FRAGMENTS OF KRIPKE–PLATEK SET THEORY AND THE METAMATHEMATICS OF α-RECURSION THEORY

SY-DAVID FRIEDMAN, WEI LI, AND TIN LOK WONG

KURT GÖDEL RESEARCH CENTER

UNIVERSITY OF VIENNA

WÄHRINGERSTRASSE 25

A-1090 WIEN

1. INTRODUCTION

Let M be a transitive set. We say M is a model of a *fragment of Kripke–Platek (*KP*) set theory*, if it satisfies the usual axiom of KP except for Foundation. We denote KP without Foundation by KP^-. The metamathematics of α-recursion theory leads us to the study of fragments of KP. More precisely, we study the following questions:

Is a theorem of α-recursion theory still valid without full foundation? How much foundation is required?

One of the motivations for the metamathematics of α-recursion theory is the metamathematics of classical recursion theory In the metamathematics of classical recursion theory, we have models of arithmetic with limited Induction, the analogue of Foundation. Paris and Kirby [?] gave a model-theoretic analysis of Induction. They showed that Σ_n Induction ($I\Sigma_n$) is strictly stronger than Σ_n Bounding $(B\Sigma_n)^1$, and that Σ_{n+1} Bounding is strictly stronger than Σ_n Induction. In fact, $B\Sigma_n$ is equivalent with $I\Delta_n$, by Slaman [?]. The metamathematics of classical recursion was started by Simpson, who made the observation that $I\Sigma_1$ is sufficient for the Friedberg–Muchnik theorem. Then Mytilinaios [?] proved that $I\Sigma_1$ is enough for a finite injury ($\mathbf{0}'$-priority) argument. Mytilinaios and Slaman [?] showed that $I\Sigma_2$ carries out an infinite injury ($\mathbf{0}''$-priority) argument. Surprisingly, $B\Sigma_2$ is sufficient for the Sacks Density theorem by Groszek and Slaman [?], though superficially, the construction in the Density theorem seems to be more complicated than infinite injury.

α-recursion theory studies the computational properties of L_α's such that $L_\alpha \models$ KP, i.e. such that α is *admissible*. Sacks and Simpson [?] showed that the Friedberg–Muchnik theorem is valid in L_α for every admissible ordinal α. The Splitting and Density theorems were established by Shore in [?] and [?]; they hold in every L_α such that α is admissible. The existence of a minimal pair is a typical example of an infinite injury argument in classical recursion theory. Yet, whether it is true in every admissible L_α is still open. Lerman, Sacks, Maass and Shore solved some cases. See [?, ?, ?].

$^1 B\Sigma_n$ implies that bounded quantifiers in front of a Σ_n formula do not change the complexity of that formula.

α-recursion theory has influenced the metamathemaics of classical recursion theory. A subset of L_α is said to be *regular* if its intersection with any α-finite set is still α-finite (where the α-finite sets are the elements of L_α). The idea of regular and non-regular sets originated from Sacks and Simpson's argument [**?**] and Shore's proof [**?**]. A *cut* is a particular non-regular set in the mematamathematics of classical recursion theory. The degree of a cut could be a minimal degree and also could form a minimal pair with some $\emptyset^{(n)}$. See [**?**]. Shore's blocking method [**?**, **?**] was introduced to solve the Splitting and Density problem in L_α. A similar method [**?**] in the mematamathematics of classical recursion theory is applied to solve the Splitting problem in $I\Sigma_1$.

There are many overlaps between the techniques and results of α-recursion and the metamathematics of classical recursion. Yet the reasons for this phenomenon are yet to be found. In this paper, our research is done in nonstandard models of set theory. These models are "between" those in nonstandard arithmetic and those in α-recursion theory. It is one attempt to search for the reason for the mysterious connections between these two areas.

The structure of the paper is as follows: Section 2 lists some basic definitions, axioms and propositions that are useful later. Section 3 applies these propositions to the Schröder-Bernstein theorem and shows that this theorem is provable in Π_1-Foundation. In Section 4 we discuss the L-hierarchy in models of fragments of KP and apply this hierarchy to separate Π_n-Foundation and Σ_n-Foundation. And Section 5 and Section 6 are devoted to the Friedberg-Muchnik theorem and the Splitting theorem respectively and prove they hold in any model of $KP^- + \Pi_1$-Foundation $+V = L$.

2. Preliminaries

2.1. **Fragments of** KP. Kripke–Platek set theory (KP) consists of the Extensionality, Foundation, Pairing and Union axioms together with Σ_0-Separation and Σ_0-Collection:

(i) Extensionality: $\forall x, y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y]$.
(ii) Foundation: If y is not a free variable in $\varphi(x)$, then $[\exists x \varphi(x) \rightarrow \exists x(\varphi(x) \land \forall y \in x \neg \varphi(y))]$.
(iii) Pairing: $\forall x, y \exists z(x \in z \land y \in z)$.
(iv) Union: $\forall x \exists y \forall z \in x \forall u \in z(u \in y)$.
(v) Σ_0-Separation: $\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \land \varphi(z)))$ for each Σ_0 formula φ.
(vi) Σ_0-Collection: $\forall x[(\forall y \in x \exists z \varphi(y, z)) \rightarrow \exists u \forall y \in x \exists z \in u \varphi(y, z)]$ for each Σ_0 formula φ.

Here, Σ_0 formulas have only bounded quantifiers.

KP does not contain the Infinity axiom. If it is necessary for our theorems, then we will state the Infinity axiom explicitly.

(vii) Infinity: $\exists x[\emptyset \in x \land \forall y \in x(y \cup \{y\} \in x)]$.

Foundation is the dual of Induction.

(viii) Induction: If y is not a free variable in $\varphi(x)$, then $[\forall x(\forall y \in x \varphi(y)) \rightarrow \varphi(x)] \rightarrow (\forall x \varphi(x))$.

Clearly, for every class Γ of formulas, Γ-Induction holds if and only if $\neg\Gamma$-Foundation holds, where $\neg\Gamma = \{\neg\varphi : \varphi \in \Gamma\}$.

We use KP^- to denote KP without Foundation (i.e. Clauses (i), (iii)–(vi)). By *fragments* of KP, we mean systems obtained from KP by restricting the foundation scheme.

Proposition 2.1. KP^- *proves the following:*

(1) Strong Pairing: $\forall x, y \exists z \, (z = \{x, y\})$.
(2) Strong Union: $\forall x \exists y \, (y = \bigcup x)$.
(3) Δ_1-*Separation and* Σ_1-*Collection.*
(4) Strong Σ_1-*Collection: Suppose f is a Σ_1 function. If $\mathsf{dom}(f)$ is a set, then $\mathsf{ran}(f)$ and $\mathsf{graph}(f)$ are sets.*
(5) Ordered Pair: $\forall x, y \exists z \, (z = (x, y))$.
(6) Cartesian Product: $\forall x, y \exists z \, (z = x \times y)$.

Proof. The usual proofs [**?**, Sections I.3 and I.4] work in KP^-. $\square$

2.2. The Lévy Hierarchy. In Proposition 2.1, Δ_1 and Σ_1 are as defined in the Lévy Hierarchy. In the Lévy Hierarchy, we usually consider *normalized* formulas, that is, formulas in the form of $Q_0 v_0 \ldots Q_{n-1} v_{n-1} \varphi$, where (a) $Q_0, \ldots, Q_{n-1}$ are alternating quantifiers, (b) $v_0 \ldots v_{n-1}$ are variables, and (c) φ is Δ_0, or equivalently, φ has only bounded quantifiers.

The Collection principle says that normalized formulas are closed under bounded quantification. Without full collection, say in KP or KP^-, such closure properties may be lost. This problem is more related to Collection than to Foundation.

Definition 2.2. We define the $*$-hierarchy of formulas here. Suppose $m \leqslant n$ are natural numbers.

$$\Sigma_0^* = \Pi_0^* = \Sigma_0 \qquad \neg \Sigma_n^* \subseteq \Pi_n^* \qquad \neg \Pi_n^* \subseteq \Sigma_n^*$$

$$\Sigma_n^* \wedge \Sigma_m^* \subseteq \Sigma_n^* \qquad \Pi_n^* \wedge \Pi_m^* \subseteq \Pi_n^* \qquad \Sigma_{n+1}^* \wedge \Pi_m^* \subseteq \Sigma_{n+1}^* \qquad \Pi_{n+1}^* \wedge \Sigma_m^* \subseteq \Pi_{n+1}^*$$

$$(\exists x \in y \Sigma_n^*) \subseteq \Sigma_n^* \quad (\exists x \in y \Pi_n^*) \subseteq \Pi_n^* \quad (\forall x \in y \Sigma_n^*) \subseteq \Sigma_n^* \quad (\forall x \in y \Pi_n^*) \subseteq \Pi_n^*$$

$$(\exists x \Sigma_n^*) \subseteq \Sigma_n^* \qquad (\exists x \Pi_n^*) \subseteq \Sigma_{n+1}^* \qquad (\forall x \Sigma_n^*) \subseteq \Pi_{n+1}^* \qquad (\forall x \Pi_n^*) \subseteq \Pi_n^*$$

A Σ_n^* (Π_n^*, resp.) formula is *normalizable* if it is equivalent to a Σ_n (Π_n, resp.) formula. KP^- proves that Σ_1^* formulas are normalizable. However, even assuming KP, there may still be a Σ_2^* formula that is not normalizable.

Proposition 2.3 (KP^-)**.** *Suppose φ and ψ are normalized formulas. Then*

(1) $\neg \varphi$, $\varphi \wedge \psi$ *and* $\varphi \vee \psi$ *are normalizable.*
(2) If φ is Σ_n (Π_n, resp.), then $\exists x \, \varphi$ and $\exists x \in y \, \varphi$ ($\forall x \, \varphi$ and $\forall x \in y \, \varphi$, resp.) are normalizable.

Proposition 2.4. $\mathrm{KP}^- + \Sigma_n$-*Collection* $\vdash$ *for any Σ_m (Π_m, resp.) formula φ, $m \leqslant n$, $\forall x \in y \varphi$ ($\exists x \in y \varphi$, resp.) is normalizable.*

Proof. For $m = 0$, it is straightforward. Now suppose $n \geqslant m > 0$, and the statement is true for $m-1$. Also, suppose u is a new variable and φ is in the form of $\exists v \psi$ ($\forall v \psi$, resp.), where ψ is normalized Π_{m-1} (Σ_{m-1}, resp.). Then $\forall x \in y \varphi \equiv \forall x \in y \exists v \psi \equiv \exists u \forall x \in y \exists v \in u \psi$ ($\exists x \in y \varphi \equiv \exists x \in y \forall v \psi \equiv \forall u \exists x \in y \forall v \in u \psi$, resp.), by Σ_n-Collection. Since $\exists v \in u \psi$ ($\forall v \in u \psi$, resp.) is Π_{m-1} (Σ_{m-1}, resp.) normalizable, $\forall x \in y \varphi$ ($\exists x \in y \varphi$, resp.) is normalizable. $\square$

Corollary 2.5. $\mathrm{KP}^- + \Sigma_n$-*Collection* $\vdash$ *all* Σ_n^* *and* Π_n^* *formulas are normalizable. In particular, assuming* KP^-, *every* Σ_1^* *or* Π_1^* *formula is respectively equivalent to a* Σ_1 *or* Π_1 *formula.*

3. Transfinite Induction and the Schröder–Bernstein Theorem

In this section, we move to the semantic aspects of fragments of KP. From now on, we always assume $M \models \mathrm{KP}^-$. And if $x \in M$, then we say x is M-*finite*.

Definition 3.1. $\alpha \in M$ is an *ordinal* if α is transitive and linearly ordered by $\in$. An ordinal of the form $\alpha \cup \{\alpha\}$, where α is an ordinal, is a *successor*. An ordinal λ is *limit* if it is nonempty and not a successor. If α is zero or a successor and no $\beta \in \alpha$ is limit, then α is *finite*.

Note that an ordinal in M must be M-finite but it may not be finite. We use Ord^M to denote the class of ordinals in M and use $<$ to denote $\in$ on the ordinals. With Σ_0-Foundation, it is possible to develop the basic properties of ordinals.

Proposition 3.2 ($\mathrm{KP}^- + \Sigma_0$-Foundation). *(1)* $0 = \emptyset$ *is an ordinal.*
(2) If α *is an ordinal, then* $\beta \in \alpha$ *is an ordinal and* $\alpha + 1 = \alpha \cup \{\alpha\}$ *is an ordinal.*
(3) $<$ *is a linear order on the ordinals.*
(4) For every ordinal α, $\alpha = \{\beta : \beta < \alpha\}$.
(5) If C *is a nonempty set of ordinals, then* $\bigcap C$ *and* $\bigcup C$ *are ordinals,* $\bigcap C = \inf C = \mu\alpha(\alpha \in C)$ *and* $\bigcup C = \sup C = \mu\alpha(\forall \beta \in C(\beta \leqslant \alpha))$.

Proof. See Jech [**?**, Chapter 2] for the usual proofs. They go through in KP^-, as the reader can verify. □

Lemma 3.3. *If* $M \models$ Infinity, *then* M *has a limit ordinal. If* $M \models \Sigma_0$-Foundation *in addition, then* M *has a least limit ordinal* ω^M.

Proof. Suppose $x \in M$ is a set witnessing the Infinity axiom. Let $C = \{\alpha \in x : \alpha$ is an ordinal$\}$. Then $\lambda = \sup C$ is an ordinal by Proposition 3.2, so that for any $\beta < \lambda$, there is an $\alpha \in x$ such that $\beta \leqslant \alpha$. Since $\alpha + 2 = \alpha \cup \{\alpha\} \cup \{\alpha \cup \{\alpha\}\} \in C$, $\beta + 1 < \alpha + 2 \leqslant \lambda$. Hence, λ is limit. □

Theorem 3.4 (Transfinite Induction along the ordinals). *Suppose* $M \models \Pi_1$-Foundation *and* $I \colon M \to M$ *is a* Σ_1 *partial function. Then the partial function* $f \colon \mathsf{Ord}^M \to M, \delta \mapsto I(f \restriction \delta)$ *is well defined and* Σ_1. *Moreover, if for all ordinals* δ *and all* M-*finite functions* $\eta \colon \delta \to M$, *we have* $\eta \in \mathsf{dom}(I)$, *then* f *is total.*

Proof. f is Σ_1 definable:

$$f(\delta) = x \leftrightarrow \exists w \,(\, w \text{ is a function with domain } \delta \cup \{\delta\}$$
$$\text{such that } \forall \delta' \leqslant \delta \,[w(\delta') = I(w \restriction \delta') \wedge w(\delta) = x]).$$

Firstly, note that Σ_1 definable set $\mathsf{dom}(f)$ is downward closed and so by Π_1-Foundation, it is either Ord^M or an ordinal in M. Suppose $\delta \in \mathsf{Ord}^M$ and $x, x' \in M$ such that $f(\delta) = x \neq x' = f(\delta)$. Then we pick witnesses w for $f(\delta) = x$ and w' for $f(\delta) = w'$. By comparing w and w', we find the least $\delta' \leqslant \delta$ such that $w(\delta') \neq w'(\delta)$. However, this contradicts the fact that $w \restriction \delta' = w' \restriction \delta'$. Hence, f is a function.

Now we suppose that $\mathsf{dom}(f)$ is not Ord^M but for all ordinals δ and all M-finite functions $\eta \colon \delta \to M$, we have $\eta \in \mathsf{dom}(I)$. Pick the least ordinal $\delta \notin \mathsf{dom}(f)$. Then $\forall \delta' < \delta \exists x'(f(\delta') = x')$. By Proposition 2.1, $\mathsf{graph}(f)$ exists. Thus, $f(\delta)$ is also defined. This is a contradiction. □

In most popular proofs of the Schröder–Bernstein theorem, for example, that in Jech [**?**, Theorem 3.2], we obtain the required bijection by an induction on ω. Such proofs normally go through in $\mathrm{KP}^- + \Pi_1$-Foundation+Infinity. Without the Axiom of Infinity, the proof breaks down because ω, although still Δ_0-definable, can no longer be used to bound quantifiers. Therefore, although the Schröder–Bernstein theorem is provable in $\mathrm{KP}^- + \Pi_1$-Foundation alone, apparently a separate argument is needed when Infinity fails.

We reduce the $\neg$Infinity case to arithmetic, in which the situation is well-known. The key to this reduction is a Σ_1-definable bijection between the universe and the ordinals, defined by $\in$-induction. As observed in Kaye–Wong [**?**], this requires the existence of *transitive closures*. Recall the *transitive closure* of a set x, denoted by $\mathrm{TC}(x)$, is the smallest transitive set that includes x.

Lemma 3.5. $\mathrm{KP}^- + \Pi_1$-Foundation $\vdash \forall x \, \exists y \, \mathrm{TC}(x) = y$.

Proof. Follow Lemma 5.3 and Proposition 5.4 in Kaye–Wong [**?**]. $\qquad\square$

Theorem 3.6 (Transfinite $\in$-induction)**.** *Let* $M \models \mathrm{KP}^- + \Pi_1$-*Foundation, and* $I \colon M \to M$ *that is* Σ_1-*definable. Then there exists a* Σ_1-*definable* $f \colon M \to M$ *satisfying* $f(x) = I(f \restriction x)$ *for every* $x \in M$.

Proof. Similar to that of Theorem 3.4. Transitive closures are used to show that such an f is total. $\qquad\square$

(The inverse of) the following bijection between the universe and the ordinals originates from Ackermann [**?**].

Theorem 3.7. *Let* $M \models \mathrm{KP}^- + \Pi_1$-*Foundation* $+ \neg$*Infinity. Then*

$$f(x) = \sum_{y \in x} 2^{f(y)}$$

defines a bijection $f \colon M \to \mathsf{Ord}^M$ *with a* Σ_1 *graph.*

Proof. A standard application of $\in$-induction shows the functionality and totality of f. The failure of the Infinity Axiom contributes to the injectivity of f. If $\alpha \in \mathsf{Ord}^M$, then $f(\mathrm{Ack}(\alpha)) = \alpha$, where

$$\mathrm{Ack}(\alpha) = \{\mathrm{Ack}(\beta) : \exists \gamma{<}\alpha \; \exists \delta{<}2^\beta \; \alpha = (2\gamma + 1)2^\beta + \delta\},$$

defined by induction on the ordinals. $\qquad\square$

In a sense, this theorem shows that $\neg$Infinity is a strong assumption over $\mathrm{KP}^- + \Pi_1$-Foundation, because it implies the Power Set Axiom, Π_1-Separation, the Axiom of Choice, and $\mathrm{V} = \mathrm{L}$. The Schröder–Bernstein theorem also follows as promised.

Theorem 3.8 ($\mathrm{KP}^- + \Pi_1$-Foundation)**.** *Let* A, B *be sets. If there are injections* $A \to B$ *and* $B \to A$, *then there is a bijection* $A \to B$.

Proof. We already mentioned that most standard proofs go through in $\mathrm{KP}^- + \Pi_1$-Foundation $+$ Infinity. So suppose $M \models \mathrm{KP}^- + \Pi_1$-Foundation $+ \neg$Infinity, and $f \colon M \to \mathsf{Ord}^M$ is the bijection given by Theorem 3.7. Take $A, B \in M$. Suppose M contains injections $A \to B$ and $B \to A$.

With Π_1-Foundation in M, we know $\mathsf{Ord}^M \models \mathrm{I}\Pi_1$ as a model of arithmetic. Via f, we may view A and B as (arithmetically) coded subsets of Ord^M. Apply $\mathrm{I}\Delta_0 + \exp$ in Ord^M to find $\alpha, \beta \in \mathsf{Ord}^M$ that are respectively bijective with A and B

in M. The hypotheses imply that there are injections $\alpha \to \beta$ and $\beta \to \alpha$ coded in Ord^M. So by the coded version of the Pigeonhole Principle, which is available in all models of $I\Delta_0$, we conclude $\alpha = \beta$. It follows that A is bijective with B. $\square$

4. THE CONSTRUCTIBLE UNIVERSE

4.1. **Basic Properties.** In this section, M always satisfies $\mathrm{KP}^- + \Pi_1$-Foundation. By a transfinite induction, we may define L^M along Ord^M:

$$L_0^M = \emptyset,$$
$$L_{\alpha+1}^M = L_\alpha^M \cup \mathrm{Def}^M(L_\alpha^M),$$
$$L_\lambda^M = \bigcup_{\alpha < \lambda} L_\alpha^M \quad \text{where } \lambda \text{ is limit.}$$

Here, $\mathrm{Def}^M(x)$ denotes the collection of all definable subsets of x in the sense of M. Let $L^M = \bigcup_{\alpha \in \mathsf{Ord}^M} L_\alpha^M$.

If Infinity holds, then we may define the function Def^M as usual. If Infinity fails, then we get Def^M using the power set axiom and Π_1-Separation given by Theorem 3.7.

Lemma 4.1 ($\mathrm{KP}^- + \Pi_1$-Foundation). *The predicate "$x \models \varphi[\vec{v}/\vec{a}]$", where x is a set, φ is a formula (in the sense of the model) and $\vec{a}$ is a sequence of sets, is Δ_1. We denote this relation by* $\mathsf{Sat}(\ulcorner \varphi \urcorner, x, \vec{a})$.

Proof. (Sketch) "$x \models \varphi[\vec{v}/\vec{a}]$" ("$x \not\models \varphi[\vec{v}/\vec{a}]$", resp.) if and only if we have an M-finite function which assigns triples ($\ulcorner \varphi' \urcorner, x', \vec{a}'$) a truth value according to the usual definition of truth such that ($\ulcorner \varphi \urcorner, x, \vec{a}$) is assigned to be true (false, respectively). From this point of view, we cannot get a conflicting truth assignment for a triple ($\ulcorner \varphi \urcorner, x, \vec{a}$). This is proved by Σ_0-Foundation on the witnesses in the above definition. Also, we may show that for every triple, we may give a truth value. This is because if not, then by Π_1-Foundation, we may pick the formula with the least length so that this claim fails (finite sequences of a set form a set), deriving a contradiction. $\square$

Note that ω^M may not be the standard ω. Nevertheless, the notion of Σ_n formulas, where n is a standard positive natural number, is absolute, in the sense that the formulas recognized in M as Σ_n are all equivalent in M to some standard Σ_n formulas. That is because we have a universal Σ_1 formula that is standard, so that in a nonstandard Σ_n formula, we may code its Δ_0 matrix into a standard Σ_1 formula, if n is odd; and we may code it into a Π_1 formula, if n is even.

Definition 4.2 ($\mathrm{KP}^- + \Pi_1$-Foundation). $V = L$ stands for $\forall x \exists \alpha (x \in L_\alpha)$.

Lemma 4.3 ($\mathrm{KP}^- + \Pi_1$-Foundation $+ V = L$). *There are*

(a) *a universal Σ_1 formula;*
(b) *a universal Σ_1 function, i.e., a recursive enumeration of all Σ_1 partial functions, and*
(c) *a universal Turing functional, i.e., a recursive enumeration of all the codes for oracle computations.*

Proof. Note that there is an effective enumeration of Σ_1 formulas (in the sense of the model). As in Corollary 4.1, we may define the universal Σ_1 formula, cf. Section 3.1 in Barwise [?]. For a universal Σ_1 function, given an index, we enumerate ordered

pairs (x, y) such that no (x, y') has appeared earlier. One can define a universal Turing functional similarly. □

However, this is not the full picture of formulas within M. We can have a Σ_n formula for a nonstandard natural number $n \in M$. Also, we have limited collection. Thus, it is possible that we have a Σ_n^* formula, where $n \in \omega$ is standard, that is not equivalent to a Σ_n formula.

Theorem 4.4 (KP$^-$ + Π_1-Foundation). *For every ordinal α, $L_\alpha^M \in M$. The function $\alpha \mapsto L_\alpha^M$ is Δ_1.*

Proposition 4.5 (KP$^-$ + Π_1-Foundation). *For every ordinal α, L_α^M is transitive and $L_\alpha^M \cap \mathsf{Ord}^M = \alpha$.*

Theorem 4.6. *If $M \models$ KP$^-$ + Π_1-Foundation, then $L^M \models$ KP$^-$ + Π_1-Foundation.*

Proof. We only need to check Π_1-Foundation and Σ_0-Collection. Pick any Π_1 formula $\forall w \varphi(x, w)$, where $\varphi \in \Delta_0$. Suppose there is an $x \in L^M$ such that $\forall w \in L^M \varphi(x, w)$. The set $\{y \in L_\alpha^M : \forall w \in L^M \varphi(y, w)\}$ is Π_1 for every ordinal α. By Π_1-Foundation in M, it is either empty or has a $\in$-least witness. Hence, L^M satisfies Π_1-Foundation.

To check Δ_0-Collection, we fix any Δ_0-formula $\psi(y, w)$ with parameters from L^M and $x \in L^M$. Suppose $\forall y \in x \exists w \in L^M \psi(y, w)$. Then we may check the L-rank of the witnesses (i.e. the least α such that $w \in L_\alpha^M$). Then $\forall y \in x \exists \alpha \exists w \in L_\alpha^M \psi(y, w)$. By Σ_1-Collection of M, there is a searching bound $\alpha^* \in M$ such that $\forall y \in x \exists \alpha < \alpha^* \exists w \in L_\alpha^M \psi(y, w)$. Therefore, $L_{\alpha^*}^M$ is the searching bound for the witness w for all $y \in x$. □

If $M \models$ KP$^-$ + Π_1-Foundation, then $L^M \models V = L$.

Lemma 4.7. *Suppose $M \models$ KP$^-$ + Π_1-Foundation + $V = L$. Then there exists a Δ_1 bijection $M \to \mathsf{Ord}^M$ that preserves the relation $\in$.*

Corollary 4.8. *Let $M \models$ KP$^-$ + Π_1-Foundation + $V = L$. Then there exists a Δ_1-definable linear order $<_{\mathrm{L}}$ on M such that M satisfies*

- *$\forall s \left(\exists x \ (x \in s) \to \exists x \ \left(x \in s \wedge \forall x' <_{\mathrm{L}} x \ (x' \notin s) \right) \right)$;*
- *$\forall x, y \ (x \in y \to x <_{\mathrm{L}} y)$; and*
- *$\forall \alpha \in \mathsf{Ord} \ \forall x \in \mathrm{L}_\alpha \ \forall v <_{\mathrm{L}} x \ (v \in \mathrm{L}_\alpha)$.*

4.2. Recursive Ordinals in Models of KP$^-$.

Lemma 4.9. *If $M \models$ KP$^-$ + Π_1-Foundation + Infinity, and $\omega^M = \omega$, then every recursive ordinal is in M.*

Proof. For the sake of a contradiction, consider the least recursive ordinal not in M and suppose Φ_k, $k < \omega$ codes a well ordering of ω isomorphic to this recursive ordinal. Then Φ_k together with its ordering is M-finite. Let $<_{\Phi_k}$ denote this ordering. Define a Σ_1 function $f : \omega \to \mathsf{Ord}^M$ as follows:

$f(n) = \gamma \leftrightarrow$

there is an order isomorphism between γ and $\{m < \omega : m <_{\Phi_k} n\}$.

If f is total, then the order type of Φ_k is in M, leading to a contradiction. Otherwise, suppose n is $<_{\Phi_k}$-least such that $n \notin \mathsf{dom}(f)$. Since $\mathsf{dom}(f) = \{m < \omega : m <_{\Phi_k} n\}$ is M-finite, $\mathsf{graph}(f)$ is M-finite. It follows that f is an isomorphism between $\mathsf{dom}(f)$ and $\mathsf{ran}(f)$, contradicting with $n \notin \mathsf{dom}(f)$. □

We may generalize recursive ordinals as in the following.

Definition 4.10 (KP^- + Π_1-Foundation + Infinity). An ordinal α is *recursive* if there is a Σ_1 (in the language of arithmetic with parameters in ω^M) linear ordering of ω^M with respect to which ω^M is order isomorphic to α.

Suppose there is an ordinal that is nonrecursive, then there must be a least one by Π_1-Foundation. We denote this ordinal by $\omega_1^{\mathsf{CK},M}$. In this case, it is straightforward to check that $L_{\omega_1^{\mathsf{CK},M}}^M$ satisfies full foundation. However, it is not clear whether it satisfies Δ_0-Collection.

Question 4.11. *If $M \models \text{KP}^- + \Pi_1$-Foundation and $\omega_1^{\mathsf{CK},M}$ exists, then is $L_{\omega_1^{\mathsf{CK},M}}^M \models \Delta_0$-Collection?*
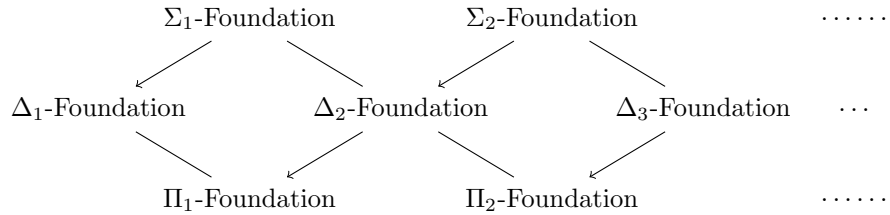
If every ordinal in Ord^M is recursive, then we say $\omega_1^{\mathsf{CK},M} = \mathsf{Ord}^M$. If we repeat the argument in Theorem 4.14 with a nonstandard model M such that its standard part of M is $L_{\omega_1^{\mathsf{CK}}}$, then the cut I could be chosen to consist of recursive ordinals in the sense of M. Using this cut I with the proof of Theorem 4.14, we obtain a model of $\text{KP}^- + \Pi_1$-Foundation but not full foundation such that every ordinal is recursive. Thus, $\text{KP}^- + \Pi_1$-Foundation + every ordinal is recursive does not imply full foundation.

4.3. **Collection, Separation and Foundation.** Collection, Separation and Foundation are closely related to each other. An immediate observation is that $\text{KP}^- + \Gamma$-Separation $+\Sigma_0$-Foundation, together with the existence of transitive closures, implies Γ-Foundation. A less obvious result is the following:

Lemma 4.12 (KP^-). *For every standard natural number n, Σ_n-Collection $\vdash \Delta_n$-Separation.*

Proof. We prove this by induction on n. Suppose we have proved the conclusion for n and Σ_{n+1}-Collection holds. Assume $\exists y \varphi(x,y)$ and $\exists y \psi(x,y)$ are formulas such that (1) φ and ψ are Π_n, (2) $\forall x \in z \, \exists y \, (\varphi(x,y) \vee \psi(x,y))$, and (3) $\neg \exists x \in z \, \exists y \, (\varphi(x,y) \wedge \psi(x,y))$. Then Σ_{n+1}-Collection implies that there is a b such that $\forall x \in z \, \exists y \in b \, (\varphi(x,y) \vee \psi(x,y))$. By Δ_n-Separation, $z' = \{x \in z : \exists y \in b \, (\varphi(x,y))\}$ and $z'' = z \setminus z'$ are sets. (Here, we need $\Pi_n^* = \Pi_n$, which is implied by Σ_{n+1}-Collection.) This shows Δ_{n+1}-Separation. $\square$

Over KP^-, the following implications hold.

$$\Sigma_1\text{-Foundation} \qquad \Sigma_2\text{-Foundation} \qquad \cdots\cdots$$

$$\Delta_1\text{-Foundation} \qquad \Delta_2\text{-Foundation} \qquad \Delta_3\text{-Foundation} \quad \cdots$$

$$\Pi_1\text{-Foundation} \qquad \Pi_2\text{-Foundation} \qquad \cdots\cdots$$

We will use the L hierarchy to show the implications indicated by double arrows above do not reverse.

Lemma 4.13 (Ramón Pino [**?**, Theorem 1.28]). *Let $n \in \mathbb{N}$. Then $\text{KP}^- + \text{Infinity} + \Sigma_{n+1}$-Collection $+ \Pi_{n+1}$-Foundation $+ \text{V} = \text{L}$ proves the following statement.*

> *For every $\delta \in \mathsf{Ord}$, there exists a sequence $(\alpha_i)_{i \leqslant \delta}$ in which $\alpha_0 = 0$*
> *and $\alpha_{i+1} = \min\{\alpha > \alpha_i : \mathrm{L}_\alpha \preceq_n \mathrm{L}\}$ for each $i < \delta$.*

Proof. If $n = 0$, then the sequence we want is just $(\alpha)_{\alpha \leqslant \delta}$. So suppose $n > 0$. With Σ_1-Induction on ω, we have a Π_n formula Π_n-Sat for the satisfaction of Π_n-formulas. We can reflect this formula arbitrarily high up in the L-hierarchy thanks to Σ_{n+1}-Collection and Σ_{n+1}-Induction on ω. This implies there are arbitrarily large $\mathrm{L}_\alpha \preceq_n \mathrm{L}$. With Σ_{n+1}-Collection and Π_{n+1}-Foundation, we can iterate this along any ordinal. $\square$

Theorem 4.14 (Ressayre [**?**, Theorem 4.6]). $\mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}$-Collection $+$ Σ_{n+1}-Foundation $+ \mathrm{V} = \mathrm{L} \nvdash \Pi_{n+1}$-Foundation *for all $n \in \mathbb{N}$.*

Proof. Start with a countable $M \models \mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}$-Collection $+ \Pi_{n+1}$-Foundation $+\mathrm{V} = \mathrm{L}$ in which $\omega^M = \omega$ but Ord^M is not well-ordered. Take a non-standard $\delta \in \mathsf{Ord}^M$. Let $(\alpha_i)_{i \leqslant \delta + \delta}$ be a sequence of ordinals given by Lemma 4.13. As δ is nonstandard, there are continuum-many initial segments of Ord^M between δ and $\delta + \delta$. So there must be one that is not definable in M. Take any initial segment $I \subseteq \mathsf{Ord}^M$ with this property. We will prove that $K = \bigcup_{i \in I} \mathrm{L}_{\alpha_i}^M$ is the model we want.

Claim 4.14.1. $K \preceq_n M$.

Proof of claim. We show by induction on $m \leqslant n$ that $K \preceq_m M$. Clearly $K \preceq_0 M$ because K is a transitive substructure of M. Let $m < n$ such that $K \preceq_m M$. Pick any $\varphi(\bar{x}, \bar{z}) \in \Sigma_m$ and $\bar{c} \in K$. Assume $K \models \forall \bar{x} \, \varphi(\bar{x}, \bar{c})$. Find some $i \in I$ such that $\bar{c} \in \mathrm{L}_{\alpha_i}^M$. Let $\bar{x} \in \mathrm{L}_{\alpha_i}^M$ be arbitrary. Then $K \models \varphi(\bar{x}, \bar{c})$. Since $K \preceq_m M$ by the induction hypothesis, and $\mathrm{L}_{\alpha_i}^M \preceq_n M$, we know $\mathrm{L}_{\alpha_i}^M \models \varphi(\bar{x}, \bar{c})$. Hence $\mathrm{L}_{\alpha_i}^M \models \forall \bar{x} \, \varphi(\bar{x}, \bar{c})$. This transfers up to M by n-elementarity, completing the induction. $\dashv$

Claim 4.14.2. $K \models \Sigma_{n+1}$-Collection.

Proof of claim. Take $a, \bar{c} \in K$ and $\varphi(x, y, \bar{z}) \in \Pi_n$ such that

$$K \models \forall x \in a \, \exists y \, \varphi(x, y, \bar{c}).$$

Pick any $j \in \delta + \delta$ above I. Let $x \in a$. Then $K \models \varphi(x, y, \bar{c})$ for some $y \in K$. Since $K \preceq_n M$ and $\mathrm{L}_{\alpha_j}^M \preceq_n M$, the same is true when the satisfaction of φ is evaluated in $\mathrm{L}_{\alpha_j}^M$ instead. Therefore, by setting $b = \mathrm{L}_\alpha^M$ for some $\alpha < \alpha_j$ above I, we see that $\mathrm{L}_{\alpha_j}^M \models \exists b \, \forall x \in a \, \exists y \in b \, \varphi(x, y, \bar{c})$.

Since the choice of $j \in \delta + \delta$ above I was arbitrary, this underspills. Let $i \in I$ and $b \in \mathrm{L}_{\alpha_i}^M$ such that $a, \bar{c} \in \mathrm{L}_{\alpha_i}^M$ and $\mathrm{L}_{\alpha_i}^M \models \forall x \in a \, \exists y \in b \, \varphi(x, y, \bar{c})$. Notice since $\mathrm{L}_{\alpha_i}^M \preceq_n M$ and $K \preceq_n M$, we have $\mathrm{L}_{\alpha_i}^M \preceq_n K$. Therefore $K \models \forall x \in a \, \exists y \in b \, \varphi(x, y, \bar{c})$ too because $\mathrm{L}_{\alpha_i}^M$ is a transitive substructure of K. $\dashv$

This claim implies $K \models \Delta_{n+1}$-Separation $+ \Delta_{n+1}$-Foundation.

Notice if $n = 0$, then we do not have Π_1-Foundation in K. Thus Corollary 4.8 does not always apply to K. Nevertheless, the model M does satisfy Π_1-Foundation, and so K can still get the conclusions of Corollary 4.8 from M.

Claim 4.14.3. $K \models \Sigma_{n+1}$-Foundation.

Proof of claim. Let $\theta(v, x)$ be a Π_n-formula that may contain undisplayed parameters from K. Suppose

$$K \models \exists x\ \exists v\ \theta(v, x) \wedge \forall x\ \big(\exists v\ \theta(v, x) \to \exists x' {\in} x\ \exists v\ \theta(v, x')\big).$$

Fix any $x_0 \in K$ such that $K \models \exists v\ \theta(v, x_0)$. Let $\eta(k, x)$ be the formula

$(x)_0 = x_0$

$$\wedge\ \forall i {\in} k \left(\begin{array}{l} (x)_{i+1} \in (x)_i \wedge \exists \alpha {\in} \mathsf{Ord} \\ \left(\begin{array}{l} (x)_i \in \mathrm{L}_\alpha \\ \wedge\ \exists v {\in} \mathrm{L}_\alpha \left(\begin{array}{l} \theta(v, (x)_{i+1}) \\ \wedge\ \forall x' {<}_{\mathrm{L}} (x)_{i+1}\ \forall v' {\in} \mathrm{L}_\alpha\ (x' \in (x)_i \to \neg \theta(v', x')) \end{array} \right) \end{array} \right) \end{array} \right),$$

which is Σ_{n+1} over M by Corollary 4.8 and Claim 4.14.2.

We show $K \models \forall k {\in} \omega\ \exists x\ \eta(k, x)$ by an external induction on k. Suppose we already have $x_0, x_1, \ldots, x_k \in K$ satisfying the inductive conditions. Take any large enough $\alpha \in \mathsf{Ord}^K$ such that $K \models x \in x_k \wedge \theta(v, x)$ for some $v, x \in \mathrm{L}_\alpha^K$. Then we can set

$$x_{k+1} = \min\nolimits_{<_{\mathrm{L}}} \big\{x \in \mathrm{L}_\alpha^K : K \models x \in x_k \wedge \exists v {\in} \mathrm{L}_\alpha^K\ \theta(v, x)\big\}.$$

This minimum exists by Π_n-Separation.

Apply Σ_{n+1}-Collection to get $s \in K$ such that $K \models \forall k {\in} \omega\ \exists x {\in} s\ \eta(k, x)$. Define $f(k) = y$ to be

$$\exists x {\in} s\ \big(\eta(k, x) \wedge (x)_k = y\big),$$

which is Σ_{n+1} over M by Σ_{n+1}-Collection. It is not hard to verify that $K \models \forall k {\in} \omega\ \exists! y\ f(k) = y$. So the set

$$\{y \in \mathrm{TC}(x_0) : K \models \exists k {\in} \omega\ f(k) = y\}$$

is Δ_{n+1}-definable but has no $\in$-minimum element. This contradicts Δ_{n+1}-Foundation in K. ⊣

Notice that $K \models \mathrm{V} {=} \mathrm{L}$ because the L-hierarchies in M and K, being Δ_1-definable, coincide.

Claim 4.14.4. $K \not\models \Pi_{n+1}$-Foundation.

Proof of claim. If $n = 0$, then K contains δ but not $\delta + \delta$, so that Π_1-Foundation fails in K. Suppose $n > 0$. Then $\delta + \delta \in K$ by Π_1-Foundation, but there can be no sequence $(\beta_i)_{i \leqslant \delta + \delta}$ in which $\beta_0 = 0$ and $\beta_{i+1} = \min\{\beta > \beta_i : \mathrm{L}_\beta \preceq_n \mathrm{L}\}$ for each $i < \delta + \delta$, because $K \preceq_n M$. So Lemma 4.13 tells us K cannot satisfy Π_{n+1}-Foundation. ⊣

□

In particular, this theorem says that if $n \in \mathbb{N}$, then $\mathrm{KP}^- + \Sigma_{n+1}$-Foundation $\nvdash$ Π_{n+1}-Foundation. We do not see how to show this without invoking the much stronger Σ_{n+1}-Collection. The use of the Infinity Axiom is necessary, because $\mathrm{KP}^- + \Sigma_{n+1}$-Foundation $+ \neg$Infinity $\vdash \Pi_{n+1}$-Foundation, as is classically known in the context of arithmetic [?]. The use of $\mathrm{V} {=} \mathrm{L}$, however, is only superficial: we may as well work with L^M if $M \models \mathrm{V} \neq \mathrm{L}$ in the proof above. Also, we may repeat the proof of Theorem 4.14 with $V = L$ replaced by $V = L[R]$ for some real R.

Theorem 4.15. $\mathrm{KP}^- + \Sigma_{n+1}$-Collection $+ \Pi_{n+1}$-Foundation $+ \mathrm{V} {=} \mathrm{L} \vdash \Sigma_{n+1}$-Foundation *for all $n \in \mathbb{N}$.*

Proof. If Infinity holds, then the proof is the same as that of Claim 4.14.2, except that now, we can use Π_{n+1}-Foundation to show $\forall k \in \omega \, \exists x \, \eta(k, x)$. If Infinity fails, then apply the equivalence between III_{n+1} and $\mathrm{I}\Sigma_{n+1}$ in arithmetic [**?**] via the bijection given by Theorem 3.7. $\qquad\square$

Question 4.16. *Let $n \in \mathbb{N}$. Does KP^-, Infinity, Σ_{n+1}-Collection, plus Π_{n+1}-Foundation prove Σ_{n+1}-Foundation?*

4.4. **Level 1-KPL.**

Definition 4.17. *Level 1*-KPL denotes $\mathrm{KP}^- + \Pi_1$-Foundation $+ \, \mathrm{V} = \mathrm{L}$.

Notice Theorem 4.15 above implies Level 1-KPL $\vdash \Sigma_1$-Foundation.

Definition 4.18 (Level 1-KPL)**.** Let I be a bounded initial segment of ordinals. We say that I is a *cut*, if there is no least ordinal $\beta \notin I$.

Note in the above definition, though I is transitive and linearly ordered by $\in$, I is not an ordinal, as otherwise, I would become the least ordinal not in I.

Lemma 4.19 (Level 1-KPL)**.** *For all $n \geqslant 1$, Σ_n-Foundation holds if and only if there is no Π_n cut. The same is true for Π_n-Foundation and Σ_n cuts if Σ_n-Collection is additionally assumed.*

Proof. If there is a Π_n cut, then Σ_n-Foundation fails, clearly. Conversely, suppose Π_n-Induction fails. That is, there is a Π_n formula $\varphi(x)$ such that $\forall x[(\forall y \in x \varphi(y)) \to \varphi(x)]$ but for some x_0, $\neg\varphi(x_0)$ holds. Let $f \colon \mathsf{Ord}^M \to M$ be the recursive bijection in Lemma 4.7. Then we check that $\forall \alpha \in \mathsf{Ord}^M[(\forall \beta < \alpha \varphi(f(\beta))) \to \varphi(f(\alpha))]$, as f preserves $\in$ of M. Now we define $I = \{\alpha \in \mathsf{Ord}^M : \forall \beta < \alpha \varphi(f(\beta))\}$. Then I is bounded Π_n and there is no least ordinal not in I. Thus, I is a Π_n cut. $\qquad\square$

Lemma 4.20 (Level 1-KPL)**.** *Every M-finite set x has a cardinality $|x|$.*

Lemma 4.21 (Level 1-KPL)**.** *If δ is an infinite cardinal, then there is an order preserving bijection from δ into δ^2, where $(a, b) \prec (c, d)$ if and only if $\max(a, b) < \max(c, d) \vee (\max(a, b) = \max(c, d) \wedge a < c) \vee (\max(a, b) = \max(c, d) \wedge a = c \wedge b < d)$.*

Proof. For the sake of a contradiction, we assume that δ is the least cardinal that fails to have this property. We define the function by Σ_1 induction along the ordinals. Note that the maximum of the two coordinates of the image of α is no more than α for any $\alpha < \delta$. Thus, the domain of the function has to be greater than δ. Let the image of δ be (a, b), where $\max(a, b) < \delta$. Considering the order preserving bijection from $|\max(a, b)|^2$ and $|\max(a, b)|$, we can get a surjection from $|\max(a, b)|$ onto δ. That is a contradiction. $\qquad\square$

Corollary 4.22 (Level 1-KPL)**.** *Suppose δ is an infinite cardinal. Then $|\delta^2| = \delta$. Thus, for x and y satisfying $|x|, |y| \leqslant \delta$, the Cartesian product $x \times y$ and the set $x^{<\omega}$ of finite sequences of x are both of cardinality at most δ. Thus, for every infinite ordinal α, $|L_\alpha| \leqslant |\alpha|$.*

Proof. Let $|x| \leqslant \delta$. Consider the sequence $\{x_n\}_{n < \omega}$. Now, by Σ_1-Induction, $|x_n| \leqslant \delta$ for all $n < \omega$. Thus, $|x^{<\omega}| \leqslant |\delta \times \omega| \leqslant \delta$.

For the sake of a contradiction, assume that α is the least infinite ordinal such that there is no injection from L_α into α. If α is a successor ordinal with predecessor α', then $|L_\alpha| \leqslant |L_{\alpha'}{}^{<\omega} \times \omega| \leqslant |\alpha' \times \omega| \leqslant |\alpha'|$, which contradicts our assumption.

Thus, α is a limit ordinal. Since for any infinite $\beta < \alpha$, $|L_\beta| \leqslant |\beta| \leqslant |\alpha|$, there is a Σ_1 injection from L_α to $\alpha \times \alpha$. Thus, $|L_\alpha| \leqslant |\alpha|$, which again is a contradiction. $\square$

5. The Friedberg–Muchnik Theorem

In this section we will show the Friedberg–Muchnik Theorem in Level 1-KPL. Again M is a model of Level 1-KPL. The Sack–Simpson construction [?] in α-recursion theory uses the Σ_2-*cofinality* (of the ordinals), i.e., the least ordinal that can be mapped to a cofinal set of ordinals by a Σ_2 function, the existence of which apparently needs much more foundation than Level 1-KPL can afford.

Question 5.1. *Is there a model of Level 1-KPL with no Σ_n cofinality for some $n \geqslant 2$?*

Lemma 5.2 (Level 1-KPL)**.** *If there is a Σ_1 injection from the universe into an ordinal, then there is the least such an ordinal. It is called the Σ_1 projectum, denoted by $\sigma 1p(M)$, or $\sigma 1p$ for short.*

Proof. Suppose $\alpha \in M$ is an ordinal such that there is a Σ_1 injection from the universe into α. We claim $|\alpha| = \sigma 1p$. Clearly, there is a Σ_1 injection from the universe into $|\alpha|$. Conversely, if we have a Σ_1 injection p from M into $\beta \leqslant |\alpha|$, then $p \upharpoonright |\alpha|$ is M-finite and is an injection into β. As $|\alpha|$ is a cardinal in M, $\beta = \alpha$. $\square$

Similarly, we may define the Σ_2 *projectum* of M, $\sigma 2p(M)$, to be the least ordinal such that there is a Σ_2 injection from the universe into it. However, it is not known whether such a projectum exists.

Question 5.3. *Is there a model of Level 1-KPL with no Σ_2 projectum?*

Corollary 5.4 (Level 1-KPL)**.** *If $\sigma 1p(M)$ exists, then $\sigma 1p(M)$ is the largest cardinal in M.*

Corollary 5.5 (Level 1-KPL)**.** *If $\sigma 1p(M)$ exists, then every Σ_1 subset of an ordinal less than $\sigma 1p(M)$ is M-finite. If $\sigma 1p(M)$ does not exist, then every Σ_1 bounded subset of Ord^M is M-finite.*

Definition 5.6 (Level 1-KPL)**.** *Suppose δ is an ordinal. We say δ is (Σ_1) stable if L_δ is a Σ_1 elementary substructure of the whole model.*

Lemma 5.7 (Level 1-KPL)**.** *For every γ such that $\omega \leqslant \gamma < \sigma 1p$, there is a stable ordinal $\delta \geqslant \gamma$ with the same cardinality as γ.*

Proof. Let γ be an ordinal such that $\omega < \gamma < \sigma 1p$ and x be the set of all finite sequences of L_γ. Suppose $f \colon M \to \mathsf{Ord}^M$ is the bijection from Lemma 4.7 and $\{\varphi_e\}$ is a universal enumeration of all Σ_1 formulas as in Lemma 4.3.

Consider the set $y = \{(e, \vec{a}) : e \in \omega, \vec{a} \in x,$ the number of free variables in φ_e is equal to the dimension of $\vec{a}$ plus one$\}$. Note that $|y| \leqslant |\omega \times |x|| \leqslant \gamma < \sigma 1p$. Thus, any Σ_1 subset of y is M-finite.

Now we define a (partial) map $g \colon y \to M$ such that $(e, \vec{a}) \mapsto$ the least v (in the order of f) such that $\varphi_e(v, \vec{a})$ holds. As $\mathsf{dom}(g)$ is M-finite, so is $\mathsf{ran}(g)$.

Let $G = \mathsf{ran}(g)$. Then $|G| \leqslant |y| \leqslant |\gamma|$. Note that $L_\gamma \subset G$. (Then $x, y \subset G$). Thus, $|G| = |\gamma|$. Suppose φ is a Σ_1 formula (in the sense of M), and $\vec{a}$ is a finite sequence in G such that the number of free variables in φ is equal to the dimension of $\vec{a}$ plus one and $M \models \exists v \varphi(v, \vec{a})$. We claim that $G \models \exists v \varphi(v, \vec{a})$. To see this,

let $\vec{\varphi}$ be an M-finite sequence of Σ_1 formulas with parameters from L_γ. Then M, and thus G, is a model of $\exists v \exists \vec{a}[\varphi(v, \vec{a})$ and each coordinate of $\vec{a}$ satisfies the corresponding coordinate in $\vec{\varphi}]$. This yields that $G \prec_1 M$.

Now we define the Mostowski collapse c of G as follows:

$c(v) = z \leftrightarrow \exists \eta(\eta$ is a function such that

$$\forall v \in \mathsf{dom}(\eta)(\eta(v) = \{\eta(v') : v' \in v \cap G\}) \text{ and } \eta(v) = z)$$

Note that c is Σ_1 definable and $\mathsf{dom}(c) = G$ by Π_1-Foundation. Let $G' = \mathsf{ran}(c)$, which is M-finite.

For every $v, v' \in G$, $v \in v' \leftrightarrow c(v) \in c(v')$ by Π_1-Foundation. Also, if $M \models v \neq v'$, then $M \models v \triangle v' \neq \emptyset$ and so $G' \models c(v) \neq c(v')$. Hence c is an isomorphism. Thus, for every ordinal in G, its image in G' is still an ordinal. Thus, $G' \subset \bigcup_{\alpha \in \mathsf{Ord}^{G'}} L_\alpha$. Conversely, $G' \supset \bigcup_{\alpha \in G'} L_\alpha$, since G' is transitive and $G \models \forall$ ordinal α, L_α exists. Let δ be the least ordinal not in G'. Then $G' = L_\delta$.

Consider the function g. Note that for every $(e, \vec{a}) \in \mathsf{dom}(g)$, $c((e, \vec{a})) = (e, \vec{a})$, and $g(e, \vec{a})$ is the least witness for $\varphi_e(v, \vec{a})$. Thus, the same is still true in G'. For this reason, $G' = G$. $\qquad\square$

5.1. **Construction.** We will construct r.e. subsets A and B of the ordinals as in the classical case.

Lemma 5.8. *Given an r.e. set A, we have a recursive enumeration of A without repetition. I.e. there is a recursive 1-1 function f such that $\mathsf{dom}(f)$ is Ord^M or an ordinal in M, and $\mathsf{ran}(f) = A$.*

Let $p: \mathsf{Ord}^M \to \sigma1p$ be a Σ_1 injection and $\{\Phi_\epsilon : \epsilon \in \mathsf{Ord}^M\}$ is a uniform sequence of Σ_1 Turing functionals. Requirements are either $\Phi_\epsilon^A \neq B$ or $\Phi_\epsilon^B \neq A$ for some ordinal ϵ. Let $\{R_\epsilon : \epsilon \in \mathsf{Ord}^M\}$ be a Σ_1 enumeration of all requirements. We say R_ϵ has *higher priority* than $R_{\epsilon'}$, if $p(\epsilon) < p(\epsilon')$. At any stage γ,

- R_ϵ *requires attention* if $\epsilon < \gamma$, R_ϵ was not satisfied prior to stage γ, and for the corresponding witness, Turing machine, and the oracle known so far, the outcome of the computation on this witness is 0 and this witness is not in the scope of any restrictions of higher-priority requirements;
- R_ϵ *receives attention* if
  (1) it requires attention;
  (2) we enumerate the witness into the corresponding set; and
  (3) we put restrictions on the usage of the computation;
- R_ϵ is *initialized* if we erase the memories of all activities of R_ϵ by stage γ and assign a new witness for it;
- R_ϵ is *satisfied* if it received attention at some previous stage, and after that until the present stage, it has not been initialized.

Suppose we are at stage $\gamma \in \mathsf{Ord}^M$. Consider $\{R_\epsilon : \epsilon < \gamma\}$. If there is a requirement requiring attention, then we satisfy the one with the highest priority, say R_ϵ and initialize all requirements in $\{R_\epsilon : \epsilon < \gamma\}$ of lower priorities. If no requirement requires attention, then we initialize all R_ϵ (together with the lower-priority requirements) with $\epsilon < \gamma$ such that some injection is enumerated into $[0, p(\epsilon))$ exactly at this stage. Then one by one, for each requirement in $\{R_\epsilon : \epsilon < \gamma\}$ that has not been satisfied nor assigned a witness not in the scope of restrictions by higher priority requirements, we assign a new witness for it.

5.2. **Verification.**

Lemma 5.9 (Level 1-KPL). *Successor infinite cardinals are regular.*

Proof. Suppose δ is a successor cardinal, its predecessor cardinal is δ^- and $\{\alpha_i : i < \beta\}$ is an M-finite sequence of ordinals such that $\alpha_i, \beta < \delta$. Then there is a Σ_1, thus M-finite, bijection from $\{(x, i) : x \in \alpha_i, i < \beta\}$ into $(\delta^-)^2$. Thus, $|\bigcup\{\alpha_i, i < \beta\}| \leqslant \delta^-$. $\square$

Lemma 5.10 (Level 1-KPL + Infinity). *Suppose $\alpha < \delta$ and δ is a regular cardinal in M. If $\{X_i : i < \alpha\}$ is a uniform r.e. sequence of M-finite sets of ordinals with cardinality less than δ. Then $\bigcup\{X_i : i < \alpha\}$ is an M-finite set of cardinality less than δ.*

Proof. Without loss of generality, we assume that the X_i's are mutually disjoint. Define a function $\gamma \mapsto$ the first (and the least, if necessary) ordinal enumerated by $\{X_i : i < \alpha\}$ but not included in the image of the function restricted to γ. This is a Σ_1 function. If there is a $\delta' < \delta$ not in its domain, then we are done. Otherwise, consider the sequences in $\{$order type of $(X_i \restriction$ enumerated before $\delta) : i < \alpha\}$. They contradict the regularity of δ. $\square$

Lemma 5.11 (Level 1-KPL). *If there is no maximum cardinal, then the cardinals are cofinal in Ord^M.*

Proof. By Lemma 5.4, $\sigma 1 p$ does not exist. Thus, Lemma 5.5 yields that every bounded r.e. set of ordinals is M-finite. For the sake of a contradiction, suppose all cardinals are bounded by γ. Then the set $\{\alpha < \gamma : \alpha$ is not a cardinal$\}$ is a bounded r.e. set and so is M-finite. Thus, $C = \{\alpha < \gamma : \alpha$ is a cardinal$\}$ is M-finite as well. Let δ be the least ordinal not in C. Then $\delta \notin C$, but it is a cardinal. $\square$

Theorem 5.12 (Level 1-KPL). *If there is no maximum cardinal or $\sigma 1 p$ is in the model, then all requirements in the construction are satisfied.*

Proof. If at some stage R_ϵ is satisfied and never initialized afterwards, then we are done. Otherwise, let γ be a stage at which all elements in $\mathsf{ran}(p) \restriction p(\epsilon)$ have been enumerated.

Let $\{S_j\}$ be the enumeration of the requirements with higher priorities than R_ϵ and R_ϵ itself with priority ordering. Then this sequence is M-finite and of length less than a regular cardinal δ in the model. Now let $I_j = \{\alpha \leqslant$ order type of stages at which S_j is initialized or assigned a new witness$\}$. Then the sequence $\{I_j\}$ is uniformly enumerable. We claim that each I_j is M-finite and less than δ. Otherwise, let j be the least such that $I_j \supseteq \delta$. By Lemma 5.10, $\bigcup_{j' < j} I_{j'}$ is M-finite and less than δ. Let ξ be the least stage such that $\bigcup_{j' < j} I_{j'}$ has been enumerated completely. Then by stage ξ, I_j cannot be more than the order type of $1 + 2 \times \bigcup_{j' < j} I_{j'}$. After stage ξ, I_j is initialized at most once. Thus, I_j is no more than the order type of $1 + 2 \times \bigcup_{j' < j} I_{j'} + 3$, not containing δ as a subset.

Thus, after some stage γ', R_ϵ is never initialized nor assigned a new witnesses. If R_ϵ requires attention, then it would be the one with highest priority and is satisfied and never injured afterwards. Otherwise, the witness would show that R_ϵ is satisfied automatically. $\square$

5.3. **Modified Construction and its Verification.** Now we consider the case that $\sigma 1p$ is not in the model and there is the maximal cardinal. We denote the maximum cardinal by $\aleph$.

The set $\{\delta > \aleph : \delta$ is not stable$\}$ is an r.e. set and so is regular. At each stage s, we say that δ is *stable at stage* s if $\aleph < \delta < \aleph + 1 + s$ and according to the information up to $\aleph + 1 + s$, we think that δ is stable. Then $\delta > \aleph$ is stable if and only if there is a stage s such that for all stages $t \geqslant s$, δ is stable at stage t. In fact, by Lemma 5.5 and Σ_1-Collection, for any $\alpha > \aleph$, there is a stage s such that after stage s, our justification of the stability of any ordinal in $(\aleph, \alpha]$ will never change.

At stage s, let $\delta_1^s < \delta_2^s < \cdots < \delta_i^s < \cdots$ be an enumeration of all stable-at-stage-s ordinals greater than $\aleph$. Let $\delta_0^s = 0$. $[\delta_i^s, \delta_{i+1}^s)$ is called *block i at stage s*. Then for every ordinal α, there is a stage s such that after stage s, all blocks below α will not be changed. For every block i at stage s, let h_i^s be the least (in the order of L) M-finite injection from block i at stage s into $\aleph$. If $\delta_1^s < \cdots < \delta_{i+1}^s$ are not changed from stage s onwards, then so are block i and h_i^s.

We do the construction of A and B as in Section 5.1 with the following priority order:

R_ϵ has *higher priority* than $R_{\epsilon'}$ if there are a stage s and blocks $i \leqslant j$ which are not changed from stage s onwards, such that

(1) ϵ is in block i and ϵ' is in block j, and
(2) either $i < j$, or $i = j$ and $h_i^s(\epsilon) < h_i^s(\epsilon')$.

This priority order is not recursive. Yet, for every ordinal α, the priority order on the set $\{R_\epsilon : \epsilon < \alpha\}$ can be recursively approximated and from some stage onwards, the approximation gives a correct order on $\{R_\epsilon : \epsilon < \alpha\}$. At each stage, we do the construction via the approximation of the priority order.

Other parts of the construction are parallel to that in Section 5.1. The rest of this section will give a detailed description. Readers familiar with this can skip to the verification.

At stage s, we say that

- R_ϵ *requires attention* if
    (1) the least stable ordinal δ at stage s such that $\epsilon < \delta < s$ exists;
    (2) there is a stage $t < s$ such that $\{\alpha \leqslant \delta : \alpha$ is stable at stage $t\} = \{\alpha \leqslant \delta : \alpha$ is stable at stage $s\}$; and
    (3) R_ϵ was not satisfied prior to stage s and for the corresponding witness, Turing machine, and the oracle known so far, the outcome of the computation on this witness is 0 and that witness is not in the scope of any restrictions of higher-priority (according to our knowledge at stage s) requirements;
- R_ϵ *receives attention* if
    (1) it requires attention;
    (2) we enumerate the witness into the corresponding set; and
    (3) we put restrictions on the usage of the computation;
- R_ϵ is *initialized* if we erase the memories of all activities of R_ϵ by stage s and assign a new witness for it;
- R_ϵ is *satisfied* if it received attention at some previous stage, and after that until the present stage, it has not been initialized.

Suppose we are at stage $s \in \mathsf{Ord}^M$. Consider $\{R_\epsilon : \epsilon < s\}$. If there is a requirement requiring attention, then we satisfy the one with the highest priority,

say R_ϵ and initialize all requirements in $\{R_\epsilon : \epsilon < s\}$ of lower priorities. If no requirement requires attention, then we initialize all R_ϵ (together with the lower-priority requirements) with $\epsilon < s$, such that its block has been changed at this stage or its map into $\aleph$ is changed at this stage, i.e. no $t, \delta < s$ satisfy

(i) $\epsilon < \delta < t$,
(ii) δ is stable at stage s (and so stable at stage t), and
(iii) $\{\alpha \leqslant \delta : \alpha \text{ is stable at stage } t\} = \{\alpha \leqslant \delta : \alpha \text{ is stable at stage } s\}$.
(iv) If ϵ is in block i at stage t, then for every $t' \in [t, s]$, $h_i^{t'} = h_i^t$.

Lastly, one by one, for each requirement in $\{R_\epsilon : \epsilon < s\}$ that has not been satisfied nor assigned a witness not in the scope of restrictions by higher priority requirements, we assign a new witness for it.

The following lemma implies that every requirement is satisfied eventually.

Lemma 5.13 (Level 1-KPL). *Suppose for all $t \geqslant s_0$ and $j \leqslant i + 2$, $\delta_j^{s_0} = \delta_j^t$. We denote $\lim_s \delta_j^s$ by δ_j, $j \leqslant i + 2$. Let $I_\epsilon = \{s : R_\epsilon$ receives attention, is assigned a new witness, or is initialized at stage $s\}$. If $\epsilon \in [\delta_i, \delta_{i+1})$, then $I_\epsilon \in L_{\delta_{i+2}}$.*

Proof. Fix an i. By the stability of δ_{i+1}, for all $s > \delta_{i+1}$, $\{\delta \leqslant \delta_{i+1}^s : \delta$ is stable at stage $s\} = \{\delta_j : j \leqslant i + 1\}$. By Σ_1-Foundation, we may let $j \leqslant i$ be the least such that there is a requirement R_{ϵ_j} in block j, such that $I_\epsilon \notin L_{\delta_{j+2}}$. Without loss of generality, we may assume that $\epsilon = \epsilon_j$. Let $s_0 > \delta_{i+1}$ be the least such that $h_i^{s_0} = h_i$ is found. Then $s_0 < \delta_{i+2}$.

By inductive hypothesis, from stage s_0 onwards, all requirements in block $< i$ will not receive attention nor be initialized. For every ϵ in block i, we consider the set $I'_\epsilon = \{\alpha : \text{the order type of } I_\epsilon \setminus s_0 \text{ is no less than } \alpha\}$. Let $\delta \leqslant \aleph$ be any infinite regular cardinal. If for every ϵ in block i with priority order, restricted to block i, less than δ, $I'_\epsilon < \delta$, then we are done. Otherwise, let ϵ be the one with the highest priority in block i such that $I' \geqslant \delta$. Then $U = \bigcup\{I_{\epsilon'} \setminus s_0 : \epsilon'$ is in block i and has higher priority than $\epsilon\}$ is a union of fewer than δ many M-finite sets, each of cardinality less than δ. By Lemma 5.10, U is M-finite with cardinality less than δ. Thus, $\eta = \sup\{I'_{\epsilon'} : \epsilon'$ in block i and has higher priority than $\epsilon\} < \delta$, and so $I'_\epsilon \leqslant 3 \times \eta + 2 < \delta$. That is a contradiction. $\square$

6. The Splitting Theorem and the Blocking Method

In this section, we show the Sacks Splitting theorem in the setting of Level 1-KPL. We fix a regular nonrecursive r.e. set X and we will split X into two r.e. sets A and B such that

(1) $A \cup B = X$,
(2) $A \cap B = \emptyset$,
(3) $X \not\leqslant_T A$, and
(4) $X \not\leqslant_T B$.

To satisfy (1) and (2), we enumerate the elements in X one by one and put them into either A or B but not both. For (3) and (4), we deal with the requirements

$$P_e : \quad \Phi_e^A \neq X$$
$$Q_e : \quad \Phi_e^B \neq X$$

for all $e \in \mathsf{Ord}^M$.

For a single requirement, we apply the classical method of preserving computation. To settle all requirements, we adopt the blocking method as in α-recursion

theory. The problem is that, within Level 1-KPL, we may not have the Σ_2 cofinality of the Ord^M. Thus, here we use a modified version that came from arithmetic [**?**]. It is a modified version of that in α-recursion theory. Here, a block is determined by its previous actions: we only stop enlarging a block when the actions of all its previous blocks terminate. The next lemma says that each block either grows to infinity or reaches to a limit at some M-finite stage.

Lemma 6.1. *For any nondecreasing recursive sequence $\{\xi_s\}_s$, either it is cofinal in Ord^M (we denote this by $\lim_s \xi_s = \infty$) or there is a stage s such that for all $t > s$, $\xi_t = \xi_s$.*

Proof. Suppose $\{\xi_s\}_s$ is bounded in Ord^M. Let δ be the least ordinal such that for all s, $\xi_s \leqslant \delta$. We note that $\forall \delta' < \delta \exists s(\xi_s > \delta')$. Then Σ_1-Collection tells us there is a stage s_0 such that $\forall \delta' < \delta \exists s < s_0(\xi_s > \delta')$. Thus, $\xi_{s_0} = \delta$ and we are done. $\square$

6.1. **Construction.** Now we construct A and B stage by stage. We may pick an enumeration of X such that at each stage s, there is at most one element less than s enumerated into X. The set of elements enumerated into X before stage s is denoted by $X_{<s}$. Similarly, we use $A_{<s}$, $B_{<s}$, etc.

We say a requirement is a *P-requirement* (a *Q-requirement*, resp.) or of *P-type* (*Q-type*, resp.), if it is of the form $\Phi_e^A \neq X$ ($\Phi_e^B \neq X$, resp.). One essential principle in the blocking method is that there is only one type of requirements in any block.

Block α at stage s is $[0, h(\alpha, s))$, where $h(\alpha, s) =$

- 1, if $\alpha = 0$; (In the rest of the definition of h, we do not consider the case $\alpha = 0$.)
- $\alpha + 1$, if $s = 0$;
- some value δ to be specified in the construction such that $\delta \geqslant h(\alpha, t)$ for all $t < s$ and $\delta > h(\beta, s)$ for all $\beta < \alpha$, if $\alpha, s > 0$.

We say α is *even* if $\alpha = \gamma + 2n$ for some limit ordinal γ and some finite ordinal n. Otherwise, α is *odd*. We always assign P-requirements to even blocks and Q-requirements to odd blocks. More precisely, for instance, suppose α is even and *stable up to stage s*, i.e., there is $t < s$ such that for all stages $t' \in [t, s)$ and all $\beta \leqslant \alpha$, $h(\beta, t') = h(\beta, t)$. Then let *the αth requirement at stage s*, which we denote by R_α^s, be $\bigwedge \{P_\lambda : \lambda$ is in Block α at stage $s\}$. For ordinals in odd blocks, Q-requirements are assigned similarly.

Also, we define the *maximum common length of R_α^s*, denoted by $\mathbf{m}(\alpha, s)$, as follows: If there is a stage $t < s$ such that $h(\alpha, t) \geqslant s$, then let $\mathbf{m}(\alpha, s) = 0$. Otherwise, suppose α is even and e is in Block α up to stage s. Then

$$m(e, s) = \sup\{l < s : \Phi_e^{A_{<s}} \upharpoonright l = X_{<s} \upharpoonright l\}.$$

Correspondingly, the *reservation of R_e^s*, denoted by $r(e, s)$, is the least ordinal $r \leqslant s$ such that, $\Phi_e^{A_{<s} \upharpoonright r} \upharpoonright m(e, s) = X_{<s} \upharpoonright m(e, s)$. Similarly, define $m(e, s)$ and $r(e, s)$ using B instead of A when e is in an odd block up to stage s. For every block α, let $\mathbf{r}(\alpha, s) = \sup\{r(e, s) : e$ is in Block α up to stage $s\}$, and $\mathbf{m}(\alpha, s) = \sup\{m(e, s) : e$ is in Block α up to stage $s\}$.

At stage $s > 0$, let R_α^s, $\mathbf{m}(\alpha, s)$, $\mathbf{r}(\alpha, s)$ be defined as above. If no element is enumerated into X, then let $A_s = A_{<s}$, $B_s = B_{<s}$ and $h(\alpha, s) = \max\{\sup_{t<s} h(\alpha, t), \sup_{\beta<\alpha}(h(\beta, s) + 1)\}$ for all α, s.

Now suppose x is enumerated into X at stage s. Let $\alpha \leqslant s$ be the least such that $x < \mathbf{r}(\alpha, s)$. If no such α exists, then enumerate x into A and $h(\alpha, s) =$

$\max\{\sup_{t<s} h(\alpha, t), \sup_{\beta<\alpha}(h(\beta, s)+1)\}$ for all α, s. Otherwise, if the requirements in Block α are of P-type, then enumerate x into B; if the requirements in Block α are of Q-type, then enumerate x into A. Let

$$h(\beta, s) = \begin{cases} \max\{\sup_{t<s} h(\beta, t), \sup_{\gamma<\beta}(h(\gamma, s) + 1)\}, & \text{if } \beta \leqslant \alpha; \\ \max\{\sup_{t<s} h(\beta, t), \sup_{\gamma<\beta}(h(\gamma, s) + 1)\} + s, & \text{if } \beta \geqslant \alpha + 1. \end{cases}$$

That is, we keep blocks up to Block α, enlarge the next block by s and move the remaining markers accordingly.

6.2. **Verification.** By Π_1-Foundation, $h(\alpha, s)$ is defined for every s and α. And by the definition of h, for every fixed α, $h(\alpha, s)$ is nondecreasing with respect to s; for every fixed s, $h(\alpha, s)$ is strictly increasing with respect to α.

In the construction, we have seen that if

$(*)$ There is an x enumerated into X at exactly stage s, and there is an $\alpha \leqslant s$ such that $\beta \geqslant \alpha + 1$ and $x < \mathbf{r}(\alpha, s)$,

then $h(\beta, s) > \sup_{t<s} h(\beta, t)$. The following lemma states that the converse is also true.

Lemma 6.2. *If $h(\beta, s) > \sup_{t<s} h(\beta, t)$, then $(*)$ holds.*

Proof. Suppose $(*)$ fails. For the sake of a contradiction, assume that β is the least such that $h(\beta, s) > \sup_{t<s} h(\beta, t)$. Then $\sup_{\gamma<\beta}(h(\gamma, s) + 1) > \sup_{t<s} h(\beta, t)$. Thus, for some $\gamma_0 < \beta$, $h(\gamma_0, s) \geqslant \sup_{t<s} h(\beta, t)$. But $h(\gamma_0, s) = \sup_{t<s} h(\gamma_0, t)$. Therefore, $\sup_{t<s} h(\gamma_0, t) \geqslant \sup_{t<s} h(\beta, t)$. Since for all $t < s$, $h(\gamma_0, t) < h(\gamma, t)$, we have (1) s is limit; (2) $\sup_{t<s} h(\gamma_0, t) = \sup_{t<s} h(\beta, t)$; and (3) $\beta = \gamma_0 + 1$.

Then $h(\beta, s) = \max\{\sup_{t<s} h(\beta, t), h(\gamma_0, s)\} = \max\{\sup_{t<s} h(\beta, t), \sup_{t<s} h(\gamma_0, t)\} = \sup_{t<s} h(\beta, t)$. That is a contradiction. $\square$

Now we define $I = \{\alpha : \exists t \forall s > t(h(\alpha, s) = h(\alpha, t))\}$. By the above lemma, I is downward closed. I might be Ord^M, an ordinal in M, or a Σ_2 cut.

Lemma 6.3. $\{\lim_s h(\alpha, s) : \alpha \in I\}$ *is regular.*

Proof. Fix $\delta \in \mathsf{Ord}^M$. By Lemma 6.1, $\{\alpha : \forall s\, (h(\alpha, s) \leqslant \delta)\} \subseteq I$. Consider its complement. By Σ_1-Foundation, there is a least ordinal, say α_0, such that $\exists s\, (h(\alpha_0, s) \geqslant \delta)$. Note that for any $s > \delta$ and $\alpha < \alpha_0$, $h(\alpha, s) = h(\alpha, \delta)$. Thus, $\{\lim_s h(\alpha, s) : \alpha \in I\} \restriction \delta = \{h(\alpha, \delta) : \alpha < \alpha_0\}$ is M-finite. $\square$

Now suppose $H = \{\lim_s h(\alpha, s) : \alpha \in I\}$ is bounded, and α_0 is the ordinal defined in the proof of Lemma 6.3. Then $\alpha_0 = I$.

Lemma 6.4. *Assume that $\{\lim_s h(\alpha, s) : \alpha \in I\}$ is bounded. Then the ordinal α_0 defined above is not limit.*

Proof. Assume that α_0 is limit. Let t be a stage such that for all $\alpha < \alpha_0$, $h(\alpha, t) = \lim_s h(\alpha, s)$. Let s be the least stage such that $h(\alpha_0, s) > h(\alpha_0, t)$. By Lemma 6.2, there is α with $\alpha + 1 < \alpha_0$ such that some $x < \mathbf{r}(\alpha, s)$ is enumerated into X at exactly stage s. Thus, $h(\alpha + 1, s) > h(\alpha + 1, t)$. This is a contradiction. $\square$

By Lemma 6.4, $\alpha_0 = \beta_0 + 1$ for some β_0. Without loss of generality, we assume that β_0 is even. Then $[0, h(\beta_0, s))$ is the limit of Block β_0 and we denote it by B. Let s_0 be the least stage such that $\lim_s h(\beta_0, s) = h(\beta_0, s_0)$.

Lemma 6.5. *Assume that $\{\lim_s h(\alpha, s) : \alpha \in I\}$ is bounded. Then X is recursive.*

Proof. By the construction, for every stage $s > s_0$, $A_{<s} \upharpoonright \mathbf{r}(\beta_0, s) = A \upharpoonright \mathbf{r}(\beta_0, s)$ and from stage $s_0 + 1$ on, both $\mathbf{r}(\beta_0, s)$ and $\mathbf{m}(\beta_0, s)$ are nondecreasing.

Since $\lim_s h(\beta_0 + 1, s) = \infty$, there are cofinally many stages such that $X_s \upharpoonright \mathbf{r}(\beta_0, s) \neq X \upharpoonright \mathbf{r}(\beta_0, s)$. Since X is regular, $\lim_s \mathbf{r}(\beta_0, s) = \infty$. Thus, $\lim_s \mathbf{m}(\beta_0, s) = \infty$. This implies that for every stage $s > s_0$, $e \in B$, $\Phi_e^{A_{<s} \upharpoonright \mathbf{r}(\beta_0, s)}[s] \upharpoonright m(e, s) = X \upharpoonright m(e, s)$.

For every δ, let $s > s_0$ be a stage such that $\mathbf{m}(\beta_0, s) > \delta$. Then $X \upharpoonright \delta = \Phi_e^{A_{<s} \upharpoonright \mathbf{r}(\beta_0, s)}[s] \upharpoonright \delta$, where $e \in B$ is such that $m(e, s) > \delta$. Therefore, $X \upharpoonright \delta = X[s] \upharpoonright \delta$. $\square$

By Lemma 6.5, $\{\lim_s h(\alpha, s) : \alpha \in I\}$ is unbounded in Ord^M. For every $\alpha \in I$, let $B_\alpha = [0, \lim_s h(\alpha, s))$, the limit of Block α.

Lemma 6.6. $X \not\leq_T A$ *and* $X \not\leq_T B$.

Proof. We only prove that $X \not\leq_T A$. The proof of $X \not\leq_T B$ is symmetric. For the sake of a contradiction suppose $X = \Phi_e^A$, $\alpha \in I$ is even and s_0 is a stage such that $e < \lim_s h(\alpha, s) = h(\alpha, s_0) < \lim_s h(\alpha + 1, s) = h(\alpha + 1, s_0) < s_0$.

By the construction, from stage s_0 on, if $\Phi_e^A[s]$ computes anything, its computation is preserved. Thus, $\Phi_e^A[s] = X \upharpoonright \mathsf{dom}(\Phi_e^A[s])$. Thus, M-finite subsets of both X and $\overline{X}$ can be effectively enumerated via $\Phi_e^A[s]$, $s > s_0$. That implies X is recursive, which is a contradiction. $\square$

Question 6.7. *Is the Sacks Density Theorem true in all models of Level 1-KPL?*