THE JOURNAL OF SYMBOLIC LOGIC
Volume 77, Number 1, March 2012

ISOMORPHISM RELATIONS ON COMPUTABLE STRUCTURES

EKATERINA B. FOKINA, SY-DAVID FRIEDMAN, VALENTINA HARIZANOV, JULIA F. KNIGHT, CHARLES MCCOY, AND ANTONIO MONTALBÁN

Abstract. We study the complexity of the isomorphism relation on classes of computable structures. We use the notion of *FF*-reducibility introduced in [9] to show completeness of the isomorphism relation on many familiar classes in the context of all Σ^1_1 equivalence relations on hyperarithmetical subsets of ω.

§1. Introduction. We develop the theory for computable structures analogous to the theory of isomorphism relations introduced by H. Friedman–Stanley in [13]. Our languages are computable, and our structures have universes contained in ω. In measuring complexity, we identify structures with their atomic diagrams. In particular, a structure is *computable* if its atomic diagram is computable.

In descriptive set theory, the study of Borel equivalence relations under Borel reducibility has developed into a rich area. The notion of Borel reducibility allows one to compare the complexity of equivalence relations on Polish spaces, for details see, for example, [15, 19, 21]. In particular, natural equivalence relations such as isomorphism and bi-embeddability on classes of countable structures have been widely studied, e.g., [13, 14, 18, 25]. An effective version of this study was introduced in [4] and [24]. The complexity of the isomorphism relation on various classes of countable structures was measured using the idea of effective transformations. In the recent work [11] the general theory of effectively Borel (i.e., Δ^1_1) equivalence relations on effectively presented Polish spaces was developed via the notion of effective Borel reducibility. The resulting structure turned out to be much more complex than in the classical case.

In computable model theory, equivalence relations have also been a subject of study, e.g., [3, 7, 23], etc. In these papers, equivalence relations of rather low complexity were studied (computable, in the Ershov hierarchy, Σ^0_1, Π^0_1). In [9] Σ^1_1 equivalence relations on computable structures were investigated. The notion of hyperarithmetical and computable reducibility of Σ^1_1 equivalence relations on ω was used to estimate the complexity of natural equivalence relations on hyperarithmetical classes of computable structures within the class of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω as a whole.

Received May 26, 2010.
The first and the second authors acknowledge the generous support of the FWF through projects M 1188 – N13 and P 19898 – N18. Fokina, Harizanov, Knight, and McCoy were partially supported by the NSF binational grant DMS-1101123. Harizanov was partially supported by the NSF grant DMS-0904101. Montalbán was partially supported by the NSF grant DMS-0901169.

0022-4812/12/7701-0008/$2.10

In this paper we continue the study of the theory of Σ^1_1 equivalence relations on computable structures. Our work here shows that this theory behaves very differently from the theory initiated in H. Friedman–Stanley [13] for isomorphism relations and further developed for arbitrary Borel equivalence relations on Polish spaces [15, 19, 21]. In particular we show that isomorphism of computable graphs is complete with respect to the chosen effective reducibility in the context of *all* Σ^1_1 equivalence relations on ω. This is false in the context of countable structures and Borel reducibility [22]: there are examples of Borel equivalence relations that are not Borel-reducible to isomorphism of graphs. We also show that the isomorphism relation on computable torsion abelian groups is complete among Σ^1_1 equivalence relations on ω, while in the classical case it is known to be incomplete among isomorphism relations on classes of countable structures [13]. The same holds for isomorphism of computable torsion-free abelian groups, which in the case of countable structures is not known to be complete for isomorphism relations.

§2. Background.

2.1. Trees. Here we give some definitions useful for describing computable trees. Our trees are isomorphic to subtrees of $\omega^{<\omega}$. For the language, we take a single unary function symbol, interpreted as the predecessor function. We write $\emptyset$ for the top node (our trees grow down), and we think of $\emptyset$ as its own predecessor. Thus, our trees are defined on ω with their structure given by the predecessor function, but we often consider them as subtrees of $\omega^{<\omega}$ and treat their elements as finite sequences.

Definition 1. Let $S, T \subseteq \omega^{<\omega}$ be trees. Define the tree $S * T$ in the following way. We think of the elements of $S * T$ as ordered pairs (σ, τ), where $\sigma \in S$, $\tau \in T$. At level 0 of $S * T$, we have $(\emptyset, \emptyset)$. For an element (σ, τ) at level k of $S * T$, σ and τ are at level k of S and T, respectively. The successors of (σ, τ) are the pairs (σ', τ'), where σ' is a successor of σ in S, and τ' is a successor of τ in T.

Definition 2. Let T be a subtree of $\omega^{<\omega}$. We define the *tree rank* of $x \in T$, denoted by $tr(x)$, by induction:

(1) $tr(x) = 0$ if x has no successor;
(2) For $\alpha > 0$, $tr(x) = \alpha$ if α is the least ordinal greater than $tr(y)$ for all successors y of x;
(3) $tr(x) = \infty$ if x does not have ordinal tree rank.

The *tree rank* of the tree T is defined to be the rank of the top node $\emptyset$.

Note that all computable trees have rank ∞ or rank some computable ordinal. Moreover, for any node $x \in T$, $tr(x) = \infty$ iff x extends to an infinite path through T [27].

Remark. The tree rank of the tree $S * T$ is the minimum of the tree ranks of S and T. In particular, $S * T$ has an infinite path iff both S and T have infinite paths. More generally, for $\sigma \in S$ and $\tau \in T$, where σ and τ lie at the same level in their respective trees, $tr((\sigma, \tau)) = \min(tr(\sigma), tr(\tau))$.

Definition 3 (rank-saturated tree). A computable subtree T of $\omega^{<\omega}$ is *rank-saturated* provided that for all x in T:

(1) If $tr(x)$ is an ordinal α, then for all $\beta < \alpha$, x has infinitely many successors z such that $tr(z) = \beta$;

(2) If $tr(x) = \infty$, then for all computable β, x has infinitely many successors z such that $tr(z) = \beta$ and x has infinitely many successors z with $tr(z) = \infty$.

LEMMA 1. *There is a computable rank-saturated tree T^∞ such that $rk(T^\infty) = \infty$.*

PROOF. In [17] Harrison proved the existence of a computable linear ordering $\mathscr{H}$ of type $\omega_1^{CK}(1+\eta)$. We let T^∞ be the set of finite sequences $((a_0, k_0), \dots, (a_n, k_n))$, where $a_0 > \cdots > a_n$ in $\mathscr{H}$ and $k_0, \dots, k_n \in \omega$. It is easy to see that if a_i corresponds to an ordinal α in $\mathscr{H}$, then $tr((a_0, k_0), \dots, (a_i, k_i)) = \alpha$, and if a_i lies in the non-well-ordered part of $\mathscr{H}$, then $tr((a_0, k_0), \dots, (a_i, k_i)) = \infty$. ⊣

PROPOSITION 1. *If T is a computable tree, then $T * T^\infty$ is a computable rank-saturated tree of the same tree rank as T.*

PROOF. The top node in $T * T^\infty$ clearly has the proper rank, by Remark 2.1. For $x \in T * T^\infty$ of rank α and $\beta < \alpha$, we show that x has infinitely many successors of rank β. Say $x = (\sigma, \tau)$. By Remark 2.1, $tr(\tau) \geq \alpha$ and because T^∞ is rank-saturated, τ has infinitely many successors τ' of rank β. Also, $tr(\sigma) \geq \alpha$, so σ has a successor σ' of rank at least β. Then for all such pairs (σ', τ'), $tr(\sigma', \tau') = \beta$. ⊣

REMARK. Computable rank-saturated trees are a special case of computable *rank-homogeneous* trees, defined in [5].

PROPOSITION 2. (1) *For every computable α, if T^α and T_1^α are computable rank-saturated trees of tree rank α, then $T^\alpha \cong T_1^\alpha$.*

(2) *If T_1^∞ is a computable rank-saturated tree of tree rank ∞, then $T^\infty \cong T_1^\infty$.*

PROOF. By induction on α. ⊣

We will fix the notation T^α for the computable rank-saturated tree of rank α, and we recall that T^∞ is a computable rank-saturated tree with infinite paths.

2.2. Σ_1^1 sets and relations. We assume the reader is familiar with basic concepts of recursion theory. However, here we list some definitions and facts that will be useful for the future proofs. Detailed information can be found, for example, in [1, 27].

DEFINITION 4. (1) A relation $S(\overline{x})$ is Σ_1^1 if there is an arithmetical relation $R(\overline{x}, u)$, on tuples of numbers, such that $\overline{x} \in S$ iff $(\exists f \in \omega^\omega)\,(\forall s)\, R(\overline{x}, f \upharpoonright s)$—we identify $f \upharpoonright s$ with its code.

(2) A relation $S(\overline{x})$ is Π_1^1 if there is an arithmetical relation $R(\overline{x}, u)$, on tuples of numbers, such that $\overline{x} \in S$ iff $(\forall f \in \omega^\omega)\,(\exists s)\, R(\overline{x}, f \upharpoonright s)$.

(3) A relation $S(\overline{x})$ is Δ_1^1 if it is both Σ_1^1 and Π_1^1.

By the Kleene–Suslin Theorem, a relation is Δ_1^1 iff it is hyperarithmetical.

If $S(\overline{x})$ is a k-place relation, we may consider the set S' of codes for k-tuples belonging to S. It is clear that S is Σ_1^1 iff S' is Σ_1^1. The next result gives familiar conditions equivalent to being Σ_1^1 [1, 27]. We identify finite sequences with their codes.

PROPOSITION 3 (Kleene). *The following are equivalent*:

(1) *S is Σ_1^1;*

(2) *There is a computable relation $R(n, u)$, on pairs of numbers, such that $n \in S$ iff $(\exists f)\,(\forall s)\, R(n, f \upharpoonright s)$;*

(3) *There is a computable sequence of computable trees $(T_n)_{n\in\omega}$ such that $n \in S$ iff T_n has an infinite path.*

THEOREM 1 (Bounding). *Let CWF denote the set of codes for computable well-founded trees on ω and for each computable ordinal α, let CWF_α denote the set of codes for computable trees of tree rank less than α. Then if F is a hyperarithmetical function from a hyperarithmetical subset of ω into CWF, there exists a computable α such that the range of F is contained in CWF_α.*

We now give a notion of effective reducibility of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω. The idea is the following. A relation E is effectively reducible to a relation E' if there is an effective procedure which allows us to answer any question about E-equivalence using information about E'-equivalence. We want to use partial computable functions as witnesses for reducibilities.

DEFINITION 5. Let E, E' be Σ^1_1 equivalence relations on hyperarithmetical subsets $X, Y \subseteq \omega$, respectively. The relation E is *FF-reducible* to E' iff there exists a partial computable function f with $X \subseteq dom(f)$, $Y \subseteq f(X)$ such that for all $x, y \in X$,

$$x\, E\, y \iff f(x)\, E'\, f(y).$$

We denote this fact by $E \leq_{FF} E'$.

The notion of *FF*-reducibility was first used in [9] where it was called "*tc*-reducibility". In the next section we will explain the relationship between *FF*-reducibility and the notion of *tc*-reducibility introduced in [4] to compare the classes of countable structures.

2.3. Computable characterization and classification. Here we review two equivalent approaches, from [16], to the problems of computable characterization and classification. The goal is to be able to measure the complexity of a set of computable structures or an equivalence relation on a set of computable structures.

The first approach is based on the notion of computable infinitary formulas. Roughly speaking, computable infinitary formulas are $L_{\omega_1\omega}$ formulas in which the infinite disjunctions and conjunctions are over c.e. sets. For a formal definition see [1]. Computable infinitary formulas form a hierarchy: a *computable* Σ_0 or Π_0 formula is a finitary quantifier-free formula. For $\alpha > 0$, a *computable* Σ_α formula is a c.e. disjunction of formulas of the form $\exists \overline{u}\psi$, where ψ is computable Π_β for some $\beta < \alpha$, and a *computable* Π_α formula is a c.e. conjunction of formulas of the form $\forall \overline{u}\psi$, where ψ is computable Σ_β for some $\beta < \alpha$.

Following [16], we say that a class K of structures closed under isomorphism *has a computable characterization* if the set K^c of its computable members consists exactly of all computable models of a computable infinitary sentence. This definition expresses the idea that the set of all computable members of K can be nicely defined among all other structures for the same language.

The second approach uses the notion of an index set. For a computable structure $\mathcal{M}$, an *index* is a number a such that $\varphi_a = \chi_{D(\mathcal{M})}$, where $(\varphi_a)_{a\in\omega}$ is a computable enumeration of all unary partial computable functions. The *index set* for $\mathcal{M}$ is the set $I(\mathcal{M})$ of all indices for computable (isomorphic) copies of $\mathcal{M}$. For a class K of structures, closed under isomorphism, the *index set* is the set $I(K)$ of all indices for computable members of K. As in [16], we say that a class K *has a computable characterization*, if its index set is hyperarithmetical.

PROPOSITION 4 (Goncharov–Knight [16]). *Let K be a class of countable structures closed under isomorphism, and let K^c be the set of computable members of K. Then the following are equivalent*:

(1) *The index set $I(K)$ of K is hyperarithmetical*;
(2) *There is a computable infinitary sentence ψ such that $K^c = Mod_\psi^c$, where Mod_ψ^c is the set of all computable models of ψ.*

For a relation E on a class K of structures, denote by $I(E, K)$ the set of pairs of indices

$$\{(m, n) \mid m, n \in I(K) \text{ and } \mathscr{M}_m \; E \; \mathscr{M}_n\}.$$

We measure the complexity of various relations on computable structures via the complexity of the corresponding sets of pairs of indices. In what follows we will often identify E with $I(E, K)$ considered as a relation on indices. Thus, it will make sense to compare relations on classes of computable structures with relations on subsets of ω. The most studied cases are that of isomorphism and bi-embeddability relations, e.g., [2, 6, 9, 16].

We are interested in studying the relations on classes that are nicely defined. For this reason we will require the index set of each class K to be hyperarithmetical. Equivalently, $K^c = Mod_\psi^c$ for some computable infinitary ψ. Let K and K' be two classes of *countable* structures, such that $K = Mod_\psi$ and $K' = Mod_{\psi'}$ for some computable infinitary ψ, ψ'. Suppose the isomorphism relation on K is *tc*-reducible to the isomorphism relation on K' in the sense of [4]. Then $I(\cong, K) \leq_{FF} I(\cong, K')$ and the reduction is exactly the restriction to computable structures of the reduction of K to K'.

§3. Isomorphism is complete among Σ_1^1 equivalence relations.

If $I(K)$ is hyperarithmetical and E is the isomorphism or bi-embeddability relation, then the corresponding equivalence relation $I(E, K)$ on indices is a Σ_1^1 set. In this section we prove completeness of the isomorphism relation on various familiar classes of structures in the context of all Σ_1^1 equivalence relations on hyperarithmetical subsets of ω under *FF*-reducibility. These results show the difference of our theory from the classical theory of Borel equivalence relations since, by [22], some Borel equivalence relations cannot be reduced to isomorphism relations.

DEFINITION 6. A relation E on a hyperarithmetical subset of ω is an *FF-complete Σ_1^1 equivalence relation* if E is Σ_1^1 and every Σ_1^1 equivalence relation E' on a hyperarithmetical subset of ω is *FF*-reducible to E.

Note that if E is an equivalence relation on a hyperarithmetical class K^c of computable structures, then it is complete if and only if for every Σ_1^1 relation E', there exists a computable sequence of computable structures $(\mathscr{M}_n)_{n\in\omega}$ from K^c such that for all $m, n \in \omega$,

$$m \; E' \; n \iff \mathscr{M}_m \; E \; \mathscr{M}_n.$$

3.1. Trees and graphs.

THEOREM 2. *The isomorphism relation on computable trees is an FF-complete Σ_1^1 equivalence relation.*

PROOF. Let E be a Σ^1_1 equivalence relation on ω. To prove that E is FF-reducible to the isomorphism relation on computable trees, we will build a computable sequence of computable trees $(T_n)_{n\in\omega}$ such that for every $m, n \in \omega$,

$$m\ E\ n \iff T_m \cong T_n.$$

By Proposition 3, since E is Σ^1_1, there exists a uniformly computable sequence of trees $(T_{m,n})_{m,n\in\omega}$ such that $\neg m\ E\ n$ if and only if $T_{m,n}$ is well founded. Then we say that $\neg m\ E\ n$ *is witnessed by stage* α if and only if $T_{m,n}$ has tree-rank less than α.

The strategy to build $(T_n)_{n\in\omega}$ is the following. First, uniformly in m, n, we will build a computable tree $T^*_{m,n}$ with the following properties:

(1) $m\ E\ n \Rightarrow T^*_{m,n} \cong T^\infty$, where T^∞ is the rank-saturated tree with an infinite path;
(2) $\neg m\ E\ n \Rightarrow T^*_{m,n} \cong T^\alpha$, where T^α is the rank-saturated tree of tree rank α, for a computable ordinal α such that for all $m' \in [m]_E$ and $n' \in [n]_E$ the relation $\neg m'\ E\ n'$ is witnessed by stage α. This α will be the least ordinal such that for all $m' \in [m]_E$, $n' \in [n]_E$ and all finite sequences $m' = a_0, a_1, \dots, a_s = n'$, $\alpha \geq \min\{tr(T_{a_i,a_{i+1}}) \colon i \leq s-1\} + 1$, where $(T_{m,n})_{m,n\in\omega}$ is the sequence fixed for E in the previous paragraph.

We start from the computable sequence of computable trees $(T_{m,n})_{m,n\in\omega}$ mentioned above: $T_{m,n}$ is well founded if and only if $\neg m\ E\ n$. For every $m, n \in \omega$, we construct (effectively and uniformly) a new tree $T'_{m,n}$ in the following way. Let $\sigma_0, \sigma_1, \dots$ be an enumeration of all finite sequences of natural numbers. Suppose $\sigma_s = (a_0, \dots, a_{l_s})$. Then under the s^{th} node on level 1 (i.e., under the element of the form $(s), s \in \omega$) of $T'_{m,n}$ we put the tree $P_s = T_{m,a_0} * T_{a_0,a_1} * \cdots * T_{a_{l_s},n}$, identifying the top node of P_s with s. Then

$$tr(T'_{m,n}) = sup\{tr(P_s) + 1 \mid s \in \omega\}.$$

If $m E n$, then $T_{m,n}$ has an infinite path, i.e., $tr(T_{m,n}) = \infty$. Thus, $tr(T'_{m,n}) = \infty$. If $\neg m E n$, then for every $\sigma = (a_0, \dots, a_l)$, $tr(T_{m,a_0} * T_{a_0,a_1} * \cdots * T_{a_l,n})$ is a computable ordinal. Indeed, fix $m, n \in \omega$ such that $\neg m E n$. For every finite sequence σ_s consider the corresponding tree $P_s = T_{m,a_0} * T_{a_0,a_1} * \cdots * T_{a_{l_s},n}$. Consider the function F from the set of finite sequences into CWF such that $F(s)$ is the code of P_s. The function F is hyperarithmetical, its domain is computable. By Bounding, there is a computable bound on the range of F. Therefore, $T'_{m,n}$ has rank α for some computable α. Note that for all $m' \in [m]_E$ and $n' \in [n]_E$, we get the same bound α. Indeed, let $m'\ E\ m, n'\ E\ n$ and let β be the computable bound on the ranks of trees constructed using finite sequences starting with m' and ending with n'. Let $P_s = T_{m,a_0} * T_{a_0,a_1} * \cdots * T_{a_{l_s},n}$ be as above. Then $tr(T_{m',m} * P_s * T_{n,n'}) = tr(P_s)$, thus $\alpha \leq \beta$. Similarly, one can show that $\beta \leq \alpha$.

Let $T^*_{m,n} = T'_{m,n} * T^\infty$. As shown in Proposition 1, the tree $T^*_{m,n}$ is a computable rank-saturated tree, $tr(T^*_{m,n}) = tr(T'_{m,n})$, and the construction is uniform.

Now we build the desired sequence $(T_n)_{n\in\omega}$. Take the tree T consisting exactly of the sequences $(m, m, \dots, m)$ of length $i \leq m$, for $m \in \omega$. Now fix n and for every m, attach $T^*_{m,n}$ to the m-th leaf of T. The resulting tree is T_n. The sequence $(T_n)_{n\in\omega}$ witnesses the reducibility: $m\ E\ n$ iff $T_m \cong T_n$. Indeed, suppose $m\ E\ n$. Then

(1) for every $k \in [m]_E = [n]_E$, $tr(T'_{k,m}) = tr(T'_{k,n}) = \infty$, thus $T^*_{k,m} \cong T^*_{k,n} \cong T^\infty$;
(2) for every $k \notin [m]_E$, $tr(T'_{k,m}) = tr(T'_{k,n}) = \alpha$, thus $T^*_{k,m} \cong T^*_{k,n} \cong T^\alpha$.

Therefore, $T_m \cong T_n$.

Suppose now that $\neg m\ E\ n$. Then $T^*_{m,m} \cong T^\infty$, while $T^*_{m,n} \cong T^\alpha$ for some computable α. Thus $T_m \not\cong T_n$. ⊣

COROLLARY 1. *The isomorphism relation on computable graphs is an FF-complete Σ^1_1 equivalence relation.*

3.2. Torsion-free abelian groups. Torsion-free abelian groups are subgroups of $\mathbb{Q}$-vector spaces. Hjorth [18] gave a transformation from trees to torsion-free abelian groups, which enabled him to show that the isomorphism relation on these groups is not Borel. Downey and Montalbán [8] built on Hjorth's ideas to show that the isomorphism problem on these groups is complete among Σ^1_1 *sets*. In this paper we use the transformation from [18] and [8] to show that the isomorphism relation on computable torsion-free abelian groups is, in fact, complete as a Σ^1_1 equivalence relation. First we describe the transformation.

We consider the elements of $\omega^{<\omega}$ as a basis for a $\mathbb{Q}$-vector space V^*. Let T be a subtree of $\omega^{<\omega}$, and let V be the subspace of V^* with basis T. Let T_n be the set of elements at level n of T. If u is at level $n > 0$, let u^- be the predecessor of u. Let $(p_n)_{n\in\omega}$ be a computable list of distinct primes. We let $G(T)$ be the subgroup of V generated by the vector space elements of the following forms:

(1) $\frac{v}{(p_{2n})^k}$, where $v \in T_n$, and $k \in \omega$,
(2) $\frac{v+v'}{(p_{2n+1})^k}$, where $v \in T_n$, v' is a successor of v, and $k \in \omega$.

THEOREM 3. *The isomorphism relation on computable torsion-free abelian groups is FF-complete among Σ^1_1 equivalence relations.*

PROOF. It follows from [12] that if we restrict the class of trees to only rank-saturated trees, then the transformation from the class of trees into torsion-free abelian groups described above is $1-1$ on isomorphism types. Thus, given a Σ^1_1 equivalence relation E for every $n \in \omega$, we first construct the sequence of rank-saturated trees $(T^*_{m,n})_{m\in\omega}$ as in Theorem 2. We want to pass effectively from the sequence to a group G_n such that $G_n \cong G_{n'}$ iff for all m, $T^*_{m,n} \cong T^*_{m,n'}$.

For $m \in \omega$, let $(p_{m,k})_{k\in\omega}$ be uniformly computable lists of primes such that for distinct m, the lists are disjoint. For each m, we apply the transformation described above, taking $T_{m,n}$ to a torsion-free abelian group $G_{m,n}$, using the list of primes $(p_{m,k})_{k\in\omega}$. The resulting sequence $(G_{m,n})_{n\in\omega}$ will satisfy the property:

$$T^*_{m,n} \cong T^*_{m',n'} \iff G_{m,n} \cong G_{m',n'}.$$

Let $G_n = \oplus_m G_{m,n}$.

Using the fact that the sequences of primes are disjoint, we can see that $G_n \cong G_{n'}$ iff for all m, $G_{m,n} \cong G_{m,n'}$. The reason is that $G_{m,n}$ is the subgroup of G_n generated by the set of elements divisible by all the powers of some prime in the list $(p_{m,k})_{k\in\omega}$ (for more details see [8] or [12]). ⊣

3.3. Abelian p-groups. Let p be a prime number. A *p-group* is a group such that each element has some power of p for its order. Countable abelian p-groups are classified up to isomorphism in terms of Ulm invariants (see [20] for details).

In this section we use the transformation from trees into abelian p-groups to get completeness of the isomorphism relation for this class. Note that in the classical theory of Borel equivalence relations the analogous result is false (see [13] and a proof for Turing computable embeddings in [12]).

THEOREM 4. *The isomorphism relation on abelian p-groups is an FF-complete Σ^1_1 equivalence relation.*

PROOF. By Theorem 2, for any Σ^1_1 equivalence relation E on ω, we have a uniformly computable sequence of trees $(T_n)_{n\in\omega}$ such that $m\ E\ n$ iff $T_m \cong T_n$. Each tree T_n is the result of combining a family of trees $T^*_{m,n}$. Each $T^*_{m,n}$ is rank-saturated, so it is really determined by its tree rank. We may modify our trees, if necessary, so that the tree rank, if it exists, is a limit ordinal.

Let $\mathscr{T} = (T^m)_{m\in\omega}$ be a sequence of rank-saturated trees. We need a transformation taking such sequences $\mathscr{T}$ to abelian p-groups $G(\mathscr{T})$, such that $G(\mathscr{T}) \cong G(\mathscr{T}')$ iff the sequences of ranks for the trees in $\mathscr{T}$ and $\mathscr{T}'$ match. We replace T^m by a tree T^m_* such that each single successor in T^m becomes a chain of p_m successors in T^m_*. Then $tr(T^m_*) = p_m\, tr(T^m)$. We form a single tree with infinitely many nodes at level 1, with a copy of T^0_* below the first, a copy of T^1_* below the second, etc. Denote the resulting tree by T. Let G be the abelian p-group generated by the elements of T in a standard way [20]: the top node is the identity, and if x' is a successor of x, then $px' = x$.

Rogers [28] described how to calculate (non-effectively, of course) the Ulm sequence for G from the tree ranks of elements in the corresponding tree T. We describe her scheme briefly. For each node of successor rank, apart from the top node, we choose a successor witnessing the rank. Now, for each α, $u_G(\alpha)$ is the number of nodes of rank α that are not chosen as witnesses. In computing $u_G(\alpha)$, we count all x at level 1 such that $tr(x) = \alpha$. Suppose x is an element at level $n > 1$, where $tr(x) = \alpha$. Let y be the predecessor of x. If $tr(y) > \alpha + 1$, then x cannot witness the rank of y, so we count x. If $tr(y) = \alpha + 1$, then x may be the chosen successor of y witnessing the rank. We count x just in case it is not chosen.

Using Rogers' scheme, we can see that our group G has the following features. For all computable α, the Ulm invariant $u_\alpha(G)$ is either ∞ or 0. For limit α, $u_\alpha(G) = 0$. If $\alpha = \omega\beta + p_m$, then $u_\alpha(G) = \infty$ iff $tr(T^m) \geq \omega\beta$. ⊣

COROLLARY 2. *The isomorphism relation on torsion abelian groups is an FF-complete Σ^1_1 equivalence relation.*

Suppose K and K' are classes of countable structures, with universe a subset of ω, closed under isomorphism. We write $K \leq_{tc} K'$ if there is a Turing computable operator $\Phi = \varphi_e$ taking the atomic diagram of each $\mathscr{A} \in K$ to the atomic diagram of some $\mathscr{B} \in K'$, such that Φ is $1-1$ on isomorphism types. This notion was introduced in [4]. If $I(K)$ and $I(K')$ are hyperarithmetical, and $K \leq_{tc} K'$, then $I(\cong, K) \leq_{FF} I(\cong, K')$. If Φ is the computable operator reducing the isomorphism relation on structures in K to that on structures in K', then for computable $\mathscr{A} \in K$, we can effectively compute an index for $\Phi(\mathscr{A})$ from an index for $\mathscr{A}$.

H. Friedman and Stanley [13] introduced the study of Borel reductions $\leq_B$ of isomorphism relations on classes of structures with universe ω. They showed that the class of undirected graphs, the class of fields of any fixed characteristic, the class of 2-step nilpotent groups, and the class of linear orderings all lie "on top" in this setting.

In [4], it was observed that the Borel transformations are all effective. Moreover, the transformations work perfectly well for structures with universe an arbitrary subset of ω. Therefore, these classes are also "on top" under the relation $\leq_{tc}$ in [4]. We have shown that for the class K of trees, the relation $I(E, K)$ (the set of pairs of indices for computable members of K that are isomorphic) lies "on top" under the relation $\leq_{FF}$ on Σ^1_1 equivalence relations on ω. From this, we immediately get the following.

THEOREM 5. *For each of the following classes K, $I(E, K)$ is an FF-complete Σ^1_1 equivalence relation*:

- *undirected graphs,*
- *fields of characteristic* 0, *or* p,
- 2-*step nilpotent groups,*
- *linear orderings.*

§4. Open problems. In [9] equivalence relations were compared not only via *FF*-reducibility but also via hyperarithmetical reducibility (h-reducibility):

DEFINITION 7. Let E, E' be Σ^1_1 equivalence relations on hyperarithmetical subsets $X, Y \subseteq \omega$, respectively. The relation E is *h-reducible* to E' iff there exists a hyperarithmetical function f such that for all $x, y \in X$,

$$x\, E\, y \Longleftrightarrow f(x)\, E'\, f(y).$$

By [14] the following theorem is true for the bi-embeddability relation on computable structures. Here we mean the standard model-theoretic notion of embeddings on structures.

THEOREM 6. *For every Σ^1_1 equivalence relation E on ω there exists a hyperarithmetical class K of structures, which is closed under isomorphism and such that E is h-equivalent to the bi-embeddability relation on computable structures from K.*

Remark 3.4 of [14] provides the result for Σ^1_1 preorders on the reals, but the result for preorders on ω follows almost immediately.

In [10] it was proved that the general structure of Σ^1_1 equivalence relations on hyperarithmetical subsets of ω (under *FF*- or h-reducibility) is rich. The above theorem states that the structure of bi-embeddability relations on hyperarithmetical classes of computable structures is as complex as the whole structure of Σ^1_1 equivalence relations under h-reducibility. It would be interesting to get the following refinement of Theorem 6:

QUESTION 1. If E is a Σ^1_1 equivalence relation on ω, does there exist a hyperarithmetical class K of structures, closed under isomorphism and such that E is *FF*-equivalent to the bi-embeddability relation on computable structures from K?

Let K be a class of structures closed under isomorphism such that the index set $I(K)$ is hyperarithmetical. Consider the following statements:

(1) $I(\cong, K)$ is properly Σ^1_1;
(2) $I(\cong, K)$ is m-complete Σ^1_1;
(3) $I(\cong, K)$ is Σ^1_1 complete under *FF*-reducibility;
(4) $I(\cong, K \restriction \mathit{highSR})$ is not hyperarithmetical within $K \restriction \mathit{highSR}$, where *highSR* is the class of structures of high (i.e., noncomputable) Scott rank;
(5) K has infinitely many non-isomorphic computable structures of high Scott rank.

The following implications are true: $(1) \Leftarrow (2) \Leftarrow (3) \Rightarrow (4) \Rightarrow (5)$.

QUESTION 2. Which of these arrows are reversible?

One of the approaches to give a negative answer to the question "$(1) \Rightarrow (3)$?" would be to positively answer the following:

QUESTION 3. Is there a hyperarithmetical class of structures with a unique (up to isomorphism) computable structure of high Scott rank?

If the answer to this question is positive, we see immediately that (1) does not imply (5). Since (3) implies (5), we also conclude that (1) does not imply (3).

REMARK. It is known that up to bi-embeddability this is true in the following sense. In the class of computable linear orderings, the equivalence class of linear orderings bi-embeddable with the rationals is Σ^1_1-complete, but every computable scattered linear ordering (i.e., not bi-embeddable with the rationals) has a hyperarithmetical equivalence class. For more information on the bi-embeddability relation in the class of countable linear orderings see [26].

This question may be also considered as a weaker version of the question from [16] where the authors asked about the existence of a computable structure with high Scott rank and a hyperarithmetical index set.

QUESTION 4. Are there isomorphism relations on hyperarithmetical classes of computable structures which are not hyperarithmetical and not *FF*-complete?

REFERENCES

[1] C. J. ASH and J. F. KNIGHT, ***Computable structures and the hyperarithmetical hierarchy***, Elsevier, 2000.

[2] W. CALVERT, ***Algebraic structure and computable structure***, PhD Dissertation, University of Notre Dame, 2005.

[3] W. CALVERT, D. CENZER, V. HARIZANOV, and A. MOROZOV, *Effective categoricity of equivalence structures*, ***Annals of Pure and Applied Logic***, vol. 141 (2006), pp. 61–78.

[4] W. CALVERT, D. CUMMINS, J. F. KNIGHT, and S. MILLER, *Comparing classes of finite structures*, ***Algebra and Logic***, vol. 43 (2004), pp. 374–392.

[5] W. CALVERT, J. KNIGHT, and J. MILLAR, *Computable trees of Scott rank ω_1^{CK} and computable approximations*, this JOURNAL, vol. 71 (2006), pp. 283–298.

[6] J. CARSON, E. FOKINA, V. S. HARIZANOV, J. F. KNIGHT, S. QUINN, C. SAFRANSKI, and J. WALLBAUM, *Computable embedding problem*, submitted.

[7] D. CENZER, V. HARIZANOV, and J. REMMEL, *Σ^0_1 and Π^0_1 equivalence structures*, ***Computability in Europe, 2009***, Lecture Notes in Computer Science, vol. 5635, 2009, pp. 99–108.

[8] R. DOWNEY and A. MONTALBÁN, *The isomorphism problem for torsion-free Abelian groups is analytic complete*, ***Journal of Algebra***, vol. 320 (2008), pp. 2291–2300.

[9] E. FOKINA and S. FRIEDMAN, *Equivalence relations on classes of computable structures*, ***Computability in Europe, 2009***, Lecture Notes in Computer Science, vol. 5635, 2009, pp. 198–207.

[10] ———, *Σ^1_1 equivalence relations on ω*, submitted.

[11] E. FOKINA, S. FRIEDMAN, and A. TÖRNQUIST, *The effective theory of Borel equivalence relations*, ***Annals of Pure and Applied Logic***, vol. 161 (2010), pp. 837–850.

[12] E. FOKINA, J. KNIGHT, A. MELNIKOV, S. QUINN, and C. SAFRANSKI, *Ulm type, and coding rank-homogeneous trees in other structures*, this JOURNAL, vol. 76 (2011), pp. 846–869.

[13] H. FRIEDMAN and L. STANLEY, *A Borel reducibility theory for classes of countable structures*, this JOURNAL, vol. 54 (1989), pp. 894–914.

[14] S. D. Friedman and L. Motto Ros, *Analytic equivalence relations and bi-embeddability*, this Journal, vol. 76 (2011), pp. 243–266.

[15] S. Gao, ***Invariant descriptive set theory***, Pure and Applied Mathematics, CRC Press/Chapman & Hall, 2009.

[16] S. S. Goncharov and J. F. Knight, *Computable structure and non-structure theorems*, ***Algebra and Logic***, vol. 41 (2002), pp. 351–373, English translation.

[17] J. Harrison, *Recursive pseudo well-orderings*, ***Transactions of the American Mathematical Society***, vol. 131 (1968), pp. 526–543.

[18] G. Hjorth, *The isomorphism relation on countable torsion-free Abelian groups*, ***Fundamenta Mathematica***, vol. 175 (2002), pp. 241–257.

[19] V. Kanovei, ***Borel equivalence relations. Structure and classification***, University Lecture Series 44, American Mathematical Society, 2008.

[20] I. Kaplansky, ***Infinite Abelian groups***, University of Michigan Press, Ann Arbor, 1954.

[21] A. Kechris, *New directions in descriptive set theory*, ***The Bulletin of Symbolic Logic***, vol. 5 (1999), no. 2, pp. 161–174.

[22] A. Kechris and A. Louveau, *The classification of hypersmooth Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 10 (1997), no. 1, pp. 215–242.

[23] B. Khoussainov, F. Stephan, and Y. Yang, *Computable categoricity and the Ershov hierarchy*, ***Annals of Pure and Applied Logic***, vol. 156 (2008), pp. 86–95.

[24] J. F. Knight, S. Miller Quinn, and M. Vanden Boom, *Turing computable embeddings*, this Journal, vol. 73 (2007), pp. 901–918.

[25] A. Louveau and C. Rosendal, *Complete analytic equivalence relations*, ***Transactions of the American Mathematical Society***, vol. 357 (2005), no. 12, pp. 4839–4866.

[26] A. Montalbán, *On the equimorphism types of linear orderings*, ***The Bulletin of Symbolic Logic***, vol. 13 (2007), pp. 71–99.

[27] H. Rogers, ***Theory of recursive functions and effective computability***, McGraw-Hill, 1967.

[28] L. Rogers, *Ulm's theorem for partially ordered structures related to simply presented Abelian p-groups*, ***Transactions of the American Mathematical Society***, vol. 227 (1977), pp. 333–343.

KURT GÖDEL RESEARCH CENTER
UNIVERSITY OF VIENNA
VIENNA, AUSTRIA
E-mail: efokina@logic.univie.ac.at
E-mail: sdf@logic.univie.ac.at

DEPARTMENT OF MATHEMATICS
GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC 20052, USA
E-mail: harizanv@gwu.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
NOTRE DAME, IN 46556, USA
E-mail: knight.1@nd.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PORTLAND
PORTLAND, OR 97203, USA
E-mail: mccoy@up.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL 60637, USA
E-mail: antonio@math.uchicago.edu