

## ON THE CONSISTENCY STRENGTH OF THE INNER MODEL HYPOTHESIS

SY-DAVID FRIEDMAN, PHILIP WELCH, AND W. HUGH WOODIN

The *Inner Model Hypothesis* (IMH) and the *Strong Inner Model Hypothesis* (SIMH) were introduced in [4]. In this article we establish some upper and lower bounds for their consistency strength.

We repeat the statement of the IMH, as presented in [4]. A sentence in the language of set theory is *internally consistent* iff it holds in some (not necessarily proper) inner model. The meaning of internal consistency depends on what inner models exist: If we enlarge the universe, it is possible that more statements become internally consistent. The *Inner Model Hypothesis* asserts that the universe has been maximised with respect to internal consistency:

*The Inner Model Hypothesis* (IMH): If a statement  $\varphi$  without parameters holds in an inner model of some outer model of  $V$  (i.e., in some model compatible with  $V$ ), then it already holds in some inner model of  $V$ .

Equivalently: If  $\varphi$  is internally consistent in some outer model of  $V$  then it is already internally consistent in  $V$ . This is formalised as follows. Regard  $V$  as a countable model of Gödel-Bernays class theory, endowed with countably many sets and classes. Suppose that  $V^*$  is another such model, with the same ordinals as  $V$ . Then  $V^*$  is an *outer model of  $V$*  ( $V$  is an *inner model of  $V^*$* ) iff the sets of  $V^*$  include the sets of  $V$  and the classes of  $V^*$  include the classes of  $V$ .  $V^*$  is *compatible with  $V$*  iff  $V$  and  $V^*$  have a common outer model.

**REMARK.** The Inner Model Hypothesis, like Lévy-Shoenfield absoluteness, is a form of absoluteness between  $V$  and arbitrary outer models of  $V$ , which need not be generic extensions of  $V$ . Formally speaking, the notion of “arbitrary outer model” does depend on the background universe in which  $V$  is situated as a countable model. However, a typical model of the IMH is minimal in the sense that for some real  $R$ , it is the smallest transitive model of Gödel-Bernays containing  $R$  (see Theorem 8 below). For minimal models, the choice of background universe is irrelevant, and if there is a model of the IMH then there is a minimal one. Thus we may in fact regard the IMH as an intrinsic hypothesis about  $V$ , independent of any background universe. An alternative way to “internalise” the IMH is to restrict the notion of outer model to class-generic extensions which preserve the axioms

---

Received September 15, 2006.

The first author was supported by FWF (Austrian Research Fund) Grants P16334-NO5 and P16790-NO4.

The third author was supported by NSF (National Science Foundation) Grant DMS-0355334.

of Gödel-Bernays. This weakened form of the IMH is expressible in the language of class theory, and the results of this paper would not be affected by this change. However, this is not in the spirit of [4], where the IMH is introduced as a form of absoluteness which is in no way limited by the notion of forcing.

**THEOREM 1.** [4] *The Inner Model Hypothesis implies that for some real  $R$ , ZFC fails in  $L_\alpha[R]$  for all ordinals  $\alpha$ . In particular, there are no inaccessible cardinals, the reals are not closed under  $\#$  and the singular cardinal hypothesis holds.*

**THEOREM 2.** *The IMH implies that there is an inner model with measurable cardinals of arbitrarily large Mitchell order.*

**PROOF.** Assume not and let  $K$  denote Mitchell’s core model for sequences of measures (see [6]). Let  $\delta$  be the maximum of  $\omega_1$  and the strict supremum of the Mitchell orders of measurable cardinals in  $K$ . By Mitchell’s Covering Theorem for  $K$  we have:

$$\text{cof}(\alpha) \geq \delta, \alpha \text{ regular in } K \rightarrow \text{cof}(\alpha) = \text{card}(\alpha). \quad (*)$$

Now iterate  $K$  by applying each measure of order 0 exactly once, i.e., if  $K_i$  is the  $i$ -th iterate of  $K$ ,  $K_{i+1}$  is formed by applying the measure of order 0 in  $K_i$  at  $\kappa_i$ , where  $\kappa_i$  is the least measurable of  $K_i$  not of the form  $\pi_{i_0 i}(\kappa_{i_0})$  for any  $i_0 < i$ . It is easy to see that  $i < j$  implies  $\kappa_i < \kappa_j$  and hence this iteration is normal. Let  $\sigma_{ij} : K_i \rightarrow K_j$  be the resulting iteration map from  $K_i$  to  $K_j$  for  $i \leq j \leq \infty$ , let  $K'$  denote  $K_\infty$  and let  $\sigma : K \rightarrow K'$  denote  $\sigma_{0\infty}$ .

- LEMMA 3.** (a) *For any  $\alpha$  and any  $i$ , there is at most one  $j \geq i$  such that  $\sigma_{j,j+1}$  is discontinuous at  $\sigma_{ij}(\alpha)$ . If there is no such  $j \geq i$ , then  $\sigma_{i\infty}$  is continuous at  $\alpha$ .*  
 (b) *For any  $\alpha$  and any  $i$ , the  $K_i$ -cardinality of  $\sigma_{i\infty}(\alpha)$  is at most  $\alpha^+$  of  $K_i$ . If  $\sigma_{i\infty}$  is continuous at  $\alpha$  and  $\alpha$  is a  $K_i$ -cardinal, then the  $K_i$ -cardinality of  $\sigma_{i\infty}(\alpha)$  equals  $\alpha$ .*  
 (c) *If  $\alpha$  is a limit cardinal of  $K_i$  then  $\alpha$  is a closure point of  $\sigma_{i\infty}$ , i.e.,  $\sigma_{i\infty}[\alpha] \subseteq \alpha$ . If in addition the  $K_i$ -cofinality of  $\alpha$  is not measurable in  $K_i$ , then  $\alpha$  is a fixed point of  $\sigma_{i\infty}$ .*

**PROOF.** (a) The map  $\sigma_{j,j+1}$  is discontinuous at  $\beta$  iff  $\beta$  has  $K_j$ -cofinality  $\kappa_j$ . Thus if the  $K_j$ -cofinality of  $\sigma_{ij}(\alpha)$  is not  $\kappa_j$  for any  $j \geq i$ , it follows that  $\sigma_{j,j+1}$  is continuous at  $\sigma_{ij}(\alpha)$  for all  $j \geq i$ . Otherwise let  $j$  be least so that  $\sigma_{ij}(\alpha)$  has  $K_j$ -cofinality  $\kappa_j$ ; then for  $k$  greater than  $j$ ,  $\sigma_{ik}(\alpha)$  has  $K_k$ -cofinality  $\sigma_{jk}(\kappa_j)$ , which by definition of the  $\kappa_k$ ’s does not equal  $\kappa_k$ . It follows that  $\sigma_{k,k+1}$  is continuous at  $\sigma_{ik}(\alpha)$  for  $k$  greater than  $j$ , as desired. The last statement is immediate.

(b) Define the ordering  $\prec$  as follows:  $(j, \alpha) \prec (k, \beta)$  iff  $j \geq k$  and  $\alpha < \sigma_{kj}(\beta)$ . The relation  $\prec$  is a well-founded partial ordering. We prove the desired property of  $(i, \alpha)$  by induction on  $\prec$ .

We may assume that  $\alpha$  is a cardinal of  $K_i$ . If  $\sigma_{ij}(\alpha)$  does not have  $K_j$ -cofinality  $\kappa_j$  for any  $j \geq i$ , then  $\sigma_{i\infty}$  is continuous at  $\alpha$  and by induction applied to pairs  $(i, \bar{\alpha})$ ,  $\bar{\alpha} < \alpha$ , we have that the  $K_i$ -cardinality of  $\sigma_{i\infty}(\alpha) = \sup \sigma_{i\infty}[\alpha]$  is equal to  $\alpha$ . Otherwise, choose the unique  $j \geq i$  so that  $\sigma_{ij}(\alpha)$  has  $K_j$ -cofinality  $\kappa_j$ ; then  $\sigma_{ij}$  is continuous at  $\alpha$  and therefore by induction applied to pairs  $(i, \bar{\alpha})$ ,  $\bar{\alpha} < \alpha$ ,  $\sigma_{ij}(\alpha)$  has  $K_i$ -cardinality  $\alpha$ . Let  $\alpha^*$  denote  $\sigma_{ij}(\alpha)$ . Now  $\sigma_{i,j+1}(\alpha) = \sigma_{j,j+1}(\alpha^*)$  has  $K_j$ -cardinality  $(\alpha^*)^+$  of  $K_j$ . And by induction applied to pairs  $(j+1, \beta)$ ,  $\beta < \sigma_{j,j+1}(\alpha^*)$ , we have that  $\sigma_{j+1,\infty}(\beta)$  has  $K_{j+1}$ -cardinality at most  $\beta^+$  of  $K_{j+1}$  for each

$\beta < \sigma_{j,j+1}(\alpha^*)$ . So as  $\sigma_{j+1,\infty}$  is continuous at  $\sigma_{j,j+1}(\alpha^*)$  and  $\sigma_{j,j+1}(\alpha^*)$  is a cardinal of  $K_{j+1}$ , it follows that the  $K_{j+1}$ -cardinality of  $\sigma_{j+1,\infty}(\sigma_{j,j+1}(\alpha^*)) = \sigma_{j\infty}(\alpha^*)$  is  $\sigma_{j,j+1}(\alpha^*)$ , and therefore the  $K_j$ -cardinality of  $\sigma_{j\infty}(\alpha^*)$  is  $(\alpha^*)^+$  of  $K_j$ . As  $\alpha^*$  has  $K_i$ -cardinality  $\alpha$ , it follows that the  $K_i$ -cardinality of  $\sigma_{i\infty}(\alpha) = \sigma_{j\infty}(\alpha^*)$  is at most  $\alpha^+$  of  $K_i$ , as desired.

(c) The first statement follows immediately from (b). For the second statement, suppose that  $\alpha$  is a closure point of  $\sigma_{i\infty}$  and the  $K_i$ -cofinality of  $\alpha$  is not measurable in  $K_i$ . We show by induction on  $j \geq i$  that  $\sigma_{ij}$  is continuous at  $\alpha$ , and therefore that  $\alpha$  is a fixed point of  $\sigma_{ij}$ : This is vacuous if  $j = i$ . If  $\alpha$  is a fixed point of  $\sigma_{ij}$  then the  $K_j$ -cofinality of  $\alpha$  is not measurable in  $K_j$  by elementarity, and therefore  $\sigma_{j,j+1}$  is continuous at  $\alpha$ ; it follows that  $\sigma_{i,j+1}$  is also continuous at  $\alpha$ . For limit  $j$ , the continuity of  $\sigma_{ij}$  at  $\alpha$  follows from the continuity of the  $\sigma_{ik}$  at  $\alpha$  for  $k < j$ .  $\dashv$

LEMMA 4. (\*) holds with  $K$  replaced by  $K'$ .

PROOF. It suffices to show by induction on  $i$  that  $(*)_i$  holds, where  $(*)_i$  is  $(*)$  with  $K$  replaced by  $K_i$ .

Base case:  $(*)_0$  is just  $(*)$ .

Successor case: Suppose that  $(*)_i$  holds and that  $\alpha$  is  $K_{i+1}$ -regular with cofinality at least  $\delta$ . We may assume that  $\alpha$  is greater than  $\kappa_i$ , else  $\alpha$  is  $K_i$ -regular and we are done by induction. If  $\alpha$  is at most  $\sigma_{i,i+1}(\kappa_i)$  then  $\alpha$  has  $K_i$ -cardinality  $\kappa_i^+$  of  $K_i$ , and, as  $K_{i+1}$  and  $K_i$  contain the same  $\kappa_i$ -sequences of ordinals,  $\alpha$  has  $K_i$ -cofinality  $\kappa_i^+$  of  $K_i$ . So  $\alpha$  and  $\kappa_i^+$  of  $K_i$  have the same cardinality and cofinality, so we are done by induction.

Now suppose that  $\alpha$  is greater than  $\sigma_{i,i+1}(\kappa_i)$ . Represent  $\alpha$  in  $K_{i+1}$ , the ultrapower of  $K_i$ , by  $[f]$  where  $f : \kappa_i \rightarrow \text{Ord}$ . We may assume that  $f$  is either constant or increasing, and also that  $f(\gamma)$  is  $K_i$ -regular and greater than  $\kappa_i$  for all  $\gamma < \kappa_i$ . If  $f$  is constant then  $\alpha = \sigma_{i,i+1}(\bar{\alpha})$  for some  $\bar{\alpha}$  which is regular in  $K_i$  and greater than  $\kappa_i$ ; then  $\alpha$  and  $\bar{\alpha}$  have the same cofinality and cardinality, so we are done by induction. Thus we may assume that  $f$  is increasing.

Now the  $K_i$ -cofinality of  $\alpha$  is at least the supremum  $\mu$  of the  $f(\gamma)$ 's, as using the regularity of the  $f(\gamma)$ 's, we can everywhere-dominate any set in  $K_i$  of  $f(\gamma)$ -many functions from  $\kappa_i$  into  $\prod_{\gamma' > \gamma} f(\gamma')$  by a single such function in  $K_i$ . As  $\mu$  is  $K_i$ -singular, the  $K_i$ -cofinality of  $\alpha$  is in fact greater than  $\mu$ . And the  $K_i$ -cardinality of  $\alpha$  is  $\mu^{\kappa_i} = \mu^+$  of  $K_i$ . It follows that  $\alpha$  and  $\mu^+$  of  $K_i$  have the same cofinality and cardinality, so we are done by induction.

Limit case: Suppose that  $i$  is a limit and  $\alpha$  is  $K_i$ -regular with cofinality at least  $\delta$ . By Lemma 3 (a), we can choose  $i_0 < i$  such that  $\alpha$  equals  $\sigma_{i_0 i}(\bar{\alpha})$ , where  $\bar{\alpha}$  is regular in  $K_{i_0}$  and  $\sigma_{i_0 i}$  is continuous at  $\bar{\alpha}$ . It follows by Lemma 3 (a) that  $\alpha$  and  $\bar{\alpha}$  have the same cardinality and cofinality, and therefore we are done by induction.  $\dashv$

LEMMA 5. If  $\lambda$  is a limit cardinal then  $\text{cof}^{K'}(\lambda)$  is not measurable in  $K'$ .

PROOF. Let  $\kappa$  denote the  $K$ -cofinality of  $\lambda$ . If  $\kappa$  is not measurable in  $K$  then by Lemma 3 (c),  $\lambda$  is a fixed point of the iteration  $\sigma : K \rightarrow K'$  and therefore the result follows by elementarity. Otherwise we claim that  $\kappa$  must equal  $\kappa_i$  for some  $i$ : If not, then as  $\kappa$  is a closure point of  $\sigma$ ,  $\kappa$  would also be a fixed point of  $\sigma$  and therefore  $\kappa$  is measurable in each  $K_i$ . By the definition of the  $\kappa_i$ 's, it must be that for each  $i$ ,  $\kappa$  is either of the form  $\sigma_{i_0 i}(\kappa_{i_0})$  for some  $i_0 < i$ , or  $\kappa_i$  is less than  $\kappa$ . But for sufficiently

large  $i$ ,  $\kappa_i$  cannot be less than  $\kappa$  and so  $\kappa$  is of the form  $\sigma_{i_0 i}(\kappa_{i_0})$  for some  $i_0 < i$ , which implies that  $\kappa$  equals  $\kappa_{i_0}$ , as  $\kappa$  is a fixed point of  $\sigma_{i_0 i}$ .

So choose  $i$  so that  $\kappa = \kappa_i$ . Then  $\kappa$  and  $\lambda$  are fixed points of  $\sigma_{0i}$  and therefore  $\lambda$  has  $K_i$ -cofinality  $\kappa_i$ . As  $K_{i+1}$  has the same  $\kappa$ -sequences as  $K_i$ , it follows that  $\lambda$  has cofinality  $\kappa$  in  $K_{i+1}$  and therefore also in  $K'$ . But since we applied the order 0 measure on  $\kappa$  to form  $K_{i+1}$ ,  $\kappa$  is not measurable in  $K_{i+1}$  and therefore not measurable in  $K'$ , as desired.  $\dashv$

Now we apply the technique of “dropping along a square sequence” from [2]. Define a function  $d : \text{Ord} \rightarrow \omega$  as follows. Fix a lightface  $K'$ -definable global  $\square$ -sequence  $\langle C_\alpha \mid \alpha \text{ singular in } K' \rangle$ :  $C_\alpha$  is closed unbounded in  $\alpha$  with ordertype less than  $\alpha$  for each  $K'$ -singular  $\alpha$  and  $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$  whenever  $\bar{\alpha}$  is a limit point of  $C_\alpha$ . If  $\alpha$  is not  $K'$ -singular then  $d(\alpha) = 0$ . Otherwise define:

$$\begin{aligned} \alpha_0 &= \alpha \\ \alpha_1 &= \text{ot}(C_{\alpha_0}) \\ \alpha_2 &= \text{ot}(C_{\alpha_1}) \\ &\vdots \\ \alpha_{n+1} &= \text{ot}(C_{\alpha_n}), \end{aligned}$$

as long as  $\alpha_n$  is  $K'$ -singular, and let  $d(\alpha)$  be the least  $n$  such that  $\alpha_n$  is not  $K'$ -singular.  $\alpha_{d(\alpha)}$  is the  $K'$ -cofinality of  $\alpha$ .

LEMMA 6 (Main Lemma, after [2]). *For each  $n$  there is a ZFC-preserving class forcing  $P_n$  that adds a CUB class  $C_n$  of singular cardinals such that for all  $\alpha$  in  $C_n$  of cofinality at least  $\delta$ ,  $d(\alpha)$  is at least  $n$ .*

PROOF. We use the following.

LEMMA 7. *Suppose  $k < m$ ,  $\alpha \geq \delta$ ,  $\alpha$  is regular and  $C$  is a closed set of ordertype  $\alpha^{+m} + 1$ , consisting of ordinals  $\geq \alpha^{+m}$  (where  $\alpha^{+0} = \alpha$ ,  $\alpha^{+(p+1)} = (\alpha^{+p})^+$ ). Then  $(C \cap \{\beta \mid d(\beta) \geq k + 1\}) \cup \text{Cof}(< \delta)$  has a closed subset of ordertype  $\alpha^{+(m-k-1)} + 1$ .*

PROOF. The proof is by induction on  $k$ , using Lemma 4.

Suppose  $k = 0$ . Let  $\beta$  be the  $\alpha^{+(m-1)}$ -st element of  $C$ . Then  $\beta$  is  $K'$ -singular since its cofinality ( $= \alpha^{+(m-1)}$ ) is at least  $\delta$  and less than its cardinality ( $\geq \alpha^{+m}$ ). Similarly, each element of  $\text{Lim}(C \cap \beta)$  of cofinality  $\geq \delta$  is  $K'$ -singular and therefore  $\text{Lim}(C \cap \beta)$  is a closed subset of  $(C \cap \{\beta \mid d(\beta) \geq 1\}) \cup \text{Cof}(< \delta)$  of ordertype  $\alpha^{+(m-1)} + 1$ , as desired.

Suppose that the Lemma holds for  $k$  and let  $m + 1 > k + 1$ ,  $C$  a closed set of ordertype  $\alpha^{+(m+1)} + 1$  consisting of ordinals  $\geq \alpha^{+(m+1)}$ . Then  $\mu = \max C$  is  $K'$ -singular, as its cofinality is at least  $\delta$  and less than its cardinality. Let  $\beta$  be the  $(\alpha^{+m} + \alpha^{+m})$ -th element of  $C \cap C_\mu$ .  $\beta$  is  $K'$ -singular as its cofinality is at least  $\delta$  and less than its cardinality. Let  $\bar{\beta}$  be the  $\alpha^{+m}$ -th element of  $C$ . Then  $\bar{C} = \{\text{ot } C_\gamma \mid \gamma \in C \cap \text{Lim } C_\beta \cap [\bar{\beta}, \beta]\}$  is a closed set of ordertype  $\alpha^{+m} + 1$  consisting of ordinals  $\geq \alpha^{+m}$ . By induction there is a closed  $\bar{D}$  contained in  $(\bar{C} \cap \{\gamma \mid d(\gamma) \geq k + 1\}) \cup \text{Cof}(< \delta)$  of ordertype  $\alpha^{+(m-k-1)} + 1$ . But then  $D = \{\gamma \in C \cap \text{Lim } C_\beta \mid \text{ot } C_\gamma \in \bar{D}\}$  is a closed subset of  $(C \cap \{\gamma \mid d(\gamma) \geq k + 2\}) \cup \text{Cof}(< \delta)$  of ordertype  $\alpha^{+(m-k-1)} + 1$ . As  $m - k - 1 = (m + 1) - (k + 1) - 1$ , we are done.  $\dashv$  LEMMA 7

Lemma 6 now follows: Let  $P_n$  consist of closed sets  $c$  of singular cardinals such that

$$\alpha \in c, \text{cof}(\alpha) \geq \delta \rightarrow d(\alpha) \geq n,$$

ordered by end-extension. Lemma 7 implies that this forcing is  $\kappa$ -distributive for every cardinal  $\kappa$ .  $\dashv$

Now for each  $n$  there is an outer model of  $V$  containing a real  $R_n$  such that in  $L[R_n]$ :

- $R_n$  codes a CUB class  $C_{R_n}$  of singular cardinals and an iterable, universal extender model  $K'_{R_n}$  such that
- $(*)_{R_n}$ 
  - a.  $d_{R_n}(\alpha) \geq n$  for  $\alpha$  in  $C_{R_n}$  of sufficiently large cofinality, where  $d_{R_n}(\alpha)$  is defined in  $K'_{R_n}$  just like  $d(\alpha)$  is defined in  $K'$ .
  - b.  $\alpha \in C_{R_n} \rightarrow \text{cof}(\alpha)$  in  $K'_{R_n}$  is not measurable in  $K'_{R_n}$ .

This is because we can use Lemma 6 to force a CUB class  $C_n$  of singular cardinals such that  $d(\alpha) \geq n$  for all  $\alpha$  in  $C_n$  of sufficiently large cofinality, and then  $L$ -code the model  $\langle V, C_n, K' \rangle$  by a real  $R_n$ . The extender model  $K'$  is universal in the extension as successors of strong limit cardinals are not collapsed and therefore weak covering holds relative to  $K'$  in the extension at all such cardinals of sufficiently large cofinality.

Applying the IMH, there are such reals  $R_n$  in  $V$ . As each  $R_n$  codes a CUB class of singular cardinals, the  $K$  of  $L[R_n]$  is universal and therefore so is the  $K_{R_n}$  arising from  $(*)_{R_n}$ . Now co-iterate the  $K_{R_n}$ 's to a single  $K^*$ , resulting in embeddings  $\pi_n : K_{R_n} \rightarrow K^*$ . As singular cardinals in  $C_{R_n}$  of sufficiently large cofinality are fixed by  $\pi_n$  (as their  $K_{R_n}$ -cofinality is not measurable in  $K_{R_n}$ ), it follows that there is a single  $\gamma$  belonging to all of the  $C_{R_n}$ 's which is fixed by all of the  $\pi_n$ 's. But then  $d^*(\gamma) \geq n$  for each  $n$ , where  $d^*(\gamma)$  is defined relative to  $K^*$  just like  $d(\gamma)$  was defined relative to  $K'$ . This is a contradiction.  $\dashv$

For each real  $x$  let  $M_x$ , if it exists, be the minimum transitive set model of ZFC containing  $x$ . Thus  $M_x$  has the form  $L_\mu[x]$  for some countable ordinal  $\mu = \mu(x)$ . If  $d$  is a Turing degree we write  $M_d, \mu(d)$  for  $M_x, \mu(x)$  ( $x$  in  $d$ ).

**THEOREM 8.** *Assume the existence of a Woodin cardinal with an inaccessible above. Then the IMH is consistent. Moreover for all  $d$  in a cone of Turing degrees,  $M_d$  exists and satisfies the IMH.*

**PROOF.** First we prove the consistency of the IMH by showing that  $M_d$  satisfies the IMH for some Turing degree  $d$  in a forcing extension of  $V$ .

Let  $\kappa$  be Woodin with an inaccessible above in  $V$ . Let  $G$  be generic over  $V$  for the Lévy collapse of  $\kappa$  to  $\omega$ . Work now in  $V[G]$ .  $\Sigma_2^1$  determinacy holds and, as there is still an inaccessible,  $M_d$  exists for each Turing degree  $d$ . It follows that the theory of  $(M_d, \in)$  is constant on a cone of Turing degrees  $d$ . Let  $d$  be a Turing degree such that the theory of  $(M_e, \in)$  is constant for Turing degrees  $e$  at least that of  $d$ .

We claim that  $M_d$ , endowed with its definable classes, witnesses the IMH. Indeed, suppose that  $\varphi$  is a sentence true in some model  $M$  of height  $\mu(d)$  compatible with  $M_d$ . By Jensen coding there is a real  $y$  such that  $d$  is recursive in  $y$ ,  $\mu(y) = \mu(d)$  and  $M$  is a definable inner model of  $M_y$ . Let  $e$  be the Turing degree of  $y$ . Then for some formula  $\psi$ ,  $M_e$  satisfies the sentence

*The inner model defined by  $\psi$  (with some choice of parameters) satisfies  $\varphi$ .*

It follows that there is an inner model of  $M_d$  which satisfies  $\varphi$ , as desired. This proves the consistency of the IMH.

To say that a countable  $M$ , together with its countable collection of definable classes, satisfies IMH is simply a  $\Pi_1^1$ -statement with a real coding  $M$  as parameter, since one only needs to quantify over outer models of  $M$  of height  $M \cap \text{Ord}$ . Thus the assertion that there exists a Turing degree  $d$  such that  $M_d$  (with its definable classes) satisfies IMH is a  $\Sigma_2^1$ -statement and hence absolute. So the existence of a Woodin cardinal with an inaccessible above implies that such an  $M_d$  exists in  $V$  (and indeed in  $L$ ).

To prove the stronger statement that in  $V$ ,  $M_d$  satisfies the IMH for a cone of  $d$ 's, one argues as follows. Say that a set of reals  $X$  is *absolutely*  $\Delta_2^1$  iff there is a pair of  $\Sigma_2^1$  formulas  $\varphi(x)$ ,  $\psi(x)$  such that  $X$  consists of all solutions to  $\varphi(x)$  in  $V$  and  $\varphi$  is equivalent to the negation of  $\psi$  both in  $V$  and all of its forcing extensions.

CLAIM. Assume that there is a Woodin cardinal. Then determinacy holds for absolutely  $\Delta_2^1$  sets.

PROOF OF CLAIM. As before let  $G$  be generic for the Lévy collapse of the Woodin cardinal to  $\omega$ . Then  $\Sigma_2^1$  determinacy holds in  $V[G]$ . By the Moschovakis Third Periodicity theorem, ([7] Theorem 6E.1), if  $X$  is  $\Sigma_2^1$  in  $V[G]$  there is a definable winning strategy in  $V[G]$  for one of the players in the game  $G_X$ . By the homogeneity of the Lévy collapse, it follows that absolutely  $\Delta_2^1$  sets are determined in  $V$ . This proves the Claim.  $\dashv$

As there is an inaccessible in  $V$ ,  $M_d$  exists for each Turing degree  $d$  in  $V$ . Now it follows from the Claim that in  $V$ , for any sentence  $\varphi$ , either for a cone of Turing degrees  $d$ ,

$$M_d \models \varphi$$

or for a cone of Turing degrees  $d$ ,

$$M_d \models \neg\varphi,$$

since the relevant games are absolutely  $\Delta_2^1$ . Therefore in  $V$  the theory of  $(M_d, \in)$  is constant for a cone of Turing degrees  $d$ . We can then apply the argument used earlier in  $V[G]$  to conclude that also in  $V$ ,  $M_d$  satisfies IMH for  $d$  in a cone of Turing degrees.  $\dashv$

**§1. Parameters and the Strong Inner Model Hypothesis.** How can we introduce parameters into the Inner Model Hypothesis? The following result shows that inconsistencies arise without strong restrictions on the type of parameters allowed.

PROPOSITION 9. [4] *The Inner Model Hypothesis with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.*

So instead we consider *absolute parameters*, as in [5]. For any set  $x$ , the *hereditary cardinality* of  $x$ , denoted  $\text{hcard}(x)$ , is the cardinality of the transitive closure of  $x$ . If  $V^*$  is an outer model of  $V$ , then a parameter  $p$  is *absolute between  $V$  and  $V^*$*  iff  $V$  and  $V^*$  have the same cardinals  $\leq \text{hcard}(p)$  and some parameter-free formula has  $p$  as its unique solution in both  $V$  and  $V^*$ .

*Inner Model Hypothesis with locally absolute parameters* Suppose that  $p$  is absolute between  $V$  and  $V^*$  and  $\varphi$  is a first-order sentence with parameter  $p$  which holds in an inner model of  $V^*$ . Then  $\varphi$  holds in an inner model of  $V$ .

For a singular cardinal  $\kappa$ , a  $\square_\kappa$  sequence is a sequence of the form  $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$  such that each  $C_\alpha$  has ordertype less than  $\kappa$  and for  $\bar{\alpha}$  in  $\text{Lim } C_\alpha$ ,  $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ . *Definable*  $\square_\kappa$  is the assertion that there exists a  $\square_\kappa$  sequence which is definable over  $H(\kappa^+)$  with parameter  $\kappa$ . We will be interested in the special case  $\kappa = \beth_\omega$ , in which case the parameter  $\kappa$  is superfluous.

**THEOREM 10.** *The Inner Model Hypothesis with locally absolute parameters is inconsistent.*

**PROOF.** We first show that definable  $\square_\kappa$  fails, where  $\kappa$  is  $\beth_\omega$ . Let  $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$  be a  $\square_\kappa$  sequence definable over  $H(\kappa^+)$  without parameters. For each  $n$  let  $S_n$  be the stationary set of all limit  $\alpha < \kappa^+$  such that the ordertype of  $C_\alpha$  is greater than  $\beth_n$ .

**CLAIM.** Let  $P_n$  be the forcing that adds a CUB subset of  $S_n$  using closed bounded subsets of  $S_n$  as conditions, ordered by end extension. Then  $P_n$  is  $\kappa^+$ -distributive, i.e., does not add  $\kappa$ -sequences.

**PROOF OF CLAIM.** It is enough to show that  $P_n$  is  $\beth_m^+$  distributive for each  $m < \omega$ . Assume that  $m$  is at least  $n$ . Suppose that  $p$  is a condition and  $\langle D_i \mid i < \beth_m \rangle$  are dense. Let  $\langle M_i \mid i < \kappa^+ \rangle$  be a continuous chain of size  $\kappa$  elementary submodels of some large  $H(\theta)$  such that  $M_0$  contains  $\kappa \cup \{ \langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle, p \}$  and for each  $i < \kappa^+$ ,  $\langle M_j \mid j \leq i \rangle$  is an element of  $M_{i+1}$ . Let  $\kappa_i$  be  $M_i \cap \kappa^+$  and  $C$  the set of such  $\kappa_i$ 's. Then a final segment  $D$  of  $C \cap \text{Lim } C_\gamma$  is contained in  $S_n$ , where  $\gamma = \kappa_{\beth_m}$ . Write  $D$  as  $\langle \kappa_{\alpha_i} \mid i < \text{ordertype } D \rangle$ . We can then choose a descending sequence  $\langle p_i \mid i < \beth_m \rangle$  of conditions below  $p$  such that  $p_{i+1}$  meets  $D_i$  and belongs to  $M_{\kappa_{\alpha_i+1}}$  for each  $i$ . Then the greatest lower bound of this sequence meets each  $D_i$ . This proves the Claim.  $\dashv$

It follows that for each  $n$  the forcing  $P_n$  does not alter  $H(\kappa^+)$ . By the Inner Model Hypothesis with locally absolute parameters  $S_n$  has a CUB subset  $C_n$  in  $V$  for each  $n$ . But this is a contradiction, as the intersection of the  $C_n$ 's is empty.

Now we refine the above argument. As not every real has a  $\#$ , there exist reals  $R$  such that  $\kappa^+$  equals  $\kappa^+$  of  $L[R]$ , where  $\kappa$  is  $\beth_\omega$ . Let  $X$  be the set of such reals and for each  $R$  in  $X$  let  $\langle C_\alpha^R \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$  be the  $L[R]$ -least  $\square_\kappa$  sequence. Now for limit  $\alpha < \kappa^+$ , define  $C_\alpha^*$  to be the intersection of the  $C_\alpha^R$ ,  $R \in X$ . Then  $\langle C_\alpha^* \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$  is definable in  $H(\kappa^+)$  without parameters and has the properties of a  $\square_\kappa$  sequence with the sole exception that  $C_\alpha^*$  is only guaranteed to be unbounded in  $\alpha$  if  $\alpha$  has cofinality greater than  $2^{\aleph_0}$ . Now repeat the above argument using  $\langle C_\alpha^* \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$  in place of  $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit} \rangle$ , to obtain a contradiction.  $\dashv$

To obtain the Strong Inner Model Hypothesis, we require more absoluteness. We say that the parameter  $p$  is (globally) absolute iff there is a parameter-free formula which has  $p$  as its unique solution in all outer models of  $V$  which have the same cardinals  $\leq \text{hcard}(p)$  as  $V$ .

*Strong Inner Model Hypothesis (SIMH).* Suppose that  $p$  is absolute,  $V^*$  is an outer model of  $V$  with the same cardinals  $\leq \text{hcard}(p)$  as  $V$  and  $\varphi$  is a first-order sentence with parameter  $p$  which holds in an inner model of  $V^*$ . Then  $\varphi$  holds in an inner model of  $V$ .

REMARK. If above we assume that the sentence  $\varphi$  holds not just in an inner model of  $V^*$  but in  $V^*$  itself, then in the conclusion we may demand that in an inner model of  $V$  witnessing  $\varphi$ ,  $p$  is definable via the same formula  $\psi$  witnessing the absoluteness of  $p$ . (This inner model may, however, fail to have the same cardinals  $\leq \text{heard}(p)$  as  $V$ .) This is because we can replace the sentence  $\varphi$  by: “ $\varphi$  holds and  $p$  is defined by  $\psi$ ”.

THEOREM 11. [4] *Assume the SIMH. Then CH is false. In fact,  $2^{\aleph_0}$  cannot be absolute and therefore cannot be  $\aleph_\alpha$  for any ordinal  $\alpha$  which is countable in  $L$ .*

THEOREM 12. *The SIMH implies the existence of an inner model with a strong cardinal.*

PROOF. Assume not, and let  $K$  be the core model below a strong cardinal (see [8]). As in the proof of Theorem 2, we let  $K'$  denote the iterate of  $K$  obtained by applying each order 0 measure exactly once. Then by the argument of Lemma 5, if  $\lambda$  is a cardinal then the  $K'$ -cofinality of  $\lambda$  is not measurable in  $K'$ . And by weak covering relative to  $K$ , if  $\lambda$  is a singular cardinal, then  $\lambda^+$  is computed correctly in  $K$  (i.e.,  $(\lambda^+)^K = \lambda^+$ ).

LEMMA 13. *For any singular cardinal  $\lambda$ ,  $\lambda^+$  is computed correctly in  $K'$ .*

PROOF OF LEMMA 13. This is clear if the  $K$ -cofinality of  $\lambda$  is not measurable in  $K$ , for then  $\lambda$  is a fixed point of the iteration from  $K$  to  $K'$  and  $(\lambda^+)^{K'} = (\lambda^+)^K = \lambda^+$ . Otherwise let  $\langle K_i \mid i \in \text{Ord} \rangle$  result from the iteration of  $K$  to  $K'$  and choose  $i$  so that the ultrapower map  $\sigma_i : K_i \rightarrow K_{i+1}$  applies the order 0 measure at  $\kappa = \text{cof}^{K_i}(\lambda)$ . If  $\langle \lambda_j \mid j < \kappa \rangle$  is a continuous and increasing sequence in  $K_i$  with supremum  $\lambda$ , then  $\lambda^+$  of  $K_{i+1}$  is represented in the ultrapower of  $K_i$  by  $\langle \lambda_j^+ \mid j < \kappa \rangle$ . In  $K_i$ , the product of the  $\lambda_j^+$ 's contains a subset of size  $(\lambda^+)^{K_i}$ , consisting of functions well-ordered by dominance on a final segment of  $\kappa$ . It follows that  $(\lambda^+)^{K_{i+1}}$  has cardinality  $\lambda^+$  and therefore  $K_{i+1}$  computes  $\lambda^+$  correctly. As  $\lambda^+$  is a fixed point of the remaining iteration from  $K_{i+1}$  to  $K'$ , it follows that  $K'$  computes  $\lambda^+$  correctly. This proves Lemma 13 □

We say that  $\lambda$  is a *cut point* of  $K'$  iff no extender on the  $K'$  sequence with critical point less than  $\lambda$  has length at least  $\lambda$  (i.e.,  $\lambda$  is *not overlapped* in  $K'$ ). The class of cut points of  $K'$  is clearly closed. If the class of cut points of  $K'$  is bounded, then for sufficiently large  $\lambda$  we can choose  $f(\lambda)$  less than  $\lambda$  which is the critical point of an extender on the  $K'$  sequence whose length is at least  $\lambda$ ; but then by Fodor's theorem, there would be a fixed  $\kappa$  and extenders on the  $K'$  sequence with critical point  $\kappa$  of arbitrarily large length, which implies that  $\kappa$  is a strong cardinal in  $K'$ . Thus as we have assumed that there is no strong cardinal in  $K'$ , the class of cut points of  $K'$  is closed and unbounded.

Let  $\langle \lambda_n \mid n \in \omega \rangle$  be the first  $\omega$ -many limit cardinals of  $V$  which are cut points of  $K'$ , and let  $\lambda_\omega$  be their supremum. Then each  $\lambda_n$ , and of course  $\lambda_\omega$ , has cofinality  $\omega$ .

LEMMA 14. *Each  $\lambda_n$ , and  $\lambda_\omega$  as well, is an absolute parameter.*

PROOF OF LEMMA 14. We first show that  $\lambda_0$  is absolute. Let  $V^*$  be an outer model of  $V$  with the same cardinals as  $V$  up to  $\lambda_0$ . Note that for some real  $R$  in  $V$ , no  $L_\alpha[R]$  satisfies ZFC and therefore  $R^\#$  does not exist in  $V^*$ . It follows that for any singular cardinal  $\lambda$  of  $V^*$ ,  $\lambda$  is singular in  $V$  and  $\lambda^+$  is computed correctly in  $V$ .



In particular,  $V^*$  and  $V$  have the same cardinals up to  $\lambda_0^+$  and the same singular cardinals up to  $\lambda_0$ .

It follows by Lemma 13 that for any singular cardinal  $\lambda$  of  $V^*$ ,  $\lambda^+$  is computed correctly in both  $K'$  and  $(K^*)'$ , where  $(K^*)'$  denotes the  $K'$  of  $V^*$ , obtained from  $K^*$ , the  $K$  of  $V^*$ , by applying each order 0 measure exactly once. As both  $K'$  and  $(K^*)'$  are universal in  $V^*$ , it follows that they coiterate simply to a common model  $W$ .  $\dashv$

CLAIM. The co-iteration of  $K'$  with  $(K^*)'$  fixes singular cardinals of  $V^*$  which are cut points either of  $K'$  or of  $(K^*)'$ .

PROOF OF CLAIM. Let  $\lambda$  be a singular cardinal of  $V^*$  (and therefore also a singular cardinal of  $V$ ). First assume that  $\lambda$  is a cut point of  $K'$ . Then as  $\lambda$  has non-measurable cofinality in  $K'$ ,  $\lambda$  is fixed by the iteration on the  $K'$ -side. And as  $\lambda$  has non-measurable cofinality in  $(K^*)'$ ,  $\lambda$  can only move on the  $(K^*)'$ -side if an extender overlapping  $\lambda$  were applied. The assumption that  $\lambda$  is not overlapped in  $K'$  implies that  $\lambda$  is not overlapped in  $W$  and therefore the least extender overlapping  $\lambda$  was applied on the  $(K^*)'$ -side. But then  $\lambda^+$  is not computed correctly in the resulting ultrapower and therefore is computed correctly neither in  $W$  nor in  $K'$ . This contradicts Lemma 13. As the same argument applies with  $K'$  and  $(K^*)'$  switched, this proves the Claim.  $\dashv$

It follows from the Claim that  $\lambda_0$  is the least limit cardinal of  $V^*$  which is a cut point of  $(K^*)'$ . As  $V^*$  is an arbitrary outer model of  $V$  with the same cardinals as  $V$  up to  $\lambda_0$ , we have shown that  $\lambda_0$  is an absolute parameter. The same argument shows that each  $\lambda_n$  is absolute, and therefore so is  $\lambda_\omega$ , the supremum of the first  $\omega$  limit cardinals which are cut points of  $K'$ . This proves Lemma 14.

Now let  $\langle C_\alpha \mid \alpha < \lambda_\omega^+, \alpha \text{ limit} \rangle$  be the least  $\square_{\lambda_\omega}$  sequence of  $K'$ ; this is also a  $\square_{\lambda_\omega}$  sequence in  $V$ , as  $(\lambda_\omega^+)^{K'} = \lambda_\omega^+$ . As in the proof of Theorem 10, there are generic extensions of  $V$  preserving  $H(\lambda_\omega^+)$  which add CUB subsets to each  $S_n = \{\alpha < \lambda_\omega^+ \mid \text{ordertype } C_\alpha > \lambda_n\}$ . It follows from the Strong Inner Model Hypothesis (and the Remark immediately following its statement) that for each  $n$  there is an inner model  $M_n$ , with the correct  $\lambda_\omega^+$  and  $\lambda_n$ , in which  $S_n^{M_n}$  contains a CUB subset  $C_n$ , where  $S_n^{M_n}$  is defined using the least  $\square_{\lambda_\omega}$  sequence of  $(K')^{M_n}$ . The latter may of course differ from the least  $\square_{\lambda_\omega}$  sequence of  $K'$ . However as  $\lambda_\omega^+$  is computed correctly in each  $(K')^{M_n}$  and  $\lambda_\omega$  is a cut point of non-measurable cofinality in each  $(K')^{M_n}$ , it follows that the  $(K' \mid \lambda_\omega^+)^{M_n}$ 's compare to a common  $K''$  of height  $\lambda_\omega^+$  with all ordinals in some CUB subset  $C$  of  $\lambda_\omega^+$  as closure points. But if  $\alpha$  is such a closure point in the intersection of the  $C_n$ 's and  $\alpha_n$  is the image of  $\alpha$  under the comparison embedding of  $(K' \mid \lambda_\omega^+)^{M_n}$  into  $K''$ , then  $C_{\alpha_n}$  as defined in  $K''$  contains elements cofinal in  $\alpha$  and therefore  $C_\alpha$  as defined in  $K''$ , an initial segment of  $C_{\alpha_n}$ , has ordertype at least that of  $C_\alpha$  as defined in  $(K' \mid \lambda_\omega^+)^{M_n}$ . It follows that  $C_\alpha$  as defined in  $K''$  has ordertype greater than  $\lambda_n$  for each  $n$ , which is a contradiction.  $\dashv$

REMARKS. (a) It is likely that Theorem 12 can be improved to obtain an inner model with a Woodin cardinal. But it is not possible to obtain an iterable inner model with a Woodin cardinal and an inaccessible above it (unless the SIMH is inconsistent): Otherwise every real would be generic for Woodin's extender algebra

defined in an iterate of such an inner model, implying that for every real  $R$  there is an inaccessible in  $L[R]$ ; this contradicts Theorem 1.

(b) David Asperó and the first author observed that the consistency of the SIMH for the parameter  $\omega_1$  follows as in the proof of Theorem 8 from that of a Woodin cardinal with an inaccessible above. In particular this yields the consistency of the natural extension of Lévy absoluteness asserting  $\Sigma_1$  absoluteness with parameter  $\omega_1$  for arbitrary  $\omega_1$ -preserving extensions.

QUESTION. Is the Strong Inner Model Hypothesis consistent relative to large cardinals?

## REFERENCES

- [1] A. BELLER, R. JENSEN, and P. WELCH, *Coding the universe*, London Mathematical Society Lecture Note Series, vol. 47, Cambridge University Press, Cambridge, 1982.
- [2] S. FRIEDMAN, *New  $\Sigma_1^1$  facts*, *Proceedings of the American Mathematical Society*, vol. 127 (1999), pp. 3707–3709.
- [3] ———, *Fine structure and class forcing*, de Gruyter Series in Logic and its Applications, vol. 3, Walter de Gruyter & Co., Berlin, 2000.
- [4] ———, *Internal consistency and the inner model hypothesis*, *The Bulletin of Symbolic Logic*, vol. 12 (2006), pp. 591–600.
- [5] ———, *Stable axioms of set theory*, *Set Theory: Centre de Recerca Matemàtica, Barcelona, 2003–2004*, Trends in Mathematics, Birkhäuser Verlag, 2006, pp. 275–283.
- [6] W. MITCHELL, *An introduction to inner models and large cardinals*, *Handbook of set theory*, Springer Verlag, to appear.
- [7] Y. MOSCHOVAKIS, *Descriptive set theory*, North-Holland Publishing Co., Amsterdam, 1980.
- [8] M. ZEMAN, *Inner models and large cardinals*, de Gruyter Series in Logic and its Applications, Walter de Gruyter & Co., Berlin, 2002.

KURT GÖEDEL RESEARCH CENTER  
 UNIVERSITY OF VIENNA  
 WÄHRINGER STRASSE 25  
 A-1090 VIENNA, AUSTRIA  
*E-mail:* sdf@logic.univie.ac.at

DEPARTMENT OF MATHEMATICS  
 THE UNIVERSITY OF CALIFORNIA  
 721 EVANS HALL # 3840  
 BERKELEY, CA 94720–3840, USA  
*E-mail:* woodin@math.berkeley.edu

SCHOOL OF MATHEMATICS  
 UNIVERSITY OF BRISTOL  
 BRISTOL BS8 1TW, UK  
*E-mail:* p.welch@bristol.ac.uk