# GENERIC CODING WITH HELP AND AMALGAMATION FAILURE

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Abstract. We show that if $M$ is a countable transitive model of ZF and if $a, b$ are reals not in $M$, then there is a $G$ generic over $M$ such that $b \in L[a, G]$. We then present several applications such as the following: if $J$ is any countable transitive model of ZFC and $M \not\subseteq J$ is another countable transitive model of ZFC of the same ordinal height $\alpha$, then there is a forcing extension $N$ of $J$ such that $M \cup N$ is not included in any transitive model of ZFC of height $\alpha$. Also, assuming $0^\#$ exists, letting $S$ be the set of reals generic over $L$, although $S$ is disjoint from the Turing cone above $0^\#$, we have that for any non-constructible real $a$, $\{a \oplus s : s \in S\}$ is cofinal in the Turing degrees.

## 1. Introduction

If $0^\#$ exists, it is not in any (set) forcing extension of $L$. On the other hand, Mostowski showed that for any real $x$, there are reals $g_1, g_2$ both Cohen generic over $L$ such that $x$ is computable from the Turing join of $g_1$ and $g_2$, written $x \leq_T g_1 \oplus g_2$ (see [4] for a proof).

In this paper we investigate the following question (assuming $0^\#$ exists): given an arbitrary $g_1 \in {}^\omega\omega$ not in $L$, is there a real $g_2$ generic over $L$ such that $0^\# \leq_T g_1 \oplus g_2$? We will see that the answer is yes. Although there is a limit to what reals are generic over $L$, there is *no limit* to what reals are constructible from a fixed non-constructible real and a real that is generic over $L$. Here is the general formulation:

**Definition 1.1.** Let $M$ be a countable transitive model of ZFC. Let $\mathbb{P} \in M$ be a poset. A real $\bar{a} \in {}^\omega\omega$ is $(\mathbb{P}, M)$-*helpful* iff for any $x \in {}^\omega\omega$, there is a $G$ that is $\mathbb{P}$-generic over $M$ such that $x \in L(\bar{a}, G)$.

Now fix a countable transitive model $M$ of ZFC. Let $\mathbb{C}$ be Cohen forcing.

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1) Fix $\mathbb{P} \in M$. No real $\bar{a} \in M$ is $(\mathbb{P}, M)$-helpful: if $x \in {}^\omega\omega$ codes the ordinal $\mathrm{Ord} \cap M$, then $x \notin L(\bar{a}, G)$ for any $G$ that is $\mathbb{P}$-generic over $M$.
2) Every real Cohen generic over $M$ is $(\mathbb{C}, M)$-helpful (see Corollary 5.5 of [4]).
3) Miha Habič (unpublished) and the first author (see [3] just after "nodes of compatibility") have independently shown that every real unbounded over $M$ is $(\mathbb{C}, M)$-helpful.
4) The first author has shown that every real Sacks generic over $M$ is $(\mathbb{C}, M)$-helpful (unpublished).
5) The central result of this paper (Theorem 1.3) is that *every* real not in $M$ is $(\mathbb{H}, M)$-helpful, where $\mathbb{H}$ is "Tree-Hechler" forcing.
6) The question of whether every real not in $M$ is $(\mathbb{C}, M)$-helpful remains open.

**Definition 1.2.** The forcing $\mathbb{H}$, called Tree-Hechler forcing, consists of all trees $T \subseteq {}^{<\omega}\omega$ such that for all $t \sqsupseteq \mathrm{Stem}(T)$ in $T$,

$$\{z \in \omega : t^\frown z \notin T\} \text{ is finite.}$$

The ordering is by inclusion.

That is, a condition in Tree-Hechler forcing has cofinite splitting beyond its stem.

Consider a tree $T \subseteq {}^{<\omega}\omega$ and a node $t \in T$. By a *successor* of $t$ we always mean some $t^\frown z \in T$ for $z \in \omega$. By $T \restriction t$ we mean the set of all $s \in T$ that are comparable to $t$. $\mathrm{Stem}(T)$ is the longest element of $T$ that is comparable with all other elements of $T$.

Let $M$ be a transitive model of ZF and suppose $G$ is $\mathbb{H}^M$-generic over $M$. Let $g = \bigcup \bigcap G$. That is, $g : \omega \to \omega$ is the union of all the stems of the trees $T \in G$. The set $G$ can be recovered from $g$ (and $M$). We will treat $g : \omega \to \omega$ as the object which is encoding information.

The poset $\mathbb{H}$ is $\sigma$-centered, because any two conditions with the same stem are comparable. Thus, $\mathbb{H}$ is c.c.c. Combining this with the fact that $|\mathbb{H}| = 2^\omega$, we have the following: there are only $2^\omega$ maximal antichains in $\mathbb{H}$. So, if $M$ is a transitive model of ZFC and $(2^\omega)^M$ is countable, then there is an $\mathbb{H}^M$-generic over $M$.

The forcing $\mathbb{H}$ is discussed in [9], along with other versions of Hechler forcing, where it is called $\mathbb{D}_{tree}$. A key ingredient for us is that $\mathbb{H}$ admits a "rank analysis" of its dense sets (see Definition 5.9 and Lemma 5.10). In [2], Jörg Brendle and Benedikt Löwe carry out a rank analysis of $\mathbb{H}$. The original rank analysis of a Hechler-like forcing was done by James Baumgartner and Peter Dordal in [1] for the non-decreasing function version of Hechler forcing (although we discovered "reachability" independently of these works).

Here is our main result:

**Theorem 1.3** (Generic Coding with Help)**.** *Let $M$ be a transitive model of ZF such that $\mathcal{P}^M(\mathbb{H}^M)$ is countable. Then given any $\bar{a}, x \in {}^\omega\omega$ such that $\bar{a} \notin M$, there is a $G$ that is $\mathbb{H}^M$-generic over $M$ such that $x \leq_T \bar{a} \oplus (\bigcup \bigcap G)$.*

Here, $\bar{a}$ is the "help" which is being used to code $x$. Theorem 1.3 has several interesting applications, which we will present first. Then for completeness we will include a proof of Theorem 1.3.

One striking application is Theorem 2.1, which shows that given two distinct countable transitive models $M, J$ of ZFC of the same height (meaning $\mathrm{Ord} \cap M = \mathrm{Ord} \cap J$), there is a forcing extension of one which does not amalgamate with the other (where two models of the same height $\alpha$ are said to amalgamate iff they are both included in a countable transitive model $W$ of ZFC of height $\alpha$). This answers a question posed by the first author [3] concerning the Hyperuniverse Program: the model $L_\alpha$ is the only node of compatibility of height $\alpha$.

Theorem 1.3 is a consequence of Lemma 5.11, the "Main Lemma". However, this was not the Main Lemma's original purpose in the literature. This lemma originated from the second author's thesis [7] where it appeared in a game theoretic form that does not explicitly refer to forcing. In that version, Players I and II play to build a descending sequence through $\mathbb{H}$, where Player I makes $\leq$-extensions but Player II makes $\leq_A$-extensions (to be defined later). The goal of this game was to prove results like Proposition 5.12. In [5] and [6] such results are proved, and the current version of the Main Lemma appears in [5]. We want to emphasize that the Main Lemma may have applications other than Theorem 1.3 and Proposition 5.12.

## 2. Amalgamation failure for C.T.M.'s

The Generic Coding with Help theorem implies in a strong way that c.t.m.'s (countable transitive models) of ZFC of the same ordinal height cannot be amalgamated:

**Theorem 2.1.** *Let $J$ be a c.t.m. of ZFC of ordinal height $\alpha < \omega_1$. Let $M \not\subseteq J$ be another c.t.m. of ZFC of height $\alpha$. Then there is a forcing extension $N$ of $J$ such that $M \cup N$ is not included in any c.t.m. of ZFC of height $\alpha$.*

*Proof.* Fix $\lambda < \alpha$ and $x \subseteq \lambda$ such that $x \in M - J$. This is possible because $J$ and $M$ are models of ZFC and $M \not\subseteq J$. That is, following the proof of Theorem 13.28 in [8], first fix $X \in M - J$. Now let $x \in M$ be a bounded subset of $\mathrm{Ord} \cap M = \alpha$ such that $X$ is in any transitive model of ZFC which contains $x$: such an $x$ can be formed by first bijecting the transitive closure $tc(X)$ of $X$ with an ordinal $\lambda' < \alpha$, and then encoding the binary relation $\in \restriction tc(X)$ as a subset of $\lambda' \times \lambda'$, and then encoding that binary relation by a single set $x \subseteq \lambda$ for some $\lambda < \alpha$. Such an $x$ cannot be in $J$.

Let $g_0'$ and $g_0''$ be mutually $\mathrm{Col}(\omega, \lambda)$-generic over $J$. Since they are mutually generic, $J[g_0'] - J$ and $J[g_0''] - J$ are disjoint. Let $g_0$ be one of $g_0'$ or $g_0''$ such that $x \notin J[g_0]$.

Now $g_0$ codes a surjection from $\omega$ to $\lambda$. Let $\tilde{x} \subseteq \omega$ be induced from this surjection and $x$. By this we mean if $W$ is any transitive model of ZFC with contains $g_0$, then $x \in W$ iff $\tilde{x} \in W$. Now $\tilde{x} \notin J[g_0]$.

Let $y \in {}^\omega\omega$ be a real that codes a well-ordering of $\omega$ of order type $\alpha$ (so $y$ cannot be in any c.t.m. of ZFC of height $\alpha$). By Theorem 1.3, let $g_1$ be $\mathbb{H}^{J[g_0]}$-generic over $J[g_0]$ such that

$$y \leq_T \tilde{x} \oplus (\bigcup \bigcap g_1).$$

Let $N = J[g_0][g_1]$. Now suppose, towards a contradiction, that there is some transitive model $W \supseteq M \cup N$ of ZFC of ordinal height $\alpha$. Because $x \in M \subseteq W$ and $g_0 \in N \subseteq W$, we have $\tilde{x} \in W$. But also $g_1 \in N \subseteq W$, so $y \in W$, which is impossible. $\square$

We say two c.t.m.'s $N, M$ of ZFC of height $\alpha$ are *compatible* iff there is a c.t.m. $W$ of ZFC of height $\alpha$ such that $N \cup M \subseteq W$.

**Corollary 2.2.** *Given any two distinct c.t.m.'s of ZFC of the same height, there is a forcing extension of one that is not compatible with the other.*

The first author asked (see [3]) if for a given $\alpha < \omega_1$, whether $L_\alpha$ was the only c.t.m. of ZFC of height $\alpha$ that was compatible with every c.t.m. of ZFC of height $\alpha$ (that is, whether $L_\alpha$ was the only *node of compatibility* of height $\alpha$ in the Hyperuniverse). Now we see the answer is yes: If $M \neq L_\alpha$ is a c.t.m. of height $\alpha$, then $M$ is not compatible with a certain forcing extension of $L_\alpha$.

*Remark* 2.3. Mostowski's result in the introduction was used by him for a result about amalgamation (see [4]): Let $J$ be a c.t.m. of ZF of ordinal height $\alpha < \omega_1$. Let $x$ be a real which codes $\alpha$. Let $c_1, c_2$ be two reals Cohen generic over $J$ such that $x \leq_T c_1 \oplus c_2$. Then $J[c_1]$ and $J[c_2]$ are not compatible.

## 3. A complex set disjoint from a Turing cone

As mentioned before, if $0^{\#}$ exists (or even just $\omega_1$ is inaccessible in $L$), then given any real $x$, there are two Cohen generics $s_1, s_2$ over $L$ such that $x \leq_T s_1 \oplus s_2$. So, let $S$ be the complement of the Turing cone above $0^{\#}$ (the Turing cone above $a \in {}^{\omega}\omega$ is the set of all $b \in {}^{\omega}\omega$ such that $b \geq_T a$). Every real generic over $L$ is in $S$. Now $S$ is small in one sense, because it is disjoint from a Turing cone. But it is large in another sense, because $\{s_1 \oplus s_2 : s_1, s_2 \in S\}$ is cofinal in the Turing degrees. We get a variation of this phenomenon using the Generic Coding with Help Theorem (1.3). Let $[x]$ denote the Turing degree of $x \in {}^{\omega}\omega$.

**Proposition 3.1.** *Assume $0^{\#}$ exists. Let $S \subseteq {}^{\omega}\omega$ be the set of all reals of the form $s = \bigcup \bigcap G$ for some $G$ that is $\mathbb{H}^L$-generic over $L$. The set $S$ is disjoint from the Turing cone above $0^{\#}$. On the other hand for any real $\bar{a}$ not in $L$, the set $S^* := \{[\bar{a} \oplus s] : s \in S\}$ is cofinal in the Turing degrees.*

*Also, if $x$ is any real such that $x \geq_T \bar{a}$ and $x$ computes a length $\omega$ enumeration of $\mathbb{R} \cap L$, then $[x] \in S^*$ (so $S^*$ contains a Turing cone).*

*Proof.* It is well known that no generic extension of $L$ contains $0^{\#}$. Hence, $0^{\#}$ is not Turing reducible to any $s = \bigcup \bigcap G$ for a $G$ that is $\mathbb{H}^L$-generic over $L$. That is, $S$ is disjoint from the Turing cone above $0^{\#}$.

Now fix a real $\bar{a}$ not in $L$. Pick any $x \in {}^{\omega}\omega$. By Theorem 1.3 there is some $G$ that is $\mathbb{H}^L$-generic over $L$ such that letting $s = \bigcup \bigcap G$, we have $x \leq_T \bar{a} \oplus s$. Hence, $S^*$ is cofinal in the Turing degrees.

For the last part, again fix a real $\bar{a} \notin L$ and let $x \geq_T \bar{a}$ be a real which computes a length $\omega$ enumeration of $\mathbb{R} \cap L$. There is some $G$ that is $\mathbb{H}^L$-generic over $L$ such that letting $s = \bigcup \bigcap G$, we have $x \leq_T \bar{a} \oplus s$. However, by the proof of Theorem 1.3, fix an $s$ like this that can be built using $\bar{a}$, $x$, and a length $\omega$ enumeration of $\mathbb{R} \cap L$ (using that the dense subsets of $\mathbb{H}^L$ in $L$ are coded by reals in $L$). So we can have $s \leq_T \bar{a} \oplus x$. We now have

$$x \leq_T \bar{a} \oplus s \leq_T \bar{a} \oplus (\bar{a} \oplus x) \leq_T \bar{a} \oplus x = x,$$

so $x =_T \bar{a} \oplus s$, and so $[x] \in S^*$. □

## 4. Larger sets are generically generic

The Generic Coding with Help theorem shows that reals not in $M$ are "helpful". The following theorem shows that *any set of ordinals not in $M$ is helpful*, provided $M$ contains the supremum of the set of ordinals and that we pass to an outer model of $V$ in which a large enough cardinal has become countable.

**Theorem 4.1.** *Let $M$ be a transitive model of ZF. Let $\lambda$ be a cardinal such that $\lambda \in M$. Let $\mathbb{P} = (Col(\omega, \lambda) * \mathbb{H})^M$. Let $\tilde{V}$ be an outer model of $V$ in which $\mathcal{P}^M(\mathbb{P})$ is countable. Let $X \in \mathcal{P}^{\tilde{V}}(\lambda)$. Let $A \in \mathcal{P}^{\tilde{V}}(\lambda) - M$. Then there is a $G$ in $\tilde{V}$ such that*

1) *$G$ is $\mathbb{P}$-generic over $M$,*
2) *$X \in L(A, G)$.*

*Proof.* Using the same mutual generic technique as in the second paragraph of the proof of Theorem 2.1, let $g_0 \in \tilde{V}$ be $\mathrm{Col}(\omega, \lambda)$-generic over $M$ so that $A \notin M[g_0]$. Let $\tilde{a} \in {}^\omega\omega$ be such that for every transitive model $N$ of ZF such that $g_0 \in N$, we have $A \in N$ iff $\tilde{a} \in N$. Now $\tilde{a} \notin M[g_0]$. Let $\tilde{x} \in {}^\omega\omega$ be such that for every transitive model $N$ of ZF such that $g_0 \in N$, we have $X \in N$ iff $\tilde{x} \in N$.

Force over $M[g_0]$ by $\mathbb{H}^{M[g_0]}$ to get $g_1$ so that $\tilde{x} \le_T \tilde{a} \oplus (\bigcup \bigcap g_1)$. Let $G := g_0 * g_1$, so $G \in \tilde{V}$ is $\mathbb{P}$-generic over $M$. $L(A, G)$ is a model of ZF and it contains $g_0$ and $A$, so it contains $\tilde{a}$. It also contains $g_1$, therefore it contains $\tilde{x}$. Since it contains $g_0$ and $\tilde{x}$, it contains $X$. □

Note that if $\lambda = \omega$ in the theorem above, then we can simply take $\mathbb{P}$ to be $\mathbb{H}^M$.

## 5. Proof of Generic Coding with Help Theorem

Theorem 1.3 follows from the Main Lemma of [5]. For completeness we give a full proof here.

### 5.1. Evasiveness and the Sticking Out Lemma.
We will now start to prove the theorem. This subsection helps to clarify how we use the hypothesis $\bar{a} \notin M$.

**Definition 5.1.** Let $M$ be a transitive model of ZF. A set $A \subseteq \omega$ is *evasive with respect to $M$* iff it is infinite and it has no infinite subsets in $M$.

**Fact 5.2.** *Given any $\bar{a} \in {}^\omega\omega$, there is a set $A \subseteq \omega$ such that $\bar{a} =_T A$ and $A$ is computable from every infinite subset of itself.*

Thus if $M$ is a transitive model of ZF and $\bar{a} \in {}^\omega\omega - M$, then if $A$ comes from the fact above, then $A$ is evasive with respect to $M$.

**Lemma 5.3** (Sticking Out Lemma)**.** *Let $M$ be a transitive model of ZF. Let $A \subseteq \omega$ be evasive with respect to $M$. Then if $B \subseteq \omega$ is infinite and in $M$, then $B - A$ is infinite.*

*Proof.* Assume towards a contradiction that $B - A$ is finite. Then $B - A \in M$. Since both $B$ and $B - A$ are in $M$, we have $B \cap A \in M$ as well. At the same time, since $B$ is infinite and $B - A$ is finite, $B \cap A$ must be infinite. So now we have shown that $B \cap A$ is an infinite subset of $A$ that is in $M$, which contradicts $A$ being evasive with respect to $M$. □

5.2. **Decoding an $x \in {}^{\omega}\omega$ from an $\mathbb{H}$ generic and an $A \subseteq \omega$.** Suppose $G$ is generic for $\mathbb{H}$. Recall that $g := \bigcup \bigcap G$ is a function from $\omega$ to $\omega$. The idea is to look at each $n \in \omega$ such that $g(n) \in A$. Which element of $A$ this $g(n)$ actually is will give us a piece of encoded information. For each $n$ such that $g(n) \notin A$, no information is being encoded. Here is what we mean precisely:

**Definition 5.4.** Fix a computable function $\theta : \omega \to \omega$ such that

$$(\forall m \in \omega)\, \theta^{-1}(m) \text{ is infinite.}$$

Given an infinite $A \subseteq \omega$, let $e_A : \omega \to A$ be the strictly increasing enumeration of $A$. Let $\eta_A : A \to \omega$ be the function $\theta \circ e_A^{-1}$.

Note that for each $m \in \omega$, $\eta_A^{-1}(m) \subseteq A$ is infinite.

**Definition 5.5.** Let $M$ be a transitive model of ZF. Let $G$ be $\mathbb{H}^M$-generic over $M$. Let $A \subseteq \omega$.

Then the real that is $A$-*encoded* by $G$ is

$$\langle \eta_A(g(n_i)) : i < \omega \rangle,$$

where $g := \bigcup \bigcap G$ and

$$n_0 < n_1 < ...$$

is the increasing enumeration of the set of $n \in \omega$ such that $g(n) \in A$. However, if there are only finitely many such $n$'s, then the real $A$-encoded by $G$ is the zero sequence.

**Observation 5.6.** *Let $x \in {}^{\omega}\omega$ be the real $A$-encoded by $G$. Then*

$$x \leq_T A \oplus (\bigcup \bigcap G).$$

5.3. **The stronger $\leq_A$ ordering and the Main Lemma.** Given $A \subseteq \omega$, there is an ordering $\leq_A$ defined on $\mathbb{H}$ which is stronger than $\leq$. Intuitively, $T' \leq_A T$ iff $T' \leq T$ and the stem of $T'$ does not "hit" $A$ any more than the stem of $T$ already does:

**Definition 5.7.** Let $A \subseteq \omega$. Then given $t, t' \in {}^{<\omega}\omega$, we write $t' \sqsupseteq_A t$ iff $t' \sqsupseteq t$ and

$$(\forall n \in \mathrm{Dom}(t') - \mathrm{Dom}(t))\, t'(n) \notin A$$

**Definition 5.8.** Let $A \subseteq \omega$. Given $T, T' \in \mathbb{H}$, we write $T' \leq_A T$ iff $T' \leq T$ and $\mathrm{Stem}(T') \sqsupseteq_A \mathrm{Stem}(T)$.

The content of the Main Lemma soon to come is that as long as $A$ is evasive with respect to $M$, we can hit dense subsets of $\mathbb{H}$ (that are in $M$) by making $\leq_A$ extensions. So, we can construct a generic without being forced to encode unwanted information. Hence, we can alternate between 1) making $\leq_A$ extensions in order to build an $\mathbb{H}$ generic but not encoding any information and 2) making non-$\leq_A$ extensions to encode information. We use a *rank analysis* to prove the Main Lemma:

**Definition 5.9.** Given $S \subseteq {}^{<\omega}\omega$ and $t \in {}^{<\omega}\omega$,

- $t$ is 0-$S$-reachable iff $t \in S$;
- $t$ is $\alpha$-$S$-reachable for some $\alpha > 0$ iff
$$\{z \in \omega : t^\frown z \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$$
is infinite;
- $t$ is $S$-reachable iff $t$ is $\alpha$-$S$-reachable for some $\alpha$.

Notice that if $t$ is not $S$-reachable, then only a finite set of successors of $t$ can be $S$-reachable.

**Lemma 5.10.** *Let $\mathcal{D} \subseteq \mathbb{H}$ be dense. Let*
$$S = \{s \in {}^{<\omega}\omega : (\exists T \in \mathcal{D})\, \mathit{Stem}(T) = s\}.$$
*Fix $t \in {}^{<\omega}\omega$. Then $t$ is $S$-reachable.*

*Proof.* Assume that some fixed $t$ is not $S$-reachable. We will construct a tree $T \in \mathbb{H}$ with stem $t$ such that no $s \sqsupseteq t$ in $T$ is in $S$. Hence, no $T' \leq T$ can be in $\mathcal{D}$.

There is only a finite set of $z \in \omega$ such that $t^\frown z$ is $S$-reachable. Let the successors of $t$ in $T$ be those $t^\frown z$ that are not $S$-reachable. Now for each $t^\frown z_0$ in $T$, there is only a finite set of $z \in \omega$ such that $t^\frown z_0^\frown z$ is $S$-reachable. Let the successors of each $t^\frown z_0$ in $T$ be those $t^\frown z_0^\frown z$ that are not $S$-reachable. Continuing this procedure $\omega$ times yields a tree $T$ such that all $s \sqsupseteq t$ in $T$ are not $S$-reachable. In particular no $s \sqsupseteq t$ in $T$ is in $S$. □

**Lemma 5.11** (Main Lemma)**.** *Let $M$ be a transitive model of ZF. Let $A \subseteq \omega$ be evasive with respect to $M$. Let $\mathbb{P} = \mathbb{H}^M$. Let $\mathcal{D} \in \mathcal{P}^M(\mathbb{P})$ be open dense (in $M$). Let $T \in \mathbb{P}$. Then there exists some $T' \leq_A T$ in $\mathcal{D}$.*

*Proof.* Let $t = \mathrm{Stem}(T)$. Let
$$S = \{s \in {}^{<\omega}\omega : (\exists T' \in \mathcal{D})\, \mathrm{Stem}(T') = s\}.$$

If we can find a $s \sqsupseteq_A t$ in $T \cap S$, then letting $T' \in \mathcal{D}$ be such that $\mathrm{Stem}(T') = s$ and letting $T'' \le T$ be $T'' = T \restriction s$, then $T' \cap T''$ is in $\mathcal{D}$ (because $\mathcal{D}$ is open), and $\mathrm{Stem}(T' \cap T'') = s$ so $T' \cap T'' \le_A T$. Hence, we will be done.

Now by the previous lemma, fix some ordinal $\alpha$ such that $t$ is $\alpha$-$S$-reachable. If $\alpha = 0$ we are done, so assume $\alpha > 0$. The set

$$B = \{z \in \omega : t^\frown z \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$$

is infinite and in $M$. Since $A$ is evasive with respect to $M$, $B - A$ must be infinite by the Sticking Out Lemma (Lemma 5.3). Thus, we may fix some $z_0 \in (B - A)$ such that $t^\frown z_0 \in T$.

Now $t^\frown z_0$ is $\beta$-$S$-reachable for some fixed $\beta < \alpha$. If $\beta = 0$ we are done, and otherwise we may find some $z_1 \in (B - A)$ such that $t^\frown z_0^\frown z_1 \in T$ and $t^\frown z_0^\frown z_1$ is $\gamma$-$S$-reachable for some $\gamma < \beta$. We may continue like this but eventually we will have some $t^\frown z_0^\frown ...^\frown z_n$ that is in $S$. $\square$

### 5.4. **Proof of Generic Coding with Help Theorem.**

*Proof of Theorem 1.3.* Let $M$ be a transitive model of ZF. Let $\mathbb{P} = \mathbb{H}^M$ and assume $\mathcal{P}^M(\mathbb{P})$ is countable. Let $x \in {}^\omega\omega$. Let $\bar{a} \in {}^\omega\omega - M$. By Fact 5.2, fix $A \subseteq \omega$ be such that $A =_T \bar{a}$ and $A$ is computable from every infinite subset of itself. Then $A$ is evasive with respect to $M$. It suffices to find a $\mathbb{P}$-generic $G$ over $M$ such that $x \le_T A \oplus (\bigcup \bigcap G)$. By Observation 5.6, it suffices to find a $\mathbb{P}$-generic $G$ over $M$ such that $x$ is the real $A$-encoded by $G$.

Since $\mathcal{P}^M(\mathbb{P})$ is countable, let $\langle \mathcal{D}_i : i < \omega \rangle$ be an enumeration of the open dense subsets of $\mathbb{P}$ in $M$. We will construct a decreasing $\omega$-sequence

$$T_0 \ge T_1 \ge ...$$

of $\mathbb{P}$-conditions such that each $T_i \in \mathcal{D}_i$. Hence

$$G := \{T \in \mathbb{P} : (\exists i)\, T \ge T_i\}$$

will be $\mathbb{P}$-generic over $M$. On the other hand, we will construct the sequence of conditions so that $x$ is the real $A$-encoded by $G$.

Since $A$ is evasive with respect to $M$, by Lemma 5.11 (the Main Lemma), let $T_0 \le_A 1_{\mathbb{P}}$ be such that $T_0 \in \mathcal{D}_0$. Now we will encode $x(0)$: let $T_0' \le T_0$ be a non-$\le_A$ extension of $T_0$, extending the stem of $T_0$ by one, such that $\mathrm{Stem}(T_0') = \mathrm{Stem}(T_0)^\frown z$ for a $z \in A$ such that $\eta_A(z) = x(0)$. This is possible because

$$\{z \in A : \eta_A(z) = x(0)\}$$

is infinite, and so must intersect

$$\{z \in \omega : \mathrm{Stem}(T_0)^\frown z \in T_0\}.$$

Next, let $T_1 \leq_A T_0'$ be such that $T_1 \in \mathcal{D}_1$. Then, let $T_1' \leq T_1$ be such that $\mathrm{Stem}(T_1') = \mathrm{Stem}(T_1)^\frown z$ for a $z \in A$ such that $\eta_A(z) = x(1)$.

Continuing this $\omega$ times, we see that $x$ is the real $A$-encoded by $G$. That is, let $g := \bigcup \bigcap G$. The only $n$'s such that $g(n) \in A$ come from when we made non-$\leq_A$ extensions. And, if $n_0 < n_1 < ...$ is the strictly increasing enumeration of these $n$'s, then we see that $\eta_A(n_i) = x(i)$ for each $i$. □

5.5. **Another application of the Main Lemma.** As described in the introduction, here is the original kind of result for which the Main Lemma was created. A proof can be found in [6].

**Proposition 5.12.** *Assume $AD^+$. Fix $a \in {}^\omega\omega$. Then there is a Borel (in fact, Baire class one) function $f_a : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is a function whose graph is disjoint from $f_a$, then*

$$a \in L[C]$$

*where $C \subseteq Ord$ is any $\infty$-Borel code for $g$.*

The function $(a, x) \mapsto f_a(x)$ is Borel as well.

## 6. HOD

By Vopěnka's Theorem, every real is generic over HOD. But one can ask if there is a single $\mathbb{P} \in$ HOD such that $(|\mathbb{P}| \leq 2^\omega)^{\mathrm{HOD}}$ and every real is $\mathbb{P}$-generic over HOD. This is relevant to our paper because by Theorem 4.1, if $\tilde{V}$ is an outer model of $V$ in which $\mathcal{P}^{\mathrm{HOD}^V}(\mathbb{H}^{\mathrm{HOD}^V})$ is countable, and $\bar{a} \in ({}^\omega\omega)^{\tilde{V}} - \mathrm{HOD}^V$ is arbitrary, then for any $x \in ({}^\omega\omega)^{\tilde{V}}$, there is a $G$ that is $\mathbb{H}^{\mathrm{HOD}^V}$-generic over $\mathrm{HOD}^V$ such that $x \in L(\bar{a}, G)$. So the question is whether the $\bar{a}$ can be removed. The answer is no:

**Proposition 6.1.** *It is consistent with ZFC that there is a real $R$ that is not $\mathbb{P}$-generic over HOD for any $\mathbb{P} \in$ HOD such that $(|\mathbb{P}| \leq 2^\omega)^{HOD}$. Moreover, this persists to any outer model of $V$. That is, if $\tilde{V}$ is an outer model of $V$, then $R$ is not $\mathbb{P}$-generic over $HOD^V$ for any $\mathbb{P} \in HOD^V$ such that $(|\mathbb{P}| \leq 2^\omega)^{HOD^V}$.*

*Proof.* Start with $L$. Let $\mathbb{C}_{\omega_2} \in L$ be the forcing to add a Cohen subset of $\omega_2$. Let $A \subseteq \omega_2$ be $\mathbb{C}_{\omega_2}$-generic over $L$. Let $X \subseteq \omega_1$ be generic over $L[A]$ by almost disjoint coding such that $A \in L[X]$. Let $R \subseteq \omega$ be generic over $L[X]$ by almost disjoint coding such that $X \in L[R]$. So now

$$L \subseteq L[A] \subseteq L[X] \subseteq L[R]$$

and $\mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^{L[A]}$ (because $\mathbb{C}_{\omega_2}$ is $<\omega_2$-closed). Let $H = \mathrm{HOD}^{L[R]}$. We will show that $L[R]$ satisfies that $R$ is not generic over $H$ by any forcing of size $(2^\omega)^H$. Moreover, fix any outer model $N$ of $L[R]$. We will show that $N$ satisfies that $R$ is not generic over $H$ by any forcing of size $(2^\omega)^H$.

The forcing $\mathbb{Q}$ to go from $L[A]$ to $L[R]$ is homogeneous [11]. This is subtle, because a three step iteration of almost disjoint coding, to code a subset of $\omega_3$ into a subset of $\omega$, may *not* be homogeneous [11]. Now because $\mathbb{Q} \in L[A]$ is homogeneous, $H \subseteq L[A]$.

Since $L \subseteq H \subseteq L[A]$ and $\mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^{L[A]}$, we have $\omega_1^L = \omega_1^H = \omega_1^{L[A]}$ and $H$ satisfies CH. Suppose towards a contradiction that there is some $\mathbb{P} \in H$ such that $R$ is in a generic extension of $H$ by $\mathbb{P}$ (meaning there is some $G \in N$ that is $\mathbb{P}$-generic over $H$ and $R \in H[G]$) and $\mathbb{P}$ has size $(2^\omega)^H = \omega_1^H$. Then because $\mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^H$, a forcing $\tilde{\mathbb{P}}$ isomorphic to $\mathbb{P}$ is in $L$. Also because $\mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^H$, all dense subsets of $\tilde{\mathbb{P}}$ in $H$ are already in $L$. Let $G \subseteq \tilde{\mathbb{P}}$ in $N$ be $\tilde{\mathbb{P}}$-generic over $L$ such that $R \in L[G]$. Note that this implies $A \in L[G]$.

By a density argument for $\mathbb{C}_{\omega_2}$, the set $A \subseteq \omega_2^L$ has no subset of size $\omega_2^L$ in $L$. On the other hand, let $\dot{A}$ be a $\tilde{\mathbb{P}}$ name for $A$. For each $\alpha \in A$, let $p_\alpha \in G$ be a condition such that $p_\alpha \Vdash \check{\alpha} \in \dot{A}$. Since $(|\tilde{\mathbb{P}}| < \omega_2)^L$, fix a $p \in G$ such that $p = p_\alpha$ for a size $\omega_2^L$ set of $\alpha \in A$. Now the set

$$\{\alpha < \omega_2^L : p \Vdash \check{\alpha} \in \dot{A}\}$$

is a size $\omega_2^L$ subset of $A$ in $L$, which is a contradiction. $\square$

We mentioned in the proof above that the three step iteration of almost disjoint coding to code a subset of $\omega_3$ into a subset of $\omega$ may not be homogeneous. The argument in the proof above also shows us why: start with $V = L$ and let $A \subseteq \omega_3$ be a Cohen subset of $\omega_3$. Let $R \subseteq \omega$ arise from the three step iteration $\mathbb{Q} \in L[A]$ of almost disjoint coding to code $A \subseteq \omega_3$ into a subset of $\omega$. Suppose towards a contradiction that $\mathbb{Q}$ is homogeneous. Then $\mathrm{HOD}^{L[R]} \subseteq L[A]$. By Vopěnka's Theorem, $R$ is generic over $\mathrm{HOD}^{L[R]}$ by a forcing of size $\omega_2$. Since $\mathcal{P}(\omega_2)^L = \mathcal{P}(\omega_2)^{L[A]}$ and $\mathrm{HOD}^{L[R]}$ is intermediate between $L$ and $L[A]$, there must be some $\tilde{\mathbb{P}} \in L$ of size $\omega_2$ such that $R$ is in a $\tilde{\mathbb{P}}$-generic extension $L[G]$ of $L$. But now since $A \in L[G]$ and $|\tilde{\mathbb{P}}| \leq \omega_2$, $A$ has a size $\omega_3$ subset in $L$. This contradicts $A$ being Cohen generic over $L$.

## 7. Questions

### 7.1. **What can replace $\mathbb{H}$?**

**Question 7.1.** Let $M$ be a c.t.m. of ZFC. What are the forcings $\mathbb{P} \in M$ such that every real $a \in {}^{\omega}\omega - M$ is $(\mathbb{P}, M)$-helpful? Does Cohen forcing work? What about a forcing which is ${}^{\omega}\omega$-bounding?

7.2. **Generically coding subsets of $\omega_1$ with help.** Given a transitive model $M$, it is natural to ask whether subsets of $\omega_1$ can be coded by generics over $M$ with help. By Theorem 4.1, this is possible as long as we pass to a sufficiently larger outer model $\tilde{V}$. We suspect that passing to $\tilde{V}$ is not necessary provided that $M$ is large enough. In terms of being large enough, note that given a forcing $\mathbb{P} \in L(\mathbb{R})$, if

- there is a surjection of ${}^{\omega}\omega$ onto $\mathbb{P}$ in $L(\mathbb{R})$,
- $\mathbb{P}$ is countably closed,
- there is a proper class of Woodin cardinals, and
- CH holds,

then there is a $\mathbb{P}$-generic over $L(\mathbb{R})$ in $V$. Here is a proof of this fact (pointed out by Paul Larson): every set of reals in $L(\mathbb{R})$ is the continuous preimage of $\mathbb{R}^{\#}$, so there are at most $2^{\omega}$ sets of reals in $L(\mathbb{R})$. But, because CH holds, there are $\omega_1$ sets of reals in $L(\mathbb{R})$. So there are $\omega_1$ dense subsets of $\mathbb{P}$ in $L(\mathbb{R})$. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all these dense sets. By the fact that $\mathbb{P}$ is countably closed, we can hit all $\omega_1$ dense sets by forming a length $\omega_1$ decreasing sequence through $\mathbb{P}$ (here we also use that ${}^{<\omega_1}\mathbb{P} \subseteq L(\mathbb{R})$).

So, we ask the following (where we have weakened arbitrary help $\bar{a} \in \mathcal{P}(\omega_1) - L(\mathbb{R})$ to some fixed help $\bar{a} \in \mathcal{P}(\mathbb{R})$):

**Question 7.2.** Assume CH and a proper class of Woodin cardinals. Is there some $\bar{a} \subseteq \mathbb{R}$ and some forcing $\mathbb{P} \in L(\mathbb{R})$ that is countably closed such that given any $X \subseteq \omega_1$, there is a $G$ that is $\mathbb{P}$-generic over $L(\mathbb{R})$ such that $X \in L(\bar{a}, G, \mathbb{R})$?

Along similar lines, Woodin has conjectured (Section 10.6 of [10]) that assuming CH and a *measurable* Woodin cardinal, then for any $X \subseteq \omega_1$, there is some $B \subseteq \mathbb{R}$ such that $L(B, \mathbb{R}) \models \mathrm{AD}^+$ and $X \in L(B, \mathbb{R})[G]$ for some $G$ that is $\mathrm{Col}(\omega_1, \mathbb{R})$-generic over $L(\mathbb{R}, B)$.

Assume Woodin's conjecture is true and assume $V$ satisfies CH and has a measurable Woodin cardinal. Let $\mathcal{C}$ be the collection of all inner models of $\mathrm{AD}^+$ containing all the reals. Then every subset of $\omega_1$ (and therefore every subset of $\mathbb{R}$ because we are assuming CH) is generic over some model in $\mathcal{C}$. Our question above asks whether the smallest model in $\mathcal{C}$, namely $L(\mathbb{R})$, is still large enough so that $(\exists \bar{a} \subseteq \mathbb{R})(\exists \mathbb{P} \in L(\mathbb{R}))(\forall X \subseteq \omega_1)(\exists G$ that is $\mathbb{P}$-generic over $L(\mathbb{R}))\, X \in L(\bar{a}, G, \mathbb{R})$.

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