Perfect trees and elementary embeddings

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Abstract

An important technique in large cardinal set theory is that of extending an elementary embedding $j : M \to N$ between inner models to an elementary embedding $j^* : M[G] \to N[G^*]$ between generic extensions of them. This technique is crucial both in the study of large cardinal preservation and of internal consistency. In easy cases, such as when forcing to make the GCH hold while preserving a measurable cardinal (via a reverse Easton iteration of α-Cohen forcing for successor cardinals α), the generic G^* is simply generated by the image of G. But in difficult cases, such as in Woodin's proof that a hypermeasurable is sufficient to obtain a failure of the GCH at a measurable, a preliminary version of G^* must be constructed (possibly in a further generic extension of $M[G]$) and then modified to provide the required G^*. In this article we use perfect trees to reduce some difficult cases to easy ones, using fusion as a substitute for distributivity. We apply our technique to provide a new proof of Woodin's theorem as well as the new result that global domination at inaccessibles (the statement that

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$d(\kappa)$ is less than 2^κ for inaccessible κ, where $d(\kappa)$ is the dominating number at κ) is internally consistent, given the existence of $0^\#$.

Let $j : M \to N$ be an elementary embedding of transitive models of ZFC. Often it is useful to extend j to an elementary embedding $j^* : M[G] \to N[G^*]$ between generic extensions of M and N. If G is P-generic over M, then G^* is chosen to be $P^* = P^N$-generic over N and such that $j[G] \subseteq G^*$. The latter property is sometimes achieved by first *constructing* an arbitrary P^*-generic G_0^*, and then *modifying* G_0^* to a P^*-generic G^* with the additional property $j[G] \subseteq G^*$. We refer to the first step as *generic construction* and the second step as *generic modification.* Two contexts in which these methods have been used are *internal consistency* and *large cardinal preservation.* We now give some background and examples for these two contexts.

Internal consistency

A statement is *internally consistent* iff it holds in an inner model, assuming the existence of large cardinals. Often to prove the internal consistency of a statement from large cardinals one forces the statement to hold over a suitable inner model and additionally verifies that a generic for the forcing used can be constructed. Examples of this can be found in [2, 3, 5, 4, 6]. To illustrate this, we discuss the following result of [5]:

Theorem 1 *Suppose that $0^\#$ exists. Then there is an inner model in which GCH fails at all regular cardinals.*

Sketch of proof. To prove his result, Easton forced over L with the Easton product of Add (α, α^{++}), α regular, to obtain a (class-) generic extension of L where GCH fails at all regulars. (Add (α, β) for an infinite regular α is the forcing that adds β-many α-Cohen sets with a product using $< \alpha$-support.) This Easton product cannot be used, as if $0^\#$ exists, there is no generic over L for Add $(\omega_1^V, 1)$. Instead we use a reverse Easton iteration; however, as Add $(\alpha, \alpha^{++}) *$ Add (α^+, α^{+++}) collapses α^{++}, we in fact need a reverse Easton iteration of products

$$\prod_n \text{Add } (\omega_n, \omega_{n+2}) * \prod_n \text{Add } (\omega_{\omega+n+1}, \omega_{\omega+n+3}) * \cdots .$$

To build a generic G for this forcing P, we build a generic $G(\leq i)$ for $P(\leq i)$, the first $i+1$ stages of this iteration, by induction on the indiscernible i. To handle limit indiscernibles of uncountable cofinality we need to ensure the coherence property: $\pi_{ij}[G(\leq i)] \subseteq G(\leq j)$ for indiscernibles $i < j$, where $\pi_{ij} : L \to L$ has critical point i and sends i to j. The key inductive step is to ensure that $\pi_{ii^*}[G(\leq i)] \subseteq G(\leq i^*)$, where i^* is the least indiscernible greater than the indiscernible i. This is equivalent to requiring $G(< i) \subseteq G(< i^*)$ and $\pi^*_{ii^*}[G(i)] \subseteq G(i^*)$, where $\pi^*_{ii^*} : L[G(< i)] \to L[G(< i^*)]$ is the canonical extension of $\pi_{ii^*} : L \to L$.

It is not difficult to *construct* a $P(\leq i^*)$-generic $G^*_0(\leq i^*)$ such that $G^*_0(< i^*)$ includes $G(< i)$. The key step is to *modify* $G^*_0(i^*)$ to a $G^*(i^*)$ which contains $\pi^*_{ii^*}[G(i)]$. The latter modification is performed by changing values of $G^*_0(i^*)$ on the range of $\pi^*_{ii^*}$ to make it agree with $\pi^*_{ii^*}[G(i)]$. Verifying the genericity of the modified $G^*(i^*)$ heavily uses the homogeneity of the forcing Add $(i^*, (i^*)^{++})$. $\square$

Large cardinal preservation

In this case we consider embeddings $j : V \to M$ of the entire universe V into an inner model M. The critical point of j is a measurable cardinal, whose measurability we wish to preserve after forcing. Thus if G is P-generic over V we wish to construct a $P^* = P^M$-generic G^* such that $j[G] \subseteq G^*$. It is important that G^* be constructed in $V[G]$, in order that the resulting embedding $j^* : V[G] \to M[G^*]$ witness the measurability of its critical point in $V[G]$. Once again, the required G^* is sometimes obtained through generic modification. We illustrate this by discussing the following result of Hugh Woodin.

Theorem 2 *(Woodin) Suppose GCH holds and κ is $P_2\kappa$ hypermeasurable (i.e., κ is the critical point of some $j : V \to M$ where $H(\kappa^{++})^V$ belongs to M). Then in a generic extension, κ is measurable and the GCH fails at κ. In a further forcing extension, the singular cardinal hypothesis fails.*

Sketch of Woodin's proof. Assume GCH and that $j : V \to M$ witnesses the $P_2\kappa$ hypermeasurability of κ via an ultrapower. Thus κ is the critical point of j, $H(\kappa^{++})^V$ belongs to M and each element of M is of the form $j(f)(a)$ for some $a \in H(\kappa^{++})^V$.

Let P be the reverse Easton iteration of Add (α, α^{++}) for α inaccessible, $\alpha \leq \kappa$, let G be $P(<\kappa)$-generic over V and let g be Add (κ, κ^{++})-generic over $V[G]$. We wish to find a generic G^* for $P^* = P^M$ over M such that $j[G * g] \subseteq G^*$. Note that the forcings P, P^* defined in V, M, respectively, are the same for the first $\kappa + 1$ stages, so we may take G^* to be of the form $G * g * H * h$, where H is generic over $M[G * g]$ for the iteration between κ and $j(\kappa)$ and h is generic over $M[G * g * H]$ for Add $(j(\kappa), j(\kappa^{++}))$.

To obtain H, Woodin uses the following trick. Let $j_0 : V \to N$ be the measure ultrapower derived from j and $k : N \to M$ so that $j = k \circ j_0$. Then as $j_0(\kappa)$ has cardinality κ^+ in V it is not difficult to build H_0 so that $G * g_0 * H_0$ is generic over N for the first $j_0(\kappa)$ stages of N's version of the iteration (g_0 is g restricted to Add $(\kappa, (\kappa^{++})^N)$). Then $k : N \to M$ extends to $k^* : N[G * g_0] \to M[G * g]$ and $k^*[H_0]$ generates the desired generic H over $M[G * g]$ for the iteration between κ and $j(\kappa)$, producing a further extension $k^{**} : N[G * g_0 * H_0] \to M[G * g * H]$ of k.

We therefore now have an extension $j^* : V[G] \to M[G * g * H]$ of j. The key part of Woodin's proof is the construction of an h which is Add $(j(\kappa), j(\kappa^{++}))$-generic over $M[G * g * H]$ and which contains $j^*[g]$. In fact, Woodin does not obtain h in $V[G * g]$, but must go to a *larger universe* $V[G * g * h_0]$ to obtain it, and then lift the resulting $j^{**} : V[G * g] \to M[G * g * H * h]$ once more to an embedding of $V[G * g * h_0]$ which is definable in $V[G * g * h_0]$.

The set h_0 is generic over $V[G * g]$ for the Add $(j_0(\kappa), j_0(\kappa^{++}))$ of $N[G * g_0 * H_0]$. (Note that h_0 cannot be constructed inside $V[G * g]$, as the forcing Add $(j_0(\kappa), j_0(\kappa^{++}))$ of $N[G * g_0 * H_0]$ has size κ^{++} but is only κ^+-closed in $V[G * g_0]$.) Woodin shows that the latter forcing preserves cardinals over $V[G * g]$. As in the construction of H, a generic h' for Add $(j(\kappa), j(\kappa^{++}))$ of $M[G * g * H]$ is generated by $k^{**}[h_0]$. But h' does not contain $j^*[g]$, as for $\alpha \in [(\kappa^{++})^N, \kappa^{++})$, the α-th κ-Cohen set chosen by g, does not belong to the model $N[G * g_0 * H_0]$ and therefore is not an initial segment of the $j^*(\alpha)$-th $j(\kappa)$-Cohen set chosen by h', whose restriction to κ does belong to $N[G * g_0 * H_0]$. Therefore h' must be *modified* to obtain the desired h, just as in the proof of Theorem 1. Once again, the homogeneity of Add $(j(\kappa), j(\kappa^{++}))$ is important for this modification. This yields $j^{**} : V[G * g] \to M[G * g *$

$H * h]$ definable in $V[G * g * h_0]$, which is extended once more to the desired embedding from $V[G * g * h_0]$ into $M[G * g * H * h * i]$ where i is generated by $j^{**}[h_0]$.

The final statement of the theorem is established using Prikry forcing. $\square$

In both of the previous results, construction of generics and generic modification were used. However, as in Woodin's proof, the construction of generics can be difficult, and moreover there are situations in which generic modification is not possible, due to the lack of homogeneity of the forcings involved. We present a new and simpler proof of Woodin's theorem as well as a new internal consistency result regarding global domination, without any need to construct generics or modify them (and without going to a larger universe). The key idea is to replace α-Cohen forcing by α-Sacks forcing, whose conditions are perfect α-trees.

An easier proof of Woodin's theorem

As before, assume GCH and that $j : V \to M$ witnesses the $P_2\kappa$ hypermeasurability of κ via an ultrapower. Thus κ is the critical point of j, $H(\kappa^{++})^V$ belongs to M and each element of M is of the form $j(f)(a)$ for some $a \in H(\kappa^{++})^V$. For the reader's convenience, we assume nothing of the above proof sketch and present a self-contained argument.

For inaccessible α, we shall force not with α-Cohen forcing, but with α-*trees*, a generalisation of Sacks forcing, whose conditions are perfect trees on ω. Such α-trees were used in higher recursion theory by J. MacIntyre ([10]) and Shore ([11]) and later in set theory by Kanamori ([9]).

For inaccessible α let Sacks(α) denote the following forcing. A condition is a subset T of $2^{<\alpha}$ (= the set of functions from an ordinal less than α into 2) such that:

1. $s \in T, t \subseteq s \to t \in T$.
2. Each $s \in T$ has a proper extension in T.
3. If $s_0 \subseteq s_1 \subseteq \cdots$ is a sequence in T of length less than α then the union of the s_i's belongs to T.
4. Let Split(T) denote the set of s in T such that both $s * 0$ and $s * 1$ belong

to T. Then for some (unique) closed, unbounded $C(T) \subseteq \alpha$, $\text{Split}(T) = \{s \in T \mid \text{length}(s) \in C(T)\}$.

Extension is defined by $S \leq T$ iff S is a subset of T. For $i < \alpha$, the *i-th splitting level of T*, $\text{Split}_i(T)$, is the set of s in T of length α_i, where $\alpha_0 < \alpha_1 < \cdots$ is the increasing enumeration of $C(T)$. Sacks(α) is an α-closed forcing of size α^+. This forcing also preserves α^+, as it obeys the following *α-fusion property.* For $\beta < \alpha$ we write $S \leq_\beta T$ iff $S \leq T$ and $\text{Split}_i(S) = \text{Split}_i(T)$ for $i < \beta$.

α-fusion: Suppose that $T_0 \geq T_1 \geq \cdots$ is a descending sequence in Sacks(α) of length α and suppose in addition that $T_{i+1} \leq_i T_i$ for each i less than α. Then the intersection of the T_i, $i < \alpha$, is a condition in Sacks(α).

α-fusion implies that α^+ is preserved, as given a condition T_0 and a name $\dot{f} : \alpha \to \alpha^+$, one can build a sequence as in the hypothesis of α-fusion so that T_i forces $\dot{f}(i)$ to belong to a subset of α^+ of size at most $2^i = i^+$; then the intersection of the T_i's forces a bound on $\dot{f}$.

We shall need a product of α-Sacks forcings. For inaccessible α let Sacks(α, α^{++}) denote the product of α^{++} copies of Sacks(α) with size α support. Thus a condition is a sequence $\vec{T} = \langle T(i) \mid i < \alpha^{++} \rangle$ whose support $\text{Supp}(\vec{T}) = \{i \mid T(i) \neq 2^{<\alpha}\}$ has size at most α, ordered component-wise. This forcing is again α-closed, and preserves α^+ via a suitable version of α-fusion, which we now describe. For $\beta < \alpha$ and $X \subseteq \alpha^{++}$ of size less than α, we write $\vec{T_0} \leq_{\beta,X} \vec{T_1}$ iff $\vec{T_0} \leq \vec{T_1}$ (i.e., $T_0(i) \leq T_1(i)$ for each $i < \alpha^{++}$) and in addition, for i in X, $T_0(i) \leq_\beta T_1(i)$.

Generalised α-fusion: Suppose that $\vec{T_0} \geq \vec{T_1} \geq \cdots$ is a descending sequence in Sacks(α, α^{++}) of length α and suppose in addition that $\vec{T}_{i+1} \leq_{i,X_i} \vec{T_i}$ for each i less than α, where the X_i's form an increasing sequence of subsets of α^{++} of size less than α whose union is the union of the supports of the $\vec{T_i}$'s. Then the $\vec{T_i}$'s have a lower bound in Sacks(α, α^{++}) (obtained by taking intersections at each component, using α-fusion).

Again this implies that α^+ is preserved, as given a condition $\vec{T_0}$ and a name $\dot{f} : \alpha \to \alpha^+$ one can build a sequence as in the hypothesis of generalised α-fusion so that $\vec{T_i}$ forces $\dot{f}(i)$ to belong to a subset of α^+ of size at most

$(2^i)^\gamma < \alpha$ for some $\gamma < \alpha$; then a lower bound of the $\vec{T}_i$'s forces a bound on f.

Sacks(α, α^{++}) also preserves α^{++}, as a Δ system argument shows that it is α^{++}-cc.

Now force over our ground model V with the reverse Easton iteration of Sacks(α, α^{++}) for α inaccessible, $\alpha \leq \kappa$. Let G denote the generic for the first κ stages of this iteration and g the generic for the κ-th stage. Thus g is generic over $V[G]$ for Sacks(κ, κ^{++}) as defined in $V[G]$.

We would like to find a suitable generic over M for M's version of the above iteration. As M contains $H(\kappa^{++})^V$, the first $\kappa + 1$ stages of the M and V iterations are the same, so we may use $G * g$ as our generic over M for the first $\kappa + 1$ stages of the M-iteration. Next we want a generic H over $M[G][g]$ for the M-iteration between κ and $j(\kappa)$; given this we obtain a lifting of $j : V \to M$ to an embedding $j^* : V[G] \to M[G][g][H]$. The last step will be to find a generic h over $M[G][g][H]$ for the $j(\kappa)$-th stage of the M-iteration, where we force with the Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$. If we also have $j^*[g] \subseteq h$ then we can lift j^* once more to an embedding $j^{**} : V[G][g] \to M[G][g][H][h]$. If H and h can be found inside $V[G][g]$ then j^{**} is definable in $V[G][g]$ and therefore witnesses the measurability of κ in the model $V[G][g]$, where GCH fails at κ.

To obtain H we argue as follows. (This argument avoids Woodin's use of the measure ultrapower N and embedding $k : N \to M$.) Note that each dense subset in $M[G][g]$ for the iteration R between κ and $j(\kappa)$ is of the form $j(f)(a)^{G*g}$ where a belongs to $H(\kappa^{++})$ and $f : H(\kappa) \to H(\kappa^+)$ belongs to V. Moreover R is κ^{+++}-closed in $M[G][g]$, as no forcing takes place between κ and the next inaccessible of M. Therefore, for each f there is a single condition in R which meets all dense sets of the form $j(f)(a)^{G*g}$, $a \in H(\kappa^{++})$. As GCH holds in V, there are only κ^+-many such f's and therefore we can build a descending κ^+-sequence of conditions in R, at stage $i + 1$ meeting all dense sets of the form $j(f_i)(a)$, $a \in H(\kappa^{++})$, where f_i is the i-th function. The existence of lower bounds at limit stages less than κ^+ follows from the κ^+-closure of R in $M[G][g]$ together with the next Lemma.

Lemma 3 *Any κ-sequence in $V[G][g]$ of elements of $M[G][g]$ belongs to $M[G][g]$.*

Proof. It follows from generalised fusion that any set x of ordinals in $V[G][g]$ of cardinality κ in $V[G][g]$ is covered by such a set in V. As any κ-sequence of ordinals in V belongs to M, it follows that x belongs to $M[y]$ where y is a subset of κ in $V[G][g]$. As all subsets of κ in $V[G][g]$ belong to $M[G][g]$, it follows that x belongs to $M[G][g]$. The same argument applies to sets x of elements of $M[G][g]$ using a wellordering in $M[G][g]$ of a sufficiently large $H(\lambda)^{M[G][g]}$. $\square$ (Lemma 3)

Thus we have extended j to an embedding $j^* : V[G] \to M[G][g][H]$.

Now we come to the construction of h. Woodin obtained h by forcing a corresponding h_0 (associated to the model N) over $V[G][g]$, lifting h_0 via (an extension of) k^* to an h', modifying h' to a generic h containing $j^*[g]$, and finally lifting the resulting embedding from $V[G][g]$ into $M[G][g][H][h]$ once more. In our proof we obtain h directly inside $V[G][g]$, without modification.

As g is a set of conditions in Sacks(κ, κ^{++}) of $V[G]$, $j^*[g]$ consists of a set of conditions in Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$. We analyse the "intersection" of the conditions in $j^*[g]$.

Lemma 4 *For $\alpha < j(\kappa^{++})$ let t be the intersection of the trees $j^*(p)(\alpha)$, p in g. If α belongs to the range of j, then t is a $(\kappa, j(\kappa))$-tuning* fork, *i.e., a subtree of $2^{<j(\kappa)}$ which is the union of two cofinal branches which split at κ. If α does not belong to the range of j, then t consists of exactly one cofinal branch through $2^{<j(\kappa)}$.*

Proof. First note that κ is the only ordinal which belongs to $j(C)$ for every $C \in V$ which is closed unbounded in κ: Clearly any ordinal in the intersection of all such $j(C)$ must be a limit cardinal of M and must be at least κ. Now suppose that β is a limit cardinal of M between κ and $j(\kappa)$. Then β is of the form $j(f)(b)$ for some $b \in H(\kappa^{++})^V$, $f : H(\kappa)^V \to \kappa$ and the set C of limit cardinals $\lambda < \kappa$ such that $f[H(\lambda)^V] \subseteq \lambda$ is closed unbounded in κ. Thus $j(C)$ is the set of limit cardinals λ of M such that $j(f)[H(\lambda)^M] \subseteq \lambda$, and $\beta = j(f)(b)$ does not belong to $j(C)$.

Write α as $j(f)(a)$ where a belongs to $H(\kappa^{++})^V$ and f has domain $H(\kappa)^V$. We can assume that $f(\bar{a})$ is an ordinal less than κ^{++} for each $\bar{a} \in H(\kappa)^V$. Let S be the range of f, a subset of κ^{++} of size at most κ in V.

Let C be a closed unbounded subset of κ. Then any condition p in Sacks(κ, κ^{++}) of $V[G]$ has an extension q such that for $i \in S$, $C(q(i))$ (= the set of splitting levels of the tree $q(i)$) is a subset of C. Choose such a q in g. Then $j^*(q) = q^*$ has the property that for all $b \in H(j(\kappa))^M$, $C(q^*(j(f)(b)))$ is a subset of $j(C)$. In particular, $C(q^*(\alpha)) = C(q^*(j(f)(a)))$ is a subset of $j(C)$. Now C was an arbitrary closed unbounded subset of κ and the intersection of the $j(C)$, C closed unbounded in κ, is $\{\kappa\}$. As $M[G][g][H]$ contains all $s : \kappa \to 2$ in $V[G][g]$, it follows that the intersection t of the $j^*(p)(\alpha)$, $p \in g$, is a subtree of $2^{<j(\kappa)}$ which is the union of at most two cofinal branches, which can only differ at κ. If $\alpha = j(\bar{\alpha})$ belongs to the range of j, then κ belongs to $j^*(C(p(\bar{\alpha}))) = C(j^*(p)(j(\bar{\alpha}))) = C(j^*(p)(\alpha))$ for each $p \in g$, and therefore if s is the unique sequence of length κ in t, both $s * 0$ and $s * 1$ belong to t. It follows in this case that t is in fact the union of two cofinal branches which do split at κ, a $(\kappa, j(\kappa))$-tuning fork.

If α does not belong to the range of j then we argue as follows. It must be that $S =$ Range (f) has size exactly κ, as otherwise the range of $j(f)$ is the pointwise image of the range of f, implying that α is in the range of j. Now let $\langle \bar{\alpha}_i \mid i < \kappa \rangle$ be a 1-1 enumeration of S, and let D be the set of conditions p in Sacks(κ, κ^{++}) of $V[G]$ such that for each $i < \kappa$, $p(\bar{\alpha}_i)$ is a tree whose first splitting level is greater than i. Then D is dense, so we may choose such a p in g. Then $j^*(\langle \bar{\alpha}_i \mid i < \kappa \rangle) = \langle \alpha_i \mid i < j(\kappa) \rangle$ enumerates $j^*(S)$ and $j^*(p)$ has the property that for each $i < j(\kappa)$, the first splitting level of $j^*(p)(\alpha_i)$ is greater than i. Now α is an element of $j^*(S)$ and therefore is equal to α_i for some unique $i < j(\kappa)$. As α is not in the range of j, i is at least κ. It follows that the first splitting level of $j^*(p)(\alpha)$ is greater than κ, and therefore t, the intersection of the $j^*(p)(\alpha)$, does not split at κ. So t consists of exactly one cofinal branch through $2^{<j(\kappa)}$. $\square$ (Lemma 4)

Now for $\alpha < j(\kappa^{++})$ in the range of j, let $(x(\alpha)_0, x(\alpha)_1)$ be the branches that make up the $(\kappa, j(\kappa))$-tuning fork at α, where $x(\alpha)_0(\kappa) = 0$, $x(\alpha)_1(\kappa) = 1$. For $\alpha < j(\kappa^{++})$ not in the range of j, let $x(\alpha)_0$ denote the unique branch through all of the $j^*(p)(\alpha)$, $p \in g$.

Lemma 5 *For any $\alpha < j(\kappa)$ and any subset S of $j(\kappa^{++})$ of size $j(\kappa)$ in $M[G][g][H]$, the sequence $\langle x(i)_0 \restriction \alpha \mid i \in S \rangle$ belongs to $M[G][g][H]$.*

Proof. Write α as $j(f_0)(a)$ where $f_0 : H(\kappa)^V \to \kappa$ and a belongs to $H(\kappa^{++})^V$. Let C consist of all limit cardinals λ less than κ such that $f_0[H(\lambda)^V] \subseteq \lambda$.

Then the least element of $j(C)$ greater than κ is also greater than α. Also note that every element of $M[G][g][H]$ is of the form $j^*(f)(b)$ for some f in $V[G]$ with domain $H(\kappa)^V$ and b in $H(\kappa^{++})^V$; this is because the class of such $j^*(f)(b)$'s forms an elementary submodel of $M[G][g][H]$ containing $M \cup \{G * g * H\}$ as a subclass and therefore equals all of $M[G][g][H]$. Thus we can write S as $j^*(f)(b)$ where $f : H(\kappa)^V \to ([\kappa^{++}]^\kappa)^{V[G]}$ belongs to $V[G]$ and b belongs to $H(\kappa^{++})^V$. In fact, we may assume that S is the image $j^*(\bar{S})$ of some subset $\bar{S}$ of κ^{++} of size κ in $V[G]$, as S is contained in $j^*(\bar{S})$ where $\bar{S}$ is the union of the $f(x)$, $x \in H(\kappa)^V$. Let $\langle \bar{\alpha}_i \mid i < \kappa \rangle$ be a 1-1 enumeration of $\bar{S}$.

Each condition p in Sacks(κ, κ^{++}) of $V[G]$ has an extension q whose support contains $\bar{S}$ such that for all $i < \kappa$, $C(q(\bar{\alpha}_i))$ is a subset of $C \setminus (i+1)$. Choose such a q in g. Then $S = j^*(\bar{S})$ is included in the support of $j^*(q)$ and S has the 1-1 enumeration $j^*(\langle \bar{\alpha}_i \mid i < \kappa \rangle) = \langle \alpha_i \mid i < j(\kappa) \rangle$. For $i < j(\kappa)$, $C(j^*(q)(\alpha_i))$ is a subset of $j(C) \setminus (i+1)$. In particular, $j^*(q)(\alpha_i)$ has no splits between κ and α for all $i < j(\kappa)$ and no splits between 0 and α for $\kappa \le i < j(\kappa)$. Note that α_i belongs to the range of j iff i is less than κ. It follows that for $i \in [\kappa, j(\kappa))$, $x(\alpha_i)_0 \restriction \alpha$ is the unique element of $j^*(q)(\alpha_i)$ of length α, and for $i < \kappa$, $x(\alpha_i)_0 \restriction \alpha$ is the unique element of $j^*(q)(\alpha_i)$ of length α which takes the value 0 at κ and extends $x(\alpha_i)_0 \restriction \kappa$. Thus the sequence $\langle x(i)_0 \restriction \alpha \mid i \in S \rangle$ can be computed from $j^*(q) \in M[G][g][H]$ together with the sequence $\langle x(\alpha_i)_0 \restriction \kappa \mid i < \kappa \rangle$. The latter is coded by a subset of κ in $V[G][g]$ and therefore belongs to $M[G][g][H]$, as $M[G][g][H]$ and $V[G][g]$ have the same subsets of κ. This proves the Lemma. $\square$ (Lemma 5)

Lemma 6 *Let h consist of all conditions p in Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ such that for each $\alpha < j(\kappa^{++})$, $x(\alpha)_0$ is contained in $p(\alpha)$. Then h is generic for Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ over $M[G][g][H]$ and contains $j^*[g]$.*

Proof. Clearly h contains $j^*[g]$, as by definition $x(\alpha)_0$ is one of two branches through the intersection of the $j^*(p)(\alpha)$, p in g.

Suppose that D is dense on Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ and belongs to $M[G][g][H]$. Write D as $j^*(f)(a)$ where f has domain $H(\kappa)^V$ and belongs to $V[G]$, and a belongs to $H(\kappa^{++})^V$. We can assume that $f(\bar{a})$ is dense on Sacks(κ, κ^{++}) of $V[G]$ for each $\bar{a}$ in $H(\kappa)^V$.

Suppose that $\bar{p}$ is a condition in Sacks(κ, κ^{++}) of $V[G]$, S is a subset of κ^{++} of size less than κ in $V[G]$ and α is less than κ. Then an (S, α)-*thinning of* $\bar{p}$ is an extension of $\bar{p}$ obtained by thinning each $\bar{p}(i)$, $i \in S$, to the subtree consisting of all nodes compatible with some particular node on the α-th splitting level of $\bar{p}(i)$. (There are $(2^{|\alpha|})^{|S|}$ such thinnings, one for each way of choosing a node on the α-th splitting level of $\bar{p}(i)$ for each $i \in S$.) We say that a condition $\bar{p}$ in Sacks(κ, κ^{++}) of $V[G]$ *reduces* $\bar{D}$, where $\bar{D}$ is dense on Sacks(κ, κ^{++}) of $V[G]$, iff for some subset S of κ^{++} of size less than κ in $V[G]$ and some $\alpha < \kappa$, any (S, α)-thinning of $\bar{p}$ meets $\bar{D}$.

If $\mathcal{D}$ is a collection of κ-many dense sets $\bar{D}$ in $V[G]$, then any condition $\bar{p}$ can be extended to a $\bar{q}$ which reduces each $\bar{D}$ in $\mathcal{D}$: List the elements of $\mathcal{D}$ as $\bar{D}_0, \bar{D}_1, \ldots$ in a κ-sequence. Extend $\bar{p} = \bar{p}_0$ to $\bar{p}_1$ meeting $\bar{D}_0$. Then choose some i_1 in the support of $\bar{p}_1$ and extend $\bar{p}_1$ to $\bar{p}_2$, without changing the 1-st splitting level of $\bar{p}_1(i_1)$, so that by simply thinning $\bar{p}_2(i_1)$ to either choice on the 1-st splitting level, $\bar{D}_1$ is met. Then choose some i_2 in the support of $\bar{p}_2$ and extend extend $\bar{p}_2$ to $\bar{p}_3$, without changing the 2-nd splitting levels of $\bar{p}_2(i_1), \bar{p}_2(i_2)$, so that by simply thinning each of $\bar{p}_3(i_1), \bar{p}_3(i_2)$ to any of the four choices of nodes on their 2-nd splitting levels, $\bar{D}_2$ is met. Continue in this way, so that at stage $\alpha + 1$, for any choice of nodes on the α-th splitting level of the trees $\bar{p}_\alpha(i)$, i one of the "first α indices", thinning to those nodes will result in a condition that meets $\bar{D}_\alpha$ (this is possible as there are only $(2^{|\alpha|})^{|\alpha|} < \kappa$ such thinnings). The indices $i_1, i_2, \ldots$ should be chosen so that after κ steps, every element of the support of the final condition $\bar{q} = \bar{p}_\kappa$ is one of the indices chosen. Then $\bar{q}$ is an extension of $\bar{p}$ that reduces each $\bar{D}$ in $\mathcal{D}$.

Thus we may choose a condition $\bar{p}$ in g which reduces each $f(\bar{a})$, $\bar{a} \in H(\kappa)^V$. Then $p = j^*(\bar{p})$ reduces each $j^*(f)(b)$, $b \in H(\kappa^{++})^V$ and therefore reduces $D = j^*(f)(a)$. In $M[G][g][H]$, choose a subset S of $j(\kappa)$ of size less than $j(\kappa)$ and $\alpha < j(\kappa)$ such that any (S, α)-thinning of p meets D. Now for each $i \in S$ thin p by choosing an initial segment of $x(i)_0$ on the α-th splitting level of $p(i)$. This sequence of choices from the α-th splitting levels of the $p(i)$, $i \in S$ belongs to $M[G][g][H]$ by Lemma 5. It follows that this thinned out condition belongs to h and meets D.

Lastly we verify that any two conditions in h are compatible. Let h' consist of all conditions q such that for some S and α, q is an (S, α)-thinning of a condition in $j^*[g]$ using initial segments of $x(i)_0$ for i in S. By the above, h' meets all dense sets for Sacks$(j(\kappa), j(\kappa^{++}))$ which belong to $M[G][g][H]$. Also, any two conditions in h' are compatible. Now for any condition p in h, consider the set D of conditions q in Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ with the property that for each $i < j(\kappa^{++})$, the tree $q(i)$ is either a subtree of $p(i)$ or shares no cofinal branch with $p(i)$. Then D is dense and belongs to $M[G][g][H]$. So there is a condition in h' which meets D and therefore extends p, as for each i, $x(i)_0$ is a branch through $p(i)$ as well as through all $q(i)$ for q in h'. Therefore every condition in h is extended by a condition in h', and as any two conditions in h' are compatible, it follows that the same holds for h.

Thus h is generic for Sacks$(j(\kappa), j(\kappa^{++}))$ of $M[G][g][H]$ over $M[G][g][H]$, as desired. $\square$ (Lemma 6)

Thus we can lift the embedding $j^* : V[G] \to M[G][g][H]$ to an embedding $j^{**} : V[G][g] \to M[G][g][H][h]$, and this lifting is definable in $V[G][g]$. So $V[G][g]$ is a model where κ is measurable and the GCH fails at κ. By adding a Prikry sequence s through κ over $V[G][g]$, we obtain a failure of the singular cardinal hypothesis, as in $V[G][g][s]$, κ is a singular strong limit cardinal of cofinality ω and $2^\kappa = \kappa^{++}$. $\square$

The above argument easily adapts to prove the following slightly stronger statement, also due to Woodin.

Theorem 7 *Assume GCH and suppose that $j : V \to M$ has critical point κ, M is closed under κ-sequences and for some $f : \kappa \to \kappa$ in V, $j(f)(\kappa) = \kappa^{++}$. Then in some generic extension of V, κ is measurable and the GCH fails at κ. In a further Prikry extension, the singular cardinal hypothesis fails.*

Proof. For any inaccessible α and any β, Sacks(α, β) denotes the product of β copies of Sacks(α) with support of size at most α. For any β, this forcing has the α^{++}-cc and obeys generalised α-fusion.

Now consider the reverse Easton iteration of length $\kappa + 1$, where at an inaccessible stage $\alpha < \kappa$ one forces with Sacks$(\alpha, f(\alpha))$ and at stage κ one forces with Sacks(κ, κ^{++}). Then the proof of the previous theorem shows that in the generic extension, κ is measurable and the GCH fails at κ. The effect of adding a Prikry sequence is the same as before. $\Box$

The relevance of this stronger version is that it yields via a theorem of Gitik ([8]) a consistent failure of the singular cardinal hypothesis just from a cardinal κ such that $o(\kappa) = \kappa^{++}$, a hypothesis weaker than $P_2\kappa$ hypermeasurability.

Global Domination

For an infinite regular cardinal κ, the *dominating number* $d(\kappa)$ is the least cardinality of a set of functions from κ to κ which are sufficient to eventually dominate any such function. The cardinal $d(\kappa)$ is greater than κ and at most 2^κ. *Global Domination* is the statement: $d(\kappa) < 2^\kappa$ for all infinite regular κ.

Cummings and Shelah ([1]) make a thorough study of the consistent global behaviours of the function $d(\kappa)$ as well as the related boundedness function $b(\kappa)$. A simple special case of their work is the following result, which makes use of Hechler forcing.

Theorem 8 *(Cummings-Shelah [1]) Con(ZFC) implies Con(ZFC + Global Domination).*

The use of Hechler forcing in the proof of this theorem seems to preclude the use of a similar technique to obtain the internal consistency of Global Domination, even restricted to inaccessibles, without assuming very large cardinals (beyond superstrong). However, using perfect tree forcing, the needed large cardinal assumption can be reduced to just $0^\#$.

Theorem 9 *Suppose that $0^\#$ exists. Then there is an inner model M in which Global Domination holds at inaccessibles, i.e., for each inaccessible cardinal κ, $d(\kappa) < 2^\kappa$.*

Remark. In fact it is possible to obtain the internal consistency of Global Domination at *all* regular cardinals, assuming only the existence of $0^\#$. The idea is to use Sacks(α, α^{++}) at inaccessible α as above, and the Cummings-Shelah method at successor α (i.e., Add (α, α^{++}) followed by an α^+-iteration of α-Hechler forcing). However, this idea does not appear to achieve Global

Domination at successors of inaccessibles; a solution to this problem appears in the forthcoming [7].

Proof. Over the ground model L, perform the reverse Easton iteration (with Easton support) of length Ord, which is trivial except at inaccessible stages α, where one forces with $\text{Sacks}(\alpha, \alpha^{++})$. P preserves cardinals, as for each regular cardinal κ, P can be factored as $P(< \kappa) * P(\kappa) * P(> \kappa)$, where $P(< \kappa)$ is κ^+-cc, $P(\kappa)$ is trivial or satisfies generalised fusion for sequences of length κ and $P(> \kappa)$ is κ^+-closed.

P also forces Global Domination at inaccessibles: If α is inaccessible, then by generalised α-fusion for $\text{Sacks}(\alpha, \alpha^{++})$, every function from α to α added by $\text{Sacks}(\alpha, \alpha^{++})$ is dominated by such a function in $V[G(< \alpha)]$, where $G(< \alpha)$ is generic for the first α-stages of the iteration. As the GCH holds at α in $V[G(< \alpha)]$, it follows that $d(\alpha) = \alpha^+$ and $2^\alpha = \alpha^{++}$ in $V[G]$ for P-generic G, as required by Global Domination at α.

Now we build a P-generic definably in $L[0^\#]$. For any α we let $P(\leq \alpha)$ denote the first $\alpha + 1$ stages of the iteration and factor $P(\leq \alpha)$ as $P(< \alpha) * P(\alpha)$, with corresponding generics $G(\leq \alpha) = G(< \alpha) * G(\alpha)$. We build a generic $G(< i)$ for $P(< i)$ by induction on the Silver indiscernible i. To facilitate the construction we ensure the following property inductively:

$$(*)\ i < j \rightarrow \pi_{ij}[G(\leq i)] \subseteq G(\leq j),$$

where π_{ij} is the unique elementary embedding from L to L whose range contains the indiscernible k iff k does not belong to the interval $[i, j)$. (Thus π_{ij} has critical point i and sends i to j.) When i is the least indiscernible, we take $G(< i)$ to be any $P(< i)$-generic, which exists due to the countability of i. For limit indiscernibles i, we take $G(< i)$ to be the union of the $G(< \bar{i})$, $\bar{i}$ a indiscernible less than i. It is easy to verify that the resulting $G(< i)$ is $P(< i)$-generic.

Now suppose that $G(< i)$ has been defined and we wish to define $G(< j)$, where j is the least indiscernible greater than i. Our first task is to define $G(i)$. Once this is accomplished, it is easy to construct the rest of $G(< j)$, as $P(i, j)$, the iteration strictly between i and j, is i^+-closed in $L[G(\leq i)]$ and using the fact that L is the definable closure in itself of the indiscernibles, the collection of dense subsets of $P(i,j)$ in $L[G(\leq i)]$ can be written as the ω-union of subcollections, each of which belongs to $L[G(\leq i)]$ and has cardinality i in that model. (For further details, see [5] or [6].)

Now we consider the construction of $G(i)$. If i is the least indiscernible then we can easily build $G(i)$ using the countability of i. If i is a limit indiscernible, then the inductively guaranteed property $(*)$ ensures that we can take $G(\leq i)$ to be the union of the $\pi_{\bar{i}i}[G(\leq \bar{i})]$, and this provides the desired generic $G(i)$.

It remains to handle the case where i is a successor indiscernible. Suppose that $\bar{i}$ is the largest indiscernible less than i and let π denote the embedding $\pi_{\bar{i}i}$. As $G(< \bar{i})$ is included in $G(< i)$, the embedding π lifts canonically to an embedding $\pi^* : L[G(< \bar{i})] \to L[G(< i)]$.

Now we follow the "tuning fork analysis" of the proof of Theorem 2 to produce the desired $G(i)$. Much of the argumentation is the same, replacing the embedding $j^* : V[G] \to M[G][g][H]$ by $\pi^* : L[G(< \bar{i})] \to L[G(< i)]$; however for the reader's convenience, we give a self-contained argument. Below, whenever we write α^+, α^{++}, etc., we mean the successors as computed in L.

Lemma 10 *For $\alpha < i^{++}$ let t be the intersection of the trees $\pi^*(p)(\alpha)$, p in $G(\bar{i})$. If α belongs to the range of π, then t is an $(\bar{i}, i)$-*tuning fork*, i.e., a subtree of $2^{<i}$ which is the union of two cofinal branches which split at $\bar{i}$. If α does not belong to the range of π, then t consists of exactly one cofinal branch through $2^{<i}$.*

Proof. First note that $\bar{i}$ is the only ordinal which belongs to $\pi(C)$ for every $C \in L$ which is closed unbounded in $\bar{i}$: Clearly the intersection of these $\pi(C)$'s contains no ordinal less than $\bar{i}$ and, by the elementarity of π, does contain the ordinal $\bar{i}$. If β is any ordinal less than i, then β can be written as $\sigma(\vec{i}_0, \bar{i}, \vec{\infty})$, where σ is an L-definable function, $\vec{i}_0$ is a finite increasing sequence of indiscernibles less than $\bar{i}$ and $\vec{\infty}$ is any finite increasing sequence of indiscernibles greater than $\bar{i}$ (of the appropriate length). Let C be the set of $\gamma < \bar{i}$ such that the largest indiscernible of $\vec{i}_0$ is less than γ and $\sigma(\vec{i}_0, \bar{\gamma}, \vec{\infty})$ is less than γ for each ordinal $\bar{\gamma} < \gamma$. Then C is a constructible club in $\bar{i}$ and $\pi(C)$ is the set of $\gamma < i$ such that the largest indiscernible of $\vec{i}_0$ is less than

γ and $\sigma(\bar{i}_0, \bar{\gamma}, \infty)$ is less than γ for each ordinal $\bar{\gamma} < \gamma$. Clearly β does not belong to $\pi(C)$ if $\bar{i}$ is less than β.

Write α as $\sigma(\bar{i}_0, \bar{i}, i, \infty)$ where $\bar{i}_0$ is a finite increasing sequence of indiscernibles less than $\bar{i}$ and ∞ is any finite increasing sequence of indiscernibles greater than i (of the appropriate length). Let S be the set of ordinals of the form $\sigma(\bar{i}_0, \gamma, \bar{i}, \infty)$, where the largest indiscernible in $\bar{i}_0$ is less than γ and γ is an ordinal less than $\bar{i}$. Then S is a set of ordinals of L-cardinality $\bar{i}$ and $\pi(S)$ contains the ordinal α. We may assume that every element of S is less than $\bar{i}^{++}$.

Let C be any constructible, closed unbounded subset of $\bar{i}$. Then any condition p in Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$ has an extension q such that for $\delta \in S$, $C(q(\delta))$ (= the set of splitting levels of the tree $q(\delta)$) is a subset of C. Choose such a q in $G(\bar{i})$. Then $\pi^*(q) = q^*$ has the property that for all $\delta \in \pi(S)$, $C(q^*(\delta))$ is a subset of $\pi(C)$. In particular, $C(q^*(\alpha))$ is a subset of $\pi(C)$. Now C was an arbitrary constructible, closed unbounded subset of $\bar{i}$ and the intersection of the $\pi(C)$'s is just $\{\bar{i}\}$. As $L[G(< i)]$ contains all $s : \bar{i} \to 2$ in $L[G(\leq \bar{i})]$, it follows that the intersection t of the $\pi^*(p)(\alpha)$, $p \in G(\bar{i})$, is a subtree of $2^{<i}$ which is the union of at most two cofinal branches, which can only differ at $\bar{i}$. If $\alpha = \pi(\bar{\alpha})$ belongs to the range of π, then $\bar{i}$ belongs to $\pi^*(C(p(\bar{\alpha}))) = C(\pi^*(p)(\pi(\bar{\alpha}))) = C(\pi^*(p)(\alpha))$ for each $p \in G(\bar{i})$, and therefore if s is the unique sequence of length $\bar{i}$ in t, both $s * 0$ and $s * 1$ belong to t. It follows in this case that t is in fact the union of two cofinal branches which do split at $\bar{i}$, an $(\bar{i}, i)$-tuning fork.

If α does not belong to the range of π then we argue as follows. It must be that S has L-cardinality exactly $\bar{i}$, as otherwise $\pi(S)$ is the pointwise image of S under π, implying that α is in the range of π. Now let $\langle \bar{\alpha}_\delta \mid \delta < \bar{i} \rangle$ be a 1-1 enumeration of S, and let D be the set of conditions p in Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$ such that for each $\delta < \bar{i}$, $p(\bar{\alpha}_\delta)$ is a tree whose first splitting level is greater than δ. Then D is dense, so we may choose such a p in $G(\bar{i})$. Then $\pi^*(\langle \bar{\alpha}_\delta \mid \delta < \bar{i} \rangle) = \langle \alpha_\delta \mid \delta < i \rangle$ enumerates $\pi(S)$ and $\pi^*(p)$ has the property that for each $\delta < i$, the first splitting level of $\pi^*(p)(\alpha_\delta)$ is greater than δ. Now α is an element of $\pi(S)$ and therefore is equal to α_δ for some unique $\delta < i$. As α is not in the range of π, δ is at least $\bar{i}$. It follows that the first splitting level of $\pi^*(p)(\alpha)$ is greater than $\bar{i}$, and therefore t, the intersection of the $\pi^*(p)(\alpha)$, does not split at $\bar{i}$. So t consists of exactly one cofinal branch through $2^{<i}$. $\Box$ (Lemma 10)

Now for $\alpha < i^{++}$ in the range of π, let $(x(\alpha)_0, x(\alpha)_1)$ be the branches that make up the $(\bar{i}, i)$-tuning fork at α, where $x(\alpha)_0(\bar{i}) = 0$, $x(\alpha)_1(\bar{i}) = 1$. For $\alpha < i^{++}$ not in the range of π, let $x(\alpha)_0$ denote the unique branch through all of the $\pi^*(p)(\alpha)$, $p \in G(\bar{i})$.

Lemma 11 *For any $\alpha < i$ and any subset S of i^{++} of size i in $L[G(< i)]$, the sequence $\langle x(\delta)_0 \restriction \alpha \mid \delta \in S\rangle$ belongs to $L[G(< i)]$.*

Proof. Write α as $\sigma(\bar{i}_0, \bar{i}, \infty)$, where σ is an L-definable function, $\bar{i}_0$ is a finite increasing sequence of indiscernibles less than $\bar{i}$ and ∞ is any finite increasing sequence of indiscernibles (of the appropriate length). Let C be a constructible, closed unbounded subset of $\bar{i}$ such that the least element of $\pi(C)$ greater than $\bar{i}$ is also greater than α. We may assume that S is the image $\pi^*(\bar{S})$ of some subset $\bar{S}$ of $\bar{i}^{++}$ of size $\bar{i}$ in $L[G(< \bar{i})]$, as if S equals $\tau(i_0, \bar{i}, i, \infty)^{G(<i)}$, where τ is an L-definable function, i_0 is a finite increasing sequence of indiscernibles less than $\bar{i}$, ∞ is any finite increasing sequence of indiscernibles greater than i of the appropriate length and $\tau(i_0, \bar{i}, i, \infty)$ is a $P(< i)$-name, then S is contained in $\pi^*(\bar{S})$ where $\bar{S}$ is the union of the set of $\tau(i_0, \delta, \bar{i}, \infty)^{G(<\bar{i})}$, $\max(i_0) < \delta < \bar{i}$. Let $\langle \bar{\alpha}_\delta \mid \delta < \bar{i}\rangle$ be a 1-1 enumeration of $\bar{S}$.

Each condition p in Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$ has an extension q whose support contains $\bar{S}$ such that for all $\delta < \bar{i}$, $C(q(\bar{\alpha}_\delta))$ is a subset of $C \setminus (\delta + 1)$. Choose such a q in $G(\bar{i})$. Then $S = \pi^*(\bar{S})$ is included in the support of $\pi^*(q)$ and S has the 1-1 enumeration $\pi^*(\langle \bar{\alpha}_\delta \mid \delta < \bar{i}\rangle) = \langle \alpha_\delta \mid \delta < i\rangle$. For $\delta < i$, $C(\pi^*(q)(\alpha_\delta))$ is a subset of $\pi(C) \setminus (\delta + 1)$. In particular, $\pi^*(q)(\alpha_\delta)$ has no splits between $\bar{i}$ and α for all $\delta < i$ and no splits between 0 and α for $\bar{i} \leq \delta < i$. Note that α_δ belongs to the range of π iff δ is less than $\bar{i}$. It follows that for $\delta \in [\bar{i}, i)$, $x(\alpha_\delta)_0 \restriction \alpha$ is the unique element of $\pi^*(q)(\alpha_\delta)$ of length α, and for $\delta < \bar{i}$, $x(\alpha_\delta)_0 \restriction \alpha$ is the unique element of $\pi^*(q)(\alpha_\delta)$ of length α which takes the value 0 at $\bar{i}$ and extends $x(\alpha_\delta)_0 \restriction \bar{i}$. Thus the sequence $\langle x(i)_0 \restriction \alpha \mid i \in S\rangle$ can be computed from $\pi^*(q) \in L[G(< i)]$ together with the sequence $\langle x(\alpha_\delta)_0 \restriction \bar{i} \mid \delta < \bar{i}\rangle$. The latter is coded by a subset of $\bar{i}$ in $L[G(\leq \bar{i})]$ and therefore belongs to $L[G(\leq \bar{i})] \subseteq L[G(< i)]$. This proves the Lemma. $\Box$ (Lemma 11)

Lemma 12 *Let $G(i)$ consist of all conditions p in Sacks(i, i^{++}) of $L[G(< i)]$ such that for each $\alpha < i^{++}$, $x(\alpha)_0$ is contained in $p(\alpha)$. Then $G(i)$ is generic for Sacks(i, i^{++}) of $L[G(< i)]$ over $L[G(< i)]$ and contains $\pi^*[G(\bar{i})]$.*

Proof. Clearly $G(i)$ contains $\pi^*[G(\bar{i})]$, as by definition, $x(\alpha)_0$ is a branch through the intersection of the $\pi^*(p)(\alpha)$, p in $G(\bar{i})$.

Suppose that D is dense on Sacks(i, i^{++}) of $L[G(< i)]$ and belongs to $L[G(< i)]$. Write D as $\pi^*(f)(i)$ where f has domain $\bar{i}$ and belongs to $L[G(< \bar{i})]$. (This is possible as D is of the form $\sigma(i_0, \bar{i}, i, \infty)^{G(<i)}$, where σ is an L-definable function, i_0 is a finite increasing sequence of indiscernibles less than $\bar{i}$, ∞ is any finite increasing sequence of indiscernibles greater than i and $\sigma(i_0, \bar{i}, i, \infty)$ is a $P(< i)$-name; now let $f(\gamma)$ be $\sigma(i_0, \gamma, \bar{i}, \infty)^{G(<\bar{i})}$, for $\gamma < \bar{i}$.) We can assume that $f(\gamma)$ is dense on Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$ for each γ less than $\bar{i}$.

Suppose that $\bar{p}$ is a condition in Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$, S is a subset of $\bar{i}^{++}$ of size less than $\bar{i}$ in $L[G(< \bar{i})]$ and α is less than $\bar{i}$. Then an (S, α)-*thinning of* $\bar{p}$ is an extension of $\bar{p}$ obtained by thinning each $\bar{p}(\delta)$, $\delta \in S$, to the subtree consisting of all nodes compatible with some particular node on the α-th splitting level of $\bar{p}(\delta)$. (There are $(2^{|\alpha|})^{|S|}$ of $L[G(< \bar{i})]$ such thinnings, one for each way of choosing a node on the α-th splitting level of $\bar{p}(\delta)$ for each $\delta \in S$.) We say that a condition $\bar{p}$ in Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$ *reduces* $\bar{D}$, where $\bar{D}$ is dense on Sacks$(\bar{i}, \bar{i}^{++})$ of $L[G(< \bar{i})]$, iff for some subset S of $\bar{i}^{++}$ of size less than $\bar{i}$ in $L[G(< \bar{i})]$ and some $\alpha < \bar{i}$, any (S, α)-thinning of $\bar{p}$ meets $\bar{D}$.

If $\mathcal{D}$ is a collection of $\bar{i}$-many dense sets $\bar{D}$ in $L[G(< \bar{i})]$, then any condition $\bar{p}$ can be extended to a $\bar{q}$ which reduces each $\bar{D}$ in $\mathcal{D}$: List the elements of $\mathcal{D}$ as $\bar{D}_0, \bar{D}_1, \ldots$ in an $\bar{i}$-sequence. Extend $\bar{p} = \bar{p}_0$ to $\bar{p}_1$ meeting $\bar{D}_0$. Then choose some δ_1 in the support of $\bar{p}_1$ and extend $\bar{p}_1$ to $\bar{p}_2$, without changing the 1-st splitting level of $\bar{p}_1(\delta_1)$, so that by simply thinning $\bar{p}_2(\delta_1)$ to either choice on the 1-st splitting level, $\bar{D}_1$ is met. Then choose some δ_2 in the support of $\bar{p}_2$ and extend extend $\bar{p}_2$ to $\bar{p}_3$, without changing the 2-nd splitting levels of $\bar{p}_2(\delta_1), \bar{p}_2(\delta_2)$, so that by simply thinning each of $\bar{p}_3(\delta_1), \bar{p}_3(\delta_2)$ to any of the four choices of nodes on their 2-nd splitting levels, $\bar{D}_2$ is met. Continue in this way, so that at stage $\alpha + 1$, for any choice of nodes on the α-th splitting level of the trees $\bar{p}_\alpha(\delta)$, δ one of the "first α indices", thinning to those nodes will result in a condition that meets $\bar{D}_\alpha$ (this is possible as there are only $(2^{|\alpha|})^{|\alpha|} < \bar{i}$ such thinnings). The indices $\delta_1, \delta_2, \ldots$ should be chosen so that after $\bar{i}$ steps, every element of the support of the final condition $\bar{q} = \bar{p}_{\bar{i}}$ is one of the indices chosen. Then $\bar{q}$ is an extension of $\bar{p}$ that reduces each $\bar{D}$ in $\mathcal{D}$.

Thus we may choose a condition $\bar{p}$ in $G(\bar{i})$ which reduces each $f(\gamma)$, $\gamma < \bar{i}$. Then $p = \pi^*(\bar{p})$ reduces each $\pi^*(f)(\gamma)$, $\gamma < i$ and therefore reduces $D = \pi^*(f)(\bar{i})$. In $L[G(< i)]$, choose a subset S of i of size less than i and $\alpha < i$ such that any (S, α)-thinning of p meets D. Now for each $\delta \in S$ thin p by choosing an initial segment of $x(\delta)_0$ on the α-th splitting level of $p(\delta)$. This sequence of choices from the α-th splitting levels of the $p(\delta)$, $\delta \in S$ belongs to $L[G(< i)]$ by Lemma 11. It follows that this thinned out condition belongs to $G(i)$ and meets D. The proof that any two conditions in $G(i)$ are compatible is as in the proof of Lemma 6. So $G(i)$ is generic for Sacks(i, i^{++}) of $L[G(< i)]$ over $L[G(< i)]$, as desired. $\Box$ (Lemma 12)

Thus we can find the desired $P(i)$-generic $G(i)$ containing $\pi_{\bar{i}i}[G(\bar{i})]$, as demanded by requirement $(*)$. This completes the inductive construction of the $G(< i)$, i an indiscernible. The union of the $G(< i)$'s is the desired P-generic. $\Box$

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