Equivalence Relations on Classes of Computable Structures*

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Abstract. If $\mathcal{L}$ is a finite relational language then all computable $\mathcal{L}$-structures can be effectively enumerated in a sequence $\{\mathcal{A}_n\}_{n\in\omega}$ in such a way that for every computable $\mathcal{L}$-structure $\mathcal{B}$ an index n of its isomorphic copy $\mathcal{A}_n$ can be found effectively and uniformly. Having such a universal computable numbering, we can identify computable structures with their indices in this numbering. If K is a class of $\mathcal{L}$-structures closed under isomorphism we denote by K^c the set of all computable members of K. We measure the complexity of a description of K^c or of an equivalence relation on K^c via the complexity of the corresponding sets of indices. If the index set of K^c is hyperarithmetical then (the index sets of) such natural equivalence relations as the isomorphism or bi-embeddability relation are Σ^1_1. In the present paper we study the status of these Σ^1_1 equivalence relations (on classes of computable structures with hyperarithmetical index set) within the class of Σ^1_1 equivalence relations as a whole, using a natural notion of hyperarithmetic reducibility.

1 Introduction

Formalization of the notion of algorithm and studies of the computability phenomenon have resulted in increasing interest in the investigation of effective mathematical objects, in particular of algebraic structures and their classes. We call an algebraic structure *computable* if its universe is a computable subset of ω and all its basic predicates and operations are uniformly computable. For a class K of structures, closed under isomorphism, we denote by K^c the set of computable members of K. One of the questions of computable model theory is to study the algorithmic complexity of such classes of computable structures and various relations on these structures. In particular, we want to have a nice way to measure the complexity of a description of K^c or to compare the complexity of relations defined on different classes of computable structures. We say that K has a *computable characterization*, if we can separate computable structures in K from all other structures (not in K or noncomputable). Possible approaches to formalize the idea of computable characterizations of classes were described

* This work was partially supported by FWF Grant number P 19375 - N18.

K. Ambos-Spies, B. Löwe, and W. Merkle (Eds.): CiE 2009, LNCS 5635, pp. 198–207, 2009.

in [10]. One such approach involves the notion of an index set from the classical theory of numberings [7]. It will be described below. The same approach can be used to formalize the question of *computable classification* of K up to some equivalence relation, i. e. the question of existence of a description of each element of K up to isomorphism, or other equivalence relation, in terms of relatively simple invariants. In this paper we will discuss different ways to measure the complexity of equivalence relations on classes of computable structures.

This work is analogous to research in descriptive set theory, where the complexity of classes of structures and relations on these classes is studied via Borel-reducibility. In this paper we will study the questions that can be considered as computable versions of questions from [12,14,8].

First of all, we introduce the necessary definitions and basic facts from computable model theory.

2 Background

2.1 Computable Sequences and Indices of Structures

Consider a sequence $\{\mathcal{A}_n\}_{n\in\omega}$ of algebraic structures.

Definition 1. *A sequence* $\{\mathcal{A}_n\}_{n\in\omega}$ *is called* computable *if each structure* $\mathcal{A}_n$ *is computable, uniformly in* n.

In other words, there exists a computable function which gives us an index for the atomic diagram of $\mathcal{A}$ uniformly in n; equivalently, we can effectively check the correctness of atomic formulas on elements of each structure in the sequence uniformly.

Definition 2. *We call a sequence* $\{\mathcal{A}_n\}_{n\in\omega}$ *of computable structures* hyperarithmetical, *if there is a hyperarithmetical function which gives us, for every* n, *an index of the atomic diagram of* $\mathcal{A}_n$.

Let $\mathcal{L}$ be a finite relational language. A result of A. Nurtazin [15] shows that there is a universal computable numbering of all computable $\mathcal{L}$-structures, i. e. there exists a computable sequence $\{\mathcal{A}_n\}_{n\in\omega}$ of computable $\mathcal{L}$-structures, such that for every computable $\mathcal{L}$-structure $\mathcal{B}$ we can effectively find a structure $\mathcal{A}_n$ which is isomorphic to $\mathcal{B}$. Fix such a universal computable numbering.

One of the approaches from [10] involves the notion of index set.

Definition 3. *An* index set *of an* $\mathcal{L}$*-structure* $\mathcal{B}$ *is the set* $I(\mathcal{B})$ *of all indices of computable structures isomorphic to* $\mathcal{B}$ *in the universal computable numbering of all computable* $\mathcal{L}$*-structures. For a class* K *of structures, closed under isomorphism, the* index set $I(K)$ *is the set of all indices of computable members in* K.

We can use a similar idea to study the computable classification of classes of structures up to some equivalence relation. There are many interesting equivalence relations from the model-theoretic point of view. We can consider classes of structures up to isomorphism, bi-embeddability, or elementary bi-embeddability, bi-homomorphism, etc. There has been a lot of work on the isomorphism problem for various classes of computable structures (see, for example, [2,3,6,10]). There also has been some work on the computable bi-embeddability problem in [5], where the relation between the isomorphism problem and the embedding problem for some well-known classes of structures is studied. The approach used in the mentioned papers follows the ideas from [10] and makes use of the following definition.

Definition 4. *Let $\{\mathcal{A}_n\}_{n\in\omega}$ be as before the universal computable numbering of all computable $\mathcal{L}$-structures. Let K be a class of $\mathcal{L}$-structures closed under isomorphism, and let $I(K)$ be its index set. Then*

- *the* isomorphism problem *for K is the set of pairs $(a,b) \in I(K) \times I(K)$ such that $\mathcal{A}_a \cong \mathcal{A}_b$;*
- *the* bi-embeddability problem *for K is the set of pairs $(a,b) \in I(K) \times I(K)$ such that there is an embedding of $\mathcal{A}_a$ into $\mathcal{A}_b$ and an embedding of $\mathcal{A}_b$ into $\mathcal{A}_a$ (for any language L, for L-structures $\mathcal{A}$ and $\mathcal{B}$ we say that $\mathcal{A}$* embedds into *$\mathcal{B}$, $\mathcal{A} \sqsubseteq \mathcal{B}$, if $\mathcal{A}$ is isomorphic to a substructure of $\mathcal{B}$).*

Generalizing this idea, every binary relation E on a class K of structures can be associated with the set $I(E,K)$ of all pairs of indices $(a,b) \in I(K) \times I(K)$ such that the structures $\mathcal{A}_a$ and $\mathcal{A}_b$ are in the relation E. We can measure the complexity of various relations on computable structures via the complexity of the corresponding sets of pairs of indices.

2.2 Computable Trees

In further constructions we will often use computable trees. Here we give some definitions useful for describing trees. Our trees are isomorphic to subtrees of $\omega^{<\omega}$. For the language, we take a single unary function symbol, interpreted as the predecessor function. We write $\emptyset$ for the top node (our trees grow down), and we think of $\emptyset$ as its own predecessor. Thus, our trees are defined on ω with their structure given by the predecessor function, but we often consider them as subtrees of $\omega^{<\omega}$ and treat their elements as finite sequences. In particular, following [14], we define a relation $\leq$ on tree nodes $s,t \in \omega^{<\omega}$ of the same length in the following way: $s \leq t$ iff for all i, the i-th coordinate of t is greater than or equal to the i-th coordinate of s. We also define the operation $s+t$ as coordinate-wise addition.

2.3 Σ^1_1 Sets

Kleene defined the *analytical hierarchy*, starting with computable relations on numbers and functions (from ω to ω) and closing under projection and complement. We need only the bottom part of this hierarchy. We use symbols $f,g,\ldots$ (possibly with indices) as function variables, and $x,y,\ldots$ (possibly, with indices) as number variables. A relation $R(\overline{x}, f)$ is *computable* if there is some e such that for all $\overline{x} \in \omega^{<\omega}$ and all $f \in \omega^{\omega}$,

$$\varphi_e^f(\overline{x}) \downarrow = \begin{cases} 1 \text{ if } R(\overline{x}, f) \\ 0 \text{ otherwise} \end{cases}$$

Definition 5. *Let $S(\overline{x})$ be a relation.*

1. *$S(\overline{x})$ is Σ_1^1 if it can be expressed in the form $(\exists f)(\forall y)\, R(\overline{x}, y, f)$, where $R(\overline{x}, y, f)$ is computable,*
2. *$S(\overline{x})$ is Π_1^1 if it can be expressed in the form $(\forall f)(\exists y)\, R(\overline{x}, y, f)$, where $R(\overline{x}, y, f)$ is computable,*
3. *$S(\overline{x})$ is Δ_1^1 if it is both Σ_1^1 and Π_1^1.*

If $S(\overline{x})$ is a k-place relation, we may consider the set S' of codes for k-tuples belonging to S. It is clear that S is Σ_1^1 iff S' is Σ_1^1. The same is true for Π_1^1 and Δ_1^1 relations. The next result gives familiar conditions equivalent to being Σ_1^1 [1,16]. We identify finite sequences with their codes.

Proposition 1 (Kleene)
The following are equivalent:

1. *S is Σ_1^1,*
2. *there is a computable relation $R(n, u)$, on pairs of numbers, such that $n \in S$ iff $(\exists f)\,(\forall s)\, R(n, f \upharpoonright s)$,*
3. *there is a c.e. relation $R(n, u)$, on pairs of numbers, such that $n \in S$ iff $(\exists f)\,(\forall s)\, R(n, f \upharpoonright s)$,*
4. *there is a computable sequence of computable trees $\{T_n\}_{n\in\omega}$ such that $n \in S$ iff T_n has a path.*

2.4 Sets of Indices

We review the formal approach of [10] to the problem of determining whether or not a class of computable structures can be nicely characterized or classified relative to some natural equivalence relation.

According to [10], we say that a class K *has a computable characterization*, if its index set is hyperarithmetical. This condition is equivalent to existence of a computable infinitary sentence φ such that K^c (the set of computable structures in K) consists exactly of all computable models of φ. This definition expresses the fact that the set of all computable members of K can be nicely defined among all other structures for the same language. Note, that if $I(K)$ is hyperarithmetical and E is the isomorphism or bi-embeddability relation, then the corresponding equivalence relation $I(E, K)$ on indices is a Σ_1^1 set. In the worst case, when this equivalence relation is properly Σ_1^1, the easiest way to say that two computable structures from K are in the relation E is to say "There are functions between the structures that are isomorphisms (embeddings) between them". Often there are easier ways to verify the relation (such as counting basis elements of vector spaces to determine the isomorphism). Within this approach we say that there is a *computable classification for* K *up to* E if the corresponding equivalence relation $I(E,K)$ on indices of computable structures from K is a hyperarithmetical set. The standard way to measure the complexity of equivalence relations on computable models from K is the following.

Definition 6. *Let* Γ *be a complexity class (e.g.,* Σ_3^0, Π_1^1*, etc.).* $I(E,K)$ *is* m-complete Γ *if* $I(E,K)$ *is* Γ *and for any* $S \in \Gamma$*, there is a computable function* f *such that*

$$n \in S \text{ iff } f(n) \in I(E,K).$$

By universality of the fixed computable numbering of $\mathcal{L}$-structures, this condition is equivalent to the condition that there is a computable sequence of pairs of computable $\mathcal{L}$-structures $\{(\mathcal{A}_n, \mathcal{B}_n)\}_{n\in\omega}$ from K for which $n \in S$ iff $\mathcal{A}_n E \mathcal{B}_n$.

3 A Complete Σ_1^1 Equivalence Relation

We can also measure the complexity of a relation E on a class of computable structures not as a set but as a relation, i. e. in a class of relations. This approach can be considered as a computable analog of the study of Borel-reducibility between analytic equivalence relations [12]. For this we need an appropriate notion of reducibility which allows us to compare relations on computable structures.

Let K be a class of structures with hyperarithmetical index set $I(K)$. As we have mentioned before, using indices we can identify every relation E on computable members of K with the corresponding relation $I(E,K)$ on natural numbers. Therefore, it is natural to restrict our attention to relations on ω.

Definition 7. *For equivalence relations* E', E'' *on* ω *we say that* E' *is* h-reducible *to* E''*,* $E' \leq_h E''$*, if there is a hyperarithmetical function* f *such that for all* x, y*,* $xE'y$ *iff* $f(x)E''f(y)$*.*

If E is an equivalence relation on K^c (the set of all computable members of K), then we often make no difference between E and $I(E,K)$ in the following sense. If E' is an arbitrary equivalence relation on ω then we say that E' *h-reduces* to E iff there exists a hyperarithmetical sequence of computable structures $\{\mathcal{A}_x\}_{x\in\omega}$ from K such that for all x, y, $xE'y$ iff $A_x E A_y$ (this is equivalent to $E' \leq_h I(E,K)$ in the sense of Definition 7).

Definition 8. *Let* $\mathcal{R}$ *be a class of relations. A relation* $R \in \mathcal{R}$ *is an* Σ_1^1 *h-*complete for $\mathcal{R}$*, if it is* Σ_1^1 *and every* Σ_1^1 *relation* $R' \in \mathcal{R}$ *h-reduces to* R*.*

Proposition 2. *Let* $\mathcal{R}$ *be the class of all isomorphism relations on classes* K^c*, where* K *is any class of structures with hyperarithmetical index set. The isomorphism relation on computable undirected graphs is* Σ_1^1 *h-complete for* $\mathcal{R}$*.*

Proof. Let K be a class of structures with hyperarithmetical index set. Using the effective transformations from [9] or [11], we get an h-reduction of the isomorphism relation on computable models of K to the isomorphism relation on computable graphs.

Theorem 1. *1. There is a class K of structures with hyperarithmetical index set and an equivalence relation on K^c which is an h-complete Σ^1_1 equivalence relation.*

2. There is a class K of structures with hyperarithmetical index set and a preorder on K^c which is an h-complete Σ^1_1 preorder.

The former statement of the theorem follows from the latter, as a complete Σ^1_1 equivalence relation results from any complete Σ^1_1 preorder. The converse is also true: the proof of the corresponding fact from [14] can be carried out effectively, and the Borel reducibility in the proof can be, in fact, substituted by h-reducibility:

Proposition 3. *Any h-complete Σ^1_1 equivalence relation on ω is induced by an h-complete Σ^1_1 preorder on ω.*

Theorem 1 can be regarded as a computable analog of results from [14]. Below we sketch the proof, which closely follows the proof of the non-effective version in [14]. First, we need a technical result on the representation of Σ^1_1 preorders.

Theorem 2. *Let R be a Σ^1_1 preorder on ω. Then there exists a computable sequence of computable trees $\{T^R_n\}_{n\in\omega}$, such that*

1. *$xRy \Leftrightarrow T^R_{\langle x,y\rangle}$ has a path;*
2. *$\forall s,t \in \omega^{<\omega}$ of the same length, such that $s \leq t$, if $s \in T^R_{\langle x,y\rangle}$ then $t \in T^R_{\langle x,y\rangle}$;*
3. *$\forall x T^R_{\langle x,x\rangle} = \omega^{<\omega}$;*
4. *If $s \in T^R_{\langle x,y\rangle}, t \in T^R_{\langle y,z\rangle}$ and $|s| = |t|$, then $s + t \in T^R_{\langle x,z\rangle}$.*

Proof. For every natural number m we define the function $\langle m\rangle$ by:

$$\langle m\rangle(x) = \begin{cases} 1 \text{ if } x = m+1 \\ 0 \text{ otherwise.} \end{cases}$$

We turn the preorder R on ω into a Σ^1_1 preorder R_0 on 2^ω in the following way:

$$xR_0y \Longleftrightarrow ((\exists m,n)x = \langle m\rangle \wedge y = \langle n\rangle \wedge mRn) \vee (x = y).$$

By [14], we get a tree S on $2 \times 2 \times \omega$, such that:

1. for all $x, y \in 2^\omega xR_0y$ iff for some $z \in \omega^\omega$, $(x|n, y|n, z|n) \in S$ for all n;
2. if $(u, v, s) \in S$ and $s \leq t$ then $(u, v, t) \in S$;
3. if $u \in 2^{<\omega}$ and $s \in \omega^{<\omega}$ have the same length, then $(u, u, s) \in S$;
4. if $(u, v, s) \in S$ and $(v, w, t) \in S$ then $(u, w, s + t) \in S$.

Note that the construction from [14] is highly effective and the resulting tree S is computable. Now we define a sequence of trees on ω using S. For every $s \in \omega^{<\omega}$ of length k, we let $s \in T^R_{\langle m,n\rangle}$ iff $(\langle m\rangle|k, \langle n\rangle|k, s) \in S$. The sequence $\{T^R_n\}_{n\in\omega}$ has all the necessary properties.

Proof (of Theorem 1). We define a computable structure $\mathcal{A}$ from K. Every such structure will code a computable sequence of computable trees with some additional property. The language for the class consists of one unary predicate symbol V and two unary function symbols g, h. In each model the function g is a successor function on $V = \{v_0, v_1, \dots, v_n, \dots\}$, and it defines on V a copy of ω. Each $v_n \in V$ is a root of a tree $T_n^{\mathcal{A}}$, with its structure given by h as a predecessor function. As we have mentioned before, we often consider $T_n^{\mathcal{A}}$ consisting of finite strings $s \in \omega^{<\omega}$. For a structure $\mathcal{A}$ of this kind to be in K we require that for all n, if $s \in T_n^{\mathcal{A}}, |s| = |t|$ and $s \leq t$ then $t \in T_n^{\mathcal{A}}$. From the definition it follows that $I(K)$ is hyperarithmetical.

Consider a preorder $\leq^*$ on K given in the following way.

$$A_1 \leq^* A_2 \Leftrightarrow \exists\varphi[\ \forall s(|s| = |\varphi(s)|) \text{ and } \forall s, t(s \preceq t \rightarrow \varphi(s) \preceq \varphi(t)) \\ \text{and } \forall s \in \omega^{<\omega}\{z | s \in T_z^{\mathcal{A}_1}\} \subseteq \{z | \varphi(s) \in T_z^{\mathcal{A}_2}\}].$$

Then $\leq^*$ is a Σ_1^1 preorder.

Consider an arbitrary Σ_1^1 preorder R on ω and prove that R is h-reducible to $\leq^*$. That is, for every x we need to hyperarithmetically build (uniformly in x) a computable structure $\mathcal{A}_x \in K$ such that $xRy \Leftrightarrow \mathcal{A}_x \leq^* \mathcal{A}_y$.

By Theorem 2, for R there is a computable sequence of computable trees T_n^R with the properties 1–4. We define the structure $\mathcal{A}_x$ as follows. For every z, the tree $T_z^{\mathcal{A}_x}$ (under the z-th element of $V^{\mathcal{A}_x}$) equals $T_{\langle z,x\rangle}^R$. Then $\{\mathcal{A}_x\}_{x\in\omega}$ is a hyperarithmetical (even computable) sequence of computable structures from K. We check that $xRy \Leftrightarrow \mathcal{A}_x \leq^* \mathcal{A}_y$.

($\Rightarrow$): Suppose xRy. Then there is a path f in the tree $T_{\langle x,y\rangle}^R$. Define a function $\varphi(s) = s + f \upharpoonright |s|$, where $s \in \omega^{<\omega}$. Then φ has the necessary properties. Let $z \in \omega$ and $s \in \omega^k$ be such that $s \in T_z^{\mathcal{A}_x}$. By definition it means that $s \in T_{\langle z,x\rangle}^R$. As xRy, we have that $f \upharpoonright k \in T_{\langle x,y\rangle}^R$. By property 4 of the sequence $\{T_n^R\}$, $s + f \upharpoonright k \in T_{\langle z,y\rangle}^R$, which gives us $\varphi(s) \in T_z^{\mathcal{A}_y}$.

($\Leftarrow$): Suppose $\mathcal{A}_x \leq^* \mathcal{A}_y$, and φ is the function witnessing this fact. Consider the tree $T_x^{\mathcal{A}_x} = T_{\langle x,x\rangle}^R$. By property 3 of $\{T_n^R\}$, the function f, defined by $f \upharpoonright k = 0^k$ is a path in $T_x^{\mathcal{A}_x}$. Therefore, by the definition of $\leq^*$, $\varphi(0^k) \in T_x^{\mathcal{A}_y}$, for all k. Hence, by property 1, xRy.

In fact, the proof of Theorem 1 gives us an even stronger result. In the definition of the reducibility (Definition 7) we can replace "hyperarithmetical" by "computable" and still get the correct statements for the new reducibility. The following definition was first introduced in [4] as an effective analog of the Borel reducibility on classes of structures (for arbitrary structures, not necessarily computable). The universe of a structure is a subset of ω, possibly finite. As above, for a class K, the structures all have the same language, and K is closed under isomorphism (modulo the restriction on the universe).

Definition 9. *1. A* Turing computable transformation *from K' to K'' is a computable operator $\Phi = \varphi_e$ such that for each $\mathcal{A} \in K'$, there exists $\mathcal{B} \in K''$ with $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$. We write $\Phi(\mathcal{A})$ for $\mathcal{B}$.*

2. *We say that* $\cong_{K'}$ tc-reduces *to* $\cong_{K''}$ *if for* $\mathcal{A}, \mathcal{A}' \in K'$,

$$\mathcal{A} \cong_{K'} \mathcal{A}' \textit{ iff } \Phi(\mathcal{A}) \cong_{K''} \Phi(\mathcal{A}').$$

We can use the same approach to compare arbitrary equivalence relations on classes of structures. Namely,

Definition 10. *For equivalence relations* E', E'' *on* K', K'' *respectively,* E' *is* tc-reducible *to* E'', $E' \leq_{tc} E''$, *if there is a computable transformation* Φ *from* K' *to* K'' *such that for* $\mathcal{A}, \mathcal{B} \in K'$, $\mathcal{A}E'\mathcal{B}$ *iff* $\Phi(\mathcal{A})E''\Phi(\mathcal{B})$.

As we study the relations on computable members of K, we can again restrict our attention on relations on ω.

Definition 11. *For equivalence relations* E', E'' *on* ω, E' *is* tc-reducible *to* E'', $E' \leq_{tc} E''$, *if there is a computable function* f *such that for all* x, y, $xE'y$ *iff* $f(x)E''f(y)$.

As before, we identify an equivalence relation E on K^c with $I(E, K)$. Then for an arbitrary equivalence relation E' on ω, we say that E' tc-reduces to E iff there exists a *computable* sequence of computable structures $\{\mathcal{A}_x\}_{x\in\omega}$ from K such that for all x, y, $xE'y$ iff $\mathcal{A}_x E \mathcal{A}_y$.

Corollary 1. *1. There is a class* K *of structures with hyperarithmetical index set and an equivalence relation on* K^c *which is a tc-complete* Σ^1_1 *equivalence relation.*
2. *There is a class* K *of structures with hyperarithmetical index set and a preorder on* K^c *which is a tc-complete* Σ^1_1 *preorder.*
3. *The isomorphism relation on computable undirected graphs is a tc-complete* Σ^1_1 *isomorphism relation.*

4 Σ^1_1 Complete Bi-Embeddability Relation

Theorem 3. *1. The embeddability relation* $\sqsubseteq$ *on the class of computable trees is an h-complete* Σ^1_1 *preorder.*
2. *The bi-embeddability relation* $\equiv$ *on the class of computable trees is an h-complete* Σ^1_1 *equivalence relation.*

Proof. We sketch the proof of the first statement of the theorem. The second statement follows from it in the obvious way. As before, we use the ideas from [14]. We have only to take care that there is enough effectiveness and uniformity of the arguments. Let K be the class of structures constructed in Theorem 1. We give a uniform effective procedure of constructing a computable tree from every computable member of K, which allows us to reduce $\leq^*$ to $\sqsubseteq$ on the class of trees.

Let G_0 be a tree defined in the following way. For every vertex x of a complete infinitely branching tree $\omega^{<\omega}$, except for the root, we add a new vertex x' between

x and its predecessor. We use G_0 as a base to construct the tree $G_{\mathcal{A}}$ corresponding to a structure $\mathcal{A} \in K^c$. To code $\mathcal{A}$ into a tree, we add to G_0 new vertices of the form $(x, s, 0^k)$ and $(x, s, 0^{2x+2}10^k)$, where $x \in \omega, s \in \omega^{<\omega}, k \in \omega$ and $s \in T_x^{\mathcal{A}}$. We connect (x, s, w) to (x, s, w') iff w' is the predecessor of w and we connect $(x, s, \emptyset)$ to s, considered as an element of G_0. The resulting tree is $G_{\mathcal{A}}$. If $\mathcal{A}$ is computable then obviously $G_{\mathcal{A}}$ is computable and the procedure is uniform.

Suppose $\mathcal{A}_1 \leq^* \mathcal{A}_2$. We prove that there is an embedding of $G_{\mathcal{A}_1}$ into $G_{\mathcal{A}_2}$. By definition of $\leq^*$, there is a function $\varphi : \omega^{<\omega} \to \omega^{<\omega}$ (which, in fact, can be taken 1-to-1 due to Property 2 of all $T_x^{\mathcal{A}_i}$). We send every element $s \in \omega^{<\omega}$ into $\varphi(s)$, and every s' into $\varphi(s)'$. Thus, we have defined an embedding of G_0 into itself. Every vertex of the form (x, s, w) we send into $(x, \varphi(s), w)$. This map will define an embedding of $G_{\mathcal{A}_1}$ into $G_{\mathcal{A}_2}$, as by definition of $\leq^*$ we have $s \in T_x^{\mathcal{A}_1} \Rightarrow \varphi(s) \in T_x^{\mathcal{A}_2}$.

On the other hand, if $G_{\mathcal{A}_1} \sqsubseteq G_{\mathcal{A}_2}$, then let g be the function witnessing the embedding. For every vertex z of $G_{\mathcal{A}_1}$, the number of its neighbors does not exceed the number of neighbors of $g(z)$, moreover the distance between any two vertices z_1, z_2 of $G_{\mathcal{A}_1}$ is the same as between $g(z_1)$ and $g(z_2)$ in $G_{\mathcal{A}_2}$. In particular, all elements of $\omega^{<\omega}$ must be sent to elements of $\omega^{<\omega}$. This map $\varphi : \omega^{<\omega} \to \omega^{<\omega}$ has the necessary properties and witnesses $\mathcal{A}_1 \leq^* \mathcal{A}_2$.

Again, all the procedures are in fact computable, thus, we get the following corollary.

Corollary 2. *1. The embeddability relation $\sqsubseteq$ on the class of trees is a tc-complete Σ^1_1 preorder.*
2. The bi-embeddability relations $\equiv$ on the class of trees is a tc-complete Σ^1_1 equivalence relation.

5 Questions

We conclude the paper with the following questions. In descriptive set theory there are many examples of Borel equivalence relations that cannot be Borel-reduced to the isomorphism relation on any class Mod_φ for any countable infinitary sentence φ. The computable analog of this statement is false, as any Δ^1_1 equivalence relation on ω is h-reducible to equality on ω. However the following remains open:

Question 1. Is there a class K with hyperarithmetical index set, such that its isomorphism relation is an h-complete (a *tc*-complete) Σ^1_1 equivalence relation?

We do not know if there exists a hyperarithmetical class of computable structures with Σ^1_1 (but not Δ^1_1) isomorphism relation which is not h-complete among all isomorphism relations on hyperarithmetical classes of computable structures. An affirmative answer to the following question may help solve this problem:

Question 2. Does there exist a *hyperarithmetical* class K of computable structures which contains a unique structure of non-computable Scott rank (up to isomorphism)?

If such a class exists then the isomorphism relation on the class of computable graphs cannot be h-reduced to the isomorphism relation on K. Indeed, there exist non-isomorphic graphs of high (i.e. $\geq \omega_1^{CK}$) Scott rank. They must be sent to non-isomorphic structures in K. However, no computable structure of high Scott rank can be sent to a computable structure of computable Scott rank under hyperarithmetical reducibility.

The main result of [8] shows that in the non-effective setting, every Σ_1^1 equivalence relation is Borel-equivalent to the bi-embeddability relation on the class of all countable models of an infinitary sentence φ.

Question 3. Let E be a Σ_1^1 equivalence relation on ω. Does there always exist a class K of structures with hyperarithmetical $I(K)$, such that E is *h-equivalent* to the bi-embeddability relation on computable members of K?

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