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AN INNER MODEL FOR GLOBAL DOMINATION

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Abstract. In this paper it is shown that the global statement that the dominating number for κ is less than 2^κ for all regular κ, is internally consistent, given the existence of $0^\#$. The possible range of values for the dominating number for κ and 2^κ which may be simultaneously true in an inner model is also explored.

§1. Introduction.

Cardinal characteristics (or invariants) have been thoroughly studied in the context of the continuum. Some, such as the dominating number, have natural extensions to other cardinals. We begin with definitions of this generalisation.

DEFINITION 1.1. Let P be a partial ordering.

1. $D \subseteq P$ is *dominating* if and only if for all $p \in P$ there is $q \in D$ such that $p \leq q$. The subset D is also known as a *dominating family* for P.
2. $d(P)$ is the least cardinality of an dominating subset of P.

DEFINITION 1.2. Let κ be an infinite cardinal.

1. If $f, g \in {}^\kappa\kappa$ then $f <^* g$ if and only if there is $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $f(\beta) < g(\beta)$. If $f <^* g$, we say that g *dominates* f.
2. Let $d(\kappa) := d(({}^\kappa\kappa, <^*))$ be the *dominating number for* κ.

The dominating number for κ was studied by Cummings and Shelah in [1]. There they determined outer models which specified values for the dominating number and the analogously defined bounding number for infinite κ. These type of covering numbers have also been studied in other contexts such as universality: when no universal model exists for a set of structures of the same size, one may consider the size of a dominating or bounding family for the partial ordering consisting of the set of structures with the embedding relation (see e.g., [9], [3]). In this paper, we explore what values the dominating number for κ can take globally in an inner model.

Cummings and Shelah proved that it is consistent that the dominating number for any regular κ can be any regular cardinal in the interval $[\kappa^+, 2^\kappa]$. This was done by fixing cardinals $F(\kappa)$ and δ such that $F(\kappa)$ is any "reasonable" value for 2^κ and

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$\kappa^+ \leq \delta \leq F(\kappa)$ is regular. They then forced to add a minimal dominating family of $({}^\kappa\kappa, <^*)$ of size δ and subsets of κ so that $2^\kappa = F(\kappa)$ in the extension.

One may achieve this using an iteration of Cohen forcing followed by a generalised version of Hechler forcing. In particular, Cohen forcing is used to add $F(\kappa)$ many subsets of κ and then over that model, Hechler forcing is iterated with $< \kappa$-support for δ steps. At each stage, the generalised Hechler forcing adds a function which dominates all functions in ${}^\kappa\kappa$ which exist at previous stages.

After this forcing is done for any infinite κ, Cummings and Shelah proceeded to globalise this result. That is, they have shown that the values of 2^κ and $d(\kappa)$ can be fixed simultaneously for all regular κ.

This paper aims to discover if this global result can be true in an inner model assuming $0^\#$. An *inner model* is a definable transitive class containing all the ordinals and satisfying the axioms of ZFC. The known method of obtaining these models involves "shifting" generics at indiscernibles and if we were to use iterated Hechler at these cardinals, the generic shift would produce functions at the next indiscernible which could not be dominated (see e.g., [4],[5], [6] for background and details about this method).

Work on a solution to this was started in [6], where an inner model for Global Domination at inaccessible cardinals was obtained. *Global Domination* is the statement: $d(\kappa) < 2^\kappa$ for all infinite regular κ. For inaccessible κ, a generalised version of Sacks forcing (also known as perfect tree forcing) was used to increase the powerset of κ while keeping the dominating number for κ the same as it was in the ground model, namely κ^+.

Here, we expand this result to achieve a full version of Global Domination in an inner model while retaining the Sacks forcing at inaccessibles and the method of constructing generics given in [5]. Namely, the generic is built over L by induction on the Silver indiscernibles, which are in particular, inaccessible in L. For the set forcings between indiscernibles, we may use the distributivity of the forcing to build any generic. Therefore, we would still like to employ the Cohen-Hechler forcing at successor regulars. However, any naive attempt to add in the Cohen-Hechler forcing fails at the successors of inaccessibles because the perfect tree forcing at the inaccessible κ is κ^{++}-cc which, together with the Cohen forcing at κ^+, may ruin the domination obtained with Hechler forcing at κ^+.

For inaccessible cardinals κ, we will combine the forcings at κ and κ^+ so that the Hechler forcing at κ^+ will also dominate the functions in ${}^{\kappa^+}\kappa^+$ which are added by the perfect tree forcing at κ. This combined forcing imposes a heavy restriction on the dominating number at inaccessibles and their successors, but in exchange, the technique of "shifting generics" at these cardinals works.

We start with a result obtainable in an inner model which sets $d(\kappa) = \kappa^+$ and $2^\kappa = \kappa^{++}$ for all regular κ. Thus we have:

Theorem 1.3. Assume $0^\#$ exists. Then there is an inner model M of V which is a class generic extension of L, preserves all cardinals and cofinalities and in which Global Domination holds.

In section 5, we investigate the range of possible values that the dominating number and powerset functions can take in an inner model. In the $0^\#$ setting, using the methods above, this range is limited at a certain class of inaccessibles and successors of inaccessibles. In order to remove these restrictions, we start with a universe which has very large cardinal assumptions, namely a combination of the properties of Woodin and supercompact cardinals. With these assumptions (defined in section 5) we get the following result:

THEOREM 1.4. Suppose that GCH holds and the universe is definably super-Woodin. Then for any definable domination and powerset pattern (with the usual ZFC restrictions on these functions) there is an inner model in which this pattern is realised.

§2. Domination at inaccessibles and their successors.

We start with a model $M \models$ GCH. Let κ be inaccessible in M. Throughout the paper we will use the term κ-support to mean that the supports have size κ.

We will define a κ-support iteration of length κ^{++}, namely

$$P(\kappa, \kappa^+) = \langle P_\alpha, \underset{\sim}{Q}_\alpha : 0 \leq \alpha < \kappa^{++} \rangle.$$

The naming convention indicates that the forcing will fix the dominating number for κ and κ^+.

The stages of the iteration will consist of three different types of forcings, two of which will be performed at every other step. The first step of the iteration, Q_0, will be used to add κ^{+++} many Cohen subsets to κ^+, using the standard κ^{++}-cc and $(< \kappa^+)$-closed forcing, denoted $\mathrm{Add}(\kappa^+, \kappa^{+++})$. In particular GCH still holds at and below κ.

At every other stage starting at step 1 of the iteration, we will force with S which is a copy of the ground model perfect tree forcing at κ using only canonical names. We call these stages "perfect tree stages". At every other stage α starting with 2, we will force with H_α which is the generalised Hechler forcing at κ^+. We denote these stages as "Hechler stages". These two forcings will be defined below.

We define $P(\kappa, \kappa^+)$ as the κ-support "iteration" of $\langle P_\alpha, \underset{\sim}{Q}_\alpha : 0 \leq \alpha < \kappa^{++} \rangle$, however as all the names from the perfect tree stages will come from the ground model, it is not a normal iteration, but rather one which has been thinned. In essence, we are dovetailing a product of perfect tree forcing in the ground model with an iteration of Hechler.

The reasons for defining the forcing in this dovetailed manner are so that cardinals are not collapsed and to ensure that the Hechler forcing has the desired effect on the dominating number. That is, if we were simply to perform the perfect tree product at κ first, 2^κ would be raised past κ^+ and then over that model, the Hechler iteration at κ^+ would collapse 2^κ. If we were to do the Hechler iteration first then it may be the case that the perfect tree forcing at κ adds functions in ${}^{\kappa^+}\kappa^+$ that the Hechler functions cannot dominate. By dovetailing the forcings, 2^κ remains κ^+ until the end of the iteration and names from an initial segment of the perfect tree product are used at the Hechler stages in order to dominate functions introduced by the perfect tree forcing.

The advantage of using a thinned-out iteration with canonical names, rather than performing a full iteration, is that an iteration of perfect tree forcings would greatly complicate the arguments, especially for the proof of Claim 4.6 (see [2] for an example of similar arguments involving perfect tree iterations). We will see that restricting the perfect tree forcing to canonical names does not harm the domination arguments, but makes it much easier to construct the generic in the global setting as will be seen in Section 4.

Perfect tree forcing. Let $\mathrm{Seq} = \bigcup_{\beta<\kappa} {}^{\beta}2$. If $q \subseteq \mathrm{Seq}$ is a tree and $s \in q$, then we say s *splits* in q if and only if $s^\frown 0 \in q$ and $s^\frown 1 \in q$. For a level β of the tree q, β is a *splitting level* of q if for all $s \in q$ of length β, we have that s splits in q. Let $\mathrm{Split}_\beta(q)$ denote the β-th splitting level of q. We say that q has *club splitting* if and only if there exists $C \subseteq \kappa$ club such that for all $s \in q$ we have that s splits in q if and only if $\mathrm{length}(s) \in C$.

Let $q \in S$ if and only if q is a closed perfect tree with club splitting, that is,

1. $q \subseteq \mathrm{Seq}$ is a tree,
2. if $s \in q$ then $s \restriction \beta \in q$ for every β,
3. if $\beta < \kappa$ is a limit ordinal and $s \restriction \gamma \in q$ for every $\gamma < \beta$ then $s \in q$,
4. q has club splitting.

Elements of S are called *perfect trees*. The ordering on S is the subset relation. That is, we say that q is stronger than p, written $q \leq p$, if and only if $q \subseteq p$.

DEFINITION 2.1. 1. If $\{p_\beta : \beta < \gamma\}$ for some $\gamma \leq \kappa$ is a decreasing sequence of conditions from S then we define the *meet* to be $p = \bigcap\{p_\beta : \beta < \gamma\}$.
2. if $\beta < \kappa$ and $p, q \in S$, then let $p \leq_\beta q$ if and only if $p \leq q$ and $\mathrm{Split}_\beta(q) = \mathrm{Split}_\beta(p)$.

The following results about S are proved in [7].

THEOREM 2.2. 1. S is $(<\kappa)$-closed and has size κ^+.
2. (Fusion) Let $\langle q_\beta : \beta < \kappa\rangle$ be a decreasing sequence in S such that $q_{\beta+1} \leq_\beta q_\beta$ for all $\beta < \kappa$ and if δ is limit, then $q_\delta = \bigcap_{\beta<\delta} q_\beta$. Then $q = \bigcap_{\beta<\kappa} q_\beta \in S$.
3. If G is S-generic over M then every $f : \kappa \to \kappa$ in the extension $M[G]$ is dominated by a function $g : \kappa \to \kappa$ in M.

Generalised Hechler forcing. Fix a cardinal κ. For $\alpha < \kappa^{++}$ let H_α denote κ^+-Hechler forcing, whose conditions are pairs (s, f) such that $s \in {}^{<\kappa^+}\kappa^+$ and $f \in {}^{\kappa^+}\kappa^+$. The ordering on conditions in H_α is as follows: $(s, f) \leq (t, f')$ (i.e., (s, f) is stronger than (t, f')) if and only if

- $\mathrm{dom}(t) \leq \mathrm{dom}(s)$ and $t = s \restriction \mathrm{dom}(t)$,
- $s(\beta) \geq f'(\beta)$ for all $\beta \in [\mathrm{dom}(t), \mathrm{dom}(s))$,
- $f(\beta) \geq f'(\beta)$ for all $\beta < \kappa^+$.

H_α is $(<\kappa^+)$-closed and satisfies the κ^{++}-cc. If G is generic for H_α then $f_G := \bigcup\{s : (\exists f)(s, f) \in G\}$ dominates all functions in ${}^{\kappa^+}\kappa^+$ of the ground model.

Properties of the mixed forcing. It was shown in [7] that a κ-support product and iteration of S of length κ^{++} has a generalised version of κ-fusion. However, as we are mixing in the κ^+-Hechler forcing, we must alter the definitions.

DEFINITION 2.3. • If $\langle p_\alpha : \alpha < \beta\rangle$ for some $\beta \leq \kappa$ is a decreasing sequence of conditions from $P(\kappa, \kappa^+)$, then define the *meet*, $\bigwedge_{\alpha<\beta} p_\alpha = p$ to be such that $\mathrm{Dom}(p) = \bigcup_{\alpha<\beta} \mathrm{Dom}(p_\alpha)$ and for each $\gamma \in \mathrm{Dom}(p)$,
  - if $\gamma = 0$ then $p(0) = \bigcup\{p_\alpha(0) : \alpha < \beta\}$,
  - if γ is a perfect tree stage let $p = \bigcap\{p_\alpha(\gamma) : \alpha < \beta\}$,

– if γ is a Hechler stage, then $p \upharpoonright \gamma \Vdash p(\gamma) = (s, F)$ such that if $p_\alpha(\gamma) = (s^\gamma_\alpha, F^\gamma_\alpha)$ then $s = \bigcup\{s^\gamma_\alpha : \alpha < \beta\}$ and F maps each δ to $\sup_{\alpha<\beta} F^\gamma_\alpha(\delta)$.

- If $p, q \in P(\kappa, \kappa^+)$ and B is a subset of the perfect tree stages of $\mathrm{Dom}(q)$ of size $< \kappa$, then let $p \leq_{B,\alpha} q$ if and only if $p \leq q$ and for every $\gamma \in B$ we have $p(\gamma) \leq_\alpha q(\gamma)$.
- Let $\langle B_\alpha : \alpha < \kappa\rangle$ be a continuous increasing sequence of subsets of κ such that for all $\alpha < \kappa$, all elements of B_α index perfect tree stages and $|B_\alpha| < \kappa$. Let $p_\alpha \in P(\kappa, \kappa^+)$ for $\alpha < \kappa$ be such that $p_{\alpha+1} \leq_{B_\alpha,\alpha} p_\alpha$ and $p_\delta = \bigwedge_{\alpha<\delta} p_\alpha$ for δ limit. In addition, assume that $\bigcup_{\alpha<\kappa} B_\alpha$ is the set of all ordinals in $\bigcup_{\alpha<\kappa} \mathrm{Dom}(p_\alpha)$ which index perfect tree stages. If for every *fusion sequence*, i.e., $\{(p_\alpha, B_\alpha) : \alpha < \kappa\}$ with the properties above, we have $\bigwedge_{\alpha<\kappa} p_\alpha \in P(\kappa, \kappa^+)$, then we say that $P(\kappa, \kappa^+)$ has the *generalised κ-fusion*.

Claim 2.4. $P(\kappa, \kappa^+)$ is $(< \kappa)$-closed, has the generalised κ-fusion and the κ^{++}-cc.

Proof. For any decreasing sequence $\{p_\alpha : \alpha < \beta\}$ such that $\beta < \kappa$, the meet $\bigwedge_{\alpha<\beta} p_\alpha$ is in $P(\kappa, \kappa^+)$ and is a lower bound, since the Cohen forcing, the perfect tree forcing and κ^+-Hechler forcing are all $(< \kappa)$-closed. Also, the intersection of canonical names is a canonical name.

Let $\{(p_\alpha, B_\alpha) : \alpha < \kappa\}$ be a fusion sequence and let p be its meet. The fact that perfect tree components of p have club splitting uses the same diagonal intersection argument as in [7]. The names at perfect tree components must also remain canonical ground model names. For Hechler components, κ^+-Hechler is $(< \kappa^+)$-closed.

In the ground model we have $2^\kappa = \kappa^+$ and $P(\kappa, \kappa^+)$ has κ support, so one may use a standard Δ system argument to show that $P(\kappa, \kappa^+)$ has the κ^{++}-cc. ⊣

Corollary 2.5. Forcing with $P(\kappa, \kappa^+)$ does not collapse cardinals.

Claim 2.6. Starting with a model where $2^\kappa = \kappa^+$ and $2^{\kappa^+} = \kappa^{++}$, forcing with $P(\kappa, \kappa^+)$ makes $2^\kappa = \kappa^{++}$ and $2^{\kappa^+} = \kappa^{+++}$ while $d(\kappa) = \kappa^+$ and $d(\kappa^+) = \kappa^{++}$.

Proof. Both the Cohen forcing at κ^+ and the κ^+-Hechler forcing do not add subsets of κ since they are $(< \kappa^+)$-closed and $2^\kappa = \kappa^+$ holds in the ground model. We will use a fusion argument to show that all functions in ${}^\kappa\kappa$ added by the mixed forcing are dominated in the ground model. To that end, assume that $\underset{\sim}{f}$ is a $P(\kappa, \kappa^+)$-name for a function in ${}^\kappa\kappa$.

We start with a condition $p \in P(\kappa, \kappa^+)$ and build a fusion sequence $\langle (p_\alpha, B_\alpha) : \alpha < \kappa\rangle$ such that $p_0 = p$, $B_0 = \emptyset$ and for each $\alpha < \kappa$, p_α decides values for $f(\alpha)$ up to a sequence of $< \kappa$ choices in the perfect tree components (as in [7, Theorem 6.2]). First let $p_1 \leq p_0$ be such that p_1 decides a value for $f(0)$. Then choose $\delta_1 \in \mathrm{Dom}(p_1)$ which indexes a perfect tree stage and such that there is $p_2 \leq p_1$ with $p_2(\delta_1) \upharpoonright \mathrm{Split}_1(p_2(\delta_1)) = p_1(\delta_1) \upharpoonright \mathrm{Split}_1(p_1(\delta_1))$ and any extension of p_2 whose first splitting level at component δ_1 is higher than that of p_2 decides a value for $f(1)$. That is, $p_2(\delta_1)$ is only thinned above the first splitting level such that the perfect trees above each node at the first splitting level are both conditions which decide a value for $f(1)$. Thus, by extending p_2 to a condition which chooses one of the two nodes at $\mathrm{Split}_1(p_2(\delta_1))$, a value is forced for $f(1)$. Now let $B_1 = \{\delta_1\}$.

Choose some δ_2 in the support of p_2 indexing a perfect tree stage and extend p_2 to p_3, without changing the second splitting levels of $p_2(\delta_1), p_2(\delta_2)$, so that by simply thinning each of $p_3(\delta_1)$, $p_3(\delta_2)$ to any of the four choices of nodes on their second splitting levels, a value for $f(2)$ is decided. Let $B_2 = \{\delta_1, \delta_2\}$. Continue in this way, so that at stage $\alpha + 1$, for any choice of nodes on the α-th splitting level of the trees $p_\alpha(\delta)$ for $\delta \in B_\alpha$, thinning to those nodes will result in a condition that decides a value for $f(\alpha)$ (this is possible as there are only $(2^{|\alpha|})^{|\alpha|} < \kappa$ such thinnings). The B_α should be chosen such that $\bigcup_{\alpha<\kappa} B_\alpha$ is exactly the set of perfect tree components in the support of $\bigcup_{\alpha<\kappa} \mathrm{Dom}(p(\alpha))$. Thus by generalised κ-fusion we have that $p_\kappa = \bigwedge_{\alpha<\kappa} p_\alpha \in P(\kappa, \kappa^+)$.

Thus, p_κ is a condition which decides $< \kappa$ many values for each $f(\alpha)$ for $\alpha < \kappa$. Define $g \in {}^\kappa\kappa$ of the ground model such that for each $\alpha < \kappa$ we have $g(\alpha)$ is the supremum of the possibilities for $f(\alpha)$ decided by p_κ plus one. Therefore, the dominating family in the ground model remains a dominating family in $V^{P(\kappa,\kappa^+)}$ and so in that model $d(\kappa) = \kappa^+$.

To show that $2^\kappa = \kappa^{++}$ in the extension, suppose $A \subseteq \kappa$ such that $A \in L[G(\kappa, \kappa^+)]$. We want to show that A depends only on the perfect tree components of $P(\kappa, \kappa^+)$, i.e., $A \in L[B]$ where B is only the perfect tree components of the forcing. Thus by a theorem of Gödel, since $B \subseteq \kappa^{++}$ we would have that $2^\kappa = \kappa^{++}$. Since $\mathrm{Add}(\kappa^+, \kappa^{+++})$ is $(< \kappa^+)$-closed, we may treat the model after Cohen forcing as if it were the ground model. We may enumerate the elements of A as $\langle A_\alpha : \alpha < \kappa \rangle$ and build a fusion sequence as above such that for each $\alpha < \kappa$, we have $p_{\alpha+1} \leq_{\alpha, B_\alpha} p_\alpha$ and $p_{\alpha+1}$ decides values for A_α up to $< \kappa$-many decisions in the perfect tree components of B_α.

Thus, the meet of this fusion sequence, p_κ is a condition where the values of A are completely determined by only by its perfect tree components.

To see that $d(\kappa^+) = \kappa^{++}$ in the end, first note that all functions ${}^{\kappa^+}\kappa^+$ in the extension are added at a stage before the end. We must show that the Hechler forcing at any Hechler stage α adds a function g_α which dominates all functions in ${}^{\kappa^+}\kappa^+$ which exist at that stage. Let $f \in {}^{\kappa^+}\kappa^+$ be a function which was added before stage α and let $p \in P_\alpha$ be such that $p \restriction \alpha \Vdash p(\alpha) = (s, F)$. Define F' which maps β to $(F(\beta) \cup f(\beta)) + 1$. Then extend p to p' (on coordinates below α) such that $p' \restriction \alpha \Vdash p'(\alpha) = (s, F')$. This forces that $f(\alpha) < g(\alpha)$ for all $\alpha \geq \mathrm{dom}(s)$. Thus, $\langle g_\alpha : \alpha < \kappa^{++} \rangle$ is a dominating family of minimal size (as $d(\kappa^+)$ must be $\geq \kappa^{++}$). $\dashv$

§3. The global forcing. For successor cardinals β, let $H(\beta)$ be the iteration $\mathrm{Add}(\beta, \beta^{++}) * \circledast\{H_\alpha : \alpha < \beta^+\}$ where $\mathrm{Add}(\beta, \beta^{++})$ is the usual Cohen forcing to add β^{++} many subsets of β and $\circledast\{H_\alpha : \alpha < \beta^+\}$ is the $< \beta$-support iteration of β-Hechler forcing as defined above.

Over the ground model L, define $\mathbb{P} = \{\mathbb{P}(\alpha) : \alpha \in \mathrm{Ord}\}$ to be a reverse Easton iteration as follows. For α inaccessible, $\mathbb{P}(\alpha) = P(\alpha, \alpha^+)$ of $V^{P(<\alpha)}$, for infinite successor cardinals α which are not successors of inaccessibles and for $\alpha = \aleph_0$, let $\mathbb{P}(\alpha) = H(\alpha)$ of $V^{P(<\lambda(\alpha))}$ where $\lambda(\alpha)$ is the largest limit cardinal $\leq \alpha$ and for all other ordinals, let $\mathbb{P}(\alpha)$ be the trivial forcing.

Notice that for each α the forcing $\mathbb{P}(\alpha)$ is defined in the model below the last limit cardinal, so the forcing in the interval $[\delta, \delta^{+\omega})$ for δ a limit cardinal is a product.

We may see that the forcing $\mathbb{P}$ preserves cardinals, as for each regular cardinal κ, $\mathbb{P}$ can be factored as follows: If κ is not inaccessible, then $\mathbb{P}$ can be factored as $\mathbb{P}(<\kappa) * \mathbb{P}(\kappa) * \mathbb{P}(>\kappa)$, where $\mathbb{P}(<\kappa) * \mathbb{P}(\kappa)$ is κ^+-cc (even when $\mathbb{P}(\kappa)$ is trivial) and $\mathbb{P}(>\kappa)$ is κ^+-distributive. If κ is inaccessible then $\mathbb{P}$ can be factored as $\mathbb{P}(<\kappa) * P(\kappa,\kappa^+) * \mathbb{P}(>\kappa^+)$, where $\mathbb{P}(<\kappa)$ is κ^+-cc, $P(\kappa,\kappa^+)$ satisfies generalised κ-fusion and is κ^{++}-cc and $\mathbb{P}(>\kappa)$ is κ^{++}-distributive.

Claim 3.1. In $L^{\mathbb{P}}$ for all regular κ we have $2^\kappa = \kappa^{++}$ and $d(\kappa) = \kappa^+$.

Proof. Fix a regular cardinal κ. If κ is inaccessible or the successor of an inaccessible, then $2^\kappa = \kappa^{++}$ is implied by Claim 2.6. If κ is some other successor, then $2^\kappa \geq \kappa^{++}$ follows from the fact that $\mathscr{P}(\kappa) \supseteq \mathrm{Add}(\kappa,\kappa^{++})$. To see that $2^\kappa \leq \kappa^{++}$, we use the factorisation of the forcing into $\mathbb{P}(<\kappa) * \mathbb{P}(\kappa) * \mathbb{P}(>\kappa)$: $L[G(<\kappa)]$ satisfies $2^\kappa = \kappa^+$, the κ-Hechler forcing adds only κ^+ many subsets of κ and $\mathbb{P}(>\kappa)$ is $(<\kappa^+)$-closed and so adds no subsets of κ.

To see that $d(\kappa) = \kappa^+$ if κ is inaccessible, note that $L[G(<\kappa)]$ satisfies $d(\kappa) = 2^\kappa = \kappa^+$ therefore, Claim 2.6 applies. If κ is the successor of an inaccessible, this is also considered in Claim 2.6. For all other regular κ, $d(\kappa) = \kappa^+$ follows from the properties of the Hechler iteration at κ and the fact that a product is taken at those cardinals. ⊣

Note that since we are taking an iteration over every limit stage, we do not have the problem of domination at successors of singulars as in [1].

§4. Finding a generic for the global forcing.

Theorem 4.1 ($0^\#$). Assume that $\mathbb{P} = \langle \mathbb{P}(\alpha) : \alpha \in \mathrm{Ord}\rangle$ is the iteration with Easton support definable in L without parameters as defined in Section 3. Then there exists G which is $\mathbb{P}$-generic over L.

Proof. In order to build the generic, we first note that any L-definable antichain in $\mathbb{P}$ is a set. This is because any L-definable club of cardinals contains an L-inaccessible and we are using Easton support: If X is an L-definable maximal antichain then for some L-inaccessible α, $X \cap L_\alpha$ is maximal in $\mathbb{P}(<\alpha)$. Then using Easton support, we may see for any $p \in P$, that $p \restriction \alpha$ is bounded in $\mathbb{P}(<\alpha)$, thus belongs to L_α. We may choose $q \in X \cap L_\alpha$ such that q is compatible with $p \restriction \alpha$ in $\mathbb{P}(<\alpha)$. Then p and q are compatible, therefore $X \cap L_\alpha$ is maximal in $\mathbb{P}$.

Let $I = \{i_\alpha : \alpha \in \mathrm{Ord}\}$ be the Silver indiscernibles for L in increasing order and let i^+ denote the L-successor of i. We will define $G(\leq i^+)$ generic for $\mathbb{P}(\leq i^+) = \mathbb{P}(<i) * P(i,i^+)$ by induction on $i \in I$. By the above argument about maximal antichains, it is clear that if we can show that $G(\leq i^+)$ is generic for $\mathbb{P}(\leq i^+)$ for each $i \in I$ then any maximal antichain X will be met by some $G(\leq i^+)$.

If α is a limit ordinal then we want $G(\leq i_\alpha^+)$ to be the "direct limit" of $G(\leq i_\beta^+)$ for $\beta < \alpha$. In order to achieve the compatibility needed to make this generic, we will use the shift map below.

Let $\beta < \beta'$. We will define $\pi_{i_\beta,i_{\beta'}}$ as follows:

$$\pi_{i_\beta,i_{\beta'}}(i_\gamma) = \begin{cases} i_\gamma & \gamma < \beta, \\ i_{\beta'+(\gamma-\beta)} & \gamma \geq \beta. \end{cases}$$

Shifting up indiscernible extends uniquely to an elementary embedding from L into L. We will abuse notation and denote this extension in the same way.

In [4] it is shown that if $\pi_{i_\beta,i_{\beta'}}[G(\leq i^+_\beta)] \subseteq G(\leq i^+_{\beta'})$ for all $\beta < \beta' < \alpha$ for α limit, then the direct limit is generic for $\mathbb{P}(\leq i^+_\alpha)$. We give this result here with proof for convenience.

Lemma 4.2. If $\pi_{i_\beta,i_{\beta'}}[G(\leq i^+_\beta)] \subseteq G(\leq i^+_{\beta'})$ for all $\beta < \beta' < \alpha$ for α limit, then the direct limit $G(\leq i^+_\alpha) = \bigcup_{\beta<\alpha} \pi_{i_\beta,i_\alpha}[G(\leq i^+_\beta)]$ is generic for $\mathbb{P}(\leq i^+_\alpha)$.

Proof. Let $\Delta = t(\vec{i}, i_\alpha, i^+_\alpha, \vec{\infty})$, where t is a definable function in L, $\vec{i}$ is a finite set of indiscernibles less than i_α and $\vec{\infty}$ is a finite set of indiscernibles greater than i^+_α, be a maximal antichain in $\mathbb{P}(\leq i^+_\alpha)$. Then $\bar{\Delta} = t(\vec{i}, i_\beta, i^+_\beta, \vec{\infty})$ is a maximal antichain in $\mathbb{P}(\leq i^+_\beta)$. If $p \in G(\leq i^+_\beta) \cap \bar{\Delta}$ and $\pi_{i_\beta,i_\alpha}[G(\leq i^+_\beta)] \subseteq G(\leq i^+_\alpha)$ then $\pi_{i_\beta,i_\alpha}(p) \in \Delta \cap G(\leq i^+_\alpha)$. ⊣

For more information about this type of argument see [5].

We need to arrange this coherence at successor indiscernibles.

Let $i < j$ be adjacent indiscernibles in I. Since the forcing at indiscernibles i is coupled with the forcing at i^+, we will assume that we have already built a generic $G(i, i^+)$ for the forcing $P(i, i^+)$. The main challenge here is to build $G(j, j^+)$ generic for the forcing $P(j, j^+)$ such that $\pi_{i,j}[G(i, i^+)] \subseteq G(j, j^+)$. (Note that here we abuse the notation $\pi_{i,j}$ to mean the canonical extension of the indiscernible shift to the model $L[G(< i)]$.)

As it turns out, simply shifting up conditions in $G(i, i^+)$ and intersecting them already generates a choice of generics for $P(j, j^+)$ so we only need to choose one. A similar argument is given in [6]; it is sketched here with the main points detailed below.

Lemma 4.3. For $\alpha < j^{++}$, an index for a perfect tree stage, let t be the meet of all $\pi_{i,j}(p)(\alpha)$ for p in $G(i, i^+)$. If α belongs to the range of $\pi_{i,j}$, then t is an (i, j)-*tuning fork*, i.e., a subtree of $2^{<j}$ which is the union of two cofinal branches which split at i. If α does not belong to the range of $\pi_{i,j}$, then t consists of exactly one cofinal branch through $2^{<j}$.

The proof, given in [6], uses the following lemma:

Lemma 4.4. For all $p \in G(i, i^+)$, for all $S \subseteq j^{++}$ a set of ordinals of size $< j$ indexing perfect tree stages and for all $\alpha < j$ there exists $p^* \leq p$ such that $p^* \in G(i, i^+)$ and for all $\delta \in S$ we have that $\pi_{i,j}(p^*)(\delta)$ has no splitting between i and $\mathrm{Split}_\alpha(\pi_{i,j}(p^*)(\delta))$. We may also arrange p^* such that for all $\delta \in S \setminus \mathrm{Ran}(\pi_{i,j})$ we have that $\pi_{i,j}(p^*)(\delta)$ has no splitting below $\mathrm{Split}_\alpha(\pi_{i,j}(p^*)(\delta))$.

Proof. We may write S as $\pi_{i,j}(f)(i)$ for some function f such that $f(\beta) = \bar{s}_\beta$ for some $\bar{s}_\beta \subseteq i^{++}$ of size $< i$ for each $\beta < i$. We may also write α as $\pi_{i,j}(g)(i)$ for some function g such that $g(\beta) = \bar{\alpha}_\beta$.

Fix some $p \in G(i, i^+)$. We define p^* such that $p^* \in G(i, i^+)$ and for each $\beta < i$ and each $\gamma \in \bar{s}_\beta$, let $p^*(\gamma)$ be an extension of $p(\gamma)$ such that $p^*(\gamma)$ has no splitting between the β-th and the $\bar{\alpha}_\beta$-th splitting level of $p(\gamma)$.

Then $\pi_{i,j}(p^*)$ has no splitting between i and the $\pi_{i,j}(g)(i) = \alpha$-th splitting level of $\pi_{i,j}(p)(\gamma)$ for each $\gamma \in \pi_{i,j}(f)(i) = S$.

To prove the second statement we need to produce such a p^* such that for $\delta \in S \setminus \mathrm{Ran}(\pi_{i,j})$ we have that $\pi_{i,j}(p^*)(\delta)$ has no splitting at levels $\leq i$. This is done similarly as above except that we arrange for all $\beta < i$ and all $\gamma \in \bar{s}_\beta$ we have

$p^*(\gamma) \leq p(\gamma)$ such that the first splitting level of $p^*(\gamma)$ is above β. Then the first splitting level of $\pi_{i,j}(p^*)(\delta)$ must be above i. ⊣

Now for $\alpha < j^{++}$, an index for a perfect tree stage in the range of $\pi_{i,j}$, let $(x(\alpha)_0, x(\alpha)_1)$ be the branches of the (i, j)-tuning fork at α, where $x(\alpha)_0(i) = 0$, $x(\alpha)_1(i) = 1$. For such $\alpha < j^{++}$ not in the range of π, let $x(\alpha)_0$ denote the unique branch through all of the $\pi_{i,j}(p)(\alpha)$ such that $p \in G(i, i^+)$.

Let $A' = \{\pi_{i,j}(p) : p \in G(i, i^+)\}$ and let $q \in A$ be the extensions of elements of $q' \in A'$ such that for all $\alpha \in \text{supp}(q') \cap \text{Ran}(\pi_{i,j})$ we have $x(\alpha)_0 \restriction i + 1 \in q(\alpha)$. Let $G(j, j^+)$ be the set of all conditions p in $P(j, j^+)$ such that there is $q \in A$ with $p \geq q$.

The following lemma, proved in [6], will be used in the proof that $G(j, j^+)$ is generic.

Lemma 4.5. For any $\alpha < j$ and any subset A of ordinals of j^{++} of size j in $L[G(< j)]$ indexing perfect tree stages, the sequence $\langle x(\delta)_0 \restriction \alpha : \delta \in A\rangle$ belongs to $L[G(< j)]$.

Claim 4.6. $G(j, j^+)$, as defined above, is generic for $P(j, j^+)$ over $L[G(< j)]$ and $\pi_{i,j}[G(i, i^+)] \subseteq G(j, j^+)$.

Proof. The second statement is clear since for each $p \in G(j, j^+)$, there is a $q \in A$ such that $\pi_{i,j}(p) \geq q$.

To show that $G(j, j^+)$ is generic, we do the same analysis of dense sets as in [6, Lemma 12]. Suppose that D is dense on $P(j, j^+)$ of $L[G(< j)]$ and belongs to $L[G(< j)]$. Write D as $\pi_{i,j}(f)(j)$ where f has domain i and belongs to $L[G(< i)]$. (This is possible as D is of the form $\sigma(\bar{\imath}, i, j, \bar{k})^{G(<j)}$, where σ is an L-definable function, $\bar{\imath}$ is a finite increasing sequence of indiscernibles less than i, $\bar{k}$ is any finite increasing sequence of indiscernibles greater than j and $\sigma(\bar{\imath}, i, j, \bar{k})$ is a $\mathbb{P}(< j)$-name; now let $f(\gamma)$ be $\sigma(\bar{\imath}, \gamma, i, \bar{k})^{G(<j)}$, for $\gamma < i$.) We can assume that $f(\gamma)$ is dense on $P(i, i^+)$ of $L[G(< i)]$ for each $\gamma < i$ (as otherwise on components γ where $f(\gamma)$ is not dense, we may extend $f(\gamma)$ to a dense set $f^*(\gamma)$ and this extension of f also has the property that $D = \pi_{i,j}(f^*)(j)$).

Suppose that p is a condition in $P(i, i^+)$ of $L[G(< i)]$, $S \subseteq i^{++}$ of size less than i in $L[G(< i)]$ and $\alpha < i$. Then $q \leq p$ is an (S, α)-*thinning of* p if and only if for all $\delta \in S$ an index for a perfect tree stage, we have $q(\delta)$ is a subtree of $p(\delta)$ consisting of all nodes compatible with some particular node on the α-th splitting level of $p(\delta)$. (There are $(2^{|\alpha|})^{|S|}$ of $L[G(< i)]$ such thinnings, one for each way of choosing a node on the α-th splitting level of $p(\delta)$ for each $\delta \in S$.)

We say that a condition p in $P(i, i^+)$ of $L[G(< i)]$ *reduces* D, where D is dense on $P(i, i^+)$ of $L[G(< i)]$, if and only if for some $S \subseteq i^{++}$ of size less than i in $L[G(< i)]$ and some $\alpha < i$, any (S, α)-thinning of p meets D.

If $\mathscr{D}$ is a collection of i-many dense sets D in $L[G(< i)]$, then any condition p can be extended to a q which reduces each D in $\mathscr{D}$: Let $\langle D_\alpha : \alpha < i\rangle$ be an enumeration of the elements of $\mathscr{D}$. We will build a fusion sequence $\langle (p_\alpha, B_\alpha) : \alpha < i\rangle$ as follows. Extend $p = p_0$ to p_1 meeting D_0. Then choose some δ_1 in the support of p_1 indexing a perfect tree stage and extend p_1 to p_2, without changing the first splitting level of $p_1(\delta_1)$, so that it is forced by $p_2 \restriction \delta_1$ that by simply thinning $p_2(\delta_1)$ to either choice on the first splitting level, D_1 is met. Let $B_1 = \{\delta_1\}$. Then choose some

δ_2 in the support of p_2 indexing a perfect tree stage and extend p_2 to p_3, without changing the second splitting levels of $p_2(\delta_1), p_2(\delta_2)$, so that by simply thinning each of $p_3(\delta_1), p_3(\delta_2)$ to any of the four choices of nodes on their second splitting levels, D_2 is met. Let $B_2 = \{\delta_1, \delta_2\}$. Continue in this way, so that at stage $\alpha + 1$, for any choice of nodes on the α-th splitting level of the trees $p_\alpha(\delta)$ for $\delta \in B_\alpha$, thinning to those nodes will result in a condition that meets D_α (this is possible as there are only $(2^{|\alpha|})^{|\alpha|} < i$ such thinnings). The B_α should be chosen such that $\bigcup_{\alpha<i} B_\alpha$ is exactly the perfect tree components of the support of the final condition $q = p_i$. Then q is an extension of p that reduces each D in $\mathscr{D}$.

Thus for any $\gamma < i$ we may choose a condition p in $G(i, i^+)$ which reduces $f(\gamma)$. Then for each $\gamma < j$ we have $p' = \pi_{i,j}(p)$ reduces $\pi_{i,j}(f)(\gamma)$, and therefore reduces $D = \pi_{i,j}(f)(i)$. In $L[G(< j)]$, choose a subset S of j of size less than j and $\alpha < j$ such that any (S, α)-thinning of p' meets D. Now for each $\delta \in S$ indexing a perfect tree stage, thin p' by choosing an initial segment of $x(\delta)_0$ on the α-th splitting level of $p'(\delta)$. This sequence of choices from the α-th splitting levels of the $p'(\delta)$, for $\delta \in S$ belongs to $L[G(< j)]$ by Lemma 4.5. It follows that this thinned out condition belongs to $G(j, j^+)$ and meets D. So $G(j, j^+)$ is generic for $P(j, j^+)$ of $L[G(< j)]$ over $L[G(< j)]$, as desired. ⊣

Now that we have seen how to build the generic at each indiscernible stage, we show how to construct the generic in between indiscernibles and the ordering of the generic construction. It is easy to build a generic for $\mathbb{P}(< i_0)$, where i_0 is the first indiscernible, since $\mathscr{P}(\mathbb{P}(< i_0))^L$ is countable. We do that first. Then, we consider the forcing strictly between i_0^+ and $i_0^{+\omega}$, which is simply a product of Cohen and iterated Hechler forcings. We may use the distributivity of this set product to build a generic here over $\mathbb{P}(< i_0)$. Then, we go back and choose a generic for $P(i_0, i_0^+)$ which is $(< i_0)$-closed. If we had done these last two steps in reverse order, we would collapse cardinals. Then we may fill in the generic for $\mathbb{P}(i_0^{+\omega}, i_1)$, the forcing between $i_0^{+\omega}$ and i_1, the next indiscernible. This may also be built over $\mathbb{P}(< i_0^{+\omega})$ using the distributivity of the forcing.

Thus, we have a generic for $\mathbb{P}(< i_1)$. Building the generic between i_1 and i_2 will be the same for any adjacent indiscernibles. Take any generic for the product strictly between i_1^+ and $i_1^{+\omega}$, which is guaranteed to exist by distributivity. Then use the method above to generate $G(i_2, i_2^+)$ using $G(i_1, i_1^+)$. Continue as before. At limit indiscernibles, choose a generic for the ω-product directly above before taking the direct limit of the generics at previous indiscernibles. ⊣

§5. Extensions.

So far we have produced an inner model in which $d(\kappa) = \kappa^+$ and $2^\kappa = \kappa^{++}$ for all regular κ. However, at successor cardinals which are not successors of inaccessibles and indeed at many inaccessibles, we need not be so restrictive. Cummings and Shelah have found outer models fixing the dominating number and powerset function with minimal restrictions. With these in mind, we define a *global domination pair* to be a pair of class functions (d, F) obeying the following:

1. The domains of d and F are the class of regular cardinals.
2. For each regular κ, $\kappa < d(\kappa) \leq F(\kappa)$, $d(\kappa)$ is regular and $F(\kappa)$ has cofinality greater than κ.
3. For $\kappa_0 < \kappa_1$ regular, $F(\kappa_0) \leq F(\kappa_1)$.

The goal of this section is to analyse which global domination pairs may be realised in an inner model. First we will demonstrate why we cannot use the Cummings-Shelah method in the $0^{\#}$ context. Then we will determine the limitations of the methods of the previous sections assuming only $0^{\#}$. Finally, we investigate other large cardinal assumptions which allow us to remove these limitations.

Notice that no inner model M of $L[0^{\#}]$ (other than $L[0^{\#}]$ itself) contains, for any regular $L[0^{\#}]$-cardinal κ, a function $f : \kappa \to \kappa$ which dominates all constructible functions in ${}^{\kappa}\kappa$. This is because for each choice of $\vec{i}$, a sequence of indiscernibles of length n which are greater than κ, the set $C_{\vec{i}} = \{\alpha < \kappa : \text{Hull}^L(\alpha \cup \{\kappa, \vec{i}\}) \cap \kappa = \alpha\}$ is constructible and the intersection of the $C_{\vec{i}}$'s is precisely the set of indiscernibles less than κ. By enumerating the elements of each $C_{\vec{i}}$ we get a function in ${}^{\kappa}\kappa$. If $f : \kappa \to \kappa$ were to dominate these enumeration functions, then by the regularity of κ in $L[0^{\#}]$, sufficiently large closure points of f would be indiscernible, implying that $0^{\#}$ belongs to M.

It follows that Global Domination in an inner model of $L[0^{\#}]$ cannot be achieved by adding a function which dominates all constructible functions.

Moreover, the method of Cummings-Shelah, which employs forcing to add dominating functions at inaccessibles, cannot be used to obtain inner models of global domination without strong large cardinal assumptions. The reason is that to obtain such inner models one is forced to extend an elementary embedding $j : V \to M$ with critical point κ to an embedding $j^* : V[G] \to M[G^*]$, where G, G^* are generic over V, M, respectively. However without strong assumptions about j, the ordinals $j(f)(\kappa)$ for $f : \kappa \to \kappa$ in M will be cofinal in $j(\kappa)$, which implies that no function $f^* : j(\kappa) \to j(\kappa)$ can dominate each $j(f)$ at κ. Therefore the desired G^* cannot exist! Indeed, more than a superstrong embedding is needed to lift to forcing extensions which add dominating functions.

For successor cardinals α which are not successors of inaccessibles, the iteration consisting of α-Cohen forcing and iterated α-Hechler forcing may be used to set the global domination pair for α with no extra restrictions. That is, using the notation from Section 3, we would need to force with the iteration $\text{Add}(\alpha, F(\alpha)) * \circledast\{H_\beta : \beta < d(\alpha)\}$ with $< \alpha$-support for any $F(\alpha)$ and $d(\alpha)$ as above.

Moreover, in order to use the method of shifting generics as in Section 4, we only need that the indiscernibles κ and their successors employ the combined $P(\kappa, \kappa^+)$ forcing defined in Section 2. That is, for any L-definable class X containing the Silver indiscernibles, we may use the Cohen/Hechler forcing for inaccessibles outside of X and their successors. Here however, there remains the issue of domination at the successors of singular cardinals and non-Mahlo inaccessibles outside of X. To deal with this, one uses the method of "Easton tail iteration" as defined in [1]. There it was shown that after forcing with the Easton tail iteration no cardinals are collapsed and the dominating number at these cardinals is as desired.

The main problem with obtaining the full globalised result with only the above restrictions in an inner model assuming only $0^{\#}$ is that of the interaction between the indiscernible and its successor. Using the methods of the combined Cohen/Sacks/Hechler forcing, it is not clear that the results of the previous sections can be extended very far. That is, one may not extend the forcing $P(\kappa, \kappa^+)$ beyond the length of κ^{++} as at the point where 2^κ becomes κ^{++}, the chain condition for the κ^+-Hechler forcing is lost. We may, however, loosen the restriction on the value of $F(\kappa^+)$, by forcing with $\mathrm{Add}(\kappa^+, F(\kappa^+))$ at the first stage of the iteration with only above restrictions on F.

However, restricting the values of one of these adjacent cardinals gives us more freedom to choose the values at the other cardinal. For instance, if at the inaccessible κ we fix $d(\kappa) = F(\kappa)$ using Cohen forcing, then we may use κ^+-Cohen forcing and iterated κ^+-Hechler forcing to set the powerset and dominating number for κ^+ with only the above restrictions. Also, by setting $d(\kappa^+) = F(\kappa^+)$ using Cohen forcing, we are free to simply use a Sacks product at κ of length $F(\kappa)$ to raise the value of 2^κ to the desired level while keeping the dominating number κ^+.

Examples of global domination pairs which are not known to hold in an inner model assuming only $0^\#$ are basically those where the values of d and F are separated for both the inaccessible and its successor (for many inaccessibles):

- $d(\kappa) < F(\kappa)$ and $\kappa^{++} < d(\kappa^+) < F(\kappa^+)$
- $d(\kappa) < F(\kappa)$, $F(\kappa) > \kappa^{++}$ and $d(\kappa^+) < F(\kappa^+)$.

As concrete examples of these combinations, we have the following:

$$d(\kappa) = \kappa^+, \qquad F(\kappa) = \kappa^{++},$$
$$d(\kappa^+) = \kappa^{+3}, \qquad F(\kappa^+) = \kappa^{+4}$$

and

$$d(\kappa) = \kappa^+, \qquad F(\kappa) = \kappa^{+3},$$
$$d(\kappa^+) = \kappa^{++}, \qquad F(\kappa^+) = \kappa^{+3}.$$

However, with a sufficiently strong large cardinal assumption, it is possible to establish the internal consistency of any definable pattern of global domination. For an inner model M we let (d^M, F^M) be the global domination pair in M realised by M, i.e., $d^M(\kappa)$ is the dominating number at κ in M and $F^M(\kappa)$ is $(2^\kappa)^M$ for each κ regular in M.

Definition 5.1. (In class theory) The universe is *super-Woodin* if and only if for each class A there exists a cardinal κ such that for each cardinal λ there is an elementary embedding $j : V \to M$ such that:

1. j has critical point κ and $j(\kappa)$ is greater than λ.
2. M is closed under λ-sequences.
3. $A \cap H(\lambda) = j(A) \cap H(\lambda)$.

The universe is *definably super-Woodin* if and only if the above holds for classes A which are definable without parameters.

Remark 5.2. An equivalent formulation of (definable) super-Woodinness is the following: For any (definable) function $f : \mathrm{Ord} \to \mathrm{Ord}$ there is a cardinal κ closed under F and an elementary embedding $j : V \to M$ with critical point κ such that M is closed under $j(f)(\kappa)$-sequences. Therefore the statement that universe is (definably) super-Woodin follows, for example, from the existence of a class of almost huge cardinals (see e.g., [8] for definition) which is stationary (with respect to definable clubs).

Theorem 5.3. Suppose GCH holds, the universe is definably super-Woodin and φ is a formula that defines a global domination pair (d, F). Then there is a pair

$W_0 \subseteq W_1$ of inner models which share the same cofinalities such that φ defines in W_0 the global domination pair (d^{W_1}, F^{W_1}) realised in W_1.

PROOF. Using definable super-Woodinness, choose cardinals $\kappa < \lambda$ with the following properties:

(a) φ defines in $H(\kappa)$ the pair $(d \upharpoonright \kappa, F \upharpoonright \kappa)$.
(b) λ is a Mahlo closure point of F greater than κ.
(c) There is an elementary embedding $j : V \to M$ with critical point κ such that M is closed under λ-sequences and (d, F) agrees with $(j(d), j(F))$ below λ.

Properties (a) and (b) can be fulfilled as property (3) in the definition of definable super-Woodinness ensures that the class of κ witnessing definable super-Woodinness for any given definable predicate A is stationary with respect to definable clubs.

We may assume that j is given by an ultrapower and therefore any inaccessible greater than λ is a fixed point of j. Choose an inaccessible δ greater than λ and a measurable γ greater than δ. Now let N be a countable elementary submodel of a large $H(\theta)$ which has $\kappa, \lambda, \delta, \gamma, d \upharpoonright \lambda, F \upharpoonright \lambda, j \upharpoonright H(\delta), M \cap H(\delta)$ and U as elements where U is a measure on γ. Let $\bar{\kappa}, \bar{\lambda}, \bar{\delta}, \bar{\gamma}, \bar{d}, \bar{F}, \bar{j}, \bar{M}$ and $\bar{U}$ be the images of these parameters under the isomorphism of N with its transitive collapse $\bar{N}$.

Now choose a generic $\bar{G}$ over $\bar{N}$ for the restriction to $\bar{\lambda}$ of the global forcing $\bar{P}$ of [1], followed by the forcing $\bar{Q}^{\bar{M}[\bar{G}(<\bar{\lambda})]}$ at stage $\bar{\lambda}$ in $\bar{j}(\bar{P})$ that realises $(\bar{j}(\bar{d}), \bar{j}(\bar{F}))$ at $\bar{\lambda}$. Thus $\bar{Q}^{\bar{M}[\bar{G}(<\bar{\lambda})]}$ is the forcing to add $\bar{j}(\bar{F})(\bar{\lambda})$-many Cohen subsets of $\bar{\lambda}$, followed by an iteration of Hechler forcing at $\bar{\lambda}$ of length $\bar{j}(\bar{d})(\bar{\lambda})$. Note that the forcing $\bar{Q}^{\bar{M}[\bar{G}(<\bar{\lambda})]}$, although defined in $\bar{M}[\bar{G}(<\bar{\lambda})]$, is $(<\bar{\lambda})$-closed and $\bar{\lambda}^+$-cc in $\bar{N}[\bar{G}(<\bar{\lambda})]$ as $\bar{Q}^{\bar{M}[\bar{G}(<\bar{\lambda})]}$ is dense in $\bar{Q}^{\bar{N}[\bar{G}(<\bar{\lambda})]}$ since the models $\bar{M}[\bar{G}(<\bar{\lambda})]$ and $\bar{N}[\bar{G}(<\bar{\lambda})]$ have the same $\bar{\lambda}$-sequences. Thus the resulting generic extension $\bar{N}[\bar{G}]$ of $\bar{N}$ has the same cofinalities as $\bar{N}$ and realises the global domination pair $(\bar{d}, \bar{F})$ below $\bar{\lambda}$. The generic $\bar{G}$ exists by the countability of $\bar{N}$. We claim that in $\bar{N}[\bar{G}]$, the embedding $\bar{j}$ from $H(\bar{\delta})^{\bar{N}}$ to $\bar{M}$ extends to an elementary embedding of $H(\bar{\delta})^{\bar{N}}[\bar{G}(<\bar{\lambda})]$ to $\bar{M}[\bar{H}]$ for some $\bar{H}$.

The generic $\bar{H}$ can be obtained as follows. Since by the second part of condition (c) on the choices of cardinals κ, λ the forcings $\bar{P}$ and $\bar{j}[\bar{P}]$ agree below $\bar{\lambda}$, we may copy the generic $\bar{G}$ below $\bar{\lambda}$ to form $\bar{H}$ below $\bar{\lambda}$. We take $\bar{H}$ at $\bar{\lambda}$ to be a copy of the generic $\bar{G}$ at stage $\bar{\lambda}$.

To obtain $\bar{H}$ on the interval $(\bar{\lambda}, \bar{j}(\bar{\lambda}))$, first note that by the GCH in $\bar{N}$, $\bar{j}(\bar{\lambda})$ has cardinality $\bar{\lambda}^+$ in $\bar{N}$ and the forcing in the interval $(\bar{\lambda}, \bar{j}(\bar{\lambda}))$ has only $\bar{\lambda}^+$-many (in the sense of $\bar{N}[\bar{G}]$) maximal antichains which belong to $\bar{M}[\bar{H}(<\bar{\lambda})]$. Now using the $\bar{\lambda}^+$-closure of the forcing $\bar{P}$ above $\bar{\lambda}$ and the fact that $\bar{M}[\bar{H}(\leq \bar{\lambda})]$ is closed under $\bar{\lambda}$-sequences in $H(\bar{\delta})^{\bar{N}}[\bar{G}]$ we can construct a filter that meets all of these antichains. But we must also ensure that the resulting generic $\bar{H}(\bar{\lambda}, \bar{j}(\bar{\lambda}))$ contains all conditions in $\bar{j}^*[\bar{G}[\bar{\kappa}, \bar{\lambda})]$ (where $\bar{j}^*$ is the canonical lifting of $\bar{j}$ to $H(\bar{\delta})^{\bar{N}}[\bar{G}(<\bar{\kappa})]$). However using the closure of $\bar{M}[\bar{H}(<\bar{j}(\bar{\kappa}))]$ under $\bar{\lambda}$-sequences in $\bar{N}[\bar{G}]$, we can form the lower bound m of the $\bar{\lambda}$-many conditions $\bar{j}^*(\bar{p})$, $\bar{p}$ in $\bar{G}[\bar{\kappa}, \bar{\lambda})$. Then any generic $\bar{H}[\bar{j}(\bar{\kappa}), \bar{j}(\bar{\lambda}))$ that contains this (master) condition m will contain every condition in $\bar{j}^*[\bar{G}[\bar{\kappa}, \bar{\lambda})]$. Thus we obtain the desired $\bar{H}$ on the interval $(\bar{\lambda}, \bar{j}(\bar{\lambda}))$.

Thus in $\bar{N}[\bar{G}]$ we have lifted to $\bar{j}$ to an elementary embedding $\bar{j}^*$ of $H(\bar{\delta})^{\bar{N}}[\bar{G}(<\bar{\lambda})]$ into $\bar{M}[\bar{H}]$. Let $\bar{D}$ be the ultrafilter derived from this embedding, i.e., the set of subsets $\bar{A}$ of $\bar{\kappa}$ in $\bar{N}[\bar{G}(<\bar{\lambda})]$ such that $\bar{\kappa}$ belongs to $\bar{j}^*(\bar{A})$. Then $\bar{D}$ belongs to $\bar{N}[\bar{G}]$ and as $H(\bar{\delta})^{\bar{N}}[\bar{G}(<\bar{\kappa})]$ and $\bar{N}[\bar{G}]$ have the same subsets of $\bar{\kappa}$, $\bar{D}$ witnesses the measurability of $\bar{\kappa}$ in $\bar{N}[\bar{G}]$.

Note that $\bar{\gamma}$ is still measurable in $\bar{N}[\bar{G}]$, as the forcing $\bar{P}$ has size less than $\bar{\gamma}$. Also, the measure $\bar{U}$ on $\bar{\gamma}$ is iterable, as $\bar{N}$ elementarily embeds into $H(\theta)$, sending $\bar{U}$ to U. As the forcing $\bar{P}$ has size less than $\bar{\gamma}$, it follows that the measure $\bar{U}$ extends to an iterable measure $\bar{U}^*$ on $\bar{\gamma}$ in $\bar{N}[\bar{G}]$. By iterating $\bar{U}^*$ through the ordinals, we see that $\bar{\kappa}$ is not only measurable in $\bar{N}[\bar{G}]$, but is in fact measurable in an inner model containing all the ordinals and therefore carries a measure in $\bar{N}[\bar{G}]$ which is iterable through all the ordinals.

Now iterate $\bar{N}[\bar{G}]$ through all the ordinals using an iterable measure on $\bar{\kappa}$. For each i let κ_i be the image of $\bar{\kappa}$ in the i-th iterate $N_i[G_i]$ and let W_0 be the inner model obtained as the union of the $H(\kappa_i)^{N_i}$, $i \in \text{Ord}$, W_1 the inner model obtained as the union of the $H(\kappa_i)^{N_i[G_i]}$, $i \in \text{Ord}$. By (a), $\bar{N}[\bar{G}]$ realises below $\bar{\kappa}$ the global domination pair defined by φ in $H(\bar{\kappa})^{\bar{N}}$; thus it follows that W_1 realises the global domination pair defined by φ in W_0, as desired. ⊣

By combining the global forcing given in Section 3 with the method of preserving a measurable cardinal in [6] we also obtain the following:

THEOREM 5.4. Suppose GCH holds and κ is a $P_2\kappa$-hypermeasurable (i.e., κ is the critical point of some elementary embedding $j : V \to M$ where $H(\kappa^{++})^V$ belongs to M). Then in a generic extension, κ is measurable and Global Domination holds.

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