}$).

For α limit, $\mathbb{P}_\alpha$ is defined following the rules of countable support iteration. We will show that $\mathbb{P}_\alpha$ has a size ω_1 dense subset and specify one such dense subset F_α later. As in the successor case, we assume $G_\alpha \cap F_\alpha \in M[G * \bar{G}]$. j_α and I_α are defined as before.

The above finishes the definition of all $\mathbb{P}_\alpha$ and I_α, assuming that it can be verified inductively that each $\mathbb{P}_\alpha$ has a size ω_1 dense subset. To prove this, we will use the flat condition argument.

Definition 3.2. *We say q is a flat condition of $\mathbb{P}_\alpha$ if there is a unique $\gamma_q < \omega_1$ and sequence $\langle A^q_i \mid i \in spt(q)\rangle$ in $V[G]$ such that $\forall i \in spt(q) \setminus 1$,*

- *if $\mathbb{Q}_i = Sh(\kappa \setminus X_i)$, then $A^q_i = A^q_{i0}$ is a closed set of ordinals with a maximal element γ_q and $q \restriction i \Vdash q(i) = A^q_i$.*
- *if $\mathbb{Q}_i = \mathbb{P}(X_i, G_0)$, then $A^q_i = (A^q_{i0}, A^q_{i1})$. A^q_{i0} is a countable order preserving partial mapping from $Sh(X_i)$ to $T(G_0)$ such that if $c \in dom(A^q_{i0})$ and $ran(c) < \gamma_q$ then there is a $c' \in dom(A^q_{i0})$ such that c' end-extends c and $ran(c') = \gamma_q$. A^q_{i1} is a countable subset of $\omega_1^{<\omega_1}$. $q \restriction i \Vdash q(i)_0 = A^q_{i0} \wedge q(i)_1 = A^q_{i1}$.*
- *Otherwise, A^q_i is the empty set and $q(i)$ is the trivial condition.*

For any $\mathbb{P}_\alpha$, we let F_α be the set of flat conditions in $\mathbb{P}_\alpha$. It is clear that $|F_\alpha| = \omega_1^{V[G]}$.

To establish the density of F_α in $\mathbb{P}_\alpha$ we introduce the "preconditions" m^t_α, where $t \in (\kappa^\kappa)^{V[G*\bar{G}]}$ and $\alpha < \omega_2$. We are particularly interested in the situation when $F_\alpha \cap G_\alpha \in V[G * \bar{G}]$ and only give the definition in this case.[3] For all $\beta < \alpha$ such that $\mathbb{Q}_\beta = \mathbb{P}(X_\beta, G_0)$, let $g^{\alpha 0}_\beta = \bigcup_{q \in F_\alpha \cap G_\alpha} A^q_{\beta 0}$ and $g^{\alpha 1}_\beta = \bigcup_{q \in F_\alpha \cap G_\alpha} A^q_{\beta 1}$. For all $\beta < \alpha$ such that $\mathbb{Q}_\beta = Sh(\kappa \setminus X_\beta)$, let $g^{\alpha 0}_\beta = \bigcup_{q \in F_\alpha \cap G_\alpha} A^q_{\beta 0}$. We define m^t_α as follows. Recall that the m^t_0 were already defined as $(\bigcup G_0) \cup \{\langle \kappa, t\rangle\}$, a condition in $j_G(\mathbb{Q}_0)$.

$$m^t_\alpha(\delta) = \begin{cases} m^t_0 & \text{if } \delta = 0. \\ \langle g^{\alpha 0}_\gamma, g^{\alpha 1}_\gamma \rangle & \text{if } j(\gamma) = \delta \wedge \mathbb{Q}_\gamma = \mathbb{P}(X_\gamma, G_0). \\ g^{\alpha 0}_\gamma \cup \{\kappa\} & \text{if } j(\gamma) = \delta \wedge \mathbb{Q}_\gamma = Sh(\kappa \setminus X_\gamma). \\ 1 & \text{otherwise.} \end{cases}$$

[3]Note that this situation is possible as we can embed $\mathbb{P}_\alpha$ into $B(\mathbb{Q})$.

Since $F_\alpha \cap G_\alpha \in V[G * \bar{G}]$, it follows that all g^α_β's are in $M[G * \bar{G}]$. The following lemma completes the definition of $\mathbb{P}_{\omega_2}$.

Lemma 3.1. *The following hold for $\alpha < \omega_2$:*

(1) *F_α is dense.*
(2) *$\mathbb{P}_\alpha$ is $< \omega_1$-distributive.*
(3) *For any $t \in \kappa^\kappa \cap V[G * \bar{G}]$ such that $\exists \xi < \kappa(t \restriction \xi \notin V[G])$, m^t_α is a j_G-G_α master condition.*

Proof. We prove (1)-(3) simultaneously by induction on α. For $\alpha = 0$, as we noted before, $\mathbb{P}_\alpha$ is isomorphic to $Add(\omega_1, 1)$ and satisfies (1)-(3). Next suppose α is a limit.

$(1)_\alpha$ Fix any condition $q' \in \mathbb{P}_\alpha$, we need to find a $q < q'$ such that $q \in F_\alpha$. Let $\bar{G}$ and G_α be such that $\bar{G}$ is $\mathbb{Q}$ generic over $V[G]$ and $G_\alpha \in V[G * \bar{G}]$ be a $\mathbb{P}_\alpha$-generic over $V[G]$ with $q \in G_\alpha$. Now for all $\beta < \alpha$, let G_β be the derived $\mathbb{P}_\beta$ generic filter. Fix a $t \in \kappa^\kappa \cap V[G * \bar{G}]$ such that $\exists \xi < \kappa(t \restriction \xi \notin V[G])$. As $G_\beta \cap F_\beta$ is in $M[G * \bar{G}]$, m^t_β can be defined. By the induction hypothesis, m^t_β is a j_G-$\mathbb{P}_\beta$ master condition and $m^t_{\beta+1}(\beta) \in M[G * \bar{G}]$. Let m'_α be defined by

$$m'_\alpha(\delta) = \begin{cases} m^t_0 & \text{if } \delta = 0. \\ m^t_{\delta+1}(\delta) & \text{if } j(\gamma) = \delta. \\ 1 & \text{otherwise.} \end{cases}$$

Since $spt(m'_\alpha) \subseteq j[\alpha]$, it follows that m'_α is a $j_G(\mathbb{P}_\alpha)$-condition. It is also clear that m'_α is a flat condition for $j_G(\mathbb{P}_\alpha)$. For any $\beta < \alpha$, as $m'_\alpha \restriction j(\beta) = m^t_\beta$ is a master condition, $m'_\alpha \restriction j(\beta) < j(q) \restriction j(\beta)$. Hence in $M[G * \bar{G}]$, $m'_\alpha < j(q')$. Then by elementarity of j_G, there is a flat condition $q < q'$ in $\mathbb{P}_\alpha$. Thus F_α is dense in $\mathbb{P}_\alpha$.

$(2)_\alpha$ Fix a condition $q' \in \mathbb{P}_\alpha$ and a sequence of dense open subsets $\vec{D} = \langle D_n \mid n < \omega\rangle$ of $\mathbb{P}_\alpha$. We need to show there is $q \in F_\alpha$ in the intersection of all D_n's extending q'. Let $\bar{G}$ and G_α be such that $\bar{G}$ is $\mathbb{Q}$ generic over $V[G]$ and $G_\alpha \in V[G * \bar{G}]$ be a $\mathbb{P}_\alpha$-generic over $V[G]$ with $q \in G_\alpha$. Now for all $n < \omega$, it follows that $G_\alpha \cap D_n \neq \emptyset$. Define m'_α as in the proof of $(1)_\alpha$. By the same argument as in the proof of $(1)_\alpha$, $m'_\alpha < j(p)$ for any $p \in G_\alpha$. Hence in $M[G * \bar{G}]$, $m'_\alpha < j(q)$ is a condition in $j(\mathbb{P}_\alpha)$ which is in all $j_G(D_n)$. Then by elementarity of j_G, there is a flat condition $q < q'$ in which is in D_n.

$(3)_\alpha$ By $(1)_\alpha$, F_α is dense. It also clear that $F_\beta = F_\alpha \restriction \mathbb{P}_\beta$ for all $\beta < \alpha$. Now by induction on $\beta < \alpha$, we can check $m^t_\beta = m^t_\alpha \restriction j(\beta)$ using the definition of m^t_α and. Hence $m^t_\alpha = m'_\alpha$ is a master condition.

Now we turn to the case $\alpha = \beta + 1$.

$(2)_{\beta+1}$ $\mathbb{Q}_\beta$ either has size ω_1 or is trivial. Also note that by Claim 3.2 and the $< \omega_1$-distributivity of the club-shooting forcing, $\mathbb{Q}_\beta$ is $< \omega_1$-distributive. It then follows that $\mathbb{P}_\alpha$ is $< \omega_1$-distributive.

$(1)_{\beta+1}$ We check that F_α is dense in $j_G(\mathbb{P}_\alpha)$. As $\mathbb{P}_\alpha$ has an ω_1-size dense subset, $\mathbb{P}_\alpha$ can be completely embedded into $B(\mathbb{Q})$ where $\mathbb{Q} = Col(\omega, j(\kappa))$. As in the proof of $(1)_\alpha$, fix a condition q' and we look for $q < q'$ in F_α. Let $\bar{G}$ and G_α be such that $\bar{G}$ is $\mathbb{Q}$ generic over $V[G]$ and $G_\alpha \in V[G * \bar{G}]$ be a $\mathbb{P}_\alpha$-generic over $V[G]$ with $q \in G_\alpha$. We define m^t_α and m^t_β accordingly. By induction hypothesis, $m^t_\beta \in M[G * G']$ is a j_G-$\mathbb{P}_\beta$ master condition. Also $\mathbb{P}_\beta$ is $< \omega_1$-distributive. Hence $D = \{q \in \mathbb{P}_\alpha \mid \exists B_q \in V[G] q \restriction \beta \Vdash q(\beta) = B_q\}$ is dense. Define m'_α by letting $m'_\alpha \restriction j(\beta) = m^t_\beta$ and

$$m'_\alpha(j(\beta)) = \begin{cases} m^t_0 & \text{if } \beta = 0. \\ \langle \bigcup_{q \in D \cap G_\alpha} (B_q)_0, \bigcup_{q \in capG_\alpha} (B_q)_1 \rangle & \text{if } \mathbb{Q}_\beta = \mathbb{P}(X_\beta, G_0). \\ \bigcup_{q \in D \cap G_\alpha} B_q \cup \{\kappa\} & \text{if } \mathbb{Q}_\beta = Sh(\kappa \setminus X_\beta). \\ 1 & \text{if } \mathbb{Q}_\beta \text{ is trivial.} \end{cases}$$

We use $\bar{m}^\alpha_\beta$ to denote $m'_\alpha(j(\beta))$.

Claim 3.4. *m'_α is a $j(\mathbb{P}_\alpha)$-condition.*

Proof. We need to show $m'_\alpha \upharpoonright j(\beta) = m^t_\beta \Vdash_{j(\mathbb{P}_\beta)} \bar{m}^\alpha_\beta$ is a $j(\mathbb{Q}_\beta)$ condition. The case for $\beta = 0$ and $\mathbb{Q}_\beta$ being trivial is clear.

If β is such that $\mathbb{Q}_\beta = Sh(\kappa \setminus X_\beta)$, then $j_G(\mathbb{Q}_\beta) = Sh(j(\kappa) \setminus j_G(X_\beta))$. Since $m^{t\prime}_\alpha \upharpoonright j(\beta) = m^t_\beta$ is a master condition. Fix any $\bar{G}_\beta$ be $j_G(\mathbb{P}_\beta)$ generic over $M[G*\bar{G}]$ such that $m^t_\beta \in \bar{G}_\beta$. Work in $V[G * \bar{G} * \bar{G}_\beta]$. By the definition of I_β, it is easy to verify that $\kappa \notin j_\beta(X_\beta)$. However, $j_\beta(X_\beta) \cap \kappa = X_\beta$. Hence $\bar{m}^\alpha_\beta$ is forced to be a closed subset of $j_\beta(X_\beta)$ and thus a $j(\mathbb{Q}_\beta)$ condition.

If β is such that $\mathbb{Q}_\beta = \mathbb{P}(X_\beta, G_0)$, then $j_G(\mathbb{Q}_\beta) = \mathbb{P}(j_G(X_\beta), \bar{G}_0)$. Since $\bigcup \bar{G}_0 \upharpoonright \kappa = \bigcup G_0$, $T(\bar{G}_0) \cap (\omega_1^{<\omega_1})^{V[G*G_0]} = T(G_0)$. Moreover $j_\beta(X_\beta) \cap \kappa = X_\beta$. Hence $g_{\bar{m}^\alpha_\beta}$ is forced to be a countable partial function from $Sh(j(X_\beta))$ to the $j(X_\beta)$-node of $T(\bar{G}_0)$, $\mathrm{dom}(g_{\bar{m}^\alpha_\beta})$ is closed under initial segment. Also $X_{\bar{m}^\alpha_\beta}$ is a countable subset of $j(\kappa)^{j(\kappa)}$. It remains to verify (c4) and (c6). Note the analogue of Claim 3.3 holds:

Claim 3.5. *For all increasing sequences $\langle c_i \mid i \in \omega\rangle$ in $dom(g_{\bar{m}^\alpha_\beta})$, either all c_i are in $dom(g_p)$ for some fixed $p \in G_\alpha$ or $\bigcup_{i\in\omega} dom(g_{\bar{m}^\alpha_\beta}(c_i)) = \sup_{i<\omega} \max(c_i) = \kappa$.*

Proof. We first assume that there is a condition $p \in H$ such that $c_i \in cl(\mathrm{dom}(g_p))$ for all $i < \omega$. Fix any c_i, as $c_{i+1} \in cl(\mathrm{dom}(g_p))$ and c_{i+1} is strictly stronger than c_i, there is a $\beta > \max c_i$ such that $c_{i+1} \upharpoonright \beta \in \mathrm{dom}(g_p)$. But then by (c5), c_i is weaker then $c_{i+1} \upharpoonright \beta \in \mathrm{dom}(g_p)$ and must be in $\mathrm{dom}(g_p)$. Now all c_i is in $\mathrm{dom}(g_p)$.

Now we turn to the case that for all p, there is a c_i not in $cl(\mathrm{dom}(g_p))$. Note that for any $\epsilon < \kappa$, the set $D_\epsilon\{p \in \mathbb{P}(S, G_0) \mid o(p) > \epsilon\}$ is a dense open set in $V[G * G_\beta]$. Now for any $\epsilon < \kappa$, there are p_0, p_1 in H such that $p_0 < p_1$, $o(p_1) > \epsilon$, $c_{i_\epsilon} \in \mathrm{dom}(g_{p_0})$ and $c_i \notin cl(\mathrm{dom}(g_{p_1}))$. It follows from the definition of compatibility that $\mathrm{dom}(g_{p_0}(c_i)) > o(p_1) > \epsilon$. Thus $\max_{i<\omega}(c_i)$ is unbounded in κ. But then $\kappa \geq \sup_{i<\omega} \mathrm{dom}(g_q(c_i)) \geq \sup_{i<\omega} \max(c_i) = \kappa$. □

For (c4), in light of the proof of Claim 3.2, we only need to prove the following.

Claim 3.6. *For all increasing sequences $\langle c_i \mid i \in \omega\rangle$ in $dom(g_{\bar{m}^\alpha_\beta})$ such that $\bigcup_{i\in\omega} dom(g_{\bar{m}^\alpha_\beta}(c_i)) = \kappa$,*

$$m'_\beta \Vdash \bigcup_{i\in\omega} g_{\bar{m}^\alpha_\beta}(c_i) \neq t.$$

Proof. As there is a successor ordinal ϵ such that $t \upharpoonright \epsilon$ is not in $V[G]$ for any i such that $\mathrm{dom}(g_{\bar{m}^\alpha_\beta})(c_i) > \epsilon$, $g_{\bar{m}^\alpha_\beta}(c_i)$ is in $V[G]$ and does not extend $t \upharpoonright \epsilon$. Hence $\bigcup_{i\in\omega} g_{\bar{m}^\alpha_\beta}(c_i) \neq t$. □

The prove of (c6) is identical the the proof of Claim 3.2. Note either all c^i are contained in $\mathrm{dom}(g_p)$ for a single condition p or $\sup_{i<\omega} \max(c_i) = \sup_{i<\omega} \mathrm{dom}(g_{\bar{m}^\alpha_\beta}(c_i)) = \kappa$.

This completes the verification that m'_α is a $j(\mathbb{P}_\alpha)$ condition. □

In $V[G * \bar{G}]$, it is clear that m'_α is a flat condition. We can check that m'_α is stronger than the image of any conditions in G_α. Fix $r \in G_\alpha$. The case for $\beta = 0$ and $\mathbb{Q}_\beta$ being trivial is clear. For $\mathbb{Q}_\beta$ is $Sh(\kappa \setminus X_\beta)$ or $\mathbb{P}(X_\beta, G_0)$, we work as follows: Since $m_\beta = m'_\alpha \upharpoonright j(\beta)$ is j_G-$\mathbb{P}_\beta$ master condition. $m_\beta < j_G(r \upharpoonright \beta)$ and forces $j_G(r(\beta)) = r(\beta) \in V[G]$. It is clear that $j_G(r(\beta)) = r(\beta) \subseteq m'_\alpha(j(\beta))$ by the definition of m'_α. Hence that $m'_\alpha(j(\beta))$ is stronger than $j_G(r(\beta)) = r(\beta)$. In particular, for any $\mathbb{P}_\alpha$ condition q, and $G_\alpha \ni q$, we can show that $m'_\alpha < j(q)$. Hence by elementarity, in $V[G]$, there is a flat condition extending q. Thus F_α is dense.

$(3)_{\beta+1}$ We only need to check that $m'_\alpha = m^t_\alpha$ as m'_α is stronger than any $j_G(q)$ with $q \in G_\alpha$ and thus a master condition. It suffices to check that $m'_\alpha(\beta) = m_\alpha(\beta)$. The case for $\beta = 0$ and $\mathbb{Q}_\beta$ being trivial is clear. For $\mathbb{Q}_\beta$ is $Sh(\kappa \setminus X_\beta)$ or $\mathbb{P}(X_\beta, G_0)$, as F_α is dense in $\mathbb{P}_\alpha$, it is routine to check that $\bigcup_{q \in D \cap G_\alpha}(B_q)_i = \bigcup_{q \in F_\alpha \cap G_\alpha} A^q_{\beta i}$ for $i = 0$ or 1. □

This completes the definition of $\mathbb{P}_{\omega_2}$. As $\mathbb{P}_{\omega_2}$ is the direct limit of the $\mathbb{P}_\alpha$, $\alpha < \omega_2$, $\mathbb{P}_{\omega_2}$ is $< \omega_1$-distributive and ω_2-c.c. For the remainder of the proof, we will not distinguish $\mathbb{P}_{\omega_2}$ from its dense subset which consists of all the flat conditions. For any $\alpha < \omega_2$, we will also view $G_\alpha \cap F_\alpha$ as G_α.

By the standard treatment of absorption and the ideal derived from lifting (see section 17 of [1]), the following holds for all $\alpha_1 < \alpha_2 < \omega_2$:

Lemma 3.2. (1) $I_{\alpha_2} \cap V[G * G_{\alpha_1}] = I_{\alpha_1}$.
(2) I_{α_1} *is a normal precipitous ideal in* $V[G * G_{\alpha_1}]$.

Proof. (1) It is routine to check that $I_{\alpha_1} \subseteq I_{\alpha_2}$. Now for $X \in I_{\alpha_2} \cap V[G * G_{\alpha_1}]$, assume that $X \notin I_{\alpha_1}$. Let $\dot{X}$ be a $\mathbb{P}_\alpha$-name exemplifying this fact, i.e, $\Vdash_{\mathbb{P}_{\alpha_1}} \dot{X} \notin I_{\alpha_1}$ and $\Vdash_{\mathbb{P}_{\alpha_2}} \dot{X} \in I_{\alpha_2}$.

By the definition of the ideal I_{α_2}, there is a condition $p * q_2 \in G * G_{\alpha_2}$ witnessing $\dot{X} \in I_{\alpha_2}$. We denote $q_2 \restriction \alpha_1$ by q_1 and $q_2 \restriction [\alpha_1, \alpha_2)$ by $q_(1,2)$. Hence,

$p \Vdash_{j(\mathbb{P})}$ For any G_{α_2} which is $\mathbb{P}_{\alpha_2}$-generic over $M[G]$ and contains q_2, for any $M[G * \bar{G}] - j(\mathbb{P}_{\alpha_2})$ generic $\bar{G}_{\alpha_2}$ which contains $j[G_{\alpha_2}, \kappa \notin j_{\alpha_2}(\dot{X}))$.

On the other hand, by the definition of the ideal I_{α_1}, for any fixed G_0, $\bar{G}_0$, G_{α_1} and $\bar{G}_{\alpha_1}$ such that:

- $G * \bar{G}$ is $j(\mathbb{P})$ generic over V.
- G_{α_1} is $\mathbb{P}_{\alpha_1}$ generic over $V[G]$, G_{α_1} in $V[G * \bar{G}]$.
- $p * q_1 \in G * G_{\alpha_1}$.

exactly one of the following holds:

a) There is no $\bar{G}_{\alpha_1}$ which is $j(\mathbb{P}_{\alpha_1})$-generic over $M[G * \bar{G}]$ such that $j[G_{\alpha_1}] \subseteq \bar{G}_{\alpha_1}$.
b) There is a $j(\mathbb{P}_{\alpha_1})$-generic filter $\bar{G}_{\alpha_1}$ over $M[G * \bar{G}]$ such that $j[G_{\alpha_1}] \subseteq \bar{G}_{\alpha_1}$ and $\kappa \in j_{\alpha_1}(\dot{X}/(G * G_{\alpha_1}))$.

We can show that a) is always false via an argument identical to the proof of Lemma 3.1(3): Whenever we choose $t \in \kappa^\kappa \cap V[G * \bar{G}]$ such that $t \notin T(G_0)$, then m^t_α defined accordingly is a master condition. Thus whenever $m^t_\alpha \in \bar{G}_{\alpha_1}$, $j[G_{\alpha_1}] \subseteq \bar{G}_{\alpha_1}$.

It follows that b) holds. Let $\bar{G}_{\alpha_1}$ be a witness. In particular, $\kappa \in j_{\alpha_1}(\dot{X}/(G * G_{\alpha_1}))$. Let $t = r(\kappa)$ for some (or any) $r \in \bar{G}_0$ with $\kappa \in \text{dom}(t)$. In what follows, we construct a generic filter G_{α_2} in $V[G * \bar{G}]$ such that $G_{\alpha_2} \restriction \mathbb{P}_{\alpha_1} = G_{\alpha_1}$ and $q_2 \ni G_{\alpha_2}$. We will also ensure that $m^t_{\alpha_2}$ is a master condition for this chosen G_{α_2}. The proof is very similar to the proof of Claim 3.2.

Claim 3.7. *There is a generic filter* G_{α_2} *in* $V[G * \bar{G}]$ *such that*
(1) $G_{\alpha_2} \restriction \mathbb{P}_{\alpha_1} = G_{\alpha_1}$ *and* $q_2 \in G_{\alpha_2}$.
(2) $m^t_{\alpha_2}$ *is a master condition.*

Proof. Note that we only need to construct G_{α_2} to be generic over $\mathcal{M} = \langle H(\kappa^+)^{V[G]}, \in \rangle$ as $\mathbb{P}_{\alpha_2}$ can be viewed as an element of $H(\kappa^+)^{V[G]}$ and thus all dense subsets are in $\mathcal{M}$. Moreover, as G_{α_1} is generic over $\mathcal{M}$, we only need to construct an H which is generic over $\mathcal{M}[G_{\alpha_1}]$ and contains $q_{(1,2)}$. Work in $V[G * \bar{G}]$, where $\mathcal{M}[G_{\alpha_1}]$ is a countable structure. The proof separates into two cases:

Case 1) For some $\alpha < \kappa$, $t \restriction \alpha$ is not in $\mathcal{M}$.

(1) Enumerate all the open dense subsets in $\mathcal{M}$ in a sequence $\langle D_n \mid n < \omega\rangle$. Define a decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$ such that $p_0 < q_{(1,2)}$ and for each n, $p_n \in D_n$. Let H be the downward closure of $\langle p_n \mid n < \omega\rangle$. Then H is a generic filter as H meets all the dense sets.

(2) We have already shown that $m^t_{\alpha_2}$ is a condition in the proof of Lemma 3.1(3).

Case 2) For all $\alpha < \kappa$, $t \restriction \alpha$ is in $\mathcal{M}$.

(1) As in the proof of case 1), we want to construct $\langle p_n \mid n < \omega\rangle$ such that $p_0 < q_{(1,2)}$ and for each n, $p_n \in D_n$. However, we will need one more requirement for the sequence $\langle p_n \mid n < \omega\rangle$ to ensure that $m^t_{\alpha_2}$ is a condition:

$$\forall n < \omega \forall \eta \in \operatorname{spt}(p_n)(\mathbb{Q}_\eta = \mathbb{P}(G_0, X_\beta) \to (\exists \gamma > o(p_n(\eta)))(t \restriction \gamma \in X_{p_{n+1}(\beta)})).$$

This can be achieved as follows: Suppose p_n has been defined, let $\gamma < \kappa$ be a successor ordinal strictly greater than $o(p_n(\eta))$ such that $\eta \in \operatorname{spt}(p_n)$ and $\mathbb{Q}_\eta = \mathbb{P}(G_0, X_\beta)$. Define p'_n by setting

$$p'_n(\beta) = \begin{cases} p_n(\beta) & \text{if } \beta = 0 \text{ or } \beta \notin \operatorname{spt}(p_n) \text{ or } \mathbb{Q}_\beta = Sh(\kappa \setminus X_\beta) \\ \langle g_{p_n(\beta)}, X_{p_n(\beta)} \cup \{t \restriction \gamma\}\rangle & \text{if } \mathbb{Q}_\beta = \mathbb{P}(X_\beta, G_0). \end{cases}$$

Clearly p'_n is a condition. Let $p_{n+1} \in D_{n+1}$ extend p'_n. Then p_{n+1} suffices. Let H be the downward closure of $\langle p_n \mid n < \omega\rangle$. It is clear that H is a generic filter.

(2) We only need to show that $m^t_{\alpha_2}$ is a condition. We follow the argument of Lemma 3.1. By induction on $\alpha < \alpha_2$, we show that $m^t_\alpha = m^t_{\alpha_2} \restriction \alpha$ is a $\mathbb{P}_\alpha$ condition. For $\eta < \alpha_1$, there is nothing to prove, as $m^t_{\alpha_1} \in \bar{G}_{\alpha_1}$.

The case when α is limit is clear by induction. When $\alpha = \beta + 1$, it is again routine to check when $\mathbb{Q}_\beta$ is not $\mathbb{P}(G_0, X_\beta)$. Now assume $\mathbb{Q}_\beta$ is $\mathbb{P}(G_0, X_\beta)$. We only need to check the requirement (c4) as all other requirements can be verified using the same argument as in the proof of Lemma $3.1(3)_{\beta+1}$. We remark that Claim 3.5 remains true here.

For (c4), as in Claim 3.6 we need to show the following: For all increasing sequences $\langle c_i \mid i \in \omega\rangle$ in $\operatorname{dom}(g_{\bar{m}^\alpha_\beta})$ such that $\bigcup_{i\in\omega} \operatorname{dom}(g_{\bar{m}^\alpha_\beta}(c_i)) = \kappa$,

$$m^t_\beta \Vdash \bigcup_{i\in\omega} g_{\bar{m}^\alpha_\beta}(c_i) \neq t.$$

By construction, there is $n \in \omega$ such that $\beta \in \operatorname{spt}(p_n)$. Hence there is a successor ordinal γ such that $t \restriction \gamma \in X_{p_{n+1}(\beta)}$. Let c_i and $n' > n$ be such that $c_i \notin \operatorname{dom}(g_{p_{n+1}})$ and $c_i \in \operatorname{dom}(g_{p\prime})$. By the definition of the forcing order, c_i cannot extend $t \restriction \gamma$. Hence $\bigcup_{i\in\omega} g_{\bar{m}^\alpha_\beta}(c_i) \neq t$. This completes the proof of Claim 3.7. $\square$

Now $G_{\alpha_2} = G_{\alpha_1} * H$ is $\mathbb{P}_{\alpha_2}$-generic over $V[G]$ and $m^t_{\alpha_2}$ is a master condition. Let $\bar{G}_{\alpha_2}$ be $j(\mathbb{P}_{\alpha_2})$ generic over $M[G*\bar{G}]$ such that $m^t_{\alpha_2} \in \bar{G}_{\alpha_2}$ and $\bar{G}_{\alpha_2} \restriction j(\mathbb{P}_{\alpha_1}) = \bar{G}_{\alpha_1}$. It follows that $\kappa \in j_{\alpha_2}(\dot{X}/(G * G_{\alpha_2})) = j_{\alpha_1}(\dot{X}/(G * G_{\alpha_1}))$ as j_{α_2} lifts j_{α_1} and $\dot{X}$ is a $\mathbb{P} * \mathbb{P}_{\alpha_1}$-name. This contradicts the fact that κ is not in $j_{\alpha_2}(\dot{X}/(G * G_{\alpha_2}))$ by the definition of I_{α_2} and the fact that $p * q_2$ is a witness to $\dot{X} \in I_{\alpha_2}$.

(2) We show that I_{α_1} is normal. Assuming otherwise, there is a function $f : \kappa \to \kappa$ such that f is regressive on a $I^+_{\alpha_1}$ set S. We need to show that there is a $\gamma < \kappa$ such that the set $S_\gamma = \{\eta < \kappa \mid f(\eta) = \gamma\} \notin I_{\alpha_1}$. Let $\bar{G} * \bar{G}_\alpha$ witness that $\kappa \in j_\alpha(S)$,[4] thus $j_\alpha(f)(\kappa) < \kappa$. Say $j_\alpha(f)(\kappa) = \gamma_r$. Then $\bar{G} * \bar{G}_\alpha$ witnesses that S_{γ_r} is not in I_{α_1}. The proof for precipitousness can be viewed as a special case of the proof of Claim 3.10 to follow and thus we omit it here. $\square$

Let G_{ω_2} be a $\mathbb{P}_{\omega_2}$ generic over $V[G]$. Let $I = \bigcup_{\alpha<\omega_2} I_\alpha$. As $\mathbb{P}_{\omega_2}$ is ω_2-c.c, any subset of ω_1 appears in $V[G_\alpha]$ for some $\alpha < \omega_2$. Hence by Lemma 3.2(2), I is a normal ideal. Also it is also clear that I contain any ground model measure 0 set

[4] Note we have already argued for the existence of such a j_{α_1} in the proof of 1).

and thus is a proper ideal. It follows that I must contain NS_{ω_1}. On the other hand, our iteration shoots clubs disjoint from any set in I. Hence, $I = NS_{\omega_1}$.

In the next lemma, we show that $T(G_0)$ is a canary tree in $V[G * G_{\omega_2}]$.

Lemma 3.3. *The following are true in $V[G * G_{\omega_2}]$:*

1. *$T(G_0)$ has no cofinal branch.*
2. *If $S \notin I \cup \bar{I}$, then there is an order-preserving function from $Sh(S)$ to $T(G_0)$.*

Proof. (1)Assume $t \in V[G * G_{\omega_2}]$ is in $\omega_1^{\omega_1}$. We need to show t is not a branch of $T(G_0)$. Let $\alpha < \omega_2$ be such that $t \in V[G * G_\alpha]$. Let $\dot{t}$ be a $\mathbb{P}_\alpha$ name of t. We prove that there are dense-many conditions in $\mathbb{P}_\alpha$ which force that $\dot{t}$ is not a branch of $T(G_0)$.

Fix a condition $q \in G * G_\alpha$. Consider the precondition m^t_α. We show that m^t_α is a $\mathbb{P}_\alpha$ condition by induction on $\beta < \alpha$. The limit cases and the case $\beta = 1$ are easy. Also the case that $\beta + 1$ is such that $\mathbb{Q}_\beta = Sh(\kappa \setminus X_\beta)$ follows from the proof of Claim 3.4.

Now assume that $\beta + 1$ is such that $\mathbb{Q}_\beta = \mathbb{P}(X_\beta, G_0)$. By the induction hypothesis, $m^t_\alpha \restriction j(\beta)$ is a $j_G(\mathbb{P}_\beta)$-condition.

Since the complement of X_β is not in I_β, by Claim 3.10, X_β is costationary. Hence in $V[G * G_\alpha]$, there is η such that $t \restriction \eta$ is a function from η to η and η is in the complement of X_β. Hence $\bar{t} = t \restriction (\eta + 1)$ is not an X_β-node. This implies that there is a condition forces $p \in G_\beta$ such that

$$p \Vdash_{\mathbb{P}_\beta} \dot{\bar{t}} \text{ is not an } X_\beta\text{-node}.$$

By the definition of $\mathbb{P}(G_0, X_\beta)$, $q \Vdash$ For all $\mathbb{P}(X_\beta, G_0)$ conditions q'', if $c \in \mathrm{dom}(g_{q''})$ such that $\mathrm{dom}(c) > \eta + 1$, then $g_{q''}(c)$ cannot extend $\bar{t}$, as $\bar{t}$ is not an X_β-node. As in the proof of Claim 3.4, we can check that $m^t_\alpha \restriction j(\beta)$ forces that $m^t_\alpha(j(\beta))$ is a condition. The only new feature is a different proof of Claim 3.6.

Claim 3.8. *For all increasing sequence $\langle c_i \mid i \in \omega \rangle$ in $dom(g_{\bar{m}^\alpha_\beta})$ such that $\bigcup_{i\in\omega} dom(g_{\bar{m}^\alpha_\beta}(c_i)) = \kappa$,*

$$m^t_\beta \Vdash \bigcup_{i\in\omega} g_{\bar{m}^\alpha_\beta}(c_i) \neq t.$$

Proof. In $V[G][G_\beta]$, whenever $\mathrm{dom}(c) > ot(\bar{t})$, $g_q(c)$ cannot extend $\bar{t}$. Hence there is a c_i such that $g(c_i)$ does not extend $\bar{t}$. Now in $V[G][G_\alpha]$, t extends $\bar{t}$. It follows that $\bigcup_{i\in\omega} m'_\alpha(j(\beta))(c_i)$ is not equal to t. □

By induction, m^t_α is a $j(\mathbb{P}_\alpha)$ condition. Note that $j(\dot{t})$ is a $j_G(\mathbb{P}_\alpha)$-name and $m^t_\alpha \Vdash j(\dot{t}) \restriction \kappa$. Moreover, $j(q) > m^t_\alpha \Vdash t = j(\dot{t}) \restriction \kappa$ is not a branch of $T(\bar{G}_0)$. By elementarity, in $V[G]$, there is a condition $q' < q$ such that $q' \Vdash \dot{t}$ is not a branch of $T(G_0)$.

(2) It follows from the assumption that there is an $\alpha < \omega_2$ such that S is a stationary costationary set in $\mathbb{P}_\alpha$, $X_\alpha = S$ and $\dot{\mathbb{Q}}_\alpha = \mathbb{P}(X_\alpha, G_0)$. Hence g^α_α is the desired order-preserving function. □

The Π_1 definition of I is as follows:

Claim 3.9. *In $V[G * G_{\omega_2}]$, $S \notin NS_{\omega_1}$ iff either S contains a club or there are $X_0 \subseteq S$, $X_1 \subseteq \bar{S}$ and g_0, g_1 such that for $i = \{0, 1\}$, g_i is an order-preserving function from $Sh(X_i)$ to $T(G_0)$.*

Proof. If $S \notin NS_{\omega_1}$, then the implication is proved in (2) of the last claim. For the other direction, we only need to show that if there is $X \subseteq \omega_1$ and g an order-preserving function from $Sh(X)$ to $T(G_0)$, then X does not contain a club. But otherwise, $\bigcup\{g(X \cap \beta) \mid \beta \in \mathrm{Lim}(X)\}$ is a cofinal branch of $T(G_0)$. □

Finally, we prove that NS_{ω_1} is a precipitous ideal, which completes the proof of Theorem 3.1.

Claim 3.10. *$I = NS_{\omega_1}$ is a precipitous ideal.*

Proof. For any I-positive set A, we need to prove there is a well-founded generic ultrapower which concentrates on A. Via a density argument, we need to prove for any $\mathbb{P} * \mathbb{P}_{\omega_2}$ condition $p * q$ and name $\dot{A}$ such that $p * q \Vdash A \notin I$, there is $G * G_{\omega_2}$ containing $p * q$ and a $P(\omega_1)/I$ generic filter D containing $\dot{A}/G * G_{\omega_2}$ such that $Ult_D V[G * \bar{G}]$ is well-founded.

Fix such a pair $p * q$ and $\dot{A}$. We can also assume that $\dot{A}$ is a $\mathbb{P} * \mathbb{P}_\alpha$ name and q is a $\mathbb{P}_\alpha$ condition for some $\alpha < \kappa^+$. Since $p * q$ forces $\dot{A} \notin I$, there exists $p_1 \in j(\mathbb{P})$ such that $p_1 < p$,

$$\begin{aligned} p_1 \Vdash_{j(\mathbb{P})} \quad & (\exists\ \mathbb{P}_\alpha\text{-generic } G_\alpha \text{ over } M[G] \text{ containing } q)(\exists\ j(\mathbb{P}_\alpha)\text{-condition } q^*) \\ & q^* \Vdash_{j(\mathbb{P}_\alpha)} j[G_\alpha] \subseteq \bar{G}_\alpha \wedge \kappa \in j_\alpha(\dot{A}). \end{aligned}$$

We choose G, $\bar{G}$, G_α and $\bar{G}_\alpha$ witnessing the above formula. Thus $\kappa \in j_\alpha(\dot{A}/G * G_\alpha)$. Let $t = r(0)(\kappa)$ for some (or any) $r \in G_\alpha$ with $\mathrm{dom}(r(0)) > \kappa$. Using the same argument of the proof of Claim 3.7, we can construct G_{ω_2} such that $G_{\omega_2} \restriction \alpha = G_\alpha$ and for any $\beta < \kappa^+$, m^t_β is a master condition in $M[G*\bar{G}]$. One possible construction is as follows: Work in a very large generic extension $V[G * \bar{G} * H]$ making κ^+ of countable cofinality. Let $\langle \beta_n \mid n < \omega \rangle$ be a cofinal sequence of κ^+ with $\beta_0 = \alpha$. By inductively applying Claim 3.7, we define the sequence $\langle G_{\beta_n} \mid n < \omega \rangle$ such that

- $G_{\beta_{n+1}} \restriction \beta_n = G_{\beta_n}$.
- $m^t_{\beta_n}$ is a master condition and $m^t_{\beta_n}$ in $M[G * \bar{G}]$.

Let $G_{\omega_2} = \bigcup_{n<\omega} G_{\beta_n}$. Then G_{ω_2} is $\mathbb{P}_{\omega_2}$-generic over $V[G * \bar{G}]$. Note although all G_β are in $M[G * \bar{G}]$, G_{ω_2} is not.

In $V[G * \bar{G} * H]$, let $\mathbb{P}^*$ be the suborder of $j_G(\mathbb{P}_{\omega_2})$ such that $r \in \mathbb{P}^*$ if $r \in \mathbb{P}_{\omega_2}$, r extend m^t_α and $\mathrm{spt}(r) \subseteq j(\alpha)$ for some $\alpha < \omega_2$. Note that $\mathbb{P}^* \restriction j(\alpha) = j_G(\mathbb{P}_\alpha)/m^t_\alpha$ as all m^t_α are in $M[G * \bar{G}] \cap j_G(\mathbb{P}_{\omega_2})$. Here $j_G(\mathbb{P}_\alpha)/m^t_\alpha = \{p \in j_G(\mathbb{P}_\alpha) \mid p < m^t_\alpha\}$. Let $\bar{G}^*$ be a $\mathbb{P}^*$ generic over $V[G * \bar{G} * H]$. Let $\bar{G}_{\omega_2} \subseteq \mathbb{P}_{\omega_2}$ be the set of conditions which are weaker than some conditions in $\bar{G}^*$. It is not difficult to check that $\bar{G}_{\omega_2}$ is $j(\mathbb{P}_{\omega_2})$ generic over $V[G*\bar{G}]$.[5] Therefore, $\bar{G}_{\omega_2}$ is also a $j(\mathbb{P}_{\omega_2})$ generic over $M[G*\bar{G}]$ and $j_G[G_{\omega_2}] \subseteq \bar{G}_{\omega_2}$. Hence we can lift j_G to $j_{\omega_2} : V[G * G_{\omega_2}] \to M[G * \bar{G} * \bar{G}_{\omega_2}]$. Note that by definition j_{ω_2} extends all j_α's where $\alpha < \omega_2$.

In summary, we can construct $G * \bar{G} * \bar{G}_{\omega_2}$ such that the induced embedding j_{ω_2} satisfies $\kappa \in j_{\omega_2}(\dot{A}/G * G_{\omega_2})$. Let $U_{\omega_2} = \{X \subseteq \omega_1 \mid \kappa \in j_{\omega_2}(X)\}$. U_{ω_2} is a normal $V[G * \bar{G}]$-ultrafilter disjoint from I_{ω_2}, the ideal dual to U_{ω_2}. It is also clear that $Ult(V[G * \bar{G}], U_{\omega_2}) = M[G * \bar{G} * \bar{G}_{\omega_2}]$ and thus well-founded. It remains to prove that U_{ω_2} is $P(\kappa)/I_{\omega_2}$ generic over $V[G * G_{\omega_2}]$.

Assume otherwise. Returning to V, let $\dot{T}$ be a $\mathbb{P} * \mathbb{P}_{\omega_2}$ name of a maximal antichain in $P(\omega_1)/I_{\omega_2}$ such that there is a $j(\mathbb{P}) * \mathbb{P}^*$ condition $s * t \in G * \bar{G} * G_{\omega_2}$ force that $\dot{T} \cap \dot{U}_{\omega_2} = \emptyset$, namely, for all $X \in T(\kappa \notin j_{\omega_2}(X))$. View $s*t$ as a $j(\mathbb{P}*\mathbb{P}_{\omega_2})$ condition, then there is a function $f : \kappa \to V$ such that $j(f)(\kappa) = \langle p, q \rangle$. Let $\dot{S}$ be the $\mathbb{P} * \mathbb{P}_{\omega_2}$ name of $\{\alpha < \kappa \mid f(\alpha) \in G * G_{\omega_2}\}$. Now $\kappa \in j_{\omega_2}(S)$ implies that $s * t \in G * \bar{G} * \bar{G}_{\omega_2}$. Hence with any choice of $\bar{G}$ and $\bar{G}_{\omega_2}$, if $\kappa \in j_{\omega_2}(S)$, then for every $X \in T$, $\kappa \notin j_{\omega_2}(X)$. It follows that $S \cap X \in I_{\omega_2}$ for all $X \in A$. However, $S \notin I_{\omega_2}$ since whenever $s*t \in G*\bar{G}*\bar{G}_{\omega_2}$, $\kappa \in j_{\omega_2}(S)$. It follows that T is maximal in $V[G * G_{\omega_2}]$ as S is a counterexample. □

This completes the proof of Theorem 3.1. □

[5] Note for any antichain A of $j_G(\mathbb{P}_{\omega_2})$, there is α such that $A \cap j_G(\mathbb{P}_\alpha)$ is a antichain of $j_G(\mathbb{P}_\alpha)$. Moreover, any $j_G(\mathbb{P}_\alpha)/m^t_\alpha$ generic filter can trivially derived a $j_G(\mathbb{P}_\alpha)$ by taking downward closure.

4. Saturation and Δ_1-definability of NS_{ω_1}

In this section, we study the consistency of "NS_{ω_1} is both saturated and Δ_1-definable". This statement follows from the statement that NS_{ω_1} is ω_1-dense. By work of Woodin, the consistency strength of the latter is ω-many Woodin cardinals. Hence we have:

Fact 4.1. *Con(ZFC+ there are ω-many Woodin cardinals)$\rightarrow$Con(ZFC + NS_{ω_1} is both saturated and Δ_1-definable).*

It is unknown whether it is possible to reduce this upper bound to one Woodin cardinal.[6] It is also unknown how to construct a model in which NS_{ω_1} is saturated but not Δ_1 definable. The following is a partial result on the latter question.

Fact 4.2. *Suppose κ is a Woodin cardinal. Then in $V^{Col(\omega_1,<\kappa)}$, NS_{ω_1} is presaturated but not Δ_1 definable.*

Proof. That NS_{ω_1} is presaturated in that model is explicitly proved in [2] and a proof of even a stronger result can be found in [5]. Since $Col(\omega_1,<\kappa) = Col(\omega_1,[\omega_2,<\kappa)) \times Add(\omega_1,\omega_2)$, there is no canary tree in $V^{Col(\omega_1,<\kappa)}$. □

We say that NS_{ω_1} is *lightface Δ_1 definable* iff it is Δ_1 definable with parameter ω_1. In the remain part of this section, we present some facts regarding the lightface Δ_1 definability and saturation of NS_{ω_1}

Recall that following definition of iterated generic ultrapower.

Definition 4.1. *Let M be a model of ZFC-Powerset Axiom. Let γ be an ordinal less than or equal to ω_1. An iteration of $(M,(NS_{\omega_1})^M)$ of length γ consists of models M_α ($\alpha \leq \gamma$), sets G_α ($\alpha \leq \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta} : M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that*

1. *$M_0 = M$,*
2. *each G_α is an M_α-generic filter for $(P(\omega_1)/(NS_{\omega_1}))^{M_\alpha}$,*
3. *each $j_{\alpha\alpha}$ is the identity mapping,*
4. *each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α,*
5. *for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system $\{M_\alpha, j_{\alpha\delta}; \alpha \leq \delta < \beta\}$, and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.*

We say M is iterable if every iterate of $(M,(NS_{\omega_1})^M)$ is wellfounded.

We need the following lemma:

Lemma 4.1. *Assuming that for all countable $X \prec H(\omega_2)$, the transitive collapse of X is iterable, then NS_{ω_1} is not lightface Δ_1 definable.*

Proof. We argue by contradiction. Fix a Σ_1-formula $\phi(x)$ such that for all stationary set S, $H(\omega_2) \models \phi(S,\omega_1)$ if and only if S is stationary. Now fix a stationary co-stationary set S. Let X be such that $S \in X$ and M be the transitive collapse of X. Let S^M be the image of S under collapsing map. Let $\langle M_\alpha, G_\alpha, j_{\alpha\beta} \mid \alpha < \beta \leq \omega_1\rangle$ be an ω_1-length iteration of $M_0 = M$ such that

$$(\forall \alpha \in \omega_1) j_{0\alpha}(S^M) \notin G_\alpha.$$

By elementarity, M_{ω_1} thinks $j_{0\omega_1}(S^M)$ is a stationary set and $\psi(j_{0\omega_1}(S^M))$ holds. Moreover, $j_{0\omega_1}(S^M)$ is nonstationary. Since M_{ω_1} is transitive such that $M_{\omega_1} \subseteq H(\omega_2)$, $H(\omega_2) \models \psi(j_{0\omega_1}(S^M))$ by Σ_1-absoluteness. Hence $j_{0\omega_1}(S^M)$ is stationary. Contradiction. □

[6]This was recently answered affirmatively by the first author and Stefan Hoffelner and will appear in Hoffelner's 2014 PhD thesis at the University of Vienna.

Corollary 4.1. *Assume that there is a proper class of Woodin cardinals. Then there is no set-forcing notion* $\mathbb{P}$ *such that* $\mathbb{P} \Vdash NS_{\omega_1}$ *is lightface* Δ_1 *definable.*

Proof. Since AD holds in $L(\mathcal{R})$, it follows from the analysis of $\mathbb{P}_{max}$ extension that for all countable $X \prec H(\omega_2)$, the transitive collapse of X is iterable. By Lemma 4.1, $L(\mathcal{R})^{\mathbb{P}_{max}} \models NS_{\omega_1}$ is not lightface Δ_1-definable. Now suppose that $\mathbb{P} \Vdash \psi(x)$ is a lightface Π_1-definition of NS_{ω_1} over $\langle H(\omega_2), \in\rangle$. Let ϕ be the sentence $\forall S(S \in NS_{\omega_1} \leftrightarrow \psi(S))$. Then $\langle H(\omega_2), NS_{\omega_1} \in\rangle^{V^{\mathbb{P}}} \models \phi$. Since there are proper class many Woodin cardinals, by Π_2-maximality of $\mathbb{P}_{max}$ (cf Theorem 4.69 of [11]), $\langle H(\omega_2), NS_{\omega_1} \in\rangle^{L(\mathcal{R})^{\mathbb{P}_{max}}} \models \phi$. This contradicts to the fact that NS_{ω_1} is not lightface Π_1 definable in this model. □

Corollary 4.2. *Assume* NS_{ω_1} *is saturated and* $P(\omega_1)^\sharp$ *exists then* NS_{ω_1} *is not lightface* Δ_1 *definable. In particular, MM implies* NS_{ω_1} *is not lightface* Δ_1 *definable.*

5. Open Question

The forcings defined in [3], [6] only give the Δ_1 definability of some restriction of the nonstationary ideal whereas [4] gives the Δ_1 definability of NS_κ only for successor cardinals κ. It is therefore natural to ask:

Question 5.1. *Assume GCH and suppose that there is a proper class of weak compacts. Is there a cardinal-preserving forcing which preserves weak compactness and forces that for uncountable regular* κ, NS_κ *is* Δ_1*-definable over* $H(\kappa^+)$ *iff* κ *is not weak compact?*

References

[1] James Cummings. Iterated forcing and elementary embeddings. In *Handbook of set theory*, volume II, pages 775–883. Springer, Dordrecht, 2010.
[2] M. Foreman, M. Magidor, and S. Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.
[3] Sy-David Friedman, Tapani Hyttinen, and Vadim Kulikov. Generalized descriptive set theory and classification theory. *to appear, Memoirs of the American Mathematical Society*, 2013.
[4] Sy-David Friedman, Liuzhen Wu, and Lyubomyr Zdomskyy. The Δ_1 definability of NS at successors. *submitted*, 2013.
[5] Noa Goldring. Woodin cardinals and presaturated ideals. *Ann. Pure Appl. Logic*, 55(3):285–303, 1992.
[6] Tapani Hyttinen and Mika Rautila. The canary tree revisited. *J. Symbolic Logic*, 66(4):1677–1694, 2001.
[7] T. Jech, M. Magidor, W. Mitchell, and K. Prikry. Precipitous ideals. *J. Symbolic Logic*, 45(1):1–8, 1980.
[8] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.
[9] Alan Mekler and Jouko Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *J. Symbolic Logic*, 58(3):1052–1070, 1993.
[10] Alan H. Mekler and Saharon Shelah. The canary tree. *Canad. Math. Bull.*, 36(2):209–215, 1993.
[11] W. Hugh Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, volume 1 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter GmbH & Co. KG, Berlin, revised edition, 2010.

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