# On $\Sigma_1^1$ Equivalence Relations over the natural numbers

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We study the structure of $\Sigma_1^1$ equivalence relations on hyperarithmetical subsets of $\omega$ under reducibilities given by hyperarithmetical or computable functions, called h-reducibility and FF-reducibility, respectively. We show that the structure is rich even when one fixes the number of properly $\Sigma_1^1$ (i.e. $\Sigma_1^1$ but not $\Delta_1^1$) equivalence classes. We also show the existence of incomparable $\Sigma_1^1$ equivalence relations that are complete as subsets of $\omega \times \omega$ with respect to the corresponding reducibility on sets. We study complete $\Sigma_1^1$ equivalence relations (under both reducibilities) and show that existence of infinitely many properly $\Sigma_1^1$ equivalence classes that are complete as $\Sigma_1^1$ sets (under the corresponding reducibility on sets) is necessary but not sufficient for a relation to be complete in the context of $\Sigma_1^1$ equivalence relations.

## 1 Introduction

In [8, 10] the notion of hyperarithmetical and computable reducibility of $\Sigma_1^1$ equivalence relations on hyper-arithmetical subsets of $\omega$ was used to study the question of completeness of natural equivalence relations on hyperarithmetical classes of computable structures as a special class of $\Sigma_1^1$ equivalence relations on $\omega$. In this paper we use this approach to study the structure of $\Sigma_1^1$ equivalence relations on $\omega$ as a whole.

In descriptive set theory, the study of definable equivalence relations under Borel reducibility has developed into a rich area. The notion of Borel reducibility allows one to compare the complexity of equivalence relations on Polish spaces, for details see e.g. [12, 15, 16]. As proved by Louveau and Velickovic in [20], the partial order of inclusion modulo finite sets on $\mathcal{P}(\omega)$ can be embedded into the partial order of Borel equivalence relations modulo Borel reducibility. Thus, the structure of Borel equivalence relations under $\leq_B$ is shown to be very rich.

In computable model theory equivalence relations have also been a subject of study, e.g. [2, 5, 17], etc. In these papers equivalence relations of rather low complexity were studied (computable, $\Sigma_1^0, \Pi_1^0$, having degree in the Ershov hierarchy). In [8] $\Sigma_1^1$ equivalence relations on computable structures were investigated. The authors used the notions of hyperarithmetical and computable reducibility of $\Sigma_1^1$ equivalence relations on $\omega$ to estimate the complexity of natural equivalence relations on hyperarithmetical classes of computable structures.

In this paper we take up the general theory of $\Sigma_1^1$ equivalence relations on hyperarithmetical subsets of $\omega$. We show that the general structure of $\Sigma_1^1$ equivalence relations on hyperarithmetical subsets of $\omega$ under reducibilities given by hyperarithmetical or computable functions is very rich. Namely, the structure of $\Sigma_1^1$ sets under hyperarithmetical many-one-reducibility ($hm$-reducibility) is embeddable into the structure of $\Sigma_1^1$ equivalence relations under reducibility given by a hypearithmetical function. Moreover, this embedding can be taken to have range within the class of $\Sigma_1^1$ equivalence relations with a unique properly $\Sigma_1^1$ equivalence class. Furthermore, we show that there are properly $\Sigma_1^1$ equivalence relations with only finite equivalence classes, and there are $\Sigma_1^1$ relations with exactly $n$ properly $\Sigma_1^1$ equivalence classes, for $n \leq \omega$. We also show that a $\Sigma_1^1$ equivalence relation with infinitely many properly (moreover, $hm$-complete) $\Sigma_1^1$ classes need not be complete with respect to the hypearithmetical reducibility.

## 2    Background

Here we list some definitions and facts that we will use throughout the paper. We assume the familiarity with the main notions from recursion theory. The standard references are [23, 25].

### 2.1    Linear orderings

**Definition 2.1**  Let $K$ be a class of structures closed under isomorphism and $K^c$ be the set of its computable members.

1. An *enumeration* of $K^c/_{\cong}$ is a sequence $(\mathcal{A}_n)_{n\in\omega}$ of elements of $K^c$ representing each isomorphism type in $K^c$ at least once.

2. An enumeration $(\mathcal{A}_n)_{n\in\omega}$ of $K^c/_{\cong}$ is *computable (hyperarithmetical)* if there is a computable (hyperarithmetical) function $f$ which, for every $n$, gives a computable index $f(n)$ for the computable structure $\mathcal{A}_n$.

As proved in [14]:

**Proposition 2.2**  *There exists a computable enumeration of all isomorphism types for computable linear orderings.*

Thus, we can consider $\omega$ as a set of effective codes for computable linear orderings. We will denote by $L_n$ the $n$-th computable linear order in this enumeration. We will abbreviate the set of codes for linear orderings as $LO$ and the set of codes for well-orderings as WO.

**Theorem 2.3**  (e.g. [23], Chapter 16, Corollary XXa)  *The set* WO *is a $\Pi_1^1$-complete set, moreover there exists a computable function $f(z,x)$ such that for every $z$, the $\Pi_1^1$ set with the $\Pi_1^1$ index $z$ is 1-reducible to* WO *by the function $\lambda x[f(z,x)]$.*

In view of Theorem 2.3 one can think about $\Pi_1^1$ sets in the following way. Let $A$ be a $\Pi_1^1$ set and let $m$ be its $\Pi_1^1$ index. Then for every $x \in A$, the ordinal isomorphic to $L_{f(m,x)}$ may be considered as "the level" at which the membership of $x$ is determined.

**Theorem 2.4**  (Bounding)  *For each computable ordinal $\alpha$, let $\mathrm{WO}_\alpha$ denote the set of codes for computable well-orderings isomorphic to an ordinal less than $\alpha$. Then if $F$ is a hyperarithmetical function from a hyperarithmetical subset of $\omega$ into* WO, *there exists a computable $\alpha$ such that the range of $F$ is contained in $\mathrm{WO}_\alpha$.*

**Theorem 2.5**  (Uniformization)  *Every $\Pi_1^1$ binary relation on $X \times Y$, where $X, Y \subseteq \omega$ are hyperarithmetical contains a $\Pi_1^1$ (hyperarithmetical) function with the same domain.*

### 2.2    Reducibilities on $\Sigma_1^1$ equivalence relations

The following definitions were introduced in [8][1]:

**Definition 2.6**  Let $E, E'$ be $\Sigma_1^1$ equivalences relations on hyperarithmetical subsets $X, Y \subseteq \omega$, respectively.

1. The relation $E$ is h-reducible to $E'$, denoted by $E \leq_{\mathrm{h}} E'$, iff there exists a hyperarithmetical function $f$ such that for all $x, y \in X$,

$$xEy \iff f(x)E'f(y).$$

2. The relation $E$ is FF-reducible to $E'$, denoted by $E \leq_{\mathrm{FF}} E'$, iff there exists a partial computable function $f$ with $X \subseteq dom(f), f[X] \subseteq Y$ such that for all $x, y \in X$,

$$xEy \iff f(x)E'f(y).$$

**Remark 2.7**  A definition analogous to that of FF-reducibility was introduced in [1] for the case of c.e. equivalence relations.

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[1] In [8], we used the term "$tc$-reducible" for "FF-reducible", by analogy with the reducibility defined in [3] for classes of countable structures. Later J. Knight suggested the term "FF-reducibility" which was used in [10]. In the current work we follow J. Knight's suggestion.

**Definition 2.8** We say that equivalence relations $E, F$ are h-equivalent (FF-equivalent), denoted by $E \equiv_{\mathrm{h}} F$ ($E \equiv_{\mathrm{FF}} F$, respectively), if $E \leq_{\mathrm{h}} F$ and $F \leq_{\mathrm{h}} E$ ($E \leq_{\mathrm{FF}} F$ and $F \leq_{\mathrm{FF}} E$, respectively).

Obviously, every $\Sigma_1^1$ equivalence relation on a hyperarithmetical subset of $\omega$ is h-equivalent to a $\Sigma_1^1$ equivalence relation on $\omega$. For FF-reducibility the situation is different:

**Fact 2.9** *There exists a $\Sigma_1^1$ equivalence relation $E$ on a hyperarithmetical subset $X$ of $\omega$ such that for no $\Sigma_1^1$ equivalence relation $E'$ on $\omega$, $E \equiv_{\mathrm{FF}} E'$.*

P r o o f. Consider an arbitrary $\Sigma_1^1$ equivalence relation on a hyperarithmetical set $X$ and suppose there exists a relation $E'$ on $\omega$ such that $E \equiv_{\mathrm{FF}} E'$. Let $f$ be a computable function which witnesses $E' \leq_{\mathrm{FF}} E$. Then $f(\omega)$ is a c.e. subset of $X$. Therefore if a $\Sigma_1^1$ equivalence relation is defined on a hyperarithmetical set without a c.e. subset, it cannot be FF-equivalent to an equivalence relation on $\omega$. $\square$

From [13], every *computable* equivalence relation on $\omega$ is FF-equivalent to one of the following:

1. For some finite $n$, the equivalence relation $x \equiv y \bmod n$, which defines a computable equivalence relation with exactly $n$ infinite equivalence classes and no finite classes.

2. The equality relation, which defines a computable equivalence relation with infinitely many classes of size one, and no other classes.

Thus, the partial ordering of the computable equivalence structures, modulo the FF-reducibility, is isomorphic to $\omega + 1$.

In the current paper we are mostly interested in properly $\Sigma_1^1$ equivalence relations, i.e. equivalence relations that are $\Sigma_1^1$ but not $\Delta_1^1$. The reason is the following:

**Fact 2.10** *Let* $\mathrm{id}_\omega$ *denote the equality on $\omega$.*

1. $\mathrm{id}_\omega \leq_{\mathrm{h}} E$ *for any $\Sigma_1^1$ equivalence relation $E$ with infinitely many equivalence classes.*

2. *Any $\Delta_1^1$ equivalence relation on a hyperarithmetical subset of $\omega$ is h-reducible to* $\mathrm{id}_\omega$.

P r o o f. Define a function $f : \omega \to X$, where $X = dom(E)$ is hyperarithmetical, in the following way:

$$f(x) = \mu y[y \in X \,\&\, \bigwedge_{z \leq x} \neg f(z) E y].$$

By its definition, $f$ is a $\Pi_1^1$ function with $dom(f) = \omega$, thus $f$ is a hyperarithmetical function. Obviously, $x = y \iff f(x) E f(y)$.

To prove the second statement, let $E$ be a $\Delta_1^1$ equivalence relation on a hyperarithmetical set $X$. Without loss of generality we assume $0 \notin X$. Consider a function $f(x)$ defined on $X$ in the following way:

$$f(x) = \mu z[x E z].$$

For $x \notin X$ define $f(x) = 0$. Then the function $f$ is hyperarithmetical and $x E y \iff f(x) = f(y) \neq 0$. $\square$

Therefore all the $\Delta_1^1$ equivalence relations on $\omega$ with infinitely many equivalence classes are h-equivalent.

The question we study in the present paper is the following:

**Question 2.11** *How complicated is the structure of all $\Sigma_1^1$ equivalence relations on $\omega$ under h-reducibility (or FF-reducibility)?*

## 2.3 Hyperarithmetical many-one reducibility on $\Sigma_1^1$ sets

In what follows we use the standard notions of $m$-reducibility and $1$-reducibility [25]:

**Definition 2.12**

1. A set $A \subseteq \omega$ is *many-one reducible* ($m$-reducible) to a set $B \subseteq \omega$, denoted by $A \leq_m B$, if there exists a computable function $f$ such that for every $n \in \omega$,

$$n \in A \iff f(n) \in B.$$

2. A set $A \subseteq \omega$ is 1-*reducible* to a set $B \subseteq \omega$, denoted by $A \leq_1 B$, if $A$ is $m$-reducible to $B$ via a $1 - 1$ computable function.

These reducibilities will be useful for the study of the structure of $\Sigma_1^1$ equivalence relations with respect to FF-reducibility.

Consider a hyperarithmetical version of the $m$-reducibility on subsets of $\omega$. It will play an important role in the investigation of complexity of the structure of $\Sigma_1^1$ equivalence relations relative to h-reducibility.

**Definition 2.13** Let $A, B$ be subsets of $\omega$. We say that $A$ is *hyperarithmetically m-reducible* to $B$, denoted by $A \leq_{hm} B$, iff there exists a hyperarithmetical function $f$ with $A \subseteq dom(f)$, such that for every $n \in \omega$,

$$n \in A \iff f(n) \in B.$$

Every equivalence relation can also be considered as a set of pairs, thus, compared to other sets via $m$- or $hm$-reducibilities. The following is straightforward:

**Fact 2.14** Let $E, F$ be $\Sigma_1^1$ equivalence relations on hyperarithmetical subsets of $\omega$.

1. *If* $E \leq_{\mathrm{FF}} F$ *then* $E \leq_m F$;

2. *if* $E \leq_{\mathrm{h}} F$ *then* $E \leq_{hm} F$.

We state that the structure of $hm$-degrees of $\Sigma_1^1$ subsets of $\omega$ is rather complicated.

**Theorem 2.15** *The countable atomless Boolean algebra may be embedded into the $hm$-degrees of $\Pi_1^1$ subsets of $\omega$.*

P r o o f.  We start as in the proof of Theorem 2.1, Chapter IX in [25]. Let $(\alpha_i)_{i \in \omega}$ be a uniformly computable sequence of computable subsets of $\omega$ which form a dense Boolean algebra under $\cup, \cap$. For each $i \in \omega$, we are going to build a $\Pi_1^1$ set $A_i$ such that the mapping

$$\alpha \mapsto A_\alpha = \{\langle i, x \rangle | i \in \alpha, x \in A_i\}$$

gives the desired embedding, i.e.,

1. $\alpha \subseteq \beta$ iff $A_\alpha \leq_{hm} A_\beta$;

2. $deg(A_{\alpha \cap \beta}) \leq deg(A_\alpha), deg(A_\beta)$;

3. $deg(A_{\alpha \cup \beta}) \geq deg(A_\alpha), deg(A_\beta)$.

Notice that the implication from left to right of the first property, as well as the second and the third properties follow from the definition of $\mathcal{A}_{\alpha_i}$. To ensure the implication from right to left of the first property, we will use the ideas of metarecursion [24]. We will build the $\Pi_1^1$ sets $A_i$'s in $\omega_1^{\mathrm{CK}}$ steps in such a way that no $A_i$ is $hm$-reducible to the set $A_{\neq i} = \{\langle k, x \rangle | k \in \omega, k \neq i, x \in A_k\}$.

The whole construction will take now $\omega_1^{\mathrm{CK}}$ steps, but as only the $\Pi_1^1$ subsets of $\omega$ are considered, there will be only $\omega$-many requirements. Thus, each of them may be injured only finitely many times. This approach is used, for example, in [24], Chapter VI, Theorems 2.1, 2.4.

Let $(f_j)_{j \in \omega}$ be a universal $\Pi_1^1$ enumeration of all $\Pi_1^1$ functions on $\omega$. Such an enumeration exists, e.g., by [23], Chapter 16.5. Recall that the hyperarithmetical functions are the total $\Pi_1^1$ functions. Then our requirements are:

$$R_{i,j} : A_i \neq f_j^{-1}[A_{\neq i}] \text{ and } A_i \text{ is co-infinite.}$$

We build our sets in stages $\sigma < \omega_1^{\mathrm{CK}}$. We assign requirements to stages in such a way that each requirement is assigned to cofinally many stages. At stage $0$ we do nothing.

At stage $0 < \sigma < \omega_1^{\mathrm{CK}}$, let $R_{i,j}$ be the current requirement. The strategy to satisfy $R_{i,j}$ is the following. Look for an $n > 2^j$ such that $f_j^\sigma(n) \downarrow \notin A_{\neq i}^\sigma$. Put $n$ into $A_i$ and restrain $f_j^\sigma(n)$ from entering $A_{\neq i}$. This may injure requirements with lower priority.

**Lemma 2.16** *For all $i, j$, the requirement $R_{i,j}$ acts only finitely many times.*

P r o o f. This is because the requirements are ordered in order type omega, and between any two stages at which the $(n + 1)$-st requirement acts, one of the first $n$ requirements must have acted. It follows by induction on $n$ that the $n$-th requirement only acts finitely many times. $\square$

**Lemma 2.17** *For all $i, j \in \omega$, $A_i \neq f_j^{-1}[A_{\neq i}]$.*

P r o o f. Assume the opposite, i.e. for some $i \in \omega$, $A_i \leq_{hm} A_{\neq i}$ via $f_j$. Choose a stage $\sigma$ where requirement $R_{i,j}$ is considered and requirements of higher priority have ceased to act; also choose an $n > 2^j$ such that $f_j^\sigma(n) \downarrow$ and $f_j^\sigma(n) \notin A_{\neq i}^\sigma$. Such a $n$ exists, as at most $2^k$ numbers less than $2^{k+1}$ are added to $A_i$ for each $k$ and therefore $A_i$ is co-infinite. But then at stage $\sigma$ a number was added to $A_i$ to violate the reduction $f_j$, contradiction. $\square$

The lemmas above prove the theorem. $\square$

**Corollary 2.18** *The countable atomless Boolean algebra may be embedded into the $hm$-degrees of $\Sigma_1^1$ subsets of $\omega$.*

Note that there are, or course, much deeper statements about the structure of c.e. $m$-degrees (e.g., [6, 19, 22]) that one could try to lift to $hm$-degrees of $\Pi_1^1$ sets. However, Corollary 2.18 provides enough evidence that the structure of $hm$-degrees of $\Sigma_1^1$ sets is rich.

# 3 A complete $\Sigma_1^1$ equivalence relation

We start the section by establishing some general properties of $\Sigma_1^1$ equivalence relations.

**Definition 3.1** An equivalence relation $E$ is *complete* in a class $\mathcal{R}$ of equivalence relations (with a specified reducibility), if $E \in \mathcal{R}$ and every equivalence relation from $\mathcal{R}$ is reducible to $E$ (with respect to the chosen reducibility).

**Theorem 3.2**    1. *There exists a universal $\Sigma_1^1$ enumeration of all $\Sigma_1^1$ equivalence relations on $\omega$.*

2. *There exists a complete $\Sigma_1^1$ equivalence relation $U$ (with respect to h- or FF-reducibility).*

P r o o f. Let $\{A_e\}_{e \in \omega}$ be the standard $\Sigma_1^1$ enumeration of all $\Sigma_1^1$ subsets of $\omega \times \omega$ (for instance, as in [23]). Define the equivalence relation $R_e$ as the reflexive transitive closure of $A_e$, i.e.

$$xR_ey \iff x = y \vee (\exists z_0, \ldots, z_k)[z_0 = x \& \ldots \& z_k = y \& (\forall i < k)(\langle z_i, z_{i+1} \rangle) \in A_e]$$
$$\vee (\exists z_0, \ldots, z_k)[z_0 = y \& \ldots \& z_k = x \& (\forall i < k)(\langle z_i, z_{i+1} \rangle) \in A_e].$$

Then every $\Sigma_1^1$ equivalence relation appears in this enumeration, moreover from the properties of the enumeration $\{A_e\}_{e \in \omega}$, the enumeration $\{R_e\}_{e \in \omega}$ is universal.

Now define an equivalence relation $R$ as follows:

$$\langle x, e \rangle R \langle y, e \rangle \iff xR_ey.$$

Then $R$ is an h- and FF-complete $\Sigma_1^1$ equivalence relation. $\square$

A useful and rather straightforward property of complete $\Sigma_1^1$ equivalence relations is the following:

**Proposition 3.3** *An h-complete (or FF-complete) $\Sigma_1^1$ equivalence relation has infinitely many properly $\Sigma_1^1$ equivalence classes.*

P r o o f. Under h- or FF-reducibility properly $\Sigma_1^1$ equivalence classes are mapped to properly $\Sigma_1^1$ equivalence classes. In Theorem 7.1 below we show that there exist $\Sigma_1^1$ equivalence relations with infinitely many properly $\Sigma_1^1$ equivalence classes. Thus, a complete $\Sigma_1^1$ equivalence relation must also have this property. $\square$

Recall the notion of $hm$-reducibility on subsets of $\omega$ introduced in Section 2.3. There exist $\Sigma_1^1$ equivalence relations with infinitely many $hm$-complete classes (e.g., as in Theorem 7.1 below). Therefore,

**Corollary 3.4** *An h-complete (FF-complete) $\Sigma_1^1$ equivalence relation must have infinitely many properly $\Sigma_1^1$ equivalence classes that are $hm$-complete ($m$-complete, respectively).*

In a following section we will show that this condition is necessary but not sufficient for a relation to be h- or FF-complete among $\Sigma_1^1$ equivalence relations.

**Remark 3.5** In [8] the authors showed that, in fact, the natural equivalence relation of bi-embeddability on the class of computable trees (here we mean the standard model-theoretic notion of embedding of structures) is FF-complete (thus, also h-complete) for the class of all $\Sigma_1^1$ equivalence relations on $\omega$, where trees are considered in the signature with one unary function symbol interpreted as the predecessor function. Furthermore, [10] showes that the isomorphism relation on many natural classes of computable structures is FF-complete among $\Sigma_1^1$ equivalence relations.

By the above results, there exist the h-degrees formed by $\Delta_1^1$ equivalence relations with exactly $n$ equivalence classes, for $n \leq \omega$, and a greatest h-degree of $\Sigma_1^1$ equivalence relations, namely, that of a complete $\Sigma_1^1$ equivalence relation. The next step is to show that the structure of h-degrees of properly $\Sigma_1^1$ equivalence relations is not trivial:

**Proposition 3.6** *There exists a $\Sigma_1^1$ equivalence relation on $\omega$ which is neither $\Delta_1^1$ nor h-complete.*

P r o o f. Let $(L_m)_{m \in \omega}$ be the numbering of all computable linear orderings on $\omega$. Consider the following equivalence relation $E_{\omega_1^{\mathrm{CK}}}$:

$$m E_{\omega_1^{\mathrm{CK}}} n \iff \quad \begin{aligned} &\text{either } L_m, L_n \text{ are not well-orders, (i.e. } m, n \notin \mathrm{WO}) \\ &\text{or } L_m \cong L_n. \end{aligned}$$

The relation $E_{\omega_1^{\mathrm{CK}}}$ is $\Sigma_1^1$ but not $\Delta_1^1$ as otherwise the equivalence class consisting of non-well-orderings would be a $\Delta_1^1$ set, a contradiction. Moreover, for every computable ordinal $\alpha$, the equivalence class of $E_{\omega_1^{\mathrm{CK}}}$ containing $\alpha$ is hyperarithmetical. The only properly $\Sigma_1^1$ equivalence class is the class consisting of the computable non well-orderings. As the complete relation $R$ constructed above has infinitely many properly $\Sigma_1^1$ equivalence classes, it cannot be reduced to $E_{\omega_1^{\mathrm{CK}}}$. Thus $E_{\omega_1^{\mathrm{CK}}}$ is not complete. □

We would like to mention another natural example of an incomplete properly $\Sigma_1^1$ equivalence relation: namely, the relation of bi-embeddability on the class of linear orders studied in [21]. Recall the notion of Scott rank: it is a measure of model theoretic complexity of countable structures. For a computable structure, the Scott rank is at most $\omega_1^{\mathrm{CK}} + 1$ (see, for instance, [4] for a definition and an overview of results about the Scott rank of computable structures). In the class of computable linear orderings with the relation of bi-embeddability, the only equivalence class that contains structures of high (i.e. non-computable) Scott rank is the class of the dense linear order $\eta$. All other equivalence classes contain only structures of computable Scott rank (see [21] for details). If bi-embeddability on linear orderings were complete, it would necessarily have infinitely many equivalence classes with structures of high Scott rank. Therefore, bi-embeddability on linear orders cannot be complete.

## 4　Embedding $\Sigma_1^1$ sets into $\Sigma_1^1$ relations

For the reasons stated in Fact 2.10 we are interested in the structure of properly $\Sigma_1^1$ equivalence relations, i.e. relations that are $\Sigma_1^1$ but not $\Delta_1^1$. In this section we will prove the following theorem:

**Theorem 4.1** *The structure of properly $\Sigma_1^1$ sets with the relation of m-reducibility is order-preservingly (and effectively) embedded into the structure of properly $\Sigma_1^1$ equivalence relations with the relation of FF-reducibility, i.e. one can assign to every properly $\Sigma_1^1$ set $A$ a properly $\Sigma_1^1$ equivalence relation $E_A$ such that for any properly $\Sigma_1^1$ sets $A, B$,*

$$A \leq_m B \iff E_A \leq_{\mathrm{FF}} E_B.$$

Before we give the proof of this theorem we will show the following:

**Theorem 4.2** *The structure of properly $\Sigma_1^1$ sets with the relation of 1-reducibility is order-preservingly (and effectively) embedded into the structure of properly $\Sigma_1^1$ equivalence relations with the relation of FF-reducibility where the reducing function is $1 - 1$.*

P r o o f. Let $A$ be a properly $\Sigma_1^1$ set.

Define the relation $E_A$ in the following way:

$$xE_Ay \iff \quad x, y \in A$$
$$\text{or } x = y.$$

The relation $E_A$ is properly $\Sigma_1^1$.

**Lemma 4.3** *For all properly $\Sigma_1^1$ sets $A, B$,*

$$A \leq_1 B \iff E_A \leq_{\mathrm{FF}} E_B,$$

*where the* FF*-reducibility is witnessed by a computable $1 - 1$ function.*

P r o o f. The direction from right to left is obvious. To prove the direction from left to right suppose $A \leq_1 B$ via a computable $1 - 1$ function $f$. Consider $x, y$ such that $xE_Ay$. By definition of $E_A, E_B$ and by properties of $f$,

$$xE_Ay \iff x, y \in A \text{ or } x = y \iff f(x), f(y) \in B \text{ or } f(x) = f(y) \iff f(x)E_Bf(y).$$

We use the fact that $f$ is injective to prove the equivalence of the $3rd$ and the $2nd$ statement. □

The lemma proves the theorem.

□

**Remark 4.4** Relations of this kind for $\Sigma_1^0$ sets were considered in [13].

**Proposition 4.5** *There exists an effective procedure which transforms a properly $\Sigma_1^1$ set $A$ into a properly $\Sigma_1^1$ set $A^*$ in such a way that*

$$A \leq_m B \Rightarrow A^* \leq_1 B^*;$$

$$A^* \leq_m B^* \Rightarrow A \leq_m B.$$

P r o o f. For every set $A$, define $A^* = A \times \omega = \{\langle x, i \rangle | x \in A, i \in \omega\}$. For every $i$, denote by $A_i$ the set $\{\langle x, i \rangle | x \in A\}$. Then $A^* = \cup_i A_i$. Note that by definition of $A^*$,

$$x \in A \iff \forall i \langle x, i \rangle \in A^* \iff \exists i \langle x, i \rangle \in A^*.$$

Suppose $A \leq_m B$ via a computable function $f$. We define a computable function h in the following way: for $x' = \langle x, i \rangle$ let $h(x') = \langle f(x), \langle x, i \rangle \rangle$, i.e. we send every $x' \in A_i$ to an element of $B_{x'}$. It guarantees that the function h is $1 - 1$. Thus we only need to show that h witnesses the 1-reduction of $A^*$ to $B^*$:

$$x' \in A^* \iff x \in A \iff f(x) \in B \iff \langle f(x), \langle x, i \rangle \rangle \in B^*.$$

Now suppose $A^* \leq_m B^*$ via a computable function h. Define $f(x) = y \iff l(h(\langle x, 0 \rangle)) = y$, i.e. $h(\langle x, 0 \rangle) = \langle y, j \rangle$, for some $j \in \omega$. Then the function $f$ $m$-reduces $A$ to $B$:

$$x \in A \iff \langle x, 0 \rangle \in A^* \iff h(\langle x, 0 \rangle) = \langle y, j \rangle \in B^* \iff y \in B.$$

□

Proof of Theorem 4.1. The proof now follows directly from Proposition 4.5 and Theorem 4.2. □

**Corollary 4.6** *For any $1 \leq n \leq \omega$, there exists an effective embedding of the structure of properly $\Sigma_1^1$ sets under $m$-reducibility into the structure of properly $\Sigma_1^1$ relations with exactly $n$ properly $\Sigma_1^1$ equivalence classes under the* FF*-reducibility.*

In Section 2.3 we introduced the notion of $hm$-reducibility on sets which is a hyperarithmetical analogue of $m$-reducibility. We showed that the structure of $hm$-degrees of $\Sigma_1^1$ sets is complicated. Consider now a hyperarithmetical version of the 1-reducibility of subsets of $\omega$:

**Definition 4.7** Let $A, B$ be subsets of $\omega$. We say that $A$ is hyperarithmetically 1-reducible to $B$, denoted by $A \leq_{h1} B$, iff there exists a hyperarithmetical $1 - 1$ function $f$, such that for every $n \in \omega$,

$$n \in A \iff f(n) \in B.$$

Using this definition and ideas from above one can show the following:

**Theorem 4.8** *The structure of properly $\Sigma_1^1$ sets with the relation of $hm$-reducibility is order-preservingly (and effectively) embedded into the structure of properly $\Sigma_1^1$ equivalence relations with the relation of h-reducibility, i.e. one can assign to every properly $\Sigma_1^1$ set $A$ a properly $\Sigma_1^1$ equivalence relation $E_A$ such that for any properly $\Sigma_1^1$ sets $A, B$,*

$$A \leq_{hm} B \iff E_A \leq_{\mathrm{h}} E_B.$$

*Moreover, for every $n \leq \omega$, there is such an embedding into the structure of properly $\Sigma_1^1$ equivalence relations with exactly $n$ properly $\Sigma_1^1$ equivalence classes.*

Thus, the structure of h-degrees of $\Sigma_1^1$ equivalence relations even with just one properly $\Sigma_1^1$ equivalence class is at least as rich as the structure of $\Sigma_1^1$ sets under $hm$-reducibility.

## 5  Properly $\Sigma_1^1$ Equivalence Relations with only hyperarithmetical equivalence classes

In this section we show that a properly $\Sigma_1^1$ equivalence relation need not contain properly $\Sigma_1^1$ equivalence classes. Moreover, the example we present contains only equivalence classes of size 1 or 2.

Let $A$ be a $\Sigma_1^1$ subset of $\omega$ which is not $\Delta_1^1$. Define the corresponding equivalence relation $F_A$ on $\omega \times 2$ in the following way:

$$(m_0, n_0) F_A(m_1, n_1) \iff \quad \begin{aligned} & m_0 = m_1 \in A \\ & \text{or } (m_0, n_0) = (m_1, n_1). \end{aligned}$$

The relation $F_A$ is $\Sigma_1^1$. The equivalence classes of $F_A$ are of the form $\{(m, n) | 1 \leq n \leq 2\}$, if $m \in A$, and $\{(m, n)\}$, if $m \notin A$. In particular, every equivalence class has size 1 or 2. Again, similar relations constructed from $\Sigma_1^0$ sets were considered in [13].

**Claim 5.1** *The equivalence relation $F_A$ is properly $\Sigma_1^1$.*

P r o o f. If $F_A$ were $\Delta_1^1$, so would be the set $A$, as $A = \{m | (m, 0) F_A(m, 1)\}$, a contradiction.  □

One can easily modify the example to get an equivalence relation with classes of size at most (and including) $k$, for $2 \leq k < \omega$.

**Definition 5.2** Following [13], we call an equivalence relation $k$-bounded if all its equivalence classes have size at most $k$.

**Theorem 5.3** *There exists a properly $\Sigma_1^1$ equivalence relation $S^{k+1}$ with all its equivalence classes containing at most $k + 1$ element such that for no $\Sigma_1^1$ equivalence relation $R$ with its equivalence classes containing at most $k$ elements do we have $R^{k+1} \leq_{\mathrm{h}} S$ (hence, for no such $R$ do we have $R^{k+1} \leq_{\mathrm{FF}} S$).*

P r o o f. As shown in [13], the analogous result is true for the case of c.e. relation. Simple transformation of this argument proves the theorem for $\Sigma_1^1$ equivalence relations.  □

## 6  Equivalence Relations with finitely many properly $\Sigma_1^1$ classes

One can modify the example from the proof of Proposition 3.6 to get, for every finite $k \geq 2$, a $\Sigma_1^1$ equivalence relation which has exactly $k$ properly $\Sigma_1^1$ equivalence classes:

**Proposition 6.1** *For every finite $k \geq 1$ there exists a $\Sigma_1^1$ equivalence relation on $\omega$ with infinitely many equivalence classes, such that exactly $k$ of them are properly $\Sigma_1^1$.*

P r o o f. Let $A_1, \ldots, A_k$ be disjoint properly $\Sigma_1^1$ sets. Consider the relation $E_{A_1,\ldots,A_k}$:

$$x E_{A_1 \ldots, A_k} y \Longleftrightarrow x = y \vee x, y \in A_1 \vee \ldots \vee x, y \in A_k.$$

Then $E_{A_1 \ldots, A_k}$ has the desired properties. $\qquad\square$

We give another example of equivalence relations with exactly $k$ properly $\Sigma_1^1$ classes, for $k \geq 1$. The reason is that in the next section we will use a generalization of this example.

Again consider $\omega$ as a set of codes for linear orders. We will define relations $F_k$, for $k \geq 1$, on pairs of linear orders. First of all, we define additional hyperarithmetical equivalence relations $E_k$ (here we identify natural numbers $k, k'$ with ordinals):

$$n_1 E_k n_2 \Longleftrightarrow \quad \begin{aligned} &\text{either } L_{n_1} \cong L_{n_2} \cong k' < k - 1 \\ &\text{or both } n_1, n_2 \text{ are not codes for well-orders of type } k' < k - 1. \end{aligned}$$

By definition, $E_k$ is hyperarithmetical and has exactly $k$ equivalence classes. We now define $F_k$ as follows: for $(m_i, n_i) \in \omega^2, i = 1, 2$,

$$(m_1, n_1) F_k (m_2, n_2) \Longleftrightarrow \quad \begin{aligned} &\text{either } (L_{m_1}, L_{m_2} \text{ are not well-orders and } n_1 E_k n_2) \\ &\text{or } (L_{m_1} \cong L_{m_2}). \end{aligned}$$

The idea is that we "cut" the properly $\Sigma_1^1$ class of $E_{\omega_1^{\mathrm{CK}}}$ (the relation defined in Proposition 3.6) into $k$ properly $\Sigma_1^1$ pieces. The relations $F_k$, $k \geq 1$, have the necessary properties. Moreover,

**Proposition 6.2** *For all $1 \leq k_1 < k_2 < \omega$, $F_{k_1} <_{\mathrm{h}} F_{k_2}$.*

P r o o f. Let $f$ be a hyperarithmetical function which witnesses $E_{k_1} <_{\mathrm{h}} E_{k_2}$. Consider the function $g(m, n) = (m, f(n))$. It is hyperarithmetical and reduces $F_{k_1}$ to $F_{k_2}$. The reduction is strict, as $F_{k_1}$ has fewer properly $\Sigma_1^1$ equivalence classes than $F_{k_2}$. $\qquad\square$

**Remark 6.3** No $F_k$, for $k \geq 1$, is complete as no $\Sigma_1^1$ equivalence relation with only finitely many properly $\Sigma_1^1$ equivalence classes can be complete for the class of $\Sigma_1^1$ equivalence relations.

## 7 Equivalence Relations with infinitely many properly $\Sigma_1^1$ classes

In this section we show that an infinite number of properly $\Sigma_1^1$ equivalence classes does not guarantee the h- or FF-completeness of a $\Sigma_1^1$ equivalence relation.

Indeed, it is easy to construct a non-complete $\Sigma_1^1$ equivalence relations with infinitely many properly $\Sigma_1^1$ equivalence classes. Take a computable sequence $(A_n)_{n \in \omega}$ of disjoint $\Sigma_1^1$ sets, such that none of them is complete and consider the relation $R_\infty$ defined as follows:

$$x R_\infty y \Longleftrightarrow x = y \vee \exists n (x, y \in A_n).$$

As the sequence $(A_i)_{i \in \omega}$ is computable, the relation $R_\infty$ is $\Sigma_1^1$. Moreover, it is not complete as, for example, the relation $R_B$ for a complete $\Sigma_1^1$ set $B$ constructed as in Section 4 is not reducible to $R_\infty$.

By Corollary 3.4, an h-complete (a FF-complete) $\Sigma_1^1$ equivalence relation must have infinitely many equivalence classes that are $hm$-complete ($m$-complete) as $\Sigma_1^1$ sets. Below we will show that this condition is not sufficient:

**Theorem 7.1** *There exists a non-h-complete (non-FF-complete) $\Sigma_1^1$ equivalence relation with infinitely many classes that are $hm$-complete ($m$-complete) among $\Sigma_1^1$ sets.*

The proof of the theorem will follow from Proposition 7.3 below.

For every computable infinite ordinal $\alpha$, we define equivalence relations $E_\alpha$ and $F_\alpha$ in the following way:

$$n_1 E_\alpha n_2 \Longleftrightarrow \quad \begin{aligned} &\text{either } L_{n_1} \cong L_{n_2} \cong \alpha' < \alpha \\ &\text{or [neither } n_1 \text{ nor } n_2 \text{ code well-orders of type } < \alpha]. \end{aligned}$$

In other words, for each $\alpha' < \alpha$, there is an equivalence class consisting of linear orders isomorphic to $\alpha'$. All the linear orders that are not isomorphic to any $\alpha' < \alpha$ form a single equivalence class. By definition, if $\alpha$ is computable, then $E_\alpha$ is hyperarithmetical with infinitely many equivalence classes, provided $\alpha$ is infinite. Indeed, for a fixed $\alpha < \omega_1^{\mathrm{CK}}$ it is hyperarithmetical to check whether or not some $n \in \omega$ is a code for a well-order of type $\alpha$ or of type $< \alpha$. Then both the first and the second line of the definition give hyperarithmetical conditions. Hence, for infinite $\alpha < \omega_1^{\mathrm{CK}}$, all $E_\alpha$ are hyperarithmetical and h-equivalent to each other. Notice that there is some non-uniformity in the definition of $E_\alpha$ for finite (defined in the previous section) and infinite $\alpha$.

Now define:

$$(m_1, n_1) F_\alpha (m_2, n_2) \iff \quad \text{either } L_{m_1}, L_{m_2} \text{ are not well-orders and } n_1 E_\alpha n_2$$
$$\text{or } L_{m_1} \cong L_{m_2}.$$

**Proposition 7.2** *For all computable infinite $\alpha_1, \alpha_2$, $F_{\alpha_1} \equiv_{\mathrm{h}} F_{\alpha_2}$.*

P r o o f. Consider the function h that witnesses the h-equivalence of the corresponding $E_{\alpha_1}, E_{\alpha_2}$. The function $h'$ which sends a pair $(m, n)$ into the pair $(m, h(n))$ gives the equivalence of $F_{\alpha_1}, F_{\alpha_2}$. $\qquad \square$

Recall the definition of the relation $E_{\omega_1^{\mathrm{CK}}}$ from Section 3:

$$m E_{\omega_1^{\mathrm{CK}}} n \iff \quad \text{either } L_m, L_n \text{ are not well-orders, (i.e. } m, n \notin \mathrm{WO})$$
$$\text{or } L_m \cong L_n.$$

Finally, we define an equivalence relation $F_{\omega_1^{\mathrm{CK}}}$ as follows:

$$(m_1, n_1) F_{\omega_1^{\mathrm{CK}}} (m_2, n_2) \iff \quad \text{either } L_{m_1}, L_{m_2} \text{ are not well-orders and } n_1 E_{\omega_1^{\mathrm{CK}}} n_2$$
$$\text{or } L_{m_1} \cong L_{m_2}.$$

Note that all $F_\alpha$, $\alpha < \omega_1^{\mathrm{CK}}$ and $F_{\omega_1^{\mathrm{CK}}}$ have infinitely many equivalence classes that are $m$-complete (thus, also $hm$-complete) among $\Sigma_1^1$ sets.

**Proposition 7.3** *For every computable $\alpha$,*

$$F_\alpha <_{\mathrm{h}} F_{\omega_1^{\mathrm{CK}}}.$$

P r o o f. Obviously, $F_\alpha \leq_{\mathrm{h}} F_{\omega_1^{\mathrm{CK}}}$: let $f$ reduce $E_\alpha$ to $E_{\omega_1^{\mathrm{CK}}}$, then $g(m, n) = (m, f(n))$ reduces $F_\alpha$ to $F_{\omega_1^{\mathrm{CK}}}$. We only need to prove that $F_{\omega_1^{\mathrm{CK}}}$ is not reducible to $F_\alpha$, for any computable $\alpha$. Suppose that for some computable $\alpha$ there were such a hyperarithmetical reduction h:

$$(m_1, n_1) F_{\omega_1^{\mathrm{CK}}} (m_2, n_2) \iff h((m_1, n_1)) F_\alpha h((m_2, n_2)).$$

Consider $n_1, n_2 \in \omega$. For every $m \notin \mathrm{WO}$ we have:

$$n_1 E_{\omega_1^{\mathrm{CK}}} n_2 \iff (m, n_1) F_{\omega_1^{\mathrm{CK}}} (m, n_2) \iff h((m, n_1)) F_\alpha h((m, n_2)) \iff$$

$$\iff L_{m_1} \cong L_{m_2} \cong \gamma, \text{ where } \gamma \text{ is an ordinal, or } [m_1, m_2 \notin \mathrm{WO} \text{ and } l_1 E_\alpha l_2],$$

where $h(m, n_i) = (m_i, l_i), i = 1, 2$. Fix this notation for the rest of the proof.

If there exists an $m \notin \mathrm{WO}$ such that for all $n_1, n_2$ the corresponding $m_1, m_2 \notin \mathrm{WO}$, then the proposition is proved. Indeed, fix such an $m$. Then for all $n_1, n_2 \in \omega$,

$$n_1 E_{\omega_1^{\mathrm{CK}}} n_2 \iff (m, n_1) F_{\omega_1^{\mathrm{CK}}} (m, n_2) \iff (m_1, l_1) F_\alpha (m_2, l_2) \iff l_1 E_\alpha l_2,$$

which gives a hyperarithmetical reduction of $E_{\omega_1^{\mathrm{CK}}}$ to $E_\alpha$, a contradiction.

Suppose now that for every $m \notin \mathrm{WO}$ there exist $n_1, n_2 \in \omega$ such that $L_{m_1} \cong L_{m_2} \cong \gamma$, for some $\gamma < \omega_1^{\mathrm{CK}}$. Define a $\Pi_1^1$ relation $R(m, n)$ as follows:

$$R(m, n) \iff \quad (m \in \mathrm{WO} \wedge m = n)$$
$$\text{or } (n \in \mathrm{WO} \wedge L_n \cong L_{m_1} \cong L_{m_2} \text{ associated to some } h(m, n_1), h(m, n_2)).$$

By Uniformization, $R$ can be uniformized by a $\Pi_1^1$ function $f$. The function $f$ is total, thus hyperarithmetical from $\omega$ to WO. By Bounding, the range of $f$ is bounded by a computable ordinal $\gamma_0$.

Consider now all $m \in$ WO, for which there exist $n_1, n_2$ such that $L_{m_1} \cong L_{m_2} \cong \gamma_0$. Then there is a computable bound $\alpha_0$ on ordinals coded by such elements $m$.

Now we have

$$\text{WO} = \{m | m \neq code(\beta) \text{ for } \beta \leq \alpha_0 \text{ and } \exists n_1, n_2 L_{m_1} \cong L_{m_2} \cong \gamma \leq \gamma_0\},$$

which gives a hyperarithmetical definition of WO, a contradiction. $\square$

**Remark.** The process above of constructing of $\Sigma_1^1$ equivalence relations may be iterated further. In particular, the relation $F_{\omega_1^{\text{CK}}}$ is not complete among $\Sigma_1^1$ equivalence relations.

## 8 More Results

The following result from [13] shows the difference between the theory of $\Sigma_1^0$ equivalence relations and that of $\Sigma_1^1$ equivalence relations:

**Theorem 8.1** *Let $A_1, \ldots, A_n$ be disjoint c.e. sets the complement of whose union is infinite. Then*

$$\text{id}_\omega \leq_{\text{FF}} R_{A_1,\ldots,A_n} \iff A_1 \cup \ldots \cup A_n \text{ is not simple.}$$

Here

$$x R_{A_1,\ldots,A_n} y \iff x = y \vee \exists i \leq n (x, y \in A_i).$$

In the case of h-reducibility and disjoint $\Sigma_1^1$ sets $A_1, \ldots, A_n$,

$$\text{id}_\omega \leq_{\text{h}} R_{A_1,\ldots,A_n}$$

always holds. Indeed, the complement $C$ of $\bigcup_{i \leq n} A_i$ is a $\Pi_1^1$ set, thus it contains a hyperarithmetical subset $B$. Then a $1 - 1$ hyperarithmetical function from $\omega$ onto $B$ witnesses the reduction.

The analogy with c.e. equivalence relations might be more complete if we considered $\Pi_1^1$ equivalence relations. Using ideas from [13] one can show the following:

**Theorem 8.2** *There exist properly $\Sigma_1^1$ equivalence relations that are $m$-complete (hm-complete) as $\Sigma_1^1$ sets but FF-incomparable (respectively, h-incomparable) as $\Sigma_1^1$ equivalence relations.*

P r o o f. Let $A$ be an $m$-complete, hence, also $hm$-complete $\Sigma_1^1$ set. Let $E_A$ be a $\Sigma_1^1$ equivalence relation built from $A$ as in Section 4. Let $F_A$ be a $\Sigma_1^1$ equivalence relation with all its equivalence classes finite built from $A$ as in Section 5. Then $E_A$ and $F_A$ are neither FF-comparable nor h-comparable.

Suppose $E_A$ is reducible to $F_A$ via a computable (or hyperarithmetical) function $f$. Fix an arbitrary $x_0 \in A$ and let $y_0 = f(x_0)$. Then $A = \{x | f(x) F_A y_0\}$, therefore $A \leq [y_0]_{F_A}$, where $[y_0]_{F_A}$ is finite. Thus $A$ is computable (hyperarithmetical), a contradiction.

Suppose now that $F_A$ is reducible to $E_A$ via $g$. Consider the set $B = \{g(x) | x \in \omega\}$. Then $B \cap A \neq \emptyset$, otherwise $F_A$ would be reducible to $\text{id}_\omega$, thus hyperarithmetical. Now let $C = \{x | g(x) \in A\}$, then $C$ is an equivalence class of $F_A$. Pick an arbitrary $y \in A$ and define $h(x)$ in the following way:

$$h(x) = \begin{cases} y, & \text{if } x \in C \\ g(x), & \text{otherwise} \end{cases}$$

All equivalence classes of $F_A$ are finite, thus h is a computable (hyperarithmetical) function which reduces $F_A$ to the equality on $\omega$. $\square$

## 9    Questions

If an equivalence relation $E$ is reducible to an equivalence relation $E'$ (under any of the two reducibilities considered here) then $E$ is reducible to $E'$ as sets (under the corresponding reducibility). On the other hand, if a $\Sigma_1^1$ equivalence relation is $m$-complete ($hm$-complete) as a $\Sigma_1^1$ set, it does not guarantee that it is FF-complete (h-complete) as a $\Sigma_1^1$ equivalence relation. Indeed, let $A$ be an $m$-($hm$-)complete $\Sigma_1^1$ set. Let $E_A$ be a $\Sigma_1^1$ equivalence relation built as in Sections 4 or 5. Then $E_A$ is not complete among $\Sigma_1^1$ relations but it is obviously complete as a $\Sigma_1^1$ set. One can also build such equivalence relations with any number of properly $\Sigma_1^1$ equivalence classes.

As it follows from Theorem 8.2, two $\Sigma_1^1$ equivalence relations may be incomparable while both being $m$-complete among $\Sigma_1^1$ sets. However in the above example one of the relations had only finite classes while the other relation had an infinite class and all the other classes of size 1. Thus the following set of questions arises naturally:

**Question 9.1** *Let $E, E'$ be $\Sigma_1^1$ equivalence relations with only finite (or hyperarithmetical) equivalence classes. Suppose $E, E'$ are both complete as sets (under $m$- or $hm$-reducibility). As follows from Theorem 5.3, it may be the case that $E < E'$. Is it possible that $E$ and $E'$ are incomparable?*

**Question 9.2** *The same for relations with a fixed number of properly $\Sigma_1^1$ ($\Sigma_1^0$) equivalence classes.*

We studied properly $\Sigma_1^1$ equivalence relations according to the number of their properly $\Sigma_1^1$ equivalence classes. We saw examples of equivalence relations with only hyperarithmetical classes, with exactly $n$ properly $\Sigma_1^1$ equivalence classes, for $n \in \omega$ and with infinitely many properly $\Sigma_1^1$ equivalence classes.

**Question 9.3** *Does there exists a properly $\Sigma_1^1$ equivalence relation on (a hyperarithmetical subset of) $\omega$ with infinitely many equivalence classes such that all its classes are properly $\Sigma_1^1$?*

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