A Null Ideal for Inaccessibles

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Abstract

In this paper we introduce a tree-like forcing notion extending some properties of the random forcing in the context of 2^κ, κ inaccessible, and study its associated ideal of null sets and notion of measurability. In particular, we answer a question of Shelah [10, Problem 0.4], about defining a forcing which is κ^κ-bounding, $< \kappa$-closed and κ^+-cc, for κ inaccessible. This also contributes to a line of research adressed in the survey paper [5].

1 Introduction

In [10, Problem 0.4], Shelah poses the following question.

Problem. Can one define a forcing which is κ^+-cc, $< \kappa$-closed and κ^κ-bounding, for κ inaccessible?

In [9], Shelah was able to provide a positive answer when κ is weakly compact. In this paper we will introduce a forcing notion with those three properties for κ inaccessible and not necessarily weakly compact. For this purpose, we will need a certain version of $\diamondsuit$, modelling our construction on work of Jensen [4] in the case $\kappa = \omega$.
Note that in the standard case, i.e., when $\kappa = \omega$, the random forcing fits with the three properties. So the first attempts to resolve such a problem might be to generalize it for κ inaccessible. Hence, since random forcing is usually defined by means of the Lebesgue measure in 2^ω, the natural way would be to define an appropriate measure in 2^κ as well. Nevertheless, there seem to be many obstacles in trying to do so. In [9], Shelah defined a tree-like forcing without any mention of a measure, and in the end he used Π^1_1-indescribability of a weakly compact cardinal to prove that the forcing is κ^κ-bounding. Our method for defining our forcing $\mathbb{F}$ will be different from Shelah's, yet we also will not use any notion of measure.
Note that even if we will not define a measure on 2^κ, we will define an ideal of $\mathbb{F}$-null sets and a notion of measurability associated to it, in a rather standard way. We will then investigate such a regularity property.
The paper is organized as follows. In section 3 we present the construction of our forcing $\mathbb{F}$. Section 4 is devoted to introducing the ideal of null sets and the notion of measurability, and to proving some (negative) results about $\mathbf{\Delta}^1_1$ sets and the club flter. A final section is then dedicated to some concluding remarks and possible further developments.
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2 Preliminaries

In this preliminary section we simply introduce the basic notions and notation which is needed throughout the paper.

- A tree T is a subset of either $2^{<\kappa}$ or $\kappa^{<\kappa}$, closed under initial segments. $\mathsf{Stem}(T)$ denotes the longest node of T compatible with all the other nodes of T; $\mathsf{succ}(t,T) := \{\xi < \kappa : t^\frown\xi \in T\}$; $\mathsf{Split}(\mathrm{T})$ is the set of splitting nodes of T (i.e., $t \in \mathsf{Split}(T)$ if both $t^\frown 0$ and $t^\frown 1 \in T$); we put $\mathsf{ht}(T) := \sup\{\alpha : \exists t \in T(|t| = \alpha)\}$, while $\textsc{Term}(T)$ denotes the *terminal* nodes of T (i.e., $t \in \textsc{Term}(T)$ if there is no $t' \supsetneq t$ such that $t' \in T$). For $\alpha < \kappa$, $T{\restriction}\alpha := \{t \in T : |t| < \alpha\}$. A *branch* through T of height κ is the limit of an increasing cofinal sequence $\{t_\xi : \xi < \kappa\}$ of nodes in T, and $[T]$ will denote the set of all branches of T. For t in a tree T, T_t is the set of nodes in T compatible with t.

- Given $t \in \mathsf{Split}(T)$, we define the rank of t as the order type of $\{\alpha < |t| : t{\restriction}\alpha \in \mathsf{Split}(T)\}$. Furthermore, we let $\mathsf{Split}_\beta(T)$ denote the set of splitting nodes in T of rank β. The forcing $\mathbb{S}^{\mathsf{Cub}}$ consists of trees T such that every node can be extended to a splitting node and for some club C in κ, the splitting nodes of T are exactly those with length in C. For T, T' in $\mathbb{S}^{\mathsf{Cub}}$ and $\gamma < \kappa$, we write $T \leq_\gamma T'$ iff T is a subtree of T' such that $\mathsf{Split}_\gamma(T) = \mathsf{Split}_\gamma(T')$; $\leq_0$ is simply denoted by $\leq$.

  Note that $\mathbb{S}^{\mathsf{Cub}}$ is closed under $\leq_\gamma$-descending sequences of length less than κ for each $\gamma < \kappa$.

- If $\{\mathbb{F}_\alpha : \alpha < \kappa\}$ is a sequence of families of trees such that $\mathbb{F}_\alpha \subseteq \mathbb{F}_{\alpha+1}$, then for every tree $T \in \bigcup_{\alpha<\kappa} \mathbb{F}_\alpha$ define $\mathsf{Rank}(T)$ to be the least $\alpha < \kappa$ such that $T \in \mathbb{F}_{\alpha+1}$. $\mathbb{F}_{<\lambda}$ denotes the union of the $\mathbb{F}_\alpha$ for $\alpha < \lambda$.

- A forcing $\mathbb{P}$ is called κ^κ-bounding iff for every $x \in \kappa^\kappa \cap V^{\mathbb{P}}$ there exists $z \in \kappa^\kappa \cap V$ such that $\Vdash \forall \alpha < \kappa(x(\alpha) < z(\alpha))$.

- In this paper $\mathbb{C}$ refers to the κ-Cohen forcing, i.e., the poset consisting of $t \in 2^{<\kappa}$, ordered by extension. The elements of 2^κ and κ^κ are called κ-reals.

Under the assumption $2^{<\kappa} = \kappa$, $\mathbb{C}$ is obviously κ^+-cc, but also adds unbounded κ-reals, which means it is not κ^κ-bounding. For κ inaccessible $\mathbb{S}^{\mathsf{Cub}}$ is κ^κ-bounding, but one loses the κ^+-cc. The next section is devoted to defining a refinement of $\mathbb{S}^{\mathsf{Cub}}$ in order to obtain the κ^+-cc and maintain κ^κ-boundedness.

3 The main construction

Fix κ to be inaccessible. As we mentioned before, our first main goal is to define a tree-forcing $\mathbb{F}$ with the following three properties: κ^+-cc, κ^κ-bounding and $<\kappa$-closure. Assume $\diamondsuit_{\kappa^+}(S^{\kappa^+}_\kappa)$, where $S^{\kappa^+}_\kappa := \{\lambda < \kappa^+ : \mathrm{cf}(\lambda) = \kappa\}$.
We construct an increasing sequence of tree forcings $\langle \mathbb{F}_\lambda : \lambda < \kappa^+ \rangle$ by induction on $\lambda < \kappa^+$. We first remark that for all $\lambda < \kappa^+$ we will maintain the following:

(P1) $\mathbb{F}_\lambda \subseteq \mathbb{S}^{\mathsf{Cub}}$ and $|\mathbb{F}_\lambda| \leq \kappa$;

(P2) $\forall T \in \mathbb{F}_{<\lambda} \forall \gamma < \kappa \exists T' \leq_\gamma T \forall T'' \leq T' (T' \in \mathbb{F}_\lambda \wedge T'' \notin \mathbb{F}_{<\lambda})$;

(P3) $\forall T \in \mathbb{F}_\lambda \forall t \in T (T_t \in \mathbb{F}_\lambda)$;

(P4) $\mathbb{F}_\lambda$ is closed under descending $< \kappa$-sequences;

(P5) $\forall \alpha < \lambda \forall T \in \mathbb{F}_\lambda \setminus \mathbb{F}_\alpha \exists \bar\gamma < \kappa \forall \gamma \geq \bar\gamma \forall t \in \mathsf{Split}_\gamma(T) \exists S \in \mathbb{F}_\alpha \setminus \mathbb{F}_{<\alpha} (T_t \subseteq S)$.

We remark that in P2 the property we are really interested in is P2bis: $\forall T \in \mathbb{F}_{<\lambda} \forall \gamma < \kappa \exists T' \leq_\gamma T (T' \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda})$; the extra requirement on all $T'' \leq T'$ is only needed to make sure that such a property will be preserved in our recursive construction.
Furthermore, P5 will be used to help ensure the κ^+-cc.

Let $\{D_\lambda : \lambda < \kappa^+\}$ be a $\diamondsuit_{\kappa^+}(S^\kappa_{\kappa^+})$-sequence. The recursive construction is developed as follows:

1. $\mathbb{F}_0 := \{(2^{<\kappa})_t : t \in 2^{<\kappa}\}$.

2. Case $\lambda + 1$: For every $T \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, pick $T' \in \mathbb{S}^{\mathsf{Cub}}$ such that $T' \leq_\gamma T$ and T' does not contain subtrees in $\mathbb{F}_\lambda$; this is possible as $\mathbb{F}_\lambda$ as cardinality κ. Then for all $t \in T'$ we add T'_t to $\mathbb{F}_{\lambda+1}$. We then close $\mathbb{F}_{\lambda+1}$ under descending $< \kappa$-sequences, i.e., for every descending $\{T^i : i < \delta\}$ in $\mathbb{F}_{\lambda+1}$, with $\delta < \kappa$, we put $T^* := \bigcap_{i<\delta} T^i$ into $\mathbb{F}_{\lambda+1}$.

3. Case $\mathrm{cf}(\lambda) < \kappa$: let $\{T^i : i < \mathrm{cf}(\lambda)\} \subseteq \mathbb{F}_{<\lambda}$ be descending with $\{\mathsf{Rank}(T^i) : i < \mathrm{cf}(\lambda)\}$ cofinal in λ.

   Then put $T^* := \bigcap_{i<\mathrm{cf}(\lambda)} T^i$ into $\mathbb{F}_\lambda$. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

4. Case $\mathrm{cf}(\lambda) = \kappa$, where $(\lambda_i : i < \kappa)$ is increasing and cofinal in λ:

   (a) Suppose $D_\lambda \subseteq \lambda$ codes a maximal antichain A_λ in $\mathbb{F}_{<\lambda}$. For every $T \in \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, construct a "κ-fusion" sequence $\{T^i : i < \kappa\}$ of trees in $\mathbb{S}^{\mathsf{Cub}}$ such that

      i. $T =: T^0 \geq_\gamma T^1 \geq_{\gamma+1} T^2 \geq_{\gamma+2} \cdots \geq_{\gamma+i} T^{i+1} \geq_{\gamma+i+1} \dots$
      ii. T^i_t belongs to $\mathbb{F}_{<\lambda}$ with $\mathsf{Rank}(T^i_t)$ at least λ_i for each t in $\mathsf{Split}_\gamma(T)$.
      iii. $T^1 := \bigcup\{S_t : t \in \mathsf{Split}_\gamma(T)\}$, where each $S_t \leq T_t$ and S_t hits A_λ, i.e., there exists $S^* \in A_\lambda$ such that $S_t \leq S^*$.

      Then add $T^* := \bigcap_{i<\kappa} T^i$ to $\mathbb{F}_\lambda$. Moreover, for every $t \in T^*$, add T^*_t to $\mathbb{F}_\lambda$ too. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

   (b) Suppose that $D_\lambda \subseteq \lambda$ codes $\{A_{i,j} : i < \kappa, j < \kappa\}$, where for each $i < \kappa$, $\bigcup_{j<\kappa} A_{i,j}$ is a maximal antichain in $\mathbb{F}_{<\lambda}$ and $j_0 \neq j_1 \Rightarrow A_{i,j_0} \cap A_{i,j_1} = \emptyset$. For every $T \in \mathbb{F}_{<\lambda}$ and $\gamma < \kappa$, build a κ-fusion sequence $\{T^i : i < \kappa\}$ of trees in $\mathbb{S}^{\mathsf{Cub}}$ such that

      i. $T =: T^0 \geq_\gamma T^1 \geq_{\gamma+1} T^2 \geq_{\gamma+2} \cdots \geq_{\gamma+i} T^{i+1} \geq_{\gamma+i+1} \dots$
      ii. T^i_t belongs to $\mathbb{F}_{<\lambda}$ with $\mathsf{Rank}(T^i_t)$ at least λ_i for t in $\mathsf{Split}_{\gamma+i}(T^i)$.
      iii. for every $i < \kappa$, $T^{i+1} := \bigcup\{S^{i+1}_t : t \in \mathsf{Split}_{\gamma+i}(T^i)\}$, where each $S^{i+1}_t \leq T^i_t$ and S^{i+1}_t hits $\bigcup_{j<\kappa} A_{i,j}$.

      Then add $T^* := \bigcap_{i<\kappa} T^i$ to $\mathbb{F}_\lambda$. Moreover, for every $t \in T^*$, add T^*_t to $\mathbb{F}^*_\lambda$ too. Finally close $\mathbb{F}_\lambda$ under descending $< \kappa$-sequences.

   (c) If D_λ neither codes a maximal antichain (case (a)) nor an instance of κ^κ-bounding (case (b)), then proceed as in case (a) without its item iii.

Finally let $\mathbb{F} := \bigcup_{\lambda<\kappa^+} \mathbb{F}_\lambda$.

Proposition 1. *The construction of $\mathbb{F}$ satisfies the five properties P1-P5.*

Proof. P1 is clear, since at any stage we only add κ many new trees which are in $\mathbb{S}^{\mathsf{Cub}}$. Also P3 and P4 follow immediately from the construction.
For P2, note that the successor case $\lambda+1$ follows easily from the construction; for $\mathrm{cf}(\lambda) < \kappa$, start with $T \in \mathbb{F}_{<\lambda}$ and use induction to build a descending $\leq_\gamma$-descending sequence $\{T^i : i < \mathrm{cf}(\lambda)\}$ such that $\{\mathsf{Rank}(T^i) : i < \mathrm{cf}(\lambda)\}$ is cofinal in λ and put $T^* := \bigcap_{i<\mathrm{cf}(\lambda)} T^i$. We may additionally require that T^i contains no subtree in $\mathbb{F}_{<\mathsf{Rank}(T^i)}$ and so T^* contains no subtree in $\mathbb{F}_{<\lambda}$ (in particular, T^* does not belong to $\mathbb{F}_{<\lambda}$); finally for the case $\mathrm{cf}(\lambda) = \kappa$ we argue similarly, using the fact that we take fusion sequences with tree-ranks cofinal in λ.
For P5, we distinguish again the three different situations. In the successor case $\lambda+1$, we can have: case 1) $\alpha < \lambda$, so simply pick some $T_0 \supseteq T$ in $\mathbb{F}_\lambda$ and use the inductive hypothesis; case 2) $\alpha = \lambda$, so pick T_0 as in case 1) and use it as the S needed to satisfy P5. In case $\mathrm{cf}(\lambda) < \kappa$, as above start with $T \in \mathbb{F}_{<\lambda}$ and use induction to build a descending γ-sequence of length $\mathrm{cf}(\lambda)$ such that $\{\mathsf{Rank}(T^i) : i < \mathrm{cf}(\lambda)\}$ is cofinal in λ and put $T^* := \bigcap_{i<\mathrm{cf}(\lambda)} T^i$. Let $\alpha < \lambda$ and for $i < \mathrm{cf}(\lambda)$ such that $\alpha < \mathsf{Rank}(T^i)$ choose γ_i sufficiently large so that for all $t \in \mathsf{Split}_{\geq\gamma_i}(T^i)$ one has T^i_t is contained in a tree of $\mathbb{F}_\alpha \setminus \mathbb{F}_{<\alpha}$; then, if $\gamma^* := \sup\{\gamma_i : i < \mathrm{cf}(\lambda)\}$, for every $t \in \mathsf{Split}_{\geq\gamma^*}(T)$ one has T_t is contained in a tree in $\mathbb{F}_\alpha \setminus \mathbb{F}_{<\alpha}$. The case $\mathrm{cf}(\lambda) = \kappa$ is treated similarly, by using a sequence with tree-ranks cofinal in λ. $\square$

Proposition 2. *$\mathbb{F}$ is $<\kappa$-closed, κ^+-cc and κ^κ-bounding.*

Proof. The $<\kappa$-closure follows from point 3 of the construction.
To prove κ^+-cc we argue as follows. Let $A \subseteq \mathbb{F}$ be a maximal antichain and pick λ such that $\mathrm{cf}(\lambda) = \kappa$ and $A \cap \mathbb{F}_{<\lambda}$ is coded by D_λ, using $\Diamond_{\kappa^+}(S^\kappa_{\kappa^+})$. By 4.(a) of the construction, for every $T \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$, there is γ' such that for every $\gamma \geq \gamma'$ for every $t \in \mathsf{Split}_\gamma(T)$, T_t is a subtree of some element of $A \cap \mathbb{F}_{<\lambda}$. By P5, if $T \in \mathbb{F} \setminus \mathbb{F}_\lambda$, there is $\gamma'' \leq \gamma'$ such that for every $\gamma \geq \gamma''$ for every $t \in \mathsf{Split}_\gamma(T)$, T_t is a subtree of some element of $\mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$. It follows that for any $T \in \mathbb{F}_\lambda \setminus \mathbb{F}_{<\lambda}$ there is $t \in T$ such that T_t is a subtree of some element of $A \cap \mathbb{F}_{<\lambda}$, and therefore $A \cap \mathbb{F}_{<\lambda}$ is a maximal antichain in $\mathbb{F}$. So $A \cap \mathbb{F}_{<\lambda} = A$, which finishes the proof as $|\mathbb{F}_{<\lambda}| = \kappa$.
For κ^κ-bounding we argue as follows. Let $\dot{x}$ be an $\mathbb{F}$-name for an element of κ^κ and $T \in \mathbb{F}$. Choose $\{A_{ij} : i < \kappa, j < \kappa\}$ so that for each $i < \kappa$, $\bigcup_{j<\kappa} A_{ij}$ is a maximal antichain and elements of A_{ij} force $\dot{x}(i) = j$. Pick $\lambda < \kappa$ such that T belongs to $\mathbb{F}_{<\lambda}$, $\mathrm{cf}(\lambda) = \kappa$ and D_λ codes such a sequence of antichains. By 4.(b) of the construction, we can then build a κ-fusion sequence in order to get $T' \leq T$ such that for each $i < \kappa$, T' forces the generic to hit $\bigcup_{j\in J_i} A_{ij}$, where each $J_i \subseteq \kappa$ has size $\leq 2^i$. Define $z \in \kappa^\kappa \cap V$ by $z(i) = \sup J_i$; then $T' \Vdash \forall i < \kappa, \dot{x}(i) \leq z(i)$.

$\square$

4 Ideal and measurability

Once we have a tree forcing notion we can introduce a related ideal of *small* sets.

Definition 3. A set $X \subseteq 2^\kappa$ is said to be $\mathbb{F}$-null iff for all $T \in \mathbb{F}$ there exists $T' \in \mathbb{F}$, $T' \leq T$ such that $[T'] \cap X = \emptyset$. Further let $\mathcal{I}_\mathbb{F}$ be the ideal consisting of all $\mathbb{F}$-null
sets. A set is $\mathbb{F}$-conull if its complement is in $\mathcal{I}_\mathbb{F}$.

Remark 4. $\mathcal{I}_\mathbb{F}$ is a κ^+-ideal; let $\{X_\alpha : \alpha < \kappa\}$ be a sequence of $\mathbb{F}$-null sets, and fix $T \in \mathbb{F}$. Build a κ-fusion sequence $\{T_\alpha : \alpha < \kappa\}$ such that for all $\alpha < \kappa$, for all $\beta \leq \alpha$, $[X_\beta] \cap [T_\alpha] = \emptyset$. Then $T' := \bigcap_{\alpha<\kappa} T_\alpha$ has the desired property.

One of the main properties of the null ideal in the standard framework is that of being *orthogonal* to the meager ideal, i.e., the space can be partitioned into a meager piece and a null piece. We now prove that the same holds for $\mathcal{I}_\mathbb{F}$.

Proposition 5. *There is $X \subseteq 2^\kappa$ such that $X \in \mathcal{M}$ and $2^\kappa \setminus X \in \mathcal{I}_\mathbb{F}$.*

Proof. Let $A := \{A_i : i < \kappa\}$ be a maximal antichain in $\mathbb{F}$. Clearly, $X := \bigcup_{i<\kappa}[A_i]$ is $\mathbb{F}$-conull, since for every $T \in \mathbb{F}$, there is $i < \kappa$ such that $A_i \parallel T$, and so there is $T' \leq A_i$ such that $T' \leq T$. It is then sufficient to show that we can find such an antichain A with the further property that any $[A_i]$ is nowhere dense. But note that by property P2, any $T \in \mathbb{F}$ can be extended to contain no subtree of the form $(2^{<\kappa})_s$ for $s \in 2^{<\kappa}$ and $[T]$ is nowhere dense for such a tree T.
Now let $\mathbb{F}^* \subseteq \mathbb{F}$ be the dense set of such trees, and pick A a maximal antichain in $\mathbb{F}^*$. Then A remains a maximal antichain in $\mathbb{F}$ as well, and it is then enough for our purpose.

□

Measurability. There are essentially two possible notions of regularity related to $\mathbb{F}$.

Definition 6. A set $X \subseteq 2^\kappa$ is said to be:

1. $\mathbb{F}$-*measurable* iff for every $T \in \mathbb{F}$ there exists $T' \in \mathbb{F}$, $T' \leq T$ such that $[T'] \setminus X \in \mathcal{I}_\mathbb{F}$ or $X \cap [T'] \in \mathcal{I}_\mathbb{F}$.
2. $\mathbb{F}$-*regular* iff there exists $B \in \mathsf{Bor}$ such that $X \triangle B \in \mathcal{I}_\mathbb{F}$.

Concerning definition 6.1, we could equivalently require "$[T'] \subseteq X$ or $[T'] \cap X = \emptyset$", as $\mathcal{I}_\mathbb{F}$ is a κ^+-ideal.

Proposition 7. *Let $X \subseteq 2^\kappa$.*
X is $\mathbb{F}$-measurable iff X is $\mathbb{F}$-regular.

Proof. The proof is just as the general case of $\mathbb{P}$-measurability in the standard case. We give it here for completeness.
$\Rightarrow$: by assumption, the set $E := \{T \in \mathbb{F} : [T] \cap X \in \mathcal{I}_\mathbb{F} \vee [T] \cap X^c \in \mathcal{I}_\mathbb{F}\}$ is dense in $\mathbb{F}$. Then pick a maximal antichain A in E and put

$$B := \bigcup \{[T] : T \in A \wedge [T] \cap X^c \in \mathcal{I}_\mathbb{F}\}.$$

Note that B is Borel ($\mathbf{\Sigma}^0_2$), since $|A| \leq \kappa$. We claim that $X \triangle B \in \mathcal{I}_\mathbb{F}$. Indeed, for every $T \in \mathbb{F}$ we have to cases: there is $S \in A$ such that $[S] \cap X \in \mathcal{I}_\mathbb{F}$, and so we find $S_0 \subseteq S$ such that $[S_0] \cap (X \triangle B) \subseteq [S_0] \cap X \in \mathcal{I}_\mathbb{F}$, where the inclusion holds as $[S] \cap B = \emptyset$; otherwise there is $S \in A$ such that $[S] \cap X^c \in \mathcal{I}_\mathbb{F}$ and so we find $S_0 \subseteq S$ such that $[S_0] \cap (X \triangle B) \subseteq [S_0] \cap X^c \in \mathcal{I}_\mathbb{F}$, where the inclusion holds as $[S] \subseteq B$.

$\Leftarrow$: by assumption, it is enough to show that any Borel set is $\mathbb{F}$-measurable. We do that by induction on the Borel hierarchy. By a straightforward generalization of a result of Brendle and Löwe (see [1]), proving that all Borel sets are $\mathbb{F}$-measurable is equivalent to proving that for every Borel set B there is $T \in \mathbb{F}$ such that $[T] \subseteq B$ or $[T] \cap B = \emptyset$. For $s \in 2^{<\kappa}$, $[s]$ is trivially $\mathbb{F}$-measurable, and if B is $\mathbb{F}$-measurable, then by symmetry B^c is $\mathbb{F}$-measurable too. Finally, if B is the union of $\leq \kappa$ many Borel sets $\{C_\alpha : \alpha < \delta\}$, with $\delta \leq \kappa$, then we have two cases: there is $\alpha < \delta$ and $T \in \mathbb{F}$ such that $[T] \subseteq C_\alpha \subseteq B$; or for all $\alpha < \delta$, $C_\alpha \in \mathcal{I}_\mathbb{F}$, and so $B \in \mathcal{I}_\mathbb{F}$ as well.

□

In [6] and [8] it was shown that for some tree forcing notions one can force all projective sets to be measurable. Nevertheless this is not the case for $\mathbb{F}$. Indeed, next result shows that there is no hope for $\mathbf{\Sigma}^1_1(\mathbb{F})$ to be consistent.

Proposition 8. *The club filter* Cub *is not* $\mathbb{F}$*-measurable.*

Proof. Standard. Given an arbitrary $T \in \mathbb{F}$ it suffices to show that $[T]$ contains branches both in Cub and in NS. We argue as follows:

- let $t_0 = \mathsf{Stem}(T)$
- for $\alpha < \kappa$ successor, pick $t_\alpha \supset t_{\alpha-1}{}^\frown 1$ such that $t_\alpha \in \mathsf{Split}(T)$ and $|t_\alpha|$ is a limit ordinal.
- for $\alpha < \kappa$ limit, pick $t_\alpha \supset (\bigcup_{\xi<\alpha} t_\xi)^\frown 1$ such that $t_\alpha \in \mathsf{Split}(T)$ and $|t_\alpha|$ is a limit ordinal.

Finally put $x := \bigcup_{\alpha<\lambda} t_\alpha$. Clearly, $x \in [T] \cap \mathsf{Cub}$.
Analogously, if in the previous choices of the t_α's we replace 1 by 0, we get $x \in [T] \cap \mathsf{NS}$.

□

We conclude by remarking that standard constrcution shows that $\mathbf{\Delta}^1_1(\mathbb{F})$ is consistently false (e.g., in $V = L$).

5 Concluding remarks and open questions

It would be interesting to prove that $\mathbf{\Delta}^1_1(\mathbb{F})$ is consistently true. The usual way for forcing such a statement for a tree-forcing $\mathbb{P}$ is to take a κ^+-iteration of $\mathbb{P}$ with κ-support. The main point is to make sure that κ^+ is preserved. For our forcing $\mathbb{F}$ it is not clear whether it is the case. So we leave the following as an open question.

Question. Does a κ^+-iteration of $\mathbb{F}$ with κ-support preserve κ^+? Or, can we modify $\mathbb{F}$ is order to let the latter work?

What remains also open is the last part of [5, Question 3.1].

Question. Can one define a tree forcing that is $< \kappa$-closed, κ^κ-bounding and κ^+-cc, for κ successor?

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