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INADMISSIBLE RECURSION THEORY

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Let β be a limit ordinal, and let S_β be stage β of Jensen's S hierachy for L (cf. [1, p. 82]). S_β is the setting for β-recursion theory, an extension of recursion theory on Σ_1 admissible, initial segments of L (so-called α-recursion theory) to rudimentarily closed, initial segments of L. A set $A \subset S_\beta$ is said to be β-recursively enumerable (β-r.e.) if it is Σ_1 definable over S_β. If β is Σ_1 admissible, i.e. S_β satisfies Σ_1 replacement, then many theorems about ordinary r.e. sets ($\beta = \omega$) remain true when r.e. is replaced by β-r.e. (cf. [3], [5] and [6]). The results below, particularly the solution to Post's problem devised by Friedman in Theorem 3, suggest that the assumption of Σ_1 admissibility is superfluous. This outcome is in the natural order of events, because arguments in the Σ_1 admissible case often consist of showing that the use of Σ_2 or Σ_3 admissibility when $\beta = \omega$ was unnecessary. It is a desirable outcome in that it implies that the methods of ordinary recursion theory can be applied to all levels of the J hierarchy for L (cf. [1]).

The definitions in this paragraph are drawn from α-recursion theory. A is β-recursive (β-rec.) if A and $S_\beta - A$ are β-r.e. A function f is β-rec. if its graph is. x is β-finite if $x \in S_\beta$. Let $F(e, x)$ be a Σ_1 formula such that $\{F(e, x) \mid e \in S_\beta\}$ is a list of all Σ_1 formulas with free variable x and parameters in S_β. For any $A \subset S_\beta$, $\{e\}^A(x)$ has y as a value if S_β satisfies

$$(Eu)(Ev)\,[u \subset A \;\&\; v \subset S_\beta - A \;\&\; F(e, \langle x, y, u, v\rangle)].$$

f is weakly β-recursive in A ($f \leqslant_{w\beta} A$) if for some e and all x, $\{e\}^A(x) = f(x)$. $B \leqslant_{w\beta} A$ if its characteristic function $C_B \leqslant_{w\beta} A$. Let B^* be the set of all β-finite $x \subset B$. B is β-recursive in A ($B \leqslant_\beta A$) if $B^* \leqslant_{w\beta} A$ and $(S_\beta - B)^* \leqslant_{w\beta} A$.

Suppose A is nonempty and β-r.e. In the admissible case an enumeration of A is a β-rec. map f of β onto A. This definition is unsuitable in the inadmissible case, because there may be a $\delta < \beta$ such that $f[\delta]$, the range of f restricted to δ, is unbounded in S_β. A notion of enumeration which suits both cases is as follows. Let $G(x, y)$ be a Δ_0 formula such that $x \in A$ iff $S_\beta \models (Ey)G(x, y)$. Let A^δ be the set of all $x \in S_\delta$ such that $S_\delta \models (Ey)G(x, y)$. Then $\{A^\delta \mid \delta < \beta\}$ is an enumeration of A, namely a nondecreasing, β-recursive sequence (of length β) of β-finite sets whose union is A.

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A is said to be tamely β-r.e. (t.r.e.) if there is an enumeration $\{A^\delta\}$ of A such that $(x)(E\delta)$ $[x \in S_\beta \;\&\; x \subset A \longrightarrow x \subset A^\delta]$. T.r.e. sets are amenable to some of the arguments of admissible recursion theory. Unfortunately there are many β's such that every t.r.e. set is β-recursive in 0, the empty set.

THEOREM 1. *Assume β is not Σ_1 admissible. Let C be the complete β-r.e. set. Then there exists a β-rec. A such that $0 <_\beta A <_\beta C$, $C \leqslant_{w\beta} A$, and every set which is either β-rec. or t.r.e. is β-rec. in A.*

The Σ_1 cofinality of β ($\sigma 1\mathrm{cf}(\beta)$) is the least γ such that $\beta = \bigcup f[\gamma]$ for some β-rec. f. The Σ_1 projectum of β ($\sigma 1 p(\beta)$) is the least γ such that some one-one β-rec. f maps β into γ. A is regular if $A \cap x \in S_\beta$ for all $x \in S_\beta$. γ is β-recursively regular if there is no β-rec. f such that $\gamma = \bigcup f[\delta]$ for some $\delta < \gamma$.

THEOREM 2. *Assume $\sigma 1 cf(\beta) \geqslant \sigma 1 p(\beta)$. Then there exist two regular, tamely β-r.e. sets such that neither is weakly β-recursive in the other.*

THEOREM 3 (S. FRIEDMAN [2]). *Assume $\sigma 1 p(\beta)$ is β-recursively regular. Then there exist two β-r.e. subsets of $\sigma 1 p(\beta)$ such that neither is weakly β-recursive in the other.*

The proof of Theorem 2 is closely tied to ideas associated with admissible recursion theory. Its converse has been proved by W. Maass [4]. The proof of Theorem 3 has several features with no antecedents in the admissible case, the most notable being the use of a β-recursive version of Jensen's $\diamondsuit$ principle [1, p. 48] to overcome the fact that the β-r.e. sets constructed cannot be required to be tame.

REFERENCES

1. K. J. Devlin, *Aspects of constructibility*, Lecture Notes in Math., vol. 354, Springer-Verlag, Berlin and New York, 1973. MR 51 #12527.

2. S. D. Friedman, *Recursion on inadmissible ordinals*, Ph. D. Thesis, Massachusetts Institute of Technology, 1976.

3. M. Lerman, *On suborderings of the α-recursively enumerable α-degrees*, Ann. Math. Logic 4 (1972), 415–443. MR 48 #5835.

4. W. Maass, *Inadmissibility, tamely* RE *sets and the admissible collapse*, Ann. Math. Logic (to appear).

5. G. E. Sacks and S. G. Simpson, *The α-finite injury method*, Ann. Math. Logic 4 (1972), 343–368. MR 51 #5277.

6. R. Shore, *The recursively enumerable α-degrees are dense*, Ann. Math. Logic 9 (1976), 123–155.

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