Classes of structures with universe a subset of ω_1

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Abstract

We continue recent work on computable structure theory in the setting of ω_1. We prove the analogue of a result from [7] saying that isomorphism of computable structures lies "on top" among Σ^1_1 equivalence relations on ω. Our equivalence relations are on ω_1. In the standard setting, Σ^1_1 sets are characterized in terms of paths through trees. In the setting of ω_1, we use a new characterization of Σ^1_1 sets that involves clubs in ω_1. Finally, we present some new results about ω_1-computable categoricity for fields.

1 Introduction

Recent work on computable structure theory has come to include the setting of ω_1-computability, in addition to the standard setting of ω-computability. In [13], there is a sample of results, including some that transfer immediately from the standard setting, some that transfer in modified form, and some that do not transfer at all. The main motivation for this work is that there are familiar uncountable structures, such as the field of real numbers and the field of complex numbers, which feel computable. With suitable assumptions, these structures actually are computable in ω_1. In [5], there is a definition of the arithmetical hierarchy in the setting of ω_1, and of "computable Σ_α" and "computable Π_α" formulas, for countable ordinals α. (These are formulas of L_{ω_2,ω_1}, as opposed to $L_{\omega_1\omega}$.) The main result of [5] is that for countable ordinals α, a relation R on a computable structure $\mathcal{A}$ is "relatively intrinsically Σ_α" iff it is definable by a computable Σ_α formula.

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In [14] and [12], there are results on computable categoricity in the setting of ω_1. In [14], it is shown that the "Zil'ber field" of size $\aleph_1$ is not computably categorical, while the "Zil'ber cover " of size $\aleph_1$ is computably categorical. The Zil'ber fields are structures that resemble the field of complex numbers, with complex exponentiation. The Zil'ber covers are a related class of structures. There is a general condition on "quasi-minimal excellent" classes, saying exactly when the member of size $\aleph_1$ will be computably categorical. In [12], there is a characterization of the linear orderings that are computably categorical in the setting of ω_1.

In the remainder of Section 1, we give a very brief summary of the basic definitions and elementary results on computability and computable structures in the setting of ω_1. For more details, we refer the reader to [13]. In Section 2 we define an analogue of the Σ^1_1 sets, and imitate a result of Kleene on Σ^1_1-completeness in order to give an analogue of a theorem from [7] concerning Σ^1_1 equivalence relations. Our result involves clubs (closed and unbounded sets) instead of paths through trees. In Section 3, we define the notion of a *computable embedding* of one class of structures (of size $\aleph_1$) into another, giving several examples. In particular, we give an analogue of a result of H. Friedman and Stanley [8], concerning universality of linear orderings. Finally, in Section 4, we study computable categoricity of fields in the context of ω_1.

1.1 Basic computability in the setting of ω_1

We assume at least that all subsets of ω are constructible, and in some places, we assume that all subsets of ω_1 are constructible. The basic definitions come from "α-recursion" theory, where $\alpha = \omega_1$ (see [23]).

Definition 1.1.

- *A set or relation on ω_1 is* computably enumerable*, or* c.e.*, if it is defined in $(L_{\omega_1}, \in)$ by a Σ_1-formula $\varphi(\bar{c}, x)$, with finitely many parameters—a Σ_1 formula is finitary, with only existential and bounded quantifiers.*
- *A set or relation is* computable *if it and its complement are both computably enumerable.*
- *A (partial) function is* computable *if its graph is c.e.*

Results of Gödel give us a $1-1$ function g from ω_1 onto L_{ω_1} such that the relation $g(\alpha) \in g(\beta)$ is computable. The function g gives us ordinal codes for sets, so that computing on ω_1 is really the same as computing on L_{ω_1}. There is also a computable function ℓ taking α to the code for L_α. Using the fact that L_{ω_1} is closed under α-sequences for any countable ordinal α, we may allow relations and functions of arity α, where α is any countable ordinal.

As in the standard setting, we have *indices* for c.e. sets. There is a c.e. set C of codes for pairs $(\varphi, \bar{c})$, representing Σ_1 definitions, where $\varphi(\bar{u}, x)$ is a Σ_1-formula and $\bar{c}$ is a tuple of parameters appropriate for $\bar{u}$. There is a computable function h mapping ω_1 onto C. A set X has index α if $h(\alpha)$ is the code for a pair $(\varphi, \overline{c})$ representing a Σ_1 formula $\varphi(\overline{c}, x)$, and X is the set defined in $(L_{\omega_1}, \in)$ by this formula. We write W_α for the c.e. set with index α. Suppose W_α is determined by the pair $(\varphi, \overline{c})$; i.e., $\varphi(\overline{c}, x)$ is a Σ_1 definition. We say that *x is in W_α at stage β*, and we write $x \in W_{\alpha,\beta}$, if L_β contains x, the parameters $\overline{c}$, and witnesses making the formula $\varphi(\overline{c}, x)$ true. The relation $x \in W_{\alpha,\beta}$ is computable. Let $U \subseteq (\omega_1)^2$ consist of the pairs (α, β) such that $\beta \in W_\alpha$. Then U is m-complete c.e. It is not computable, since the "halting set" $K = \{\alpha : \alpha \in W_\alpha\}$ is c.e. and not computable.

In the setting of ω_1, we have a good notion of relative computability.

Definition 1.2.

- *A relation is* c.e. relative to X *if it is Σ_1-definable in $(L_{\omega_1}, \in, X)$.*
- *A relation is* computable relative to X *if it and its complement are both c.e. relative to X.*
- *A (partial or total) function is* computable relative to X *if the graph is c.e. relative to X.*

A *c.e. index* for R relative to X is an ordinal α such that $g(h(\alpha)) = (\varphi, \overline{c})$, where φ is a Σ_1 formula (in the language with $\in$ and a predicate symbol for X), and $\varphi(\overline{c}, x)$ defines R in $(L_{\omega_1}, \in, X)$. We write W_α^X for the c.e. set with index α relative to X.

As in the standard setting, we have a universal c.e. set of partial computations using oracle information. Let U consist of the codes for triples (σ, α, β) such that $\sigma \in 2^\rho$ (for some countable ordinal ρ), and for X with characteristic function extending σ, $\beta \in W_\alpha^X$. Then U is c.e.

Definition 1.3. *The* jump *of X is the set $X' = \{(\alpha, x) : x \in W_\alpha^X\}$.*

We can iterate the jump function through countable levels. We let $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$, and for limit α, $X^{(\alpha)}$ is the set of codes for pairs (β, x) such that $\beta < \alpha$ and $x \in X^{(\beta)}$. As L_{ω_1} is closed under countable sequences, it follows that for countable limit λ, $X^{(\lambda)}$ is the least upper bound of the $X^{(\alpha)}$ for $\alpha < \lambda$, in the ordering of relative computability.

1.2 A little computable structure theory

We consider structures with universe a subset of ω_1. As in the standard setting, we usually identify a structure with its atomic diagram. A structure is *computable* if the atomic diagram is computable. A structure is *decidable* if the complete diagram is computable. We mention some simple examples, taken from [13]. The ordered field of reals has a computable copy with universe ω_1. If we think of the reals as a subset of L_{ω_1}, where each number is represented by a rational cut, this is a computable structure. The field of complex numbers has a computable copy. We may even add exponential functions such as exp, noting that any analytic function is determined by the countable sequence of coefficients of a power series.

In the standard setting, Morley [20] and Millar [19] showed that for any countable complete decidable elementary first order theory T, there is a decidable saturated model iff there is a computable enumeration of the complete types consistent with T. In the setting of ω_1, we have the following (see [13]).

Proposition 1.4. *For any countable complete elementary first order theory T (with infinite models), T has a decidable saturated model with universe ω_1.*

In the standard setting, the first non-computable ordinal, ω_1^{CK}, is the next admissible ordinal after ω. In the setting of ω_1, the first non-computable ordinal comes much before the next admissible after ω_1. This was well-known in the 1970's. The simple proof[1] is found in [9] and [13]. In the standard setting, the *Harrison ordering* is a computable ordering of type $\omega_1^{CK} \cdot (1+\eta)$. This ordering has initial segments isomorphic to all computable well orderings. In the setting of ω_1, we have the following (see [13]).

Theorem 1.5 (Greenberg-Knight-Shore)**.** *There is a computable ordering $\mathcal{H}$ with initial segments isomorphic to all computable ordinals.*

Sketch of proof. We take a uniformly computable list of linear orderings, representing all computable isomorphism types, and carry out a finite-injury priority construction to produce $\mathcal{H}$ with an initial segment that is a sum of intervals representing the well ordered $\mathcal{A}_\alpha$, in order, followed by various other intervals that are not well ordered.

□

In the standard setting the property of being well ordered is complete Π_1^1. We have seen that in the setting of ω_1 being well-ordered is relatively simple. The following result holds in the standard setting [7].

Theorem 1.6 (Fokina-S.Friedman-Harizanov-Knight-McCoy-Montalbán)**.** *For every Σ_1^1 equivalence relations E on ω, there is a uniformly computable sequence of trees $(T_n)_{n\in\omega}$ (subtrees of $\omega^{<\omega}$) such that*

$$mEn \Leftrightarrow T_m \cong T_n \ .$$

In the proof of Theorem 1.6, in the standard setting, the isomorphism type of each tree T_n is determined by an ω-sequence of computable ordinals, or ∞.

In [7], the result for trees is used to show that isomorphism on computable members of certain other classes lies on top in the same way: notably, torsion-free Abelian groups and Abelian p-groups.

[1] Here is the argument: Let α be the least admissible after ω_1. Then the set of computable well-orderings of ω_1 is an element of L_α and the function f that takes such a well-ordering to its length is Σ_1 definable over L_α; it follows that the range of f is bounded in α.

In the next section, we shall lift Theorem 1.6 to the setting of ω_1. Since there are so few computable ordinals, we shall need some fresh ideas.

Theorem 1.7. *Assume $V = L$. For any Σ^1_1 equivalence relation E on ω_1, there is a uniformly computable sequence of structures $M^*(\alpha)_{\alpha<\omega_1}$ (with universe ω_1) such that $\alpha E \beta$ iff $M^*(\alpha) \cong M^*(\beta)$.*

2 Σ^1_1 sets

Recall that in the standard setting, a set $S \subseteq \omega$ is Σ^1_1 if there is a computable relation $R(x, u)$ such that

$$n \in S \Leftrightarrow (\exists f \in \omega^\omega)\,(\forall s \in \omega)\, R(n, f \upharpoonright s)\ .$$

Kleene showed the following.

Theorem 2.1 (Kleene)**.** *If S is Σ^1_1, then there is a uniformly computable sequence of subtrees $(T_x)_{x\in\omega}$ of $\omega^{<\omega}$ such that $x \in S$ iff T_x has a path.*

In the standard setting, a computable tree with no path has a tree rank that is a computable ordinal. The ordinal tree ranks were crucial to the proof of Theorem 1.6. In our setting, we do not have enough computable ordinals, so we will need a new idea. We take the following as our definition of Σ^1_1 subset of ω_1.

Definition 2.2. *A set $S \subseteq \omega_1$ is Σ^1_1 if there is a computable relation R, on ordinals and functions in $\omega_1^{<\omega_1}$, such that $x \in S$ iff $(\exists f \in \omega_1^{\omega_1})\,(\forall \beta \in \omega_1)\, R(x, f \upharpoonright \beta)$.*

Thus Σ^1_1 sets are projections of "co-c.e." subsets of $\omega_1^{\omega_1}$, defined using Π_1 formulas and parameters from L_{ω_1}. As in the standard setting, we can replace Π_1 by Π_n here for any finite n, using Skolem functions to replace alternating quantification over L_{ω_1} with existential quantification over functions in $\omega_1^{\omega_1}$ (for the details see Chapter 16 of [22]).

A *subtree T of $\omega_1^{<\omega_1}$* is a subset of $\omega_1^{<\omega_1}$ which is closed under initial segments (i.e., if σ belongs to T then so does $\sigma \upharpoonright \beta$ for all β). Two such trees are *isomorphic* if there is a bijection between them which preserves the initial segment relation $\sigma \subseteq \tau$. The same definitions apply to subtrees of $A^{<\omega_1}$ for any set A.

Lemma 2.3. *For any Σ^1_1 set $S \subseteq \omega_1$, there is a uniformly computable sequence $(T_x)_{x<\omega_1}$ of subtrees of $\omega_1^{<\omega_1}$ such that $x \in S$ iff T_x has an ω_1-branch.*

Proof. We do just what Kleene did. Let T_x consist of those $\sigma \in \omega_1^{<\omega_1}$ such that $\forall \beta < \mathit{length}(\sigma)\, R(x, \sigma \upharpoonright \beta)$.

□

The structures that we produce for our main result (Theorem 1.7) are not members of any familiar class. The structures in the range of our embedding will each code a sequence of sets $(X_\beta)_{\beta<\omega_1}$, up to an equivalence relation $\sim$, which is defined as follows.

Definition 2.4. *For $X, Y \subseteq \omega_1$, $X \sim Y$ iff $X \triangle Y$ is not stationary, where $X \triangle Y$ denotes the symmetric difference of X and Y.*

Lemma 2.5. *For any Σ^1_1 set $X \subseteq \omega_1$, there is a uniformly computable sequence $(S_\alpha)_{\alpha<\omega_1}$ of subsets of ω_1 such that $\alpha \in X$ iff S_α contains a club.*

Proof. Choose a uniformly computable sequence of trees $(T_\alpha)_{\alpha<\omega_1}$ as in Lemma 2.3. Thus $\alpha \in S$ iff T_α has an ω_1-branch. Choose a parameter $p \in L_{\omega_1}$ and a Σ_1 formula φ so that σ belongs to T_α iff $L_{\omega_1} \vDash \varphi(p, \alpha, \sigma)$. This is possible as the uniformly computable sequence of T_α's is Σ_1 definable with some parameter over L_{ω_1}.

For ordinals $\alpha < \beta \leq \omega_1$ such that p belongs to L_β we let T^β_α be the interpretation of the tree T_α in L_β, i.e. $\{\sigma \mid L_\beta \vDash \varphi(p, \alpha, \sigma)\}$. This may not be a tree for all such pairs $\alpha < \beta$. By definition we have that $T^{\omega_1}_\alpha = T_\alpha$.

Now let S_α be the set of countable ordinals $\beta > \alpha$ such that for some countable $\gamma > \beta$,

1. $L_\gamma \models ZF^-$ (ZF minus Power Set),
2. $\omega_1^{L_\gamma} = \beta$,
3. T^β_α is a tree which has a branch of length β in L_γ.

First, suppose that T_α has an ω_1-branch b. We show that S_α contains a club.

Suppose that M is a countable elementary substructure of L_{ω_2} such that $b \in M$. Then the transitive collapse, denoted by $\overline{M}$, has the form L_γ. Let $\beta = \omega_1^{\overline{M}}$. Since b is an ω_1-branch through the tree $T_\alpha = T^{\omega_1}_\alpha$, $b \restriction \beta$ is a β-branch through the tree T^β_α that belongs to L_γ and therefore γ witnesses that β belongs to S_α.

Now form a continuous chain $(M_i)_{i<\omega_1}$ of countable elementary substructures of L_{ω_2}. Let $\overline{M}_i$ be the transitive collapse of M_i. Then $\overline{M}_i = L_{\gamma_i}$, for some countable ordinal γ_i. Let $\beta_i = \omega_1^{L_{\gamma_i}}$. Then the sequence $(\beta_i)_{i<\omega_1}$ enumerates a club C in ω_1. For each i, the image of b under the transitive collapse of M_i, $\pi_i(b)$, is a β_i-branch through $T^{\beta_i}_\alpha$ belonging to L_{γ_i}, witnessing that β_i belongs to S_α. Thus C is the required club.

Conversely, we show that if T_α has no ω_1-branch, then S_α does not contain a club.

Suppose that C is a club and we will show that some element of C does not belong to S_α. Let M be the least elementary substructure of L_{ω_2} such that $C, \alpha, \omega_1 \in M$. In L_{ω_2}, $T_\alpha = T^{\omega_1}_\alpha$ has no ω_1-branch, so the same holds in M. Again, we take the transitive collapse $\pi(M) = \overline{M} = L_{\overline{\gamma}}$. We have $\beta = \omega_1^{\overline{M}} \in C$ and T^β_α has no β-branch in $L_{\overline{\gamma}}$. We claim that β does not belong to S_α. Indeed, suppose otherwise and that the ordinal γ witnesses this. Then γ must be greater than $\overline{\gamma}$, as T^β_α has no β-branch in $L_{\overline{\gamma}}$. But if γ is greater than $\overline{\gamma}$, then β is countable in L_γ: as M was chosen to be the least elementary substructure of L_{ω_2} containing the parameters C, α, ω_1, it follows that $\overline{M}$ is the least elementary substructure of $\overline{M}$ containing the parameters $\pi(C), \pi(\alpha), \pi(\omega_1)$ and therefore $\overline{M}$, as well as $\beta \in \overline{M}$, is countable in $L_{\bar{\gamma}+2}$. We have reached the desired contradiction.

□

Let E be a Σ^1_1 equivalence relation on ω_1. We identify pairs of ordinals with single ordinals and let S be as above, so that $\alpha E \beta$ iff $S_{\alpha,\beta}$ contains a club. For any $X \subseteq \omega_1$, let $L(X)$ be the $\aleph_1$-like linear order formed by stacking ω_1 many copies of the rational order, and at limit stage α putting in a supremum iff $\alpha \in X$. More precisely: Let $\mathcal{Q}$ denote the rational order and $\mathcal{Q}_0$ the rational order together with a least element; then $L(X)$ is obtained from the ordering $(\omega_1, <)$ by replacing α with a copy of $\mathcal{Q}_0$ if α is a limit ordinal in X and otherwise replacing α by a copy of $\mathcal{Q}$.

Lemma 2.6. *For $X, Y \subseteq \omega_1$, $L(X) \cong L(Y)$ iff $X \sim Y$ (as in Definition 2.4).*

Proof. Suppose $L(X)$ is isomorphic to $L(Y)$ via the isomorphism π. For countable α let $L(X)\restriction\alpha$ be the initial segment of $L(X)$ obtained from the order $(\alpha, <)$ by replacing $i < \alpha$ by $\mathcal{Q}_0$ if i is a limit ordinal in X and by $\mathcal{Q}$ otherwise. Then for club-many α, the restriction of π to $L(X)\restriction\alpha$ is an isomorphism from $L(X)\restriction\alpha$ onto $L(Y)\restriction\alpha$. For such α, α belongs to X iff it belongs to Y, as otherwise the restriction of π to $L(X)\restriction\alpha$ would not be extendible to an isomorphism from $L(X)$ onto $L(Y)$. Thus X, Y agree on a club and $X \sim Y$.

Conversely, suppose that $X \sim Y$ and choose a club C on which X, Y agree. By induction on α in C, build an isomorphism between $L(X)\restriction\alpha$ and $L(Y)\restriction\alpha$: The base case is easy, as there is a unique countable dense linear order without endpoints. The limit cases are trivial, as the limit of isomorphisms is an isomorphism. For the case where α is the C-successor to $\beta \in C$, use the fact that X, Y agree at β to conclude that the ordinal β is replaced by the same ordering in $L(X)\restriction\alpha$ as it is in $L(Y)\restriction\alpha$.

□

We use ideas from [7]. For any finite chain $c = (\alpha, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta)$, let

$$S(c) = S_{\alpha,\gamma_1} \cap S_{\gamma_1,\gamma_2} \cap \ldots \cap S_{\gamma_n,\beta}$$

If $\alpha' E \alpha$, then $S_{\alpha',\alpha}$ contains a club. Therefore, for each finite chain c from α to β, $(S_{\alpha',\alpha} \cap S(c)) \sim S(c)$. It follows that if we define $S^*(\alpha, \beta)$ to be the set of the $S(c)$, where c is a chain starting with α and ending with β, and $\alpha E \alpha'$, then $S^*(\alpha, \beta)$ agrees with $S^*(\alpha', \beta)$, in the sense that they have the same elements modulo the ideal of nonstationary sets. Let $M(\alpha, \beta)$ be the structure that is the "free union" of ω_1 copies of the linear orders $L(X)$ for $X \in S^*(\alpha, \beta)$. One way to make this precise is to let $M(\alpha, \beta)$ consist of two disjoint sets A, B of size ω_1, with a relation $R(a, b_0, b_1)$ for a in A and b_0, b_1 in B so that for each fixed a, $R(a, -, -)$ defines a linear order of B isomorphic to one of the $L(X)$, for $X \in S^*(\alpha, \beta)$, and each such order occurs for exactly ω_1-many such a in A.

Alternatively, we may let $M(\alpha, \beta)$ have an equivalence relation with an ordering on each equivalence class, so that for each set $X \in S^*(\alpha, \beta)$, the ordering

$L(X)$ is copied in uncountably many equivalence classes, and for each equivalence class, the ordering on the equivalence class is isomorphic to $L(X)$ for some $X \in S^*(\alpha, \beta)$. We note that in either case, the language of the structures $M(\alpha, \beta)$ is finite.

Lemma 2.7.

1. *If $\alpha E \alpha'$, then for all β, $M(\alpha, \beta) \cong M(\alpha', \beta)$.*
2. *If it is not the case that $\alpha E \alpha'$, then $M(\alpha, \alpha) \ncong M(\alpha', \alpha)$.*

Proof. (1) is clear. For (2), we note that if it is not the case that $\alpha E \alpha'$, then there is no set $X \in S^*(\alpha, \alpha')$ that contains a club, but there is such a set in $S^*(\alpha, \alpha)$. From this it follows that $M(\alpha, \alpha)$ is not isomorphic to $M(\alpha, \alpha')$.

□

The structures $M(\alpha, \beta)$ have a finite language. Finally, let $M^*(\alpha)$ be the *sequence* (not the free union) of the structures $M(\alpha, \beta)$, for $\beta < \omega_1$. To make this precise we could add to the language a disjoint family of unary predicates $(U_\beta)_{\beta < \omega_1}$, where U_β is the universe of a copy of $M(\alpha, \beta)$; but as we would like to keep the language finite, we instead let $M^*(\alpha)$ be a structure that includes a copy of ω_1 with the usual ordering, and has a predicate associating to each $\beta < \omega_1$ one of a family of sets, disjoint from ω_1 and disjoint from each other. We put a copy of $M(\alpha, \beta)$ on the set associated with β.

Lemma 2.8. *For all α, α', $\alpha E \alpha'$ iff $M^*(\alpha) \cong M^*(\alpha')$.*

Proof. If $\alpha E \alpha'$, then for all β, $M(\alpha, \beta) \cong M(\alpha', \beta)$. Then $M^*(\alpha) \cong M^*(\alpha')$. If it is not the case that $\alpha E \alpha'$, then $M(\alpha, \alpha) \ncong M(\alpha', \alpha)$. Then we have $M^*(\alpha) \ncong M^*(\alpha')$.

□

This completes the proof of Theorem 1.7.

Our structures $M^*(\alpha)$ are in a finite relational language, and we may use standard coding tricks to transform them into undirected graphs. We represent each element by a point attached to a triangle. For an n-place relation symbol R, we represent each n-tuple of elements by a special point, attached by chains of length $1, 2, \ldots, n$ to the points representing the elements. Therefore the structures $M^*(\alpha)$ of Theorem 1.7 can be chosen to be undirected graphs.

3 Turing computable embeddings

H. Friedman and Stanley [8] introduced the notion of Borel embedding for comparing classification problems for classes of countable structures. The notion of Turing computable embedding [2] allows some finer distinctions. Here we define the analogue of Turing computable embedding for structures with universe a subset of ω_1. We have said that in the setting of ω_1, $A \leq_T B$ if there is some

α such that $\varphi_\alpha^A = \chi_B$. We begin by saying something about indices for partial computable functions, and for partial computable functions relative to an oracle. We write φ_α for the partial computable function f derived in the following way from the c.e. set W_α. We order the elements according to their Gödel codes. We let $f(x) = y$ if for the first γ such that there is a pair $(x, u) \in W_{\alpha,\gamma}$, y is the first such u—if there is no pair (x, u) in W_α, then $f(x)$ is undefined. Similarly, φ_α^X is the partial function f derived from W_α^X such that $f(x) = y$ if for the first γ such that there is some pair (x, u) in W_γ^X, y is the first such u. If we let X vary, then we obtain an operator $\Phi = \varphi_\alpha$ taking each set X to the partial function φ_α^X.

Definition 3.1.

1. *Let K and K' be classes of structures with universe a subset of L_{ω_1}. A* computable transformation *from K to K' is a computable operator $\Phi = \varphi_\alpha$ such that for each $\mathcal{A} \in K$, there is some $\mathcal{B} \in K'$ such that $\varphi_\alpha^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$. We write $\Phi(\mathcal{A}) = \mathcal{B}$.*

2. *We write $K \leq_{tc} K'$ if there is a computable transformation $\Phi = \varphi_\alpha$ from K to K' such that $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$.*

In the standard setting, the class of undirected graphs lies on top among classes of countable structures under $\leq_{tc}$. The same is true in our setting. Let L be a computable relational language—L may be uncountable, and it may include symbols of arity α for computable ordinals α. When we say that the language is computable, we mean that the set of relation symbols is computable, and we have a computable function assigning a countable ordinal arity to each symbol. Let $Mod(L)$ be the class of L-structures with universe a subset of ω_1. Let UG be the class of undirected graphs.

Proposition 3.2. $Mod(L) \leq_{tc} UG$

Proof. We first give a transformation that replaces the language L by one with just finitely many relations of finite arity. We have a predicate U, for elements of the structure M. We have a predicate O with an ordering of type ω_1. We have another predicate S for special points, representing triples (R, α, σ), where R is a predicate symbol R in L of arity α and σ is an α-tuple from M. There is a relation Q that holds of $x \in O$, $p \in S$ and $a \in U$ if p is the special point corresponding to a triple (R, α, σ) such that x is the β^{th} element of O, and $\sigma(\beta) = a$. We let T be the set of special points p in S representing atomic facts that are true in M; i.e., p is the special point corresponding to (R, α, σ), where R is a relation symbol of arty α, $\sigma \in M^\alpha$, and $M \models R(\sigma)$. The unary predicates U, O, and S are disjoint, and the universe of our structure M^* is the union. Beyond these, we have a binary relation—the ordering on O, the ternary relation Q, and the set $T \subseteq S$. So, the language is finite. It is not difficult to see that $M_1 \cong M_2$ iff $M_1^* \cong M_2^*$.

Marker included in his basic model theory text [16] a simple way of coding an arbitrary structure (for a finite relational language L) in an undirected graph.

(There are further coding methods that accomplish the same thing, described by Lavrov, Nies, and others.) In Marker's transformation, for each element b of the input structure, the graph has an element g_b, which we mark by making it one vertex of a triangle. For each relation symbol R_j in the finite language L, $j = 0, \ldots$, we designate a pair of shapes, a $(2j+4)$-gon, and a $(2j+5)$-gon, which we use to indicate that the relation holds, or does not hold. For each ordered n-tuple of elements $b_1, \ldots, b_n$, we introduce a special point $t_{R_j, b_1, \ldots, b_n}$. This point is connected to g_{b_i} by a chain of length i. The special point $t_{R_j, b_1, \ldots, b_n}$ is one vertex of a $(2j+4)$-gon if $R(b_1, \ldots, b_n)$ holds, and a $(2j+5)$-gon, otherwise. The n-gons have no points in common, aside from the special points. We can easily make this transformation into a computable embedding in the setting of ω_1.

We start with a large ω_1-computable graph $\mathcal{G}^*$, such that there are $\aleph_1$ elements that are the special vertex of a triangle, for each n-ary relation symbol R_j in L and each n-tuple of special vertices of triangles $v_1, \ldots, v_n$, there is a special point $g_{R_j, b_1, \ldots, b_n}$, connected to b_i by a chain of length i. This point $g_{R_j, b_1, \ldots, b_n}$ is one vertex of both a $(2j + 4)$-gon, and a $(2j + 5)$-gon. We note that the elementary first order theory of the desired $\mathcal{G}^*$ is totally categorical. It follows that there is an ω_1-computable model $\mathcal{G}^*$ of size $\aleph_1$—see [13], or [14]. The set V of special vertex elements is computable. To see this, note that there is a Σ_1 definition of V—saying that the element is part of a triangle and is connected to further elements. There is also a Σ_1 formula defining the elements not in V—a disjunction of formulas indicating the position in an n-gon, or on a chain from a vertex of a triangle to a vertex of some other n-gon. For each n-ary relation symbol R_j, there is a computable function taking n-tuples $b_1, \ldots, b_n$ of points in V to the special point $g_{R_j, b_1, \ldots, b_n}$. There are obvious Σ_1 definitions of the graph. We have a computable function f taking α to the α^{th} element of V, which we call b_α. The functions are defined by Σ_1 recursion. We have a Σ_1 formula saying how $f(\alpha)$ is obtained from $f \upharpoonright \alpha$.

Now, for an input L-structure $\mathcal{A}$, with universe that we can computably identify with a subset of ω_1, $\Phi(\mathcal{A})$ is the subgraph of G^* consisting of the following elements: b_α, for $\alpha \in \mathcal{A}$, the special points $g_{R_j, b_{\alpha_1}, \ldots, b_{\alpha_n}}$, for R_j an n-ary relation symbol and a n-tuple $\alpha_1, \ldots, \alpha_n$, together with the rest of the $(2j + 4)$-gon if $\mathcal{A} \models R_j(\alpha_1, \ldots, \alpha_n)$ and the $(2j + 5)$-gon otherwise. $\square$

In [8], H. Friedman and Stanley give a Borel embedding of countable undirected graphs into fields (of arbitrary characteristic). The embedding requires a modification for characteristic 2. Some of the algebraic number theory behind the result for finite characteristic was completed in later work of Shapiro. This embedding is effective. It is used in [2]. It is also used, with minor modification, in [4] and [3]. Here the modification helped us keep track of Scott rank. The basic embedding, and the modification from [4], both transfer directly to the uncountable setting, with no fresh ideas. The coding idea was used already in the uncountable setting for a result of Hirschfeldt that is included in [13]. We describe the embedding below. We state the result just for characteristic 0, although it is true also for any finite characteristic.

Proposition 3.3. *If K is the class of undirected graphs, and K' is the class of fields of characteristic 0, then $K \leq_{tc} K'$. (The same is true for finite characteristic.)*

Proof. Let F^* be a large algebraically closed field of characteristic,0 with independent transcendentals b_α, for $\alpha < \omega_1$. Since the theory is $\aleph_0$-categorical, we get an ω_1-computable model F^* of size ω_1—see [13]. We may suppose that the universe is ω_1. Checking independence of a countable tuple is computable. Let $f(\alpha)$ be the first element not algebraic over $\{f(\beta) : \beta < \alpha\}$. Then $ran(f)$ is a transcendence basis for F^*. This function is computable. We write b_α for $f(\alpha)$. For an input graph G, with universe a subset of ω_1, we let $\Phi(G)$ be the subfield of F^* generated by the elements b_α, for $\alpha \in G$, elements algebraic over a single b_α, and elements $\sqrt{b_\alpha + b_\beta}$, if in G there is an edge between α and β. It may seem clear that for graphs G and G', $G \cong G'$ iff $\Phi(G) \cong \Phi(G')$. For characteristic $p \neq 2$, the construction is the same, except that F^* has characterstic p. For characteristic 2, it is necessary to use a different coding to indicate the presence of an edge—we may use the cube root of $b_\alpha + b_\beta$ instead of the square root. The hardest part of Friedman and Stanley's proof is showing $\sqrt{b_\alpha + b_\beta}$ do not get in by accident. The proof in [8] works for characteristic 0. There is a reference in [10] to work of Shapiro [24] (from algebraic number theory) that completes the proof. For characteristic 0, there is a geometric proof, due to Dwyer, which was included in Calvert's thesis [6].

$\square$

In [8], H. Friedman and Stanley give a Borel embedding of undirected graphs into linear orderings. This embedding is effective. We can use the same idea, with some modifications, to give an embedding in our uncountable setting.

Proposition 3.4. *If K is the class of undirected graphs and K' is the class of linear orderings, then $K \leq_{tc} K'$.*

Proof. By Proposition 1.4, a countable complete theory has a computable saturated model Q with universe ω_1. We could apply this result to the theory of dense linear orderings without endpoints. We want a theory of dense linear orderings without endpoints, partitioned into infinitely many disjoint dense subsets. Let T be the theory of a structure whose universe is the disjoint union of predicates U and V, where on V, there is a dense linear ordering without endpoints, U is infinite, and there is a function f from V onto U such that for each $u \in U$, $f^{-1}(u)$ is dense in V. We consider a computable saturated model Q^* of this theory, with universe ω_1. We have a type ω_1 ordering on the elements of U^{Q^*}, inherited from the ordering on ω_1. This is not part of the structure Q^*.

We identify the elements of U^{Q^*} with the countable ordinals. We write Q for V^{Q^*}. Let Q_α be $f^{-1}(u)$, where u is the α^{th} element of U. Now, Q, with the ordering from Q^*, is a saturated dense linear ordering without endpoints. The sets $(Q_\alpha)_{\alpha<\omega_1}$ partition Q into dense subsets. The elements of Q_α represent the element α from an input graph G, but only as part of a tuple of some countable arity β. The sets Q_0 and Q_1 will play another role as well, indicating that there is more to the tuple, or that it is coming to an end. We make a list of the atomic types of countable graphs t_α, $\alpha < \omega_1$. The types are given in terms of a tuple of variables $(x_i)_{i<\beta}$, where β is a countable ordinal—the types indicate, for $i \neq j$, whether there is an edge between x_i and x_j. Consider the lexicographic ordering on $Q^{<\omega_1}$. The ordering corresponding to a given input graph G will be a sub-ordering $\Phi(G)$ of this ordering, consisting of the sequences σ of length $2\beta + 2$ such that for some β-tuple $\overline{a}$ from G, satisfying the atomic type t_α, we have

1. for $\gamma < \beta$, $\sigma(2\gamma) \in Q_0$, and $\sigma(2\gamma+1) \in Q_{2+a_\gamma}$,
2. $\sigma(2\beta) \in Q_1$,
3. $\sigma(2\beta+1)$ is an element of U identified with an ordinal less than α.

This is the construction. In [8], Q is a countable dense linear ordering without endpoints, partitioned into disjoint dense sets $(Q_n)_{n\in\omega}$.

Let $\sigma \in \Phi(G)$. We say that σ *represents* the tuple $\overline{a}$ if σ is related to $\overline{a}$ in the way described. The ordering $\Phi(G)$ is made up of intervals having the order type α, for the various atomic types t_α realized in G. It takes effort to show that $G_1 \cong G_2$ iff $\Phi(G_1) \cong \Phi(G_2)$. In [8], the details are omitted. We say a little here. First, suppose $G_1 \cong G_2$ via f. To show that $\Phi(G_1) \cong \Phi(G_2)$, we consider the set $\mathcal{F}$ of countable partial isomorphisms p, where for some countable family of $\sigma_i \in \Phi(G_1)$,

1. if $p(\sigma_i) = \tau_i$, where σ_i represents the tuple $\overline{a}_i$ (of countable arty), then τ_i represents the corresponding tuple $f(\overline{a}_i)$,
2. if σ_i represents a tuple $\overline{a}_i$ realizing type t_α, then p maps the full interval of type α containing σ_i to the corresponding interval of type α containing τ_i.

Clearly, $\mathcal{F}$ is closed under unions of countable chains. The fact that Q^* is saturated allows us to show that it has the back-and-forth property. Say $p \in \mathcal{F}$ maps σ_i representing $\overline{a}_i$ to τ_i representing $\overline{b}_i = f(\overline{a}_i)$. We indicate how to go forth, with a further element σ of $\Phi(G_1)$, representing $\overline{c}$. We need an image τ representing $\overline{d} = f(\overline{c})$. If σ has length $2\beta+2$, we choose τ, of the same length, representing $f(\overline{c})$. For $\gamma < \beta$, $\tau(2\gamma+1) \in Q_{f(a)}$, located to the left, or right, of $\sigma(2\gamma+1)$ is located to the left, or right, of $\sigma_i(2\gamma+1)$. Similarly, for $\gamma < \beta$, $\tau(2\gamma) \in Q_0$, $\tau(2\beta) \in Q_1$, located in the proper relation to $\tau_i(2\gamma)$, or $\tau_i(2\beta)$. Finally, $\tau(2\beta+1)$ matches $\sigma(2\beta+1)$. Going back, with a further element of $\Phi(G_2)$ is similar.

Next, suppose $\Phi(G_1) \cong \Phi(G_2)$ via f. We define an isomorphism g from G_1 onto G_2. The universes of G_1 and G_2 are subsets of ω_1, so we have lists of elements $(a_\alpha)_{\alpha<\omega_1}$ and $(b_\alpha)_{\alpha<\omega_1}$. At step α, we have a countable partial isomorphism g_α. At stage $\alpha+1$, we add an element to the domain or the range. Take the first β such that a_β is not in the domain or b_β is not in the range. If

a_β is not in the domain, then we add a_β to the domain, and otherwise, we add b_β to the range.

The partial isomorphisms will form a continuous chain. Of course $g_0 = \emptyset$. For g_1, we must add a_0 to the domain. Take $\sigma \in \Phi(G_1)$ representing the sequence (a_0), of atomic type t_δ. Then σ has the form $r_0 q_0 r_1 \beta$, where $r_0 \in Q_0$, $q_0 \in Q_{2+a_0}$, $r_1 \in Q_1$, and $\beta < \delta$. Note that σ is part of a maximal well-ordered interval of type δ, and t_δ is the atomic type of the sequence of length 1. Now, $f(\sigma)$ must be in the same position in a maximal well-ordered interval of type δ. Therefore, $f(\sigma)$ also represents a 1-tuple—it is a sequence of the form r_0', q_0', r_1', β, where $r_0' \in Q_0$, $q_0' \in Q_d$, for $d \in G_2$, and $r_1' \in Q_1$. We let $g_1(a_0) = d$. Assuming that $b_0 \neq d$, we define g_2 with b_0 in the range. For this, we take $\tau \in \Phi(G_2)$ of length 6, representing (d, b_0), such that τ extends $r_0' q_0'$. Let $t_{\alpha'}$ be the atomic type of d, b_0. Then τ has the form $r_0', q_0', r_1'', q_1'', r_2'', \delta'$, where $r_1'' \in Q_1$, $q_1'' \in Q_{b_0}$, $r_1'' \in Q_1$, and $\delta' < \alpha'$. Note that in the interval between $f(\sigma)$ and τ, every element represents an extension of (d)—there is no sequence representing the type of the empty sequence. Since τ lies in a maximal well-ordered interval of type α', the same is true of $f^{-1}(\tau)$. Between $f(\sigma)$ and τ, there is no interval of type γ, where γ is the atomic type of the empty set, so all elements of this interval represent extensions of (q_0). We can see that $f^{-1}(\tau)$ will have the form $(r_0, q_0, r_1'', q_1, r_2, \delta'')$, where $q_1 \in Q_c$, for some $c \in G_1$, and $r_2 \in Q_1$, and $\delta'' < \alpha'$. We let $g_1(c) = b_0$.

We continue in this way. At each stage α, we have g_α mapping a countable sequence $(c_\beta)_{\beta<\alpha}$ from G_1 to a countable sequence $(d_\beta)_{\beta<\alpha}$ from G_2. We also have sequences $(\sigma_\beta)_{\beta<\alpha}$ in $\Phi(G_1)$ and $(\tau_\beta)_{\beta<\alpha}$ in $\Phi(G_2)$, such that σ_β, of length $2\beta+2$, represents the sequence $(c_\gamma)_{\gamma<\beta}$, and τ_β, also of length $2\beta+2$, represents the sequence $(d_\gamma)_{\gamma<\beta}$. We arrange that between σ_β and $\sigma_{\beta+1}$ in $\Phi(G_1)$, all elements represent extensions of the sequence represented by σ_β and between τ_β and $\tau_{\beta+1}$ in $\Phi(G_1)$, all elements represent extensions of the sequence represented by τ_β. (We choose one side, and the isomorphism f takes care of the other side.) For limit α, we let $g_\alpha = \cup_{\beta<\alpha} g_\beta$, as required. We let σ_α be the sequence of length $2\alpha + 2$ such that for $\beta < \alpha$, σ_α agrees with σ_β on the first 2β terms. Let t_ρ be the atomic type of the domain, and range, of g_α. At the end of σ_α, we put two last terms $r_{2\alpha} \in Q_1$ and $\delta < \rho$. We let $\tau_\alpha = f(\sigma_\alpha)$. This extends the initial part of τ_β of length 2β.

□

4 Results on fields

Here we consider arbitrary ω_1-computable fields of characteristic 0. The domain of each field is either ω_1 or possibly just ω, and the field operations are all ω_1-computable. We believe that our first results carry over equally well to fields of positive characteristic, and so we denote the prime subfield (either $\mathbb{Q}$ or $\mathbb{F}_p$) of a field F by Q.

Lemma 4.1. *Every ω_1-computable field F has a computable transcendence basis over its prime subfield Q. (Q itself is ω_1-computable, being countable.)*

Proof. This follows from the proof of [13, Lemma 4.6], although it is stated there only for ω_1-computable presentations of $\mathbb{C}$. For each $\alpha \in F$ we define $\alpha \in B$ iff

$$(\forall \langle \beta_1, \ldots, \beta_n \rangle \in \alpha^{<\omega})(\forall p \in Q[X_1, \ldots, X_n, Y])$$
$$[p(\beta_1, \ldots, \beta_n, \alpha) = 0 \implies p(\beta_1, \ldots, \beta_n, Y) = 0].$$

This statement quantifies only over countable sets which we can enumerate uniformly and know when we have finished enumerating each one. It says that α lies in B iff α satisfies no nonzero polynomial over the subfield $Q(\beta : \beta < \alpha)$ generated by all elements $< \alpha$. Clearly this B is a transcendence basis for F. $\square$

Corollary 4.2. *The field $\mathbb{C}$ of complex numbers is relatively ω_1-computably categorical.*

Proof. Given any two ω_1-computable fields $E \cong F \cong \mathbb{C}$, use the lemma to find computable transcendence bases B for E and C for F. Let f be any computable bijection from B onto C (for instance, let $f(\alpha)$ be the least element of C which is $> f(\beta)$ for every $\beta < \alpha$). We now extend this f effectively to an isomorphism from E onto F. For each element $\xi \in E$ in order, find any polynomial $q \in Q(B)[X]$ satisfied by ξ and take the finite subset $B_0 \subseteq B$ of those elements of B actually used in the coefficients of q. Let E_0 be the countable subfield of E generated by $B_0 \cup \xi$ (with ξ now denoting the set of those ordinals $< \xi$, so that E_0 lies within the domain on which f has already been defined), and find the minimal polynomial $p(X)$ of the field element ξ over E_0. (This can be done by brute force, just by checking all of the countably many polynomials in $E_0[X]$.) Every coefficient in $p(X)$ lies in E_0, hence already has an image in F under f, and we choose $f(\xi)$ to be the least root in F of the image of this polynomial $p(X)$ in $F[X]$ under f, which must exist, F being algebraically closed. Thus we recursively build a field embedding $f : E \to F$. But since f maps B onto the transcendence basis C for F, f must map E onto all of F: every $\eta \in F$ has a minimal polynomial $p(X) \in Q(C)[X]$ of some degree d, and the roots $\xi_1, \ldots, \xi_d$ of its preimage in $E[X]$ must map one-to-one to the d-many roots of $p(X)$ in F, forcing $\eta \in \text{ran}(f)$.

The foregoing proof relativizes to the degree of any field $E \cong \mathbb{C}$, yielding relative ω_1-computable categoricity. $\square$

For our next results, it is useful to have an ω_1-computable bijection $f : \omega_1 \to (\omega_1)^2$. Say that the pair (α, β) lies on the diagonal D_γ in $(\omega_1)^2$ if $\alpha + \beta = \gamma$. (Geometrically, this is a misnomer: for instance, the "diagonal" D_ω contains the pairs $(0, \omega), (1, \omega), \ldots$ and $(\omega, 0)$.) Let $f(0) = (0, 0)$, and to define $f(\delta)$ recursively, find the least γ for which D_γ is not a subset of $\text{ran}(f \restriction \delta)$, and the least $\alpha \leq \gamma$ for which the (unique) pair (α, β) in D_γ is not in $\text{ran}(f \restriction \delta)$, and set $f(\delta) = (\alpha, \beta)$. If $\delta = \theta + 1$ is a successor and $f(\theta) = (\alpha, \beta)$, this defines

$$f(\theta + 1) = \begin{cases} (0, \alpha + \beta + 1), & \text{if } \beta = 0; \\ (\alpha + 1, \beta - 1), & \text{if } 0 < \beta < \omega; \\ (\alpha + 1, \beta), & \text{otherwise.} \end{cases}$$

One checks that this f really is a bijection, and then uses angle brackets to write $\langle \alpha, \beta \rangle = f^{-1}((\alpha, \beta)) \in \omega_1$, borrowing the notation from ω-computability. Moreover, f is clearly computable, hence allows us to partition ω_1 effectively into ω_1-many uniformly computable disjoint subsets of size ω_1:

$$\omega_1^{[\alpha]} = \{\langle \alpha, \beta \rangle : \beta \in \omega_1\}.$$

One guesses that every countably generated extension of the field of complex numbers should be computably categorical. First, we show that there is a computable copy.

Proposition 4.3. *Let F be countably generated extension of the field $\mathbb{C}$ of complex numbers; i.e., $F = \mathbb{C}(a_0, a_1, \ldots)$, for some countable sequence $a_0, a_1, \ldots$. Then F has an ω_1-computable copy.*

Proof. First, let C_0 be a countable algebraically closed subfield of $\mathbb{C}$ such that if a_i is algebraic over $\mathbb{C}((a_j)_{j<i})$, then it is algebraic over $C_0((a_j)_{j<i})$; i.e., the minimal polynomial for a_i over $\mathbb{C}((a_j)_{j<i})$ has coefficients in $C_0((a_j)_{j<i})$. Define $A_0 = C_0(a_0, a_1, \ldots)$ using the same minimal polynomials as in F; this is all countable, hence can be done ω_1-effectively. If C_1 is an algebraically closed extension of C_0 of transcendence degree 1 over C_0, we set $A_1 = C_1((a_i)_{i \in \omega})$. Thus, if a_i is algebraic over $C_1((a_j)_{j<i})$, its minimal polynomial has coefficients in $C_0((a_j)_{j<i})$. We continue, building C_α and A_α, for all countable ordinals α, with $C_0 \subseteq C_\alpha \subseteq \mathbb{C}$, where each C_α is the algebraic closure of the first α-many elements in a computable transcendence basis for $\mathbb{C}$ over C_0. We obtain $C_{\alpha+1}$ and $A_{\alpha+1}$ from C_α and A_α in the same way we obtained C_1 and A_1 from C_0 and A_0. For limit α, we let $C_\alpha = \cup_{\beta<\alpha} C_\beta$, and $A_\alpha = \cup_{\beta<\alpha} A_\beta$. Finally, let $C = \cup_{\alpha<\omega_1} C_\alpha$ and $A = \cup_{\alpha<\omega_1} A_\alpha$. Clearly, C is isomorphic to $\mathbb{C}$, via an isomorphism extending the (identity) map from the subfield C_0 of the copy of $\mathbb{C}$ within F to the copy of C_0 within A_0. Moreover, if a_i is algebraic over $C((a_j)_{j<i})$, then its minimal polynomial has coefficients in $C_0((a_j)_{j<i})$. Therefore, $A \cong F$. $\square$

We can prove computable categoricity, modulo a conjecture, for every field F such as described in Proposition 4.3. We believe that the conjecture holds, but the proof seems delicate. Notice that within each such field F, although there are many subfields isomorphic to $\mathbb{C}$, one such subfield must contain all the others, and indeed this largest one is the only copy of $\mathbb{C}$ over which F is countably generated. (If $\mathbb{C}_1 \not\subseteq \mathbb{C}_2$ are copies of $\mathbb{C}$ within a larger field, then $\mathbb{C}_1$ cannot be countably generated over $\mathbb{C}_2$: each $z_1 \in \mathbb{C}_1 \setminus \mathbb{C}_2$ must be transcendental over $\mathbb{C}_2$, so for each z_2 in a transcendence basis for $\mathbb{C}_2$, a separate generator is required to produce $\sqrt{z_1 + z_2}$.)

Conjecture 4.4. *Suppose F is a countably generated extension of a subfield C isomorphic to $\mathbb{C}$, and let C_0 and A_0 be as in the previous proof. Then C is definable in F by a computable Σ_1-formula with a countable tuple of parameters from A_0, and hence is relatively intrinsically ω_1-c.e.*

We believe that it is possible to fix an element $z_0 \in \mathbb{C} \setminus C_0$ such that, for $x \in F$, $x \in \mathbb{C}$ if and only if the algebraic closure of the countable set $\{x\} \cup C_0(z_0)$ is contained within F. This containment is a Σ_1 statement, as the algebraic closure would have to be contained within some countable initial segment σ of $\mathrm{dom}(f) = \omega_1$, and so this would prove Conjecture 4.4. The details remain elusive, and here we leave this claim as a conjecture. Once shown to be true, it will imply the following theorem.

Theorem 4.5 (modulo Conjecture 4.4)**.** *If F is a countably generated extension of $\mathbb{C}$, then F is relatively ω_1-computably categorical.*

Proof using Conjecture 4.4. For simplicity, we identify F with one of its computable copies, and let F' be another computable copy. Let C_0 and A_0 be the countable parts of F from the proof of Proposition 4.3. There is a noncomputable isomorphism ρ from F onto F', and we write C_0' and A_0' for the images of C_0 and A_0 under ρ. Let C and $C' = \rho(C)$ be the largest subfields isomorphic to $\mathbb{C}$ within F and F', respectively. Note that $A_0 = C_0(a_0, a_1, \ldots)$, and if $a_i' = \rho(a_i)$, then $A_0' = C_0'(a_0', a_1', \ldots)$.

We build a computable isomorphism f from F onto F' recursively. We define, by Σ_1 recursion a chain of countable subfields $A_\alpha \subseteq F$ and $A_\alpha' \subseteq F'$ with a chain of functions such that f_α is an isomorphism from A_α onto A_α'. We will have $F = \cup_\alpha A_\alpha$ and $F' = \cup_\alpha A_\alpha'$. Then $f = \cup_\alpha f_\alpha$ will be a computable isomorphism from F onto F'. Inside A_α and A_α', we will have algebraically closed subfields C_α and C_α' such that $C_\alpha \subseteq C$ and $C_\alpha' \subseteq C'$. For each α, A_α will be the field generated by the elements of C_α and the elements a_i. Similarly, A_α' will be the field generated by the elements of C_α' and the elements a_i'. Once we have f_α taking C_α isomorphically onto C_α', and knowing that $f_\alpha(a_i) = a_i'$, the rest of f_α is determined.

To start off, we have A_0, C_0, A_0', and C_0'. We let f_0 be the restriction of ρ to A_0. Given A_α, C_α, A_α', C_α', with the isomorphism f_α taking C_α to C_α and taking a_i to a_i', we extend as follows. Applying Conjecture 4.4, we let c be the first element that we find in $C \setminus A_\alpha$. Since C_α is algebraically closed, c is not algebraic over C_α. It is also not algebraic over $A_\alpha = C_\alpha(a_0, a_1, \ldots)$. Similarly, let c' be the first element that we find in $C' \setminus A_\alpha'$. This is not algebraic over A_α. Let $C_{\alpha+1}$ be the algebraic closure of $C_\alpha(c)$ in C, and let $C_{\alpha+1}'$ be the algebraic closure of $C_\alpha(c')$ in C'. We let $f_{\alpha+1}$ be the extension of f_α taking c to c', and taking $C_{\alpha+1}$ isomorphically onto $C_{\alpha+1}'$. Extend in the obvious way to an isomorphism from $A_{\alpha+1}$ onto $A_{\alpha+1}'$. For limit ordinals α, C_α, A_α, C_α', A_α', and f_α are all defined by taking limits.

It is clear that no element of C can be left out of the domain of f, and no element of C' can be left out of the range. The a_i are all in the domain, and the a_i' are all in the range. Therefore, the domain includes all of F and the range includes all of F', so f is the desired computable isomorphism from F onto F'.

Relative ω_1-computable categoricity again follows just by relativizing the argument to the degree of any $F \cong E$. □

It is natural to ask whether Theorem 4.5 would hold for fields of countable transcendence degree over $\mathbb{C}$. Such fields need not be countably generated over $\mathbb{C}$, so the theorem does not apply to them directly, and indeed such a field need not be computably categorical.

Theorem 4.6. *There exists an ω_1-computable field F, with transcendence degree 1 over a computable subfield isomorphic to $\mathbb{C}$, such that F is not ω_1-computably categorical.*

Proof. We use a computable listing $\{\varphi_\alpha : \alpha \in \omega_1\}$ of all partial ω_1-computable functions from ω_1 into ω_1. With this listing, we build the following two computable fields E and F, with $E \cong F$ and each with $\mathbb{C}$ as a computable subfield, but diagonalizing to satisfy requirements $\mathcal{R}_\alpha$:

$$\mathcal{R}_\alpha : \quad \varphi_\alpha \text{ is not an isomorphism from } E \text{ onto } F.$$

Set $E_{-2} = F_{-2} = \mathbb{C}$ to be a computable copy of $\mathbb{C}$ with a computable transcendence basis $\{z_\alpha : \alpha \in \omega_1\}$, and let $E_{-1} = F_{-1} = \mathbb{C}(x)$ with x purely transcendental over $\mathbb{C}$. We then build $E_0 = F_0$ by adjoining every $\sqrt{x + z_\alpha}$, for every α, so that $E_0 = E_{-1}(\sqrt{x + z_\alpha} : \alpha \in \omega_1) = F_0$. (Of course, adjoining a square root also adjoins its conjugate. When we write $\sqrt{x + z_\alpha}$ below, we will always mean the root actually adjoined to E here, which is taken to be a lesser element than its conjugate in the domain ω_1 of E.) With these fields, we are ready to begin diagonalizing. It is important to note that E_0 can be viewed as a subfield of the real numbers $\mathbb{R}$ (by considering the elements inside the square roots to be positive), and that henceforward every E_σ will also embed into $\mathbb{R}$, by the same trick. Hence E will not contain any square root of -1.

At stage 0, we *initialize* every requirement $\mathcal{R}_\alpha$, by declaring it unsatisfied and setting $y_{\alpha,0} = 0$. We also set every $\tau_\alpha = 0$, and define $f_0 : E_0 \to F_0$ to be the identity map.

At stage $\sigma + 1$, we have $E_\sigma \cong F_\sigma$ via an isomorphism f_σ. Find the least $\alpha \leq \sigma$ such that

- $\mathcal{R}_\alpha$ is currently unsatisfied; and
- $\varphi_{\alpha,\sigma}$ respects the addition and multiplication operations in E_σ and F_σ (at all inputs from E_σ for which $\varphi_{\alpha,\sigma}$ converges); and
- for some $y \in F_\sigma$, $\varphi_{\alpha,\sigma}(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})\downarrow = y$ and $\varphi_{\alpha,\sigma}(x + z_{\langle\alpha,\tau_\alpha\rangle})\downarrow = y^2$; and
- $\varphi_{\alpha,\sigma}(\xi)\downarrow$, where ξ is the least element of ω_1 such that $\varphi_{\alpha,\sigma'}(\xi)\uparrow$ at the last stage $\sigma' \leq \sigma$ at which $\mathcal{R}_\alpha$ either was initialized or received attention.

If there is no such $\alpha \leq \sigma$, then do nothing. Otherwise, $\mathcal{R}_\alpha$ *receives attention* according to the following instructions.

1. If $y = \varphi_\alpha(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$ is algebraically dependent over $\{y_{\beta,\sigma} : \beta < \alpha\}$, and the set $\{\varphi_\alpha(\sqrt{x + z_{\langle\alpha,\rho\rangle}}) : \rho \leq \tau_\alpha\}$ is algebraically independent in F_σ, then increment τ_α by 1, and do nothing else.

2. If this y is dependent over $\{y_{\beta,\sigma} : \beta < \alpha\}$, and the set $\{\varphi_\alpha(\sqrt{x + z_{\langle\alpha,\rho\rangle}}) : \rho \leq \tau_\alpha\}$ is algebraically dependent in F_σ, then declare $\mathcal{R}_\alpha$ satisfied, set $y_{\alpha,\sigma} = y$, and do nothing else.

3. If y is algebraically independent over $\{y_{\beta,\sigma} : \beta < \alpha\}$, then check whether y and/or $(-y)$ has a square root in F_σ. If either one does, then declare $\mathcal{R}_\alpha$ satisfied (since $x_{\langle\alpha,\tau_\alpha\rangle}$ has no square root in E_σ). If neither of $\pm y$ has a square root in F, then:

   (a) adjoin to E_σ a square root a of $\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}}$, and adjoin to F_σ a square root b of $f_\sigma(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$ (in which case $f_{\sigma+1} \supseteq f_\sigma$, with $f_{\sigma+1}(a) = b$), provided that the adjoinment of this square root in F does *not* generate a square root of $\varphi_\alpha(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$, nor of $\varphi_\beta(\sqrt{x + z_{\langle\beta,\tau_\beta\rangle}})$ for any $\beta < \alpha$; or

   (b) adjoin to E_σ a square root a of $\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}}$, and adjoin to F_σ a square root b of $-f_\sigma(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$ (in which case $f_{\sigma+1}(a) = b$ and $f_{\sigma+1} \upharpoonright F_\sigma = f_\sigma \circ \psi$, where ψ is the automorphism of $F_{\sigma+1}$ interchanging $\pm\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}}$ and fixing everything else), provided that the adjoinment of this square root in F does *not* generate a square root of $\varphi_\alpha(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$, nor of $\varphi_\beta(\sqrt{x + z_{\langle\beta,\tau_\beta\rangle}})$ for any $\beta < \alpha$; or

   (c) increment τ_α by 1, if neither (a) nor (b) holds.

   If (3a) or (3b) applied, then $\mathcal{R}_\alpha$ is declared satisfied, with $y_{\alpha,\sigma+1} = \varphi_\alpha(\sqrt{x + z_{\alpha,\tau_\alpha}})$, and every $\mathcal{R}_\beta$ with $\beta > \alpha$ is injured at this stage: $\mathcal{R}_\beta$ is initialized, with τ_β being incremented by 1 (instead of being reset to 0).

Only in Steps (3a) and (3b) is any element adjoined to either E or F. When one of these applies at a stage $\sigma + 1$, $f_{\sigma+1}$ is redefined only on $\pm\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}}$, not on E_{-1} nor on any $\sqrt{x + z_\beta}$ with $\beta \neq \langle\alpha, \tau_\alpha\rangle$. f will never subsequently be redefined on $\pm\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}}$ (since either $\mathcal{R}_\alpha$ remains satisfied forever, or else it is subsequently injured and τ_α is incremented). Therefore $f(x) = \lim_\sigma f_\sigma(x)$ exists for all $x \in E$. Since every $f_\sigma : E_\sigma \to F_\sigma$ was an isomorphism, this limit f is an isomorphism from E onto F.

However, we claim by induction on α that every $\mathcal{R}_\alpha$ holds, and that it injures the requirements $\mathcal{R}_\beta$ with $\beta > \alpha$ at only countably many stages. Assume that this holds for all $\alpha' < \alpha$ (so that τ_α is incremented on account of injury at only countably many stages).

If φ_α is not total, then eventually the construction finds an ξ on which it diverges, and thereafter it never receives attention again. Likewise, if φ_α fails to respect the field operations, then at some stage we will discover this and $\mathcal{R}_\alpha$ will never again receive attention. (Of course, in both of these cases, φ_α cannot be an isomorphism.) So suppose that φ_α is a field embedding from E into F. Once the injuries by higher-priority requirements have ceased (according to our inductive hypothesis), τ_α can only be incremented by Steps (1) or (3c) at stages where $\mathcal{R}_\alpha$ receives attention. But if there are uncountably many such stages, then uncountably many algebraically independent elements $\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}}$

in E are mapped to elements algebraically dependent over the countable subset $\{y_{\beta,\sigma} : \beta < \alpha\}$ of F. No field embedding can do this, so by assumption, Step (1) applies to $\mathcal{R}_\alpha$ at only countably many stages, and therefore the construction must eventually reach Step (2) or Step (3).

Of course, if Step (2) ever happens, then $\mathcal{R}_\alpha$ is satisfied right then and never again becomes unsatisfied. (Indeed, in this case φ_α cannot have been a field embedding, since it maps an algebraically independent set to an algebraically dependent set.) So assume that eventually we reach Step (3) for $\mathcal{R}_\alpha$. If we execute either Step (3a) or (3b) there, then $\sqrt{x + z_{\langle\alpha,\tau\rangle}}$ has a square root in E, but $\varphi_\alpha(\sqrt{x + z_{\langle\alpha,\tau\rangle}})$ has no square root in F, which will ensure that φ_α is not an isomorphism. So we must show that Step (3c) cannot apply forever. But this is easy. First, the proposed adjoinments of square roots in Steps (3a) and (3b) cannot both generate square roots of $\varphi_\alpha(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$: the two proposed square roots do not generate each other (as neither E nor F contains any $\sqrt{-1}$), so the two proposed square roots generate distinct field extensions, and since each of these proposed extensions has degree 2, they intersect only in the ground field. Therefore, when (3c) applies, one or the other proposed square root must be algebraic over $\{\varphi_\beta(\sqrt{x + z_{\beta,\tau_\beta}}) : \beta < \alpha\}$. If Step (3c) applied at uncountably many stages $\sigma + 1$, then for uncountably many distinct values of τ_α, $f(\sqrt{x + z_{\langle\alpha,\tau_\alpha\rangle}})$ would be dependent over $\{\varphi_\beta(\sqrt{x + z_{\beta,\tau_\beta}}) : \beta < \alpha\}$ which (since all τ_β converge to limits) would mean that φ_α would map an uncountable, algebraically independent set to a set of elements all of which are dependent over a countable set. No field embedding can do this, so in this case φ_α would eventually show itself not to be an embedding. Therefore, eventually either Step (3a) or Step (3b) must apply, at which stage $\mathcal{R}_\alpha$ is declared satisfied and never again receives attention. Subsequent adjoinments to E and F are done only in Steps (3a) or (3b) by lower-priority requirements, which are always careful not to adjoin elements which would cause $\mathcal{R}_\alpha$ to become unsatisfied. (This is the reason for the existence of Step (3c).) Our induction is now complete, and the theorem is proven. $\square$

At the other extreme from algebraically closed fields, namely fields purely transcendental over Q, computable categoricity fails again.

Proposition 4.7. *If $F = \mathbb{Q}(X_\alpha : \alpha \in \omega_1)$ is an ω_1-computable field and is purely transcendental over the rationals with transcendence degree ω_1, then F is not ω_1-computably categorical.*

Proof. We take F itself to be a presentation with the transcendence basis $\{X_\alpha : \alpha < \omega_1\}$ computable. (Lemma 4.1 only guarantees the existence of some computable transcendence basis, not necessarily of one generating the entire field.) We build a computable field $E \cong F$ with no computable isomorphism from E onto F. X_α will be our witness that the computable function φ_α is not such an isomorphism.

At the start, we build E_0 to be F itself, although we only use the elements of $\omega_1^{[0]}$ to do so. (Let E_0 be the isomorphic image of F under the map $\lambda + n \mapsto$

$\lambda + 2n$ for all limit ordinals λ.) We write $y_\alpha \in E_0$ for the image of x_α under this map. Then, for each α, we wait for $\varphi_\alpha(y_\alpha)$ to converge, say to some $z_\alpha \in F$. When this happens, we find $\beta_1, \ldots, \beta_n$ such that $z_\alpha \in \mathbb{Q}(x_{\beta_1}, \ldots, x_{\beta_n})$, and ask whether the polynomial $p(X) = X^2 - z_\alpha$ factors over the subfield $\mathbb{Q}(x_{\beta_1}, \ldots, x_{\beta_n})$. (Kronecker gives a splitting algorithm for this field in [15], since we know the elements x_{β_i} to be algebraically independent over $\mathbb{Q}$.) If so, then z_α has a square root in F, and so we do not change anything in E, but define $y'_\alpha = y_\alpha$. If not, then we adjoin to E a new element y'_α whose square in E is y_α, and use the next row of currently unused elements to close E under the field operations. (This must happen at ω_1-many stages, so all rows eventually get used.) Formally, the existing field E_σ is extended to $E_{\sigma+1} = E_\sigma[X]/(X^2 - y_\alpha)$, which is a field because the quadratic polynomial $(X^2 - y_\alpha)$, having no roots in E_σ, must be irreducible in $E_\sigma[X]$. This completes the construction.

Now $E = \mathbb{Q}(y'_\alpha : \alpha < \omega_1)$ is isomorphic to F via the map $y'_\alpha \mapsto x_\alpha$. However, if $\varphi_\alpha(y_\alpha)\downarrow$, then y_α has a square root in E iff $\varphi_\alpha(y_\alpha)$ has no square root in F. Thus no φ_α can be an isomorphism from E onto F. $\square$

References

[1] C. J. Ash, J. F. Knight, M. Manasse, and T. Slaman, "Generic copies of countable structures," *APAL*, vol. 42 (1989), pp. 195-205.

[2] W. Calvert, D. Cummins, J. F. Knight, and S. Miller, "Comparing classes of finite structures," *Algebra and Logic*, vol. 43 (2004), pp. 365-373.

[3] W. Calvert, S. S. Goncharov, and J. F. Knight, "Computable structures of Scott rank ω_1^{CK} in familiar classes," *Advances in Logic (Proceedings of the North Texas Logic Conference, October 8–10, 2004)*, *Contemporary Mathematics* vol. 425 (2007), American Mathematical Society, pp. 49-66.

[4] W. Calvert, J. F. Knight, and J. Millar, "Computable trees of Scott rank ω_1^{CK}, and computable approximation," *J. Symbolic Logic* vol. 71 (2006) 1, pp. 283-298.

[5] J. Carson, J. Johnson, J. F. Knight, K. Lange, C. McCoy, and J. Wallbaum, "The arithmetical hierarchy in the setting of ω_1," to appear in *Computability*.

[6] J. Chisholm, "Effective model theory vs. recursive model theory," *J. Symb. Logic* vol. 55 (1990), pp. 1168-1191.

[7] E. Fokina, S-D. Friedman, V. Harizanov, J. F. Knight, A. Montalbán, and C. McCoy, "Isomorphism relations on computable structures," *J. Symb. Logic*, vol. 77 (2012), pp. 122-132.

[8] H. Friedman and L. Stanley, "A Borel reducibility theory for classes of countable structures," *J. Symb. Logic*, vol. 54 (1989), pp. 894-914.

[9] S-D. Friedman,"Uncountable Admissibles I: Forcing," *TAMS*, vol. 270 (1982), pp. 61-73.

[10] A. Fröhlich and J. C. Shepherdson, "Effective procedures in field theory," *Philos. Trans. Royal Soc. London, Ser. A.*, vol. 248 (1956), pp. 407-432.

[11] S. S. Goncharov and A. T. Nurtazin, "Constructive models of complete decidable theories," *Algebra and Logic*, vol. 12 (1973), pp. 67-77.

[12] N. Greenberg, A. Kach, S. Lempp, and D. Turetsky, "Computability and uncountable linear orders I: Computable categoricity," pre-print.

[13] N. Greenberg and J. F. Knight, "Computable structure theory using admissible recursion theory on ω_1," to appear in *Effective Mathematics of the Uncountable*, eds. N. Greenberg, J. D. Hamkins, D. Hirschfeldt, & R. Miller, ASL *Lecture Notes in Logic* (Cambridge University Press, 2013).

[14] J. Johnson, *Computable Model Theory for Uncountable Structures*, PhD thesis at Notre Dame University.

[15] L. Kronecker, "Grundzüge einer arithmetischen Theorie der algebraischen Größen," *J. f. Math.*, vol. 92 (1882), pp. 1-122.

[16] D. Marker, *Model Theory: An Introduction*, Springer Graduate Texts in Mathematics, 2000.

[17] G. Metakides and A. Nerode, "Recursively enumerable vector spaces," *Annals of Math. Logic*, vol. 11 (1977), pp. 141-171.

[18] G. Metakides and A. Nerode, "Effective content of field theory," *Annals of Math. Logic*, vol. 17 (1979), pp. 289-320.

[19] T. S. Millar, "Foundations of recursive model theory," *APAL*, vol. 13 (1978), pp. 45-72.

[20] M. Morley, "Decidable models," *Israel J. of Math.*, vol. 25 (1976), pp. 233-240.

[21] A. T. Nurtazin, *Completable Classes and Criteria for Autostability*, PhD thesis, Alma-Ata, 1974.

[22] H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[23] G. Sacks, *Higher Reecursion Theory*, Perspectives in Mathematical Logic, Springer-Verlag, 1990.

[24] D. Shapiro, "Composites of algebraically closed fields," *Journal of Algebra*, vol. 130 (1990), pp. 176-190.

[25] B. van der Waerden, "Eine Bemerkung über die Unzerlegbarkeit von Polynomen," *Math. Ann.*, vol. 1-2 (1930), pp. 738-739.

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