FUNDAMENTA
MATHEMATICAE
Online First version

On the complexity of classes of uncountable structures: trees on $\aleph_1$

by

Sy-David Friedman and **Dániel T. Soukup** (Wien)

Abstract. We analyse the complexity of the class of (special) Aronszajn, Suslin and Kurepa trees in the projective hierarchy of the higher Baire space $\omega_1^{\omega_1}$. First, we show that none of these classes have the Baire property (unless they are empty). Moreover, under $V = L$, (a) the class of Aronszajn and Suslin trees is Π_1^1-complete, (b) the class of special Aronszajn trees is Σ_1^1-complete, and (c) the class of Kurepa trees is Π_2^1-complete. We achieve these results by finding nicely definable reductions that map subsets X of ω_1 to trees T_X so that T_X is in a given tree class $\mathcal{T}$ if and only if X is stationary/non-stationary (depending on the class $\mathcal{T}$). Finally, we present models of CH where these classes have lower projective complexity.

1. Introduction. We set out to investigate the complexity of certain well-studied classes of $\aleph_1$-trees on ω_1. In particular, under various set-theoretic assumptions, we determine the Borel/projective complexity of the class of Aronszajn, Suslin and Kurepa trees as a subset of the higher Baire space $\omega_1^{\omega_1}$. In several cases, we prove the existence of nicely definable reductions between these tree classes and the stationarity relation on $\mathcal{P}(\omega_1)$. As the latter is Π_1^1-complete under $V = L$, we get the parallel completeness of the tree classes. In the case of Kurepa trees, we use a different coding argument.

The general setting of our paper is *the higher Baire space* ([1]) on $\omega_1^{\omega_1}$ and 2^{ω_1}. Basic open sets correspond to countable partial functions, which, in turn, give rise to an ω_1-Borel structure on $\omega_1^{\omega_1}$, 2^{ω_1} and so $\mathcal{P}(\omega_1)$ as well. This allows us to measure the complexity of subsets of $\omega_1^{\omega_1}$ or, equivalently, of families of natural combinatorial structures on ω_1. In this paper, we will focus on models of CH, i.e., $2^{\aleph_0} = \aleph_1$. This is a fairly natural assumption in

2020 *Mathematics Subject Classification*: 03E15, 03E35.
Key words and phrases: Aronszajn tree, Suslin tree, Kurepa tree, Borel, projective, definable, reduction.
Received 4 June 2019; revised 8 April 2020.
Published online *.

([1]) The name "generalized Baire space" is also commonly used.

this higher Baire setting which, in particular, ensures that $\omega_1^{\omega_1}$ has a basis of size $\aleph_1$. This, of course, is analogous to the standard Baire space ω^ω having a countable basis.

Our primary interest lies in the set of $\aleph_1$-*trees*: partial orders $T = (\omega_1, <_T)$ on ω_1 such that (1) the set of predecessors of each node is well-ordered ([2]), and (2) for any ordinal ξ, the set of nodes T_ξ of height ξ is countable and non-empty if $\xi < \omega_1$ and empty otherwise.

Following [17], we will use TO to denote the class of trees without uncountable branches; so we allow trees with uncountable levels here. A tree T in TO is called *Aronszajn* if T is also an $\aleph_1$-tree (i.e., all levels are countable). We call T a *Suslin tree* if it is an Aronszajn tree without uncountable antichains. On the other hand, an $\aleph_1$-tree T is *Kurepa* if it has at least $\aleph_2$ uncountable branches. We will denote these classes of trees by AT, ST and KT, respectively. An Aronszajn tree is *special* if it is the union of countably many antichains. The latter collection will be denoted by sAT. These are the main classes of trees we will be analysing in detail. Let us refer the reader to the classical set theory textbooks [11, 14] and to [23] for a nice introduction to trees of height $\aleph_1$; the latter survey emphasizes the connection of trees to topology and linear orders.

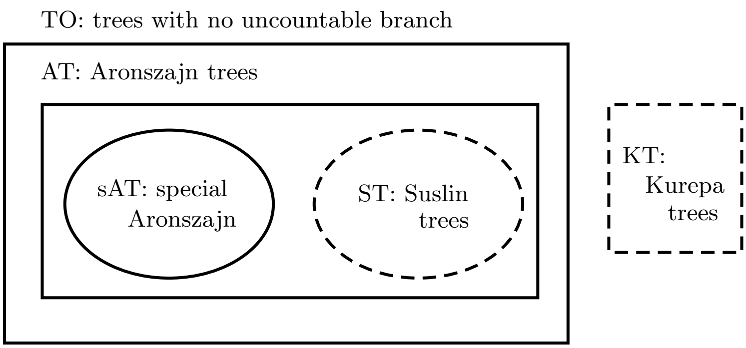

Fig. 1. Classes of trees of height ω_1

Recall that special Aronszajn trees exist in ZFC, however, even assuming the Continuum Hypothesis, ST and KT may be empty. In fact, as proved by R. Jensen, CH is consistent with AT = sAT [5] and so ST = $\emptyset$ in this model ([3]). The consistency of no Kurepa trees was proved by Silver [21].

On the other hand, under $V = L$ (or just assuming strong enough diamonds), both Suslin and Kurepa trees exist [11].

Let us present some results that will place our paper in the context of past research. Trees have played a significant role in the study of both the

([2]) This allows us to define a height function on T and the levels T_ξ of T.

([3]) We mention that AT $\neq$ sAT does not imply the existence of Suslin trees [19].

standard and higher Baire space [6, 12, 16, 17, 24]. Recall that the set of trees on ω without infinite branches is complete co-analytic. Analogously, a classical result from higher descriptive set theory is the following theorem of A. Mekler and J. Väänänen.

THEOREM 1.1 ([17]). *CH implies that* TO*, the set of all trees on* ω_1 *without uncountable branches, is* Π^1_1*-complete.*

We mention that TO is Σ^1_1 if $\mathrm{MA}_{\aleph_1}$ holds; indeed, $\mathrm{MA}_{\aleph_1}$ implies that TO is exactly the set of special trees on ω_1 [3] which is easily verified as a Σ^1_1 definition.

Yet another subclass of TO is the following: a *canary tree* T is a tree of size continuum with no uncountable branches and with the property that in any extension W of the universe V with $\mathbb{R}^V = \mathbb{R}^W$, if a stationary set of V is no longer stationary in W then T has an uncountable branch in W. Now, canary trees give a simple definition for a subset of ω_1 to be stationary.

THEOREM 1.2 ([10, 16]). *There is a canary tree iff* $\mathrm{Stat} \subset \omega_1^{\omega_1}$*, the set of all stationary subsets of* ω_1*, is* Σ^1_1*. Moreover, the existence of canary trees is independent of GCH.*

If $V = L$ then there are no canary trees, and in fact the following polar opposite result holds, which appears implicitly in [7].

THEOREM 1.3 ([7]). *If* $V = L$ *then* Stat *is* Π^1_1*-complete.*

We will use this theorem to show that certain classes of trees are complete in their complexity class. Finally, let us mention that there are strong connections between infinitary logic, trees and the complexity questions that our paper is concerned with [7, 20, 24]. We only included the results most relevant for our study but we would like to refer the reader to the survey [24] and the book [8] for more details.

First, we will show that none of the classes AT, sAT, ST and KT have the Baire property and hence they are non-Borel (unless ST and KT are empty, in which case they are trivially Borel). Moreover, we will prove the following results about the complexity of these classes:

	AT	sAT	ST	KT
ZFC	$\Pi^1_1 \setminus$ Borel	$\Sigma^1_1 \setminus$ Borel	$\Pi^1_1 \setminus$ Borel or $\emptyset$	$\Pi^1_2 \setminus$ Borel or $\emptyset$
$V = L$	Π^1_1-complete	Σ^1_1-complete	Π^1_1-complete	Π^1_2-complete
Abraham–Shelah model	$\Delta^1_1 \setminus$ Borel	$\Delta^1_1 \setminus$ Borel	$\Delta^1_1 \setminus$ Borel	?
$\mathrm{MA}_{\aleph_1}$ or $\mathrm{PFA}(S)[S]$	$\Delta^1_1 \setminus$ Borel	$\Delta^1_1 \setminus$ Borel	$\emptyset$	?

Fig. 2. A summary of complexity

Some of these results are easy consequences of known theorems (such as the results regarding the Abraham–Shelah model which we will describe shortly). However, the completeness of the classes under $V = L$ requires significant work and new ideas. We present the ZFC results and facts about the Abraham–Shelah model in Section 2. Then, in Section 3, we will show that stationarity can be reduced (in a Borel way) to the classes AT and ST. The results on $V = L$ will follow easily then. Finally, we deal with Kurepa trees in Section 4. Note that under $\mathrm{MA}_{\aleph_1}$ and $\mathrm{PFA}(S)[S]$, AT = sAT so the last line of Figure 2 follows by the definitions [4]. We end our paper with some remarks and open problems in Section 5.

1.1. Preliminaries. We defined the tree classes already but let us review the most important descriptive-set-theoretic notions that we need. The family of *Borel sets* in $\omega_1^{\omega_1}$ is the smallest family containing all open sets which is closed under taking complements and unions/intersections of size $\aleph_1$. It is easy to see that the set of $\aleph_1$-trees forms a Borel set with an appropriate coding of the order into a subset of ω_1.

Now, a subset $\mathcal{T}$ of $\omega_1^{\omega_1}$ is Π^1_1 (and is called *co-analytic*) if there is an open $B \subset \omega_1^{\omega_1} \times \omega_1^{\omega_1}$ such that $T \in \mathcal{T}$ if and only for all $g \in \omega_1^{\omega_1}$, $(T, g) \in B$. Complements of Π^1_1 sets, denoted by Σ^1_1, are called *analytic* sets. In Section 2, the reader can see elementary applications of this definition.

Finally, a subset $\mathcal{T}$ of $\omega_1^{\omega_1}$ is *complete* for a complexity class Γ if $\mathcal{T} \in \Gamma$ and for any $\mathcal{S} \in \Gamma$, there is a continuous $\pi : \omega_1^{\omega_1} \to \omega_1^{\omega_1}$ such that $T \in \mathcal{S}$ if and only if $\pi(T) \in \mathcal{T}$. That is, no matter how we pick $\mathcal{S}$ in Γ, we can completely decide $\mathcal{S}$ by our single fixed set $\mathcal{T}$ and using an appropriate continuous map. In Section 3, we shall see this definition at work.

Let us also recall some classical guessing principles: $\diamondsuit^+$ asserts the existence of a sequence $\underline{A} = \{\mathcal{A}_\alpha : \alpha < \omega_1\}$ of countable sets such that for any $X \subset \omega_1$, there is a club $C \subset \omega_1$ with $C \cap \alpha, X \cap \alpha \in \mathcal{A}_\alpha$ for any $\alpha \in C$. In this situation, we say that $\underline{A}$ *witnesses* $\diamondsuit^+$.

We say that $\underline{N} = (N_\alpha)_{\alpha<\omega_1}$ is a $\diamondsuit^+$*-oracle over* P if

(1) $\underline{N}$ is an increasing sequence of countable elementary submodels of $H(\aleph_2)$,
(2) $P, (N_\alpha)_{\alpha<\beta} \in N_\beta$ for all $\beta < \omega_1$, and
(3) $\underline{N}$ witnesses $\diamondsuit^+$.

Clearly, if $\diamondsuit^+$ holds then for any $P \in H(\aleph_2)$, there is a $\diamondsuit^+$-oracle over P. Also, recall that $\diamondsuit^+$ implies that ST and KT are non-empty [11].

For later reference, we state a few consistency results, the first being a now classical theorem of R. Jensen.

([4]) In fact, in the latter model, all Aronszajn trees are club-isomorphic [25].

THEOREM 1.4 ([5]). *Consistently, CH holds and all Aronszajn trees are special.*

Jensen's argument was built on an elaborate ccc forcing (in fact, a completely new iteration technique). A more mainstream proof of this theorem is due to S. Shelah [19] using countable support iteration of proper posets.

Given two trees S, T, a *club-embedding of T into S* is an order preserving injection f defined on $T\restriction C = \bigcup\{T_\alpha : \alpha \in C\}$, where $C \subset \omega_1$ is a *club* (closed and unbounded subset), with range in S. A *derived tree of S* is a level product of the form $\prod_{i<n} S \cap s_i^{\uparrow}$ where the s_i are distinct nodes from the same level of S (5). A *fully Suslin tree* is a Suslin tree with the property that all its derived trees are Suslin as well.

We will refer to the model in the next theorem as the *Abraham–Shelah model.*

THEOREM 1.5 ([1]). *Consistently, CH holds and there is a fully Suslin tree R and a special Aronszajn tree U such that, for any Aronszajn tree T, either*

(1) *T club-embeds into U, or*
(2) *there is a derived tree of R that club-embeds into T.*

Moreover, there are only $\aleph_1$-many Suslin trees modulo club-isomorphism (6).

In essence, the above theorem says that any Aronszajn tree is either special or embeds a Suslin tree closely associated to R.

Finally, the fact that there might be no Kurepa trees was proved by J. Silver in 1971.

THEOREM 1.6 ([21]). *If a strongly inaccessible cardinal is Lévy collapsed to ω_2 then in the resulting model, there are no Kurepa trees.*

2. Aronszajn and Suslin trees.

To avoid some technicalities, from now on let us restrict our attention to certain *regular trees* only: those trees T which are rooted, every node in T has at least two immediate successors, and T is *pruned*, i.e., for any $s \in T$ of height α and any $\beta < \omega_1$ above α, there is some $t \in T$ of height β that extends s. These are simple Borel conditions and we assume that our classes AT, ST and later KT consist of regular trees only.

Now, let us start the complexity analysis of these classes. Our first observation follows from the definitions immediately.

(5) Here, $S \cap s_i^{\uparrow} = \{t \in S : t \geq s_i\}$.

(6) It is an intriguing open problem whether one can find *a model with a single Suslin tree* (modulo club-isomorphism).

OBSERVATION 2.1.

(1) *The set of all* $\aleph_1$*-trees on* ω_1 *is Borel.*
(2) AT *and* ST *are both* Π^1_1 *sets.*
(3) sAT *is* Σ^1_1.

Proof. The proof is a fairly standard exercise in descriptive set theory. To demonstrate the definitions, we prove that AT is Π^1_1, and leave the rest to the interested reader. We need to find an open $B \subset \omega_1^{\omega_1} \times \omega_1^{\omega_1}$ such that $T \in \mathrm{AT}$ if and only for all $g \in \omega_1^{\omega_1}$, $(T, g) \in B$. Indeed, let B denote the set of pairs (T, g) such that T is an $\aleph_1$-tree on ω_1 and $g \in \omega_1^{\omega_1}$ does not code an uncountable branch in T. Now, B is open in the product of codes for $\aleph_1$-trees (a Borel set) and $\omega_1^{\omega_1}$. Indeed, if g does not code an uncountable branch then either g codes a countable branch in T (i.e., there is a level of T without any element of g), or g codes two incomparable elements. Both cases can be witnessed by fixing a countable initial segment of T, and g and hence B is open. Now, an $\aleph_1$-tree T is Aronszajn if and only if for any g, $(T, g) \in B$. ∎

So in models where AT = sAT e.g., in Jensen's model of CH from Theorem 1.4, we get the following.

COROLLARY 2.2. *Consistently, CH holds and* $\mathrm{AT} = \mathrm{sAT} \in \Delta^1_1$ *and hence* $\mathrm{ST} = \emptyset$ [7].

Our next goal is to show that none of the classes AT, sAT and ST are Borel (unless $\mathrm{ST} = \emptyset$, in which case it is trivially Borel). We will apply the following well-known fact.

LEMMA 2.3. *Suppose that* T, T' *are countable, rooted, binary branching, and pruned trees of height* $\beta < \omega_1$. *Then* T *and* T' *are isomorphic. In fact, any isomorphism* $f : T_{\leq\alpha} \to T'_{\leq\alpha}$ *with* $\alpha < \beta$ *extends to an isomorphism* $\bar{f} : T \to T'$.

The proof is an easy back-and-forth argument that we omit. For future use we note that the above lemma also applies with "binary branching" replaced by "ω-branching".

LEMMA 2.4. *Suppose that* T *is a regular* $\aleph_1$*-tree and* $\mathcal{U}$ *is somewhere co-meager in the set of all regular trees on* ω_1. *Then*

(1) *there is an isomorphic copy* S *of* T *in* $\mathcal{U}$, *and*
(2) $\mathcal{U}$ *contains a tree with an uncountable branch.*

Proof. (1) Suppose that $\mathcal{U} = [S^0] \setminus \bigcup\{Y_\xi : \xi < \omega_1\}$ where each Y_ξ is a nowhere dense set of trees. That is, any countable tree S has a countable end-extension S' such that any extension of S' into a tree on ω_1 is not in Y_ξ. Here, $[S^0]$ denotes the (basic open) set of all trees extending S^0.

---

[7] We remark here that $\mathrm{MA}_{\aleph_1}$ implies AT = sAT so they are both Δ^1_1 but CH fails.

Now, we construct an increasing sequence $(S^\xi)_{\xi<\omega_1}$ of countable trees and isomorphisms $f_\xi : S^\xi \to T_{<\delta_\xi}$. Given $(S^\xi)_{\xi<\zeta}$, we look at $S^{<\zeta} = \bigcup_{\xi<\zeta} S^\xi$. The latter is isomorphic to $T_{<\delta}$ where $\delta = \sup_{\xi<\zeta} \delta_\xi$ is witnessed by

$$f_{<\zeta} = \bigcup_{\xi<\zeta} f_\xi : S^{<\zeta} \to T_{<\delta}.$$

Define an end-extension $\overline{S^{<\zeta}}$ of $S^{<\zeta}$ of height $\delta+1$ by adding upper bounds to exactly those branches $b \subset S^{<\zeta}$ for which the chain $f_{<\zeta}[b]$ has an upper bound in T_δ. Clearly, there is an isomorphism $\overline{f_{<\zeta}} : \overline{S^{<\zeta}} \to T_{\delta+1}$ that extends $f_{<\zeta}$. Now, let S^ζ be an end-extension of $\overline{S^{<\zeta}}$ which cannot be extended to a tree on ω_1 that is in Y_ζ. This can be done since Y_ζ is nowhere dense. Finally, apply Lemma 2.3 to extend $\overline{f_{<\zeta}}$ to some isomorphism $f_\zeta : S^\zeta \to T_{<\delta_\zeta}$.

This finishes the construction and the tree $S = \bigcup\{S^\zeta : \zeta < \omega_1\}$ is as desired.

(2) Since there is a regular $\aleph_1$-tree T which contains an uncountable branch, we can apply (1). ∎

We shall use the fact that any non-meager set with the Baire property is somewhere co-meager. The previous lemma and latter fact immediately yield the following corollaries.

Corollary 2.5.

(1) *The isomorphism class of any regular $\aleph_1$-tree T is everywhere non-meager.*
(2) *Suppose that T, S are non-isomorphic $\aleph_1$-trees. Then their isomorphism classes cannot be separated by sets with the Baire property.*
(3) *The set of trees isomorphic to a fixed tree without an uncountable branch is Σ^1_1 but does not have the Baire property and hence is not Borel.*
(4) *The sets* AT *and* sAT *do not have the Baire property. In turn,* AT *and* sAT *are not Borel.*
(5) *If* ST $\neq \emptyset$ *then* ST *does not have the Baire property and so* ST *is not Borel.*
(6) *If* KT $\neq \emptyset$ *then* KT *does not have the Baire property and so* KT *is not Borel.*

Proof. (1) and (2) are immediate from Lemma 2.4(1).

(3) If such an isomorphism class has the Baire property then there is a somewhere co-meager set of trees all isomorphic to a fixed tree with no uncountable branch. This is not possible by Lemma 2.4(2).

(4), (5) and (6) again follow from Lemma 2.4(2): these classes are closed under isomorphism classes so must be everywhere non-meager. If they are Baire then they are somewhere comeager and hence contain a tree with an uncountable branch and also a special Aronszajn tree. This leads to a contradiction in case of any of these classes. ∎

So, whenever there is a Suslin tree then the set of all Suslin trees is not Borel (but always Π^1_1). Could it be analytic too? We show that this is independent (even assuming CH).

PROPOSITION 2.6. *In the Abraham–Shelah model,* $\mathrm{AT} \neq \mathrm{sAT} \in \Delta^1_1$ *and* $\mathrm{ST} \in \Delta^1_1$ *as well.*

Proof. Indeed, there are non-special Aronszajn trees (even Suslin trees) and a tree T is special if and only if it club-embeds no derived subtree of a fixed Suslin tree S. Since there are only $\aleph_1$-many such derived subtrees, this gives $\mathrm{sAT} \in \Pi^1_1$ and so $\mathrm{sAT} \in \Delta^1_1$ (using Observation 2.1).

In the Abraham–Shelah model, there are only $\aleph_1$-many Suslin trees modulo club-isomorphism, so fix a representative of each class and collect them as $\mathcal{S}$. Now, being Suslin is characterized by being club-isomorphic to some element of $\mathcal{S}$, which in turn implies $\mathrm{ST} \in \Sigma^1_1$ and $\mathrm{ST} \in \Delta^1_1$ as well (using again Observation 2.1). ∎

In the next section, we show that both AT and ST are Π^1_1-complete if we assume $V = L$.

3. Reductions between subsets of ω_1, and $\aleph_1$-trees. Our first theorem in this section establishes a continuous reduction between stationarity and ST in a strong form.

THEOREM 3.1. *Suppose* $\diamondsuit^+$. *There is a map* $X \mapsto T^X$ *from subsets of* ω_1 *to the set of downward closed* $\aleph_1$*-subtrees of* $2^{<\omega_1}$ *such that*

(1) *if* $X \cap \alpha = Y \cap \alpha$ *then* $T^X \restriction \alpha = T^Y \restriction \alpha$,
(2) *if* X *is stationary then* T^X *is Suslin, and*
(3) *if* X *is non-stationary then* T^X *has an uncountable branch.*

Note that by CH, $2^{<\omega_1}$ has size $\aleph_1$ so we can easily transform our trees to live on ω_1. We immediately get the following corollary by Theorem 1.3.

COROLLARY 3.2. *If* $V = L$ *then* AT *and* ST *are both* Π^1_1*-complete.*

Proof of Theorem 3.1. Let us start by fixing a $\diamondsuit^+$-oracle $\bar{N}$, i.e., a sequence of elementary submodels $(N_\alpha)_{\alpha<\omega_1}$ of $(H(\aleph_2), \in, \prec)$ [8] such that $(N_\alpha)_{\alpha<\beta} \in N_\beta$ and $\bar{N}$ witnesses $\diamondsuit^+$.

Given $X \subset \omega_1$, we construct the downward closed subtree $T^X \subset 2^{<\omega_1}$ level by level in an induction, so that $(T^X_\alpha)_{\alpha<\beta} \in N_{\beta+1}$ for all $\beta < \omega_1$. At each step, we shall add a new countable level to the tree constructed so far. At successor steps $\beta = \alpha + 1 < \omega_1$, we simply take the binary extension $T^X_\beta = \{s^\frown i : s \in T^X_\alpha,\ i < 2\}$ of T^X_α.

[8] Here $\prec$ denotes a fixed wellorder of $H(\aleph_2)$.

Now, assume $\beta \in \omega_1$ is a limit ordinal. If $\beta \in \omega_1 \setminus X$ then we let

$$T^X_\beta = \{b \in 2^\beta \cap N_\beta : b\restriction\alpha \in T^X_\alpha \text{ for all } \alpha < \beta\}.$$

In other words, we continue all those branches through $T^X_{<\beta}$ which are in N_β. Since N_β is countable, this is a valid extension and is defined in $N_{\beta+1}$. Moreover, any $s \in T^X_{<\beta}$ has an extension in T^X_β (i.e., the tree remains pruned).

Second, if $\beta \in X$ then, working in $N_{\beta+1}$, we make sure that

- any $s \in T^X_{<\beta}$ has an extension in T^X_β, and
- for any $A \in N_\beta$ such that $A \subset T^X_{<\beta}$ is a maximal antichain, any new element $t \in T^X_\beta$ is above a node in A.

Since N_β is countable and $N_\beta, T^X_{<\beta} \in N_{\beta+1}$, the level T^X_β can be constructed in $N_{\beta+1}$ (just as in the classical construction of Suslin trees [14]).

This induction certainly defines an $\aleph_1$-tree T^X for any $X \subset \omega_1$. The next two claims will conclude the proof of the theorem.

CLAIM 3.3. *If X is stationary then T^X is Suslin.*

Proof. Suppose that $A \subset T^X$ is a maximal antichain. Since $\bar{N}$ guesses A at club many points, we can find some $\beta \in X$ such that $A \cap T^X_{<\beta} \in \mathcal{N}_\beta$ and $A \cap T^X_{<\beta}$ is a maximal antichain in $T^X_{<\beta}$. So, at stage β, we made sure that any $t \in T^X_\beta$ is above some element of $A \cap T^X_{<\beta}$. In turn, we must have $A \subset T^X_{<\beta}$ and so A is countable. ∎

CLAIM 3.4. *If X is non-stationary then T^X has an uncountable branch.*

Proof. There is some club B that is disjoint from X, and there is a club $C \subset \omega_1$ such that $C \cap \alpha, B \cap \alpha, X \cap \alpha \in N_\alpha$ whenever $\alpha \in C$. In particular, $C \cap B \cap \alpha \in N_\alpha$ for any $\alpha \in C \cap B$. Let $\{\beta_\alpha : \alpha < \omega_1\}$ be the increasing enumeration of $C \cap B$.

We construct $t_\alpha \in T^X_{\beta_\alpha}$ so that

(1) $t_{\alpha'} < t_\alpha$ for all $\alpha' < \alpha < \omega_1$,
(2) $(t_{\alpha'})_{\alpha'<\alpha} \in N_{\beta_\alpha}$, and
(3) t_α is the $\prec$-minimal (9) element of 2^{β_α} that extends all elements in the chain $(t_{\alpha'})_{\alpha'<\alpha}$.

At limit steps, note that $\bar{N}\restriction B \cap C \cap \beta_\alpha = (N_{\beta_{\alpha'}})_{\alpha'<\alpha} \in N_{\beta_\alpha}$. So the sequence $(t_{\alpha'})_{\alpha'<\alpha}$ is in N_{β_α} too since it can be uniquely defined from $\bar{N}\restriction B \cap C \cap \beta_\alpha$ and $X \cap \beta_\alpha$. As $\beta_\alpha \notin X$, we sealed all branches that are in N_{β_α}, so $t_\alpha \in T^X_{\beta_\alpha}$ as well. In turn, $(t_{\alpha'})_{\alpha'\le\alpha} \in N_{\beta_\alpha+1}$. ∎

This proves the theorem. ∎

(9) Recall that $\prec$ denotes our fixed wellorder of $H(\aleph_2) \supseteq H(\aleph_1)$.

It would be interesting to see whether a single Suslin tree suffices to construct such a reduction or if weaker reductions (say between stationarity and AT) exist under weaker assumptions than $\diamondsuit^+$.

Next, we present a variant that reduces non-stationarity to the set of $\mathbb{R}$-special, ω-branching trees. A tree T is $\mathbb{R}$-*special* if there is a monotone map from its nodes into the linear order $(\mathbb{R}, <)$ ([10]). Such trees are Aronszajn but not Suslin, as forcing with them will collapse ω_1.

THEOREM 3.5. *Suppose $\diamondsuit^+$. There is a map $X \mapsto T^X$ from subsets of ω_1 to the set of downward closed $\aleph_1$-subtrees of $\omega^{<\omega_1}$ such that*

(1) *if $X \cap \alpha = Y \cap \alpha$ then $T^X \restriction \alpha = T^Y \restriction \alpha$,*
(2) *if X is stationary then T^X is Suslin, and*
(3) *if X is non-stationary then T^X is an $\mathbb{R}$-special Aronszajn tree.*

Proof. The idea is very similar: we build T^X level by level and aim for a Suslin tree at stages $\beta \in X$. However, if $\beta \in \omega_1 \setminus X$ then we shall try to make T^X special. In fact, we will add the new level T^X_β so that any nice enough monotone map $\varphi : T^X_{<\beta} \to \mathbb{R}$ that is also in N_β has an extension to T^X_β. We will need to make sure that any new node at level β works simultaneously for all such specializing maps, which inspires the definition of a *specializing pair* below. Intuitively, we not just assign a real number $\varphi(s)$ to a tree node s but also a promise (in the form of a positive number $\delta(s)$) to keep all values $\varphi(t)$ close to $\varphi(s)$ whenever t is above s. The details follow below.

Working with specializations appears to require us to work not with binary trees but with ω-branching trees, because we want our specializing function to increase its value at a node by an arbitrarily small amount at the successors of that node. And it also appears to require that we specialize not with rational-valued monotone functions but with real-valued monotone functions, as this enables us to assign the supremum of the values assigned to predecessors to nodes of limit height.

Fix a $\diamondsuit^+$-oracle $\bar{N}$. Given X, we construct the downward closed subtree $T^X \subset 2^{<\omega_1}$ so that $(T^X_\alpha)_{\alpha<\beta} \in N_{\beta+1}$ for all $\beta < \omega_1$. If β is successor then we extend each node on level $\beta - 1$ to ω-many nodes on level β. If β is a limit ordinal in X then, working in $N_{\beta+1}$, we repeat the construction in the previous theorem. We make sure that

- any $s \in T^X_{<\beta}$ has an extension in T^X_β, and
- for any $A \in N_\beta$ such that $A \subset T^X_{<\beta}$ is a maximal antichain, any new element $t \in T^X_\beta$ is above a node in A.

([10]) For more about such trees, see [23, Section 9].

This will certainly make sure that T^X is Suslin whenever X is stationary. Let us turn to the construction when β is a limit ordinal from $\omega_1 \setminus X$.

Given any tree T of height $\beta \le \omega_1$, we say that (φ, δ) is a *specializing pair* on T if

(i) $\varphi : T \to \mathbb{R}$ is monotone, in particular, T is $\mathbb{R}$-special,
(ii) if $s < t$ in T then $|\varphi(s) - \varphi(t)| < \delta(s)$, and
(iii) if $\alpha < \beta < \operatorname{ht} T$, $s \in T_\alpha$ and $\Delta > 0$ then there is some $t \in T_\beta$ above s such that $|\varphi(s) - \varphi(t)| < \Delta$ and $\delta(t) < \Delta$.

Our first goal is the following: given a countable tree T of limit height β and a countable family Φ of specializing pairs for T, we show that there is a cofinal branch b through T such that any $(\varphi, \delta) \in \Phi$ can be extended to $t^* = \bigcup b$.

Lemma 3.6. *Suppose that $T \subset \omega^{<\omega_1}$ is a countable tree of limit height β, and Φ is a countable family of specializing pairs for T. Fix some $s \in T$, $(\varphi^*, \delta^*) \in \Phi$ and $\Delta > 0$. Then there is a cofinal, downward closed branch $b \subset T$ containing s such that*

$$\sup_{s \le t \in b} |\varphi^*(s) - \varphi^*(t)| < \Delta,$$

and for any $t' \in b$ and $(\varphi, \delta) \in \Phi$,

$$\sup_{t' \le t \in b} |\varphi(s) - \varphi(t)| < \delta(t').$$

Indeed, for β a limit ordinal, if the lemma holds and we set $t^* = \bigcup b$ and define $\varphi(t^*) = \sup_{t \in b} \varphi(t)$ for $(\varphi, \delta) \in \Phi$, then (φ, δ) will satisfy (i) and (ii) from the definition of specializing pair on the extra node t^*. Note that $\varphi(t^*)$ may not be rational. Moreover if we set $\delta(t^*) = \Delta/2$ then for this particular s and Δ we made sure that condition (iii) is witnessed by t^*. By repeating the procedure of this lemma for all elements of Φ with all possible rational $\Delta > 0$ we get the following.

Lemma 3.7. *Suppose that $T \subset 2^{<\omega_1}$ is a countable tree of limit height β, and Φ is a countable family of specializing pairs for T. Then T has a pruned end-extension T^* of height $\beta + 1$ such that any $(\varphi, \delta) \in \Phi$ can be extended to a specializing pair for T^*.*

A useful corollary to Lemma 3.7 is that there does exist an ω_1-tree T with a specializing pair: Define $T_{\le\beta}$ and a specializing pair $(\varphi_\beta, \delta_\beta)$ for it by induction on the limit ordinal β (together with 0). For $\beta = 0$, $T_{\le 0}$ has just one node $\emptyset$ and we choose (φ_0, δ_0) arbitrarily. If $T_{\le\beta}$ and $(\varphi_\beta, \delta_\beta)$ are defined then let $T_{<\beta+\omega}$ extend $T_{\le\beta}$ in the obvious way and apply Lemma 3.7 to extend $(\varphi_\beta, \delta_\beta)$ to a specializing pair $(\varphi_{\beta+\omega}, \delta_{\beta+\omega})$ for a pruned end-extension $T_{\le\beta+\omega}$ of $T_{\le\beta}$. If β is a limit of limits then let $T_{<\beta}$ be the union of the $T_{\bar\beta}$

for limit $\bar{\beta} < \beta$ and (φ, δ) the union of the $(\varphi_{\bar{\beta}}, \delta_{\bar{\beta}})$ for limit $\bar{\beta} < \beta$ and apply Lemma 3.7 to produce the desired $T_{\leq\beta}$ and $(\varphi_\beta, \delta_\beta)$ extending $T_{<\beta}$ and (φ, δ) respectively.

The same argument shows that any specializing pair for a countable, pruned, ω-branching tree T of successor height can be extended to a specializing pair for a pruned, ω-branching tree of height ω_1 which end-extends T.

Proof of Lemma 3.6. Given $s < t \in T$ and $\mathbf{p} \in \Phi$, let $\Delta_{\mathbf{p}}(s,t) = \delta(s) - |\varphi(s) - \varphi(t)|$. This measures how much slack we have after jumping from s to t. List all $\mathbf{p} \in \Phi$ and $t \in T$ as $(\mathbf{p}_n, t_n)$, each infinitely often. We define $s < s_0 < s_1 < s_2 < \cdots$ so that $s_n \in T_{\alpha_n}$ for some fixed cofinal sequence $(\alpha_n)_{n\in\omega}$ in α. First, we pick s_0 so that

$$\varphi^*(s_0) - \varphi^*(s) < \Delta/2 \quad \text{and} \quad \delta^*(s_0) < \Delta/2.$$

This can be done by (iii). Moreover, note that no matter how we pick t above s_0, we will always have $\varphi^*(t) - \varphi^*(s_0) < \Delta/2$ and so, for any cofinal branch b above s_0,

$$\sup_{s\leq t\in b} |\varphi^*(t) - \varphi^*(s)| < \Delta$$

by the triangle inequality.

Given s_n, we pick s_{n+1} as follows: look at $(\mathbf{p}_n, t_n)$ and assume $t_n < s_n$ (if the latter fails, pick s_{n+1} arbitrarily). Now, look at $\tilde{\Delta} = \Delta_{\mathbf{p}_n}(t_n, s_n)$ and pick s_{n+1} so that

$$\varphi_n(s_{n+1}) - \varphi_n(s_n) < \tilde{\Delta}/2 \quad \text{and} \quad \delta_n(s_{n+1}) < \tilde{\Delta}/2$$

where $\mathbf{p}_n = (\varphi_n, \delta_n)$. This is again possible by (iii). As before, for any t above s_{n+1}, we will always have $\varphi_n(t) - \varphi_n(s_{n+1}) < \tilde{\Delta}/2$ and so for any cofinal branch b above s_{n+1},

$$\sup_{s_{n+1}\leq t\in b} \varphi_n(t) - \varphi_n(t_n) < \delta_n(t_n)$$

by the triangle inequality and unwrapping the definition of $\tilde{\Delta}$.

The final branch b is given by the downward closure of $(s_n)_{n<\omega}$. Since any $t' \in b$ and specializing pair $\mathbf{p} \in \Phi$ were considered infinitely often during the construction, we clearly satisfied the requirements. ■

Finally, we can describe what happens in the construction of T^X_β at limit steps $\beta \in \omega_1 \setminus X$. Working in $N_{\beta+1}$, we consider the tree $T^X_{<\beta}$ and $\Phi_\beta = \{(\varphi, \delta) \in N_\beta : (\varphi, \delta)$ is a specializing pair for $T^X_{<\beta}\}$. Now, applying Lemma 3.7, we add a new level to $T^X_{<\beta}$ so that any specializing pair from Φ_β extends to $T^X_{\leq\beta}$.

This ends the construction of T^X and we are left to prove the following.

Claim 3.8. *If X is non-stationary then T^X is an $\mathbb{R}$-special Aronszajn tree.*

Proof. In fact, we prove that T^X has a specializing pair. By our assumption on X, we can find a club $C \subset \omega_1 \setminus X$ consisting of limit ordinals such that for any $\beta \in C$, $X \cap \beta, C \cap \beta \in N_\beta$. Let $\{\beta_\alpha : \alpha < \omega_1\}$ be the increasing enumeration of C and define $(\varphi_\alpha, \delta_\alpha)$ for $\alpha < \omega_1$ so that

- $(\varphi_\alpha, \delta_\alpha) \in N_{\beta_\alpha+1}$ is a specializing pair on $T^X_{\le\beta_\alpha}$ uniquely definable from $X \cap \beta_\alpha, C \cap \beta_\alpha, \bar{N}\restriction\beta_\alpha$,
- for $\alpha < \alpha' < \omega_1$, $(\varphi_{\alpha'}, \delta_{\alpha'})$ extends $(\varphi_\alpha, \delta_\alpha)$.

For limit α, we use Lemma 3.7 to define the specializing pair $(\varphi_\alpha, \delta_\alpha)$ extending the union of the specializing pairs $(\varphi_{\bar{\alpha}}, \delta_{\bar{\alpha}})$ for $\bar{\alpha} < \alpha$. For α equal to 0 or a successor, as remarked after the proof of Lemma 3.7, the specializing pair $(\varphi_{\alpha-1}, \delta_{\alpha-1})$ can be extended to a specializing pair for an end-extension T^* of $T_{\le\beta_{\alpha-1}}$ of height ω_1. By Lemma 2.3 (applied to ω-branching trees), $T^*_{\le\beta_\alpha}$ is isomorphic to $T^X_{\le\beta_\alpha}$ via an isomorphism which is the identity on $T^X_{\le\beta_{\alpha-1}}$ and therefore we can extend $(\varphi_{\alpha-1}, \delta_{\alpha-1})$ to a specializing pair for $T^X_{\le\beta_\alpha}$.

As before, $N_{\beta_{\alpha'}}$ has all the information to reconstruct the sequence $((\varphi_\alpha, \delta_\alpha))_{\alpha<\alpha'}$ and so

$$\Big(\bigcup_{\alpha<\alpha'} \varphi_\alpha, \bigcup_{\alpha<\alpha'} \delta_\alpha\Big) \in \Phi_{\beta_{\alpha'}}.$$

Since $\beta_{\alpha'} \notin X$, we made sure that this specializing pair has an extension to level $T^X_{\beta_{\alpha'}}$ which gives $(\varphi_{\alpha'}, \delta_{\alpha'})$. ∎

This proves the theorem. ∎

If we strengthen our hypothesis to $V = L$, we can work with subtrees of $2^{<\omega_1}$ which are special in the standard sense.

Theorem 3.9. *Suppose $V = L$. There is a map $X \mapsto T^X$ from subsets of ω_1 to the set of downward closed $\aleph_1$-subtrees of $2^{<\omega_1}$ such that*

(1) *if $X \cap \alpha = Y \cap \alpha$ then $T^X\restriction\alpha = T^Y\restriction\alpha$,*
(2) *if X is stationary then T^X is Suslin, and*
(3) *if X is non-stationary then T^X is a special Aronszajn tree.*

Proof. As in the previous proof, we take steps at limit ordinal stages in X to make T^X Suslin and at limit ordinal stages outside of X to make T^X special. The main difference is that because of the condensation properties of L it will suffice to extend at most one specialization of $T^X_{<\beta}$ to $T^X_{\le\beta}$ for limit ordinals β outside of X, whereas in the previous argument we extended countably many, leading to the consideration of specializing pairs and irrational values of the specializing function.

In this proof, by *tree* we mean a rooted, binary, pruned subtree of $2^{<\omega_1}$ with countable levels, of some ordinal height at most ω_1. A *good specialization* of such a tree T is a monotone function $\varphi : T \to \mathbb{Q}$ with the property that if $\alpha < \beta \leq \operatorname{ht} T$, β is limit, $s \in T_\alpha$ and $\Delta > 0$ then s has an extension t in T_β such that $\varphi(t) < \varphi(s) + \Delta$. (As we are working now with binary trees, we cannot require this if β is a successor.) The standard construction of a binary special Aronszajn tree yields a tree of height ω_1 with a good specialization. It then follows from Lemma 2.3 that for any tree T, if $\alpha + 1 < \operatorname{ht} T$ then any good specialization of $T_{\leq\alpha}$ can be extended to a good specialization of T.

A key fact for the present proof is that for α a countable limit ordinal, any good specialization φ of a tree T of height α can be extended to a good specialization φ^* of a tree T^* of height $\alpha+1$ which end-extends T. The αth level of T^* is obtained by choosing, for each $s \in T$ and $\Delta > 0$, a cofinal branch b through T containing s such that $\sup_{t\in b} \varphi(t)$ is less than Δ and then assigning $\varphi^*(\bigcup b)$ to be a rational between this supremum and Δ. (The difficulty in the previous proof is that it is not possible to do this in general for more than one specialization simultaneously.)

To ensure that T^X is Suslin when X is stationary we need slightly less in this proof than $\diamondsuit^+$. The principle $\diamondsuit^*$ asserts the existence of a sequence $\underline{A} = \{\mathcal{A}_\alpha : \alpha < \omega_1\}$ of countable subsets of ω_1 such that for any $X \subset \omega_1$ there is a club $C \subset \omega_1$ such that $X \cap \alpha \in \mathcal{A}_\alpha$ for any $\alpha \in C$. In this situation, we say that $\underline{A}$ *witnesses* $\diamondsuit^*$.

Fix a witness $\underline{A}$ to $\diamondsuit^*$. We construct $T^X_{\leq\beta}$ by induction on β, where β ranges over the countable limit ordinals together with 0. In addition we will choose a good specialization φ_β of $T^X_{\leq\beta}$ for certain countable ordinals β not in X, called *good* ordinals. There is a natural tree order $\prec$ on good ordinals and we design our specializations so that if $\alpha \prec \beta$ are good then φ_β extends φ_α. Moreover, if X is not stationary, there will be a club C of good ordinals with the property that $\alpha < \beta$ in C implies $\alpha \prec \beta$; we then get a specialization of T^X by taking the union of the φ_α for α in C.

An ordinal $\alpha \leq \omega_1$ is *good* if for some limit ordinal $\beta > \alpha$, α is the ω_1 of $L_\beta[X\cap\alpha, \underline{A}\restriction\alpha]$ and in this model there is a club in α disjoint from X. If β is least with this property then note that the Skolem hull of α in $L_\beta[X\cap\alpha, \underline{A}\restriction\alpha]$ is all of $L_\beta[X\cap\alpha, \underline{A}\restriction\alpha]$. As $\alpha = \omega_1$ in this model, it follows that the set C_α of $\bar\alpha < \alpha$ such that $\bar\alpha$ equals the intersection with α of the Skolem hull of $\bar\alpha$ in $L_\beta[X\cap\alpha, \underline{A}\restriction\alpha]$ is a closed subset of α. We write $\bar\alpha \prec \alpha$ if $\bar\alpha$ belongs to this closed set C_α. Using condensation it is easy to check that if $\bar\alpha$ belongs to C_α then $\bar\alpha$ is good and $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$. So $\prec$ is a tree order.

Now by induction on β we define $T_{\leq\beta}$ and, if β is good, the good specialization φ_β of $T^X_{\leq\beta}$.

If $\beta = 0$ then T^X_β has just the one element $\emptyset$. If β is a successor then we end-extend $T^X_{<\beta}$ to $T^X_{\leq\beta}$ by providing each node on level $\beta - 1$ with two successors on level β.

If β is a limit and belongs to X, then we extend the union $T^X_{<\beta}$ of the trees $T^X_{\leq\bar\beta}$ for $\bar\beta < \beta$ to $T^X_{\leq\beta}$ so as to guarantee that each node on level β lies above an element of each maximal antichain of $T^X_{<\beta}$ which belongs to $\mathcal{A}_\beta$. This will guarantee that T^X is Suslin if X is stationary.

Finally, if β is good then we proceed as follows. If C_β, the set of ordinals α such that $\alpha \prec \beta$, is bounded in β then let $\bar\beta$ be the maximum of C_β (or 0 if C_β is empty), let $T_{\leq\beta}$ be any tree of height $\beta + 1$ end-extending $T_{<\beta}$ and let φ_β be any good specialization of $T_{\leq\beta}$ extending $\varphi_{\bar\beta}$ (or any good specialization of $T_{\leq\beta}$ if $\bar\beta = 0$). If C_β is unbounded in β then let $\varphi_{<\beta}$ be the union of the good specializations $\varphi_{\bar\beta}$, $\bar\beta \prec \beta$, and let $T_{\leq\beta}$, φ_β extend $T_{<\beta}$, $\varphi_{<\beta}$ respectively.

This completes the construction of T^X, the union of the $T^X_{\leq\beta}$, $\beta < \omega_1$.

If X is stationary then T^X is Suslin as at limit stages in X we have used the $\diamondsuit^*$ sequence $\underline{A}$ to seal antichains. If X is non-stationary then ω_1 is a good ordinal and $C = C_{\omega_1}$ is club in ω_1. Moreover, $\alpha < \beta$ in C implies that α belongs to C_β and so, by construction, φ_β extends φ_α. Therefore T^X has the good specialization $\varphi =$ the union of the φ_β, β in C. ∎

Corollary 3.10. *If $V = L$ then* sAT *is Σ^1_1-complete.*

Another application of the method used in the proof of Theorem 3.9 is that assuming $V = L$, the conclusion (3) of Theorem 3.1 can be strengthened from "has an uncountable branch" to "has a unique uncountable branch": Instead of guranteeing the existence of a particular good specialization of T^X when X is non-stationary we guarantee the existence of a particular uncountable branch b of T^X and use a $\diamondsuit$ sequence $(D_\beta \mid \beta < \omega_1)$ to ensure that T^X has no other uncountable branches by choosing T^X_β to have no node above the $\diamondsuit$ guess D_β if D_β is a cofinal branch through $T_{<\beta}$ distinct from $b\restriction\beta$.

4. Kurepa trees. Our goal in this section is to show the following.

Theorem 4.1. ($V = L$) *The set* KT *of all Kurepa trees is Π^1_2-complete.*

We prove the above result through a series of lemmas. First, we will build on the following representation of Π^1_2 sets.

Lemma 4.2. *Assume $V = L$. If A is a Π^1_2 subset of $\omega_1^{\omega_1}$ then for some Σ_1 formula ϕ and some parameter $P \in \omega_1^{\omega_1}$, the following are equivalent:*

(1) *$X \in A$,*
(2) $\sup\{i < \omega_2 : L_{\omega_2}[X] \models \phi(X, P, i)\} = \omega_2$.

Proof. Since A is Π^1_2, we can find a Borel set B such that X is in A if and only if

$$\forall Y\ \exists Z\ (X, Y, Z) \in B.$$

Let $f : \omega_2 \to \omega_1^{\omega_1}$ be a bijection which is Σ_1 over L_{ω_2} and choose a Σ_1 formula ψ with parameter P in $\omega_1^{\omega_1}$ such that $(X, Y, Z) \in B$ if and only if $L_{\omega_2} \models \psi(X, Y, Z, P)$.

Then $X \in A$ if and only if

$$L_{\omega_2} \models \forall j\ \exists k\ \psi(X, f(j), f(k), P). \tag{4.1}$$

Let $\phi(X, P, i)$ be the formula

$$X, P \in L_i \wedge L_i \models \forall j\ \exists k\ \psi(X, f(j), f(k), P).$$

Then (4.1) (and so $X \in A$ as well) is equivalent to

$$\sup\{i < \omega_2 : L_{\omega_2} \models \phi(X, P, i)\} = \omega_2,$$

as desired. ∎

Fix some $X \subset \omega_1^{\omega_1}$ with corresponding Σ_1 formula ϕ and parameter P. First, note that

$$\{i < \omega_2 : L_{\omega_2} \models \phi(X, P, i)\} = \{i < \omega_2 : \exists \beta < \omega_2\ (L_\beta \models \phi(X, P, i))\}.$$

Our plan is to form an $\aleph_1$-tree $T = T_X$ consisting of triples $(\bar{\alpha}, \bar{i}, \bar{\beta})$ from ω_1 which resemble a triple (ω_1, i, β) from ω_2 with $L_\beta \models \phi(X, P, i)$. Distinct uncountable branches in the tree T will correspond to distinct triples (ω_1, i, β). In turn, whether T has ω_2-many branches (i.e., if T is Kurepa) will characterize whether $X \in A$. This will prove that KT is Π^1_2-complete.

We will say that a triple (α, i, β) from ω_2 is *good* (with respect to X, ϕ and P) if

(1) $\alpha < i < \beta < \omega_2$,
(2) $L_\beta \models \alpha = \omega_1$,
(3) β is the least limit ordinal such that
  (a) $X \cap \alpha, P \cap \alpha \in L_\beta$,
  (b) $L_\beta \models |i| = \alpha$,
  (c) $L_\beta \models \phi(X \cap \alpha, P \cap \alpha, i)$.

Note that $\alpha \le \omega_1$; in case of equality, $X \cap \alpha = X$ and $P \cap \alpha = P$. If $\alpha < \omega_1$ then $\beta < \omega_1$ as well, by (2).

Claim 4.3. *If (α, i, β) is good then the Skolem hull of $\alpha \cup \{\alpha, X \cap \alpha, P \cap \alpha, i\}$ in L_β is all of L_β.*

Proof. Let H be the above Skolem hull and $\pi : H \simeq L_{\bar{\beta}}$ its transitive collapse map. As L_β contains a map from α onto i as an element, so does H. Therefore since α is contained in H, so is i and hence $\pi(i) = i$. Also $\pi(\alpha) = \alpha$,

$\pi(X\cap\alpha)=X\cap\alpha$ and $\pi(P\cap\alpha)=P\cap\alpha$. It follows that α, i, $X\cap\alpha$ and $P\cap\alpha$ have the same properties in $L_{\bar\beta}$ that they have in L_β and therefore by the minimality of β, $\bar\beta=\beta$. As $L_{\bar\beta}$ is the Skolem hull of $\alpha\cup\{\alpha,X\cap\alpha,P\cap\alpha,i\}$ in $L_{\bar\beta}$, this is also true of L_β. ∎

Next, we define an ordering on good triples: we write

$$(\bar\alpha,\bar i,\bar\beta)\lhd(\alpha,i,\beta)$$

if $\bar\alpha<\alpha$ and there is a (unique) elementary embedding

$$\psi:L_{\bar\beta}\hookrightarrow L_\beta$$

such that

- $\psi\restriction\bar\alpha$ is the identity,
- $\psi(\bar\alpha)=\alpha$,
- $\psi(X\cap\bar\alpha)=X\cap\alpha$, $\psi(P\cap\bar\alpha)=P\cap\alpha$, and
- $\psi(\bar i)=i$.

Note that $L_{\bar\beta}=\psi^{-1}(L_\beta)$ is the transitive collapse of

$$H=\mathrm{Hull}(\bar\alpha\cup\{\alpha,X\cap\alpha,P\cap\alpha,i\})$$

in L_β. In turn, for a given good triple (α,i,β) and $\bar\alpha<\alpha$, there are $\bar i,p\bar\beta$ such that $(\bar\alpha,\bar i,\bar\beta)<(\alpha,i,\beta)$ iff for the above hull H, $H\cap\alpha=\bar\alpha$.

CLAIM 4.4.

(1) *The relation $\lhd$ is transitive.*

Moreover, for any good triple (α,i,β),

(2) *for any $\bar\alpha<\alpha$, there is at most one choice of $\bar i,\bar\beta$ such that $(\bar\alpha,\bar i,\bar\beta)\lhd(\alpha,i,\beta)$,*

(3) *the set $\{\bar\alpha<\alpha:\exists\bar i,\bar\beta\ (\bar\alpha,\bar i,\bar\beta)\lhd(\alpha,i,\beta)\}$ is closed in α.*

Proof. (1) Suppose $(\alpha_0,i_0,\beta_0)\lhd(\alpha_1,i_1,\beta_1)\lhd(\alpha_2,i_2,\beta_2)$. Then $H_2=$ the Skolem hull of $\alpha_1\cup\{\alpha_2,X\cap\alpha_2,P\cap\alpha_2,i_2\}$ in L_{β_2} with parameters α_2, $X\cap\alpha_2,P\cap\alpha_2,i_2$ is isomorphic to L_{β_1} with parameters $\alpha_1,X\cap\alpha_1,P\cap\alpha_1,i_1$, and the Skolem hull of $\alpha_0\cup\{\alpha_1,X\cap\alpha_1,P\cap\alpha_1,i_1\}$ in L_{β_1} with parameters $\alpha_1,X\cap\alpha_1,P\cap\alpha_1,i_1$ is isomorphic to L_{β_0} with parameters $\alpha_0,X\cap\alpha_0$, $P\cap\alpha_0,i_0$. But the Skolem hull of $\alpha_0\cup\{\alpha_2,X\cap\alpha_2,P\cap\alpha_2,i_2\}$ in L_{β_2} with parameters $\alpha_2,X\cap\alpha_2,P\cap\alpha_2,i_2$ is isomorphic to the Skolem hull of $\alpha_0\cup\{\alpha_1,X\cap\alpha_1,P\cap\alpha_1,i_1\}$ in L_{β_1} with parameters $\alpha_1,X\cap\alpha_1,P\cap\alpha_1,i_1$ and therefore is isomorphic to L_{β_0} with parameters $\alpha_0,X\cap\alpha_0,P\cap\alpha_0,i_0$, which implies that $(\alpha_0,i_0,\beta_0)\lhd(\alpha_2,i_2,\beta_2)$.

(2) If $(\bar\alpha,\bar i,\bar\beta)\lhd(\alpha,i,\beta)$ then $L_{\bar\beta}$ is the transitive collapse of the Skolem hull of $\bar\alpha\cup\{\alpha,X\cap\alpha,P\cap\alpha,i\}$ in L_β and $\bar i$ is the image of i under the transitive collapse map of this hull. So $\bar i,\bar\beta$ are uniquely determined by $\bar\alpha$ and the triple (α,i,β).

(3) The set in question is the same as the set of $\bar{\alpha} < \alpha$ such that $H(\bar{\alpha}) =$ the Skolem hull of $\bar{\alpha} \cup \{\alpha, X \cap \alpha, P \cap \alpha, i\}$ in L_β has the property $H(\bar{\alpha}) \cap \alpha = \bar{\alpha}$. If $\bar{\alpha} < \alpha$ is the supremum of the set X of γ less than $\bar{\alpha}$ with this property then $\bar{\alpha}$ also has this property, as the union of the $H(\gamma)$ for γ in X equals $H(\bar{\alpha})$. ■

Claim 4.5. *The relation* $\lhd$ *is a tree order on good triples.*

Proof. Suppose that we are given good triples (α_k, i_k, β_k) for $k < 3$ such that

$$(\alpha_0, i_0, \beta_0), (\alpha_1, i_1, \beta_1) \lhd (\alpha_2, i_2, \beta_2).$$

If $\alpha_0 = \alpha_1$ then we have $(\alpha_0, i_0, \beta_0) = (\alpha_1, i_1, \beta_1)$; indeed, this follows from Claim 4.4. So we can assume $\alpha_0 < \alpha_1$. Now, if $\psi_j : L_{\beta_j} \hookrightarrow L_{\beta_2}$ witnesses that $(\alpha_j, i_j, \beta_j) \lhd (\alpha_2, i_2, \beta_2)$ for $j = 0, 1$ then $\psi = \psi_1^{-1} \circ \psi_0$ witnesses $(\alpha_0, i_0, \beta_0) \lhd (\alpha_1, i_1, \beta_1)$. ■

For some technical reasons, instead of taking the tree of good triples, we will look at functions associated to good triples and the tree formed by them. For each good triple (α, i, β), define a function $f_{(\alpha,i,\beta)}$ with domain $\alpha + 1$ as follows:

$$f_{(\alpha,i,\beta)}(\bar{\alpha}) = \begin{cases} (\alpha, i, \beta) & \text{if } \bar{\alpha} = \alpha, \\ (\bar{\alpha}, \bar{i}, \bar{\beta}) & \text{if } (\bar{\alpha}, \bar{i}, \bar{\beta}) \lhd (\alpha, i, \beta) \text{ for some } \bar{i}, \bar{\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined by Claim 4.4.

Claim 4.6. *For any* $(\bar{\alpha}, \bar{i}, \bar{\beta}) \lhd (\alpha, i, \beta)$, $f_{(\alpha,i,\beta)} \restriction \bar{\alpha} + 1 = f_{(\bar{\alpha},\bar{i},\bar{\beta})}$.

Proof. This follows immediately from the fact that $\lhd$ is a tree order. ■

We let T_X be the set of functions $f_{(\alpha,i,\beta)} \restriction (\bar{\alpha} + 1)$ where (α, i, β) is a good triple (with respect to X, ϕ, P) of countable ordinals and $\bar{\alpha} \leq \alpha$.

Claim 4.7. T_X *is a tree of height at most* ω_1 *and countable levels.*

Proof. We prove by induction that every level of T_X is countable. Elements of T_X on level $\bar{\alpha}$ are of the form $f = f_{(\alpha,i,\beta)} \restriction (\bar{\alpha} + 1)$. These functions satisfy either (i) $f(\bar{\alpha}) = 0$, or (ii) $f(\bar{\alpha}) = (\bar{\alpha}, \bar{i}, \bar{\beta})$. In case (i), f must be constantly 0 on an end-segment of $\bar{\alpha}$ and so f is completely determined by the previous levels, so by induction there are only countably many choices for f. In case (ii), we note that $f = f_{(\bar{\alpha},\bar{i},\bar{\beta})}$ by Claim 4.6. Finally, note that there are only countably many possibilities for the value of $\bar{\beta}$ since for any large enough $\gamma < \omega_1$, $L_\gamma \models |\alpha| \leq \aleph_0$. This proves that level $\bar{\alpha}$ of T_X must be countable. ■

The next claim will conclude the proof of the theorem.

Claim 4.8. *T_X has $\aleph_2$-many uncountable branches if and only if*

$$\sup\{i < \omega_2 : L_{\omega_2}[X] \models \phi(X, P, i)\} = \omega_2.$$

Proof. First, assume that $L_{\omega_2} \models \phi(X, P, i)$ for some $i > \omega_1$. Pick the minimal β such that $L_\beta \models \phi(X, P, i)$ and so (ω_1, i, β) is a good triple with respect to X, ϕ, P.

Subclaim 4.8.1. *$B = \{f_{(\omega_1,i,\beta)}\restriction(\alpha+1) : \alpha < \omega_1\}$ is an uncountable branch in T_X.*

Moreover, the branch B uniquely determines the triple (ω_1, i, β) by the following claim.

Subclaim 4.8.2. *(L_β, X, P, i) is the direct limit of $(L_{\bar\beta}, X \cap \bar\alpha, P \cap \bar\alpha, \bar i)$ for $(\bar\alpha, \bar i, \bar\beta) \lhd (\omega_1, i, \beta)$.*

Thus distinct good triples (ω_1, i, β) correspond to different branches in T_X.

Conversely, suppose that we have a branch B in T_X, which is not eventually 0. Now, the direct limit of the $(L_{\bar\beta}, X \cap \bar\alpha, P \cap \bar\alpha, \bar i)$ for $(\bar\alpha, \bar i, \bar\beta) \in \operatorname{ran} \bigcup B$ yields some (L_β, X, P, i) and a good triple (ω_1, i, β). Moreover, B is the restriction of $f_{(\omega_1,i,\beta)}$. Distinct ω_1-branches yield distinct triples (ω_1, i, β), and since β is uniquely determined by i, we get ω_2-many i such that $\phi(X, P, i)$ holds in L_{ω_2}. ∎

This proves the theorem. ∎

5. Open problems and future goals. A positive answer to the following question would show that there are no ZFC reductions between stationarity and AT.

Question 5.1. Is it consistent with CH that all Aronszajn trees are special and there are no canary trees?

Regarding Kurepa trees, the following remain open.

Question 5.2. Can KT be Δ^1_2 and non-empty?

Question 5.3. What is the complexity of KT under $\mathrm{MA}_{\aleph_1}$ (given such trees exist)?

The following would also be very interesting.

Question 5.4. Find a *natural* class of structures $\mathcal{X}$ in $\Sigma^1_1 \setminus \Delta^1_1$ or $\Pi^1_1 \setminus \Delta^1_1$ which is *not* complete for its complexity class.

The reason we ask for a natural class is that under $V = L$, one can build such *artificial* examples (and for inaccessible cardinals there are even natural examples) but we wonder if there are more combinatorial examples on ω_1. Also, Harrington proved that consistently no such intermediate classes exist.

Yet another axiom to consider in more detail is PFA(S) for coherent Suslin trees S (see e.g. [22]). Such models allow the existence of Suslin trees while sharing many properties with models of the proper forcing axiom.

QUESTION 5.5. How does PFA(S) affect the complexity of the classes AT, sAT and ST?

Once we force with the Suslin tree S over a model of PFA(S), the resulting extension has no Suslin trees any more, and in fact any two Aronszajn trees will be club-isomorphic [25]. Hence, the complexity of AT and sAT is Δ^1_1 as mentioned in Figure 2.

It would certainly be interesting to see to what extent our results generalize to higher cardinals above ω_1. In particular, we mention a recent result of Krueger [13] on the club-isomorphism of higher Aronszajn trees that could substitute the Abraham–Shelah model. The construction schemes developed by Brodsky, Lambie-Hanson and Rinot [4,15,18] for higher Suslin and Aronszajn trees also seem rather relevant. The theorem of Jensen on CH and all Aronszajn trees being special was recently generalized to higher cardinals in a breakthrough result by Asperó and Golshani [2]. Definability of NS_κ on successor cardinals was investigated by Friedman, Wu and Zdomskyy [9].

We believe that a similar analysis of other classes of structures on ω_1 is well worth exploring. To name the most natural candidates, we would be interested in the following:

(1) *Graphs and more generally colourings* $c : [\omega_1]^2 \to r$ with $r \leq \omega_1$. One might look at graphs with chromatic or colouring number ω_1, or strong colourings that witness the failure of square bracket relations (i.e., $\omega_1 \not\to [\omega_1]^2_{\omega_1}$). Hypergraphs and various set-systems are also natural candidates.
(2) *Ladder systems on* ω_1. Natural classes are ladder systems with various guessing properties (i.e., $\clubsuit$ sequences or club guessing sequences), and ladder systems with or without the uniformization property.
(3) *Linear orders on* ω_1. For Aronszajn and Suslin lines, our analysis most likely yields the appropriate complexities but we did not address the important class of *Countryman lines.*
(4) *Forcing notions.* We can consider various classes of forcing posets on ω_1 such as ccc, Knaster, σ-centred, σ-linked or proper partial orders.

Finally, it will be very natural to consider the classical *equivalence relations* on these classes and to find Borel definable reductions between them. To mention a few, we name the order isomorphism of trees and linear orders; the notion of club-isomorphism between trees; graph isomorphism; biembeddability of various structures; or forcing equivalence of posets.

Acknowledgements. The authors would like to thank the Austrian Science Fund (FWF) for the generous support through Grant I1921. The second author was also supported by NKFIH OTKA-113047.

References

[1] U. Abraham and S. Shelah, *Isomorphism types of Aronszajn trees*, Israel J. Math. 50 (1985), 75–113.

[2] D. Asperó and M. Golshani, *The special Aronszajn tree property at $\aleph_2$ and GCH*, arXiv:1809.07638, 2018.

[3] J. Baumgartner, J. Malitz and W. Reinhardt, *Embedding trees in the rationals*, Proc. Nat. Acad. Sci. U.S.A. 67 (1970), 1748–1753.

[4] A. M. Brodsky and A. Rinot, *A microscopic approach to Souslin-tree constructions, Part I*, Ann. Pure Appl. Logic 168 (2017), 1949–2007.

[5] K. J. Devlin and H. Johnsbraten, *The Souslin Problem*, Lecture Notes in Math. 405, Springer, 2006.

[6] M. Džamonja and J. Väänänen, *A family of trees with no uncountable branches*, Topology Proc. 28 (2004), 113–132.

[7] E. Fokina, S.-D. Friedman, J. Knight and R. Miller, *Classes of structures with universe a subset of ω_1*, J. Logic Comput. 23 (2013), 1249–1265.

[8] S.-D. Friedman, T. Hyttinen and V. Kulikov, *Generalized descriptive set theory and classification theory*, Mem. Amer. Math. Soc. 230 (2014), no. 1081, vi+80 pp.

[9] S.-D. Friedman, L. Wu and L. Zdomskyy, *δ_1-Definability of the non-stationary ideal at successor cardinals*, Fund. Math. 229 (2015), 231–254.

[10] T. Hyttinen and M. Rautila, *The canary tree revisited*, J. Symbolic Logic 66 (2001), 1677–1694.

[11] T. Jech, *Set Theory. The Third Millennium Edition, Revised and Expanded*, Springer, 2003.

[12] A. S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Math. Soc. Lecture Note Ser. 128, Cambridge Univ. Press, 1987.

[13] J. Krueger, *Club isomorphisms on higher Aronszajn trees*, Ann. Pure Appl. Logic 169 (2018), 1044–1081.

[14] K. Kunen, *Set Theory: An Introduction to Independence Proofs*, Elsevier, 2014.

[15] C. Lambie-Hanson, *Aronszajn trees, square principles, and stationary reflection*, Math. Logic Quart. 63 (2017), 265–281.

[16] A. H. Mekler and S. Shelah, *The canary tree*, Canad. Math. Bull. 36 (1993), 209–215.

[17] A. Mekler and J. Väänänen, *Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$*, J. Symbolic Logic 58 (1993), 1052–1070.

[18] A. Rinot, *Higher Souslin trees and the GCH, revisited*, Adv. Math. 311 (2017), 510–531.

[19] S. Shelah, *Proper and Improper Forcing*, Springer, 1998.

[20] S. Shelah and J. Väänänen, *Stationary sets and infinitary logic*, J. Symbolic Logic 65 (2000), 1311–1320.

[21] J. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, in: Axiomatic Set Theory, Proc. Sympos. Pure Math. 13, Part I, Amer. Math. Soc., 1971, 383–390.

[22] F. D. Tall, PFA$(S)[S]$ *for the masses*, Topology Appl. 232 (2017), 13–21.

[23] S. Todorčević, *Trees and linearly ordered sets*, in: Handbook of Set-Theoretic Topology, North-Holland, 1984, 235–293.
[24] J. Väänänen, *Games and trees in infinitary logic: a survey*, in: Quantifiers: Logics, Models and Computation, Vol. 1: Surveys, Kluwer, 1995, 105–138.
[25] T. Yorioka, *Club-isomorphisms of Aronszajn trees in the extension with a Suslin tree*, Notre Dame J. Formal Logic 58 (2017), 381–396.

Sy-David Friedman, Dániel T. Soukup
Kurt Gödel Research Center for Mathematical Logic
Universität Wien
1090 Wien, Austria
E-mail: sdf@logic.univie.ac.at
daniel.soukup@univie.ac.at
URL: http://www.logic.univie.ac.at/~sdf/
http://www.logic.univie.ac.at/~soukupd73/