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DEGREE THEORY ON $\aleph_\omega$

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0. Introduction

Singular cardinals are of particular interest in the study of recursion theory on the ordinals. On the one hand many familiar techniques from Turing degree theory such as the minimal degree construction break down completely when applied to many singular cardinals, due to their Σ_2-inadmissibility. Moreover it is known that even though the Friedberg–Muchnik solution to Post's problem can be adapted to all admissible ordinals [3] its relativized version will fail for singular cardinals of uncountable cofinality [2].

On the other hand sometimes the singularity of a cardinal can be an aid rather than a handicap in constructing recursively enumerable sets. In Friedman [1] it was shown that if κ is a limit cardinal, then the sets $S(\lambda)=\{\gamma<\kappa \mid \kappa\text{-cofinality}(\gamma)=\lambda\}$ occupy distinct intermediate κ-RE degrees as λ varies over infinite regular κ-cardinals. This result solves Post's problem for the κ-degrees without a priority argument.

In this paper we focus on the first singular L-cardinal, $\aleph_\omega^L$. The first two sections answer two questions left open from Friedman [1] concerning the sets $S(\aleph_n^L)$ described above. In Section 1 we use the infinite injury priority method to construct an incomplete $\aleph_\omega^L$-RE degree greater than the $\aleph_\omega^L$-degrees of the sets $S(\aleph_n^L)$. Section 2 provides a natural example of a nonzero $\aleph_\omega^L$-RE degree below the $\aleph_\omega^L$-degrees of both $S(\aleph_n^L)$ and $S(\aleph_m^L)$, for any n, m. The methods used here are an elaboration of the Gödel collapse methods used in Friedman [1]. Finally in Section 3 the notion of *character* of an $\aleph_\omega^L$-RE set is defined and studied. Using it an order-preserving embedding of the partial-ordering $\langle\mathcal{P}(\omega)/\text{Finite}, \subseteq\rangle$ into the $\aleph_\omega^L$-RE degrees is obtained, without the use of a priority argument.

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1. An incomplete upper bound for the sets $S(\aleph_n^L)$

Let α denote $\aleph_\omega^L$. Let

$$S = \{(\gamma, \delta) \mid \gamma, \delta \text{ are limit ordinals } <\alpha \text{ and } \alpha\text{-cofinality}(\gamma) = \alpha\text{-cofinality}(\delta)\}.$$

S is α-RE as $(\gamma, \delta) \in S$ iff $\exists f \in L_\alpha$ (f is an order-preserving function from an unbounded subset of γ onto an unbouded subset of δ). Moreover $S(\aleph_n^L) \leqslant_\alpha S$ for each n as $S(\aleph_n^L) = (S)_{\aleph_n^L} = \{\delta \mid (\aleph_n^L, \delta) \in S\}$. Unfortunately $S =_\alpha 0'$ as $\{\aleph_n^L \mid n \in \omega\} = \{\gamma \mid \gamma \text{ is a limit ordinal } <\alpha \text{ and } (S)_\gamma \cap \gamma = \emptyset\}$ has α-degree $0'$. Our result in this section is that there is an incomplete α-RE thick subset of S. A set $A \subseteq S$ is *thick* if for each $\gamma < \alpha$, $(S)_\gamma - (A)_\gamma$ is bounded in α (where $(S)_\gamma = \{\delta \mid (\gamma, \delta) \in S\}$). Clearly any thick subset of S is an upper bound (in the sense of $\leqslant_\alpha$) for the sets $S(\aleph_n^L)$.

Theorem 1.1. *S has an α-RE thick subset A of α-degree $<0'$.*

The original Thickness lemma for classical recursion theory was established by Shoenfield (see [4]). It its simplest form it states that if $B \subseteq \omega \times \omega$ is RE, $(B)_n$ is recursive for each n and C is nonrecursive, then B has a thick RE subset A such that $C \not\leqslant_T A$. The corresponding result for α is false. For, it is easy to construct an α-RE $B \subseteq \alpha \times \alpha$ such that any thick α-RE $A \subseteq B$ is high ($A' =_\alpha 0''$). But then $A =_\alpha 0'$ as Shore [5] showed that any incomplete α-RE set A is low ($A' =_\alpha 0'$).

There are two key properties of S used in the proof of Theorem 1.1. Let $\alpha_n = \aleph_n^L$. The first fact is that for any n, $S \cap (\alpha_n \times \alpha_n)$ has incomplete α_n-RE degree. For any $\gamma < \alpha$ let $(S)_{<\gamma} = \{(\gamma', \delta) \in S \mid \gamma' < \gamma\}$. The second fact is that if $\gamma < \alpha_{n-1}$, then α_n is $(S)_{<\gamma}$-stable; i.e., $\langle L_{\alpha_n}, (S)_{<\gamma} \cap (\alpha_n \times \alpha_n)\rangle$ is a Σ_1-elementary substructure of $\langle L_\alpha, (S)_{<\gamma}\rangle$.

To demonstrate the first fact note that $S \cap (\alpha_n \times \alpha_n)$ is α_n-RE as α_n is α-stable and S has a parameter-free $\Sigma_1(L_\alpha)$ definition. Note that $\{\beta \mid \beta \text{ is an } \alpha_n\text{-cardinal}\}$ is finite and hence α_n-finite. So $S \cap (\alpha_n \times \alpha_n)$ is in fact α_n-recursive since if γ, δ are limit ordinals $<\alpha_n$, $(\gamma, \delta) \notin S$ iff $\exists \alpha_n$-cardinals κ, λ s.t. $(\kappa, \gamma), (\lambda, \delta) \in S$ and $\kappa \neq \lambda$.

As for the second fact note that is suffices to establish the T-stability of α_n where $T = S(\omega) \vee S(\aleph_1^L) \vee \cdots \vee S(\aleph_{n-2}^L)$. But this follows from the remark made between Theorems 1 and 2 of Friedman [1].

Our proof follows the same outline as the proof of Shoenfield's Thickness Lemma given in Soare [7]. The facts above are used to bound the lim inf of the restraint imposed by a proper initial segment of negative requirements.

We use Soare's notation. Let $\Phi_{e,\sigma}(X; y)$ be the result, if any, of performing the eth partial α-recursive reduction with oracle X to argument y through stage $\sigma < \alpha$. Also let $\Phi_e(X) = \bigcup_\sigma (\lambda y \Phi_{e,\sigma}(X; y))$. We have in mind the α-recursive enumeration of our desired α-RE set A, written as $\{A^\sigma \mid \sigma < \alpha\}$. $A^{<\sigma} = \bigcup \{A^{\sigma'} \mid \sigma' < \sigma\}$. The definition of this enumeration is guided by some auxiliary functions:

$$u(e,x,\sigma)=\begin{cases}\min\{z \mid \Phi_{e,\sigma}(A^\sigma[z];x) \text{ is defined}\} & \text{if } z \text{ exists,}\\ 0 & \text{otherwise,}\end{cases}$$

$$a_\sigma=\begin{cases}\text{least } x[x\in A^\sigma-A^{<\sigma}] & \text{if } A^\sigma-A^{<\sigma}\neq\emptyset,\\ \sup(A^\sigma\cup\{\sigma\}) & \text{otherwise,}\end{cases}$$

$$\hat{\Phi}_{e,\sigma}(A^\sigma;x)=\begin{cases}\Phi_{e,\sigma}(A^\sigma;x) & \text{if defined and } u(e,x,\sigma)<a_\sigma,\\ \text{undefined} & \text{otherwise,}\end{cases}$$

$$\hat{u}(e,x,\sigma)=\begin{cases}u(e,x,\sigma) & \text{if } \hat{\Phi}_{e,\sigma}(A^\sigma;x) \text{ is defined,}\\ 0 & \text{otherwise.}\end{cases}$$

Thus we use the modified computation function $\hat{\Phi}$. It has the property that at a *true* stage σ any apparent computation $\hat{\Phi}_{e,\sigma}(A^\sigma;x)$ is a true computation $\Phi_e(A;x)$. The set of true stages T is defined by:

$$T=\{\sigma \mid A^\sigma[a_\sigma]=A[a_\sigma]\}.$$

We also use:

$$\hat{l}(e,\sigma)=\sup\{x \mid \forall\, y<x\ (K^\sigma(y)=\hat{\Phi}_{e,\sigma}(A^\sigma;y))\},$$
$$\hat{r}(e,\sigma)=\sup\{\hat{u}(e,x,\sigma) \mid x\leqslant\hat{l}(e,\sigma)\},$$
$$\hat{R}(e,\sigma)=\sup\{\hat{r}(e',\sigma) \mid e'\leqslant e\}.$$

These are the length of agreement function, restraint function and full restraint function, respectively. The set K is the complete α-RE set defined by $K=\{(e,x) \mid \Phi_{e,\sigma}(\emptyset,x) \text{ is defined for some } \sigma\}$. Thus $\hat{l}$ measures the length of agreement between K and $\Phi_e(A)$ at stage σ. $\hat{r}$ indicates how large an initial segment of A must remain unchanged for the sake of preserving $\Phi_e(A)$ through this length of agreement.

The element x *injures* e at stage σ if $x\in A^\sigma-A^{<\sigma}$ and $x\leqslant\hat{r}(e,\sigma)$. So we are thinking of e synonymously with the requirement $N_e: K\neq\Phi_e(A)$ (where K is identified with its characteristic function). The strategy for achieving N_e is to preserve agreements between K and $\Phi_e(A)$. If x injures e, then x is interfering with this strategy. We define the *injury sets*

$$\hat{I}_{e,\sigma}=\{x \mid \exists\,\sigma'\leqslant\sigma\,[x\leqslant\hat{r}(e,\sigma') \text{ and } x\in A^{\sigma'}-A^{<\sigma'}1\},$$
$$\hat{I}_{e,<\sigma}=\bigcup\{\hat{I}_{e,\sigma'} \mid \sigma'<\sigma\},\qquad \hat{I}_e=\bigcup\{\hat{I}_{e,\sigma} \mid \sigma<\alpha\}.$$

In terms of the above definitions our construction proceeds as follows: First choose an α-recursive enumeration $\{S^\sigma \mid \sigma<\alpha\}$ of the α-RE set S. We enumerate x into $(A^\sigma)_\gamma$ if $x\in(S^\sigma)_\gamma$ and $x>\hat{r}(i,\sigma)$ for all $i\leqslant\gamma$. Let $A=\bigcup\{A^\sigma \mid \sigma<\alpha\}$.

We must show that $(A)_\gamma$ contains a final segment of $(S)_\gamma$ and $K\neq\Phi_\gamma(A)$, for each $\gamma<\alpha$. This is done by establishing the following claims by induction on $\gamma<\alpha$:

(i) For $\gamma<\alpha_n$, $\lim\{\hat{R}(\gamma,\sigma)\mid\sigma\in T_\gamma\}<\alpha_{n+1}$ where

$$T_\gamma=\{\sigma\mid(A)_{<\gamma}[a_\sigma]=(A^\sigma)_{<\gamma}[a_\sigma]\}$$
$$=\text{true stages for enumeration of } (A)_{<\gamma}.$$

(ii) For $\gamma<\alpha_n$, $(S)_\gamma-(A)_\gamma$ is a bounded subset of α_{n+1}.

Suppose (i), (ii) hold for all $\gamma'<\gamma$ and we seek to establish (i), (ii) for $\gamma\geqslant 0$. (Thus the base case of the induction is included here.) Note that for $\gamma'<\gamma$, $T_{\gamma'}\supseteq T_\gamma$. So $\lim\{\sup_{\gamma'<\gamma}\hat{R}(\gamma',\sigma)\mid\sigma\in T_\gamma\}<\alpha_{n+1}$ by the regularity of α_{n+1}. But then $(S)_{<\gamma}-(A)_{<\gamma}$ is a bounded subset of α_{n+1} since at a stage $\sigma\in T_\gamma$ nothing prevents $x\in(S^\sigma)_{<\gamma}$ from entering $(A^\sigma)_{<\gamma}$ if x is greater than the above limit. It follows that $\hat{I}_\gamma$ is α-recursive in $(S)_{<\gamma}$ since $(A)_{<\gamma}\leqslant_\alpha(S)_{<\gamma}$ and:

$$x\in\hat{I}_\gamma \quad\text{iff } x\in(A)_{<\gamma} \text{ and } x\in\hat{I}_{\gamma,\sigma} \text{ where } \sigma \text{ is the stage}$$
$$\text{s.t. } x\in(A^\sigma)_{<\gamma}-(A^{<\sigma})_{<\gamma}.$$

Note that the reduction $\hat{I}_\gamma\leqslant_\alpha(S)_{<\gamma}$ only uses parameters $<\alpha_{n+1}$ and we have that $\hat{I}_{\gamma,<\alpha_{n+1}}\leqslant_{\alpha_{n+1}}(S)_{<\gamma}\cap(\alpha_{n+1}\times\alpha_{n+1})$.

We claim now that $K\cap\alpha_{n+1}\neq\Phi_\gamma(A)\cap\alpha_{n+1}$. First note that $K\cap\alpha_{n+1}\neq\Phi_{\gamma,<\alpha_{n+1}}(A^{<\alpha_{n+1}})$. For otherwise, as $K\cap\alpha_{n+1}=K^{<\alpha_{n+1}}$ by the α-stability of α_{n+1}, we would have $\lim\{\hat{l}(\gamma,\sigma)\mid\sigma<\alpha_{n+1}\}=\alpha_{n+1}$. But now we can compute $K\cap\alpha_{n+1}$ in an α_{n+1}-recursive way from $\hat{I}_{\gamma,<\alpha_{n+1}}$: To compute $K(\rho)$ for $\rho<\alpha_{n+1}$ find some stage $\sigma<\alpha_{n+1}$ such that $\hat{l}(\gamma,\sigma)>\rho$ and

$$\forall x\leqslant\rho\,\forall z\,[z\leqslant u(\gamma,x,\sigma)\rightarrow(z\notin\hat{I}_{\gamma,<\alpha_{n+1}} \text{ or } z\in A^\sigma)].$$

Then $K(\rho)=\Phi_{\gamma,\sigma}(A^\sigma;\rho)$: To see this it suffices to argue that for $\tau>\sigma$, $\tau<\alpha_{n+1}$ we have $\hat{l}(\gamma,\tau)>\rho$ and $\hat{r}(\gamma,\tau)\geqslant\sup\{u(\gamma,x,\sigma)\mid x\leqslant\rho\}$. If τ were the least counterexample to this claim, then we must have $K^\tau(x)\neq K^{<\tau}(x)$ for some $x\leqslant\rho$. But then $K(x)=K^\tau(x)\neq\Phi_{\gamma,<\alpha_{n+1}}(A^{<\alpha_{n+1}})(x)$ as the information about A^σ used in establishing $\Phi_{\gamma,\sigma}(A^\sigma;x)$ cannot change before stage α_{n+1}. We have therefore shown $K\cap\alpha_{n+1}\leqslant_{\alpha_{n+1}}\hat{I}_{\gamma,<\alpha_{n+1}}$. But then $K\cap\alpha_{n+1}\leqslant_{\alpha_{n+1}}(S)_{<\gamma}\cap(\alpha_{n+1}\times\alpha_{n+1})$ which is impossible since $K\cap\alpha_{n+1}=\{(e,x)\mid\Phi_{e,\sigma}(\emptyset;x)$ is defined for some $\sigma<\alpha_{n+1}\}$ is complete α_{n+1}-RE and $(S)_{<\gamma}\cap(\alpha_{n+1}\times\alpha_{n+1})$ is α_{n+1}-recursive.

Second, we argue that $\Phi_{\gamma,<\alpha_{n+1}}(A^{<\alpha_{n+1}})=\Phi_\gamma(A)\cap\alpha_{n+1}$. For the α-stability of α_{n+1} implies that $A^{<\alpha_{n+1}}=A\cap\alpha_{n+1}$ so clearly $\Phi_{\gamma,<\alpha_{n+1}}(A^{<\alpha_{n+1}})\subseteq\Phi_\gamma(A)$. But conversely, if $\Phi_\gamma(A)(x)=y$ where $x,y<\alpha_{n+1}$, then we have:

$$\exists\sigma<\alpha[\Phi_{\gamma,\sigma}(A^\sigma;x)=y \text{ and } \forall z(z\leqslant u(\gamma,x,\sigma)\rightarrow(z\notin\hat{I}_\gamma \text{ or } z\in A^\sigma))].$$

So since α_{n+1} is $(S)_{<\gamma}$-stable and $\hat{I}_\gamma\leqslant_\alpha(S)_{<\gamma}$ (with parameter $<\alpha_{n+1}$ for the reduction):

$$\exists\sigma<\alpha_{n+1}[\Phi_{\gamma,\sigma}(A^\sigma;x)=y \text{ and } \forall z(z\leqslant u(\gamma,x,\sigma)\rightarrow(z\notin\hat{I}_\gamma \text{ or } z\in A^\sigma))].$$

But then $\Phi_{\gamma,<\alpha_{n+1}}(A^{<\alpha_{n+1}};x)=y$.

Now we can establish (i), (ii), for γ. Define ρ = least x s.t. $K(x) \neq \Phi_\gamma(A)(x)$. Then $\rho < \alpha_{n+1}$. By the preceding paragraph we can choose $\sigma < \alpha_{n+1}$ such that for all $\tau \geq \sigma$: $\forall x \leq \rho\ (K(x) = K^\tau(x))$ and $\forall x < \rho\ (\hat{\Phi}_{\gamma,\tau}(A^\tau; x) = \Phi_\gamma(A)(x))$. If for all $\tau \in T_\gamma$, $\tau \geq \sigma$ $\hat{\Phi}_{\gamma,\tau}(A^\tau; \rho)$ is undefined, then for any $\tau \geq \sigma, \tau \in T_\gamma \rightarrow \hat{l}(\gamma, \tau) = \rho$ and $\hat{r}(\gamma, \tau) = \sup\{\hat{u}(\gamma, x, \sigma) \mid x < \rho\}$. So $\lim\{\hat{r}(\gamma, \tau) \mid \tau \in T_\gamma\}$ exists and is less than α_{n+1}. If $\hat{\Phi}_{\gamma,\tau}(A^\tau; \rho)$ is defined for some $\tau \geq \sigma$, $\tau \in T_\gamma$, then for any $\sigma' \geq \tau$ $K(\rho) \neq \hat{\Phi}_{\gamma,\sigma'}(A^{\sigma'}; \rho)$ since the computation $\hat{\Phi}_{\gamma,\tau}(A^\tau; \rho) = y$ cannot be injured at any stage $\sigma' \geq \tau$ (as τ is a true stage for the enumeration of $(A)_{<\gamma}$). Thus for all $\sigma' \geq \tau$ $\hat{l}(\gamma, \sigma') = \rho$ and $\hat{r}(\gamma, \sigma') = \hat{r}(\gamma, \tau)$. So once again $\lim\{\hat{r}(\gamma, \sigma') \mid \sigma' \in T_\gamma\}$ exists. But note that τ can be chosen to be less than α_{n+1} by the $(S)_{<\gamma}$-stability of α_{n+1}, since $T_\gamma \leq_\alpha (A)_{<\gamma} \leq_\alpha (S)_{<\gamma}$ and all of these reductions only involve parameters less than α_{n+1}. We have therefore established (i) for γ.

We can now easily conclude (ii) for γ since at a stage $\sigma \in T_\gamma$ nothing can prevent $x \in (S^\sigma)_\gamma$ from entering $(A^\sigma)_\gamma$ provided $x > \lim\{\hat{R}(\gamma, \sigma') \mid \sigma' \in T_\gamma\}$ and this last ordinal is less than α_{n+1} by (i) for γ.

Finally note that in the course of establishing (i), (ii) we demonstrated that $K \neq \Phi_\gamma(A)$ for each $\gamma < \alpha$. Also (ii) immediately implies that A is a thick subset of S. This completes the proof of Theorem 1.1.

2. $S(\aleph_n^L)$, $S(\aleph_m^L)$ do not form a minimal pair

To establish this result we follow a strategy communicated to us by Carl Jockusch. If A, B are RE sets of integers, choose recursive onto functions $f: \omega \rightarrow A$, $g: \omega \rightarrow B$. An RE set recursive simultaneously in A and B is $C = \{(x, y) \mid \text{for some } n, x = f(n), y = g(n)\}$. This can be a useful way of showing that A, B do not form a minimal pair.

Thus let $A = S(\aleph_n^L) = \{\beta < \aleph_\omega^L \mid L\text{-cof}(\beta) = \aleph_n^L\}$, $B = S(\aleph_m^L) = \{\beta < \aleph_\omega^L \mid L\text{-cof}(\beta) = \aleph_m^L\}$. In order to produce C as above we first make a definition.

Definition. If $\beta \in S(\aleph_i^L)$ let $\hat{\beta}$ be the least ordinal γ such that there is a cofinal $f: \aleph_i^L \rightarrow \beta$ which is definable over L_γ.

Then set $C = \{(\beta_1, \beta_2) \mid \beta_1 \in S(\aleph_n^L), \beta_2 \in S(\aleph_m^L) \text{ and } \hat{\beta}_1 = \hat{\beta}_2\}$. It is easy to see that C is $\aleph_\omega^L$-recursive in each of $S(\aleph_n^L)$, $S(\aleph_m^L)$. For, given (β_1, β_2) first check if $\beta_1 \in S(\aleph_n^L)$; if not, then $(\beta_1, \beta_2) \notin C$. If so, then compute $\hat{\beta}_1$ effectively and check if $L_{\hat{\beta}_1+1}$ contains a cofinal increasing $f: \aleph_m^L \rightarrow \beta_2$. If it does but $L_{\hat{\beta}_1}$ does not, then $(\beta_1, \beta_2) \in C$. Otherwise $(\beta_1, \beta_2) \notin C$. This shows that C is $\aleph_\omega^L$-recursive in $S(\aleph_n^L)$. A similar argument works for $S(\aleph_m^L)$.

It remains to show that C is not $\aleph_\omega^L$-recursive. Suppose it is and let $\exists y\, \phi(y, \beta_1, \beta_2)$ define the complement of C over $\langle L_{\aleph_\omega^L}, \in \rangle$ where ϕ is Δ_0 with parameter p. Choose k so large that $m, n < k$ and $p \in L_{\aleph_k^L}$. Note that $(\aleph_{k+1}^L, \aleph_{k+2}^L) \notin C$ so we can choose $y'' \in L_{\aleph_\omega^L}$ such that $\phi(y'', \aleph_{k+1}^L, \aleph_{k+2}^L)$. We get our desired contradiction by producing $\beta_1, \beta_2, y \in L_{\aleph^L_{k+1}}$ such that $(\beta_1, \beta_2) \in C$ but $\phi(y, \beta_1, \beta_2)$.

The desired β_1, β_2, y are obtained by applying a Gödel collapse argument to $\aleph^L_{k+1}$, $\aleph^L_{k+2}$, y''. The construction is very similar to the proof of Theorem 1 in Friedman [1].

Choose a Σ_2-admissible $\alpha'' < \aleph^L_\omega$ large enough so that $y'' \in L_{\alpha''}$, $\aleph^L_{k+2} < \alpha''$. Define an $\aleph^L_m$-sequence by:

$$H_0 = \Sigma_1 \text{ Skolem hull of } \aleph^L_{k+1} \cup \{y'', \aleph^L_{k+2}\} \text{ in } L_{\alpha''},$$

$$\gamma_0 = H_0 \cap \aleph^L_{k+2},$$

$$H_{\delta+1} = \Sigma_1 \text{ Skolem hull of } \aleph^L_{k+1} \cup \{y'', \aleph^L_{k+2}, \gamma_\delta\} \text{ in } L_{\alpha''},$$

$$\gamma_{\delta+1} = H_{\delta+1} \cap \aleph^L_{k+2},$$

$$H_\lambda = \bigcup \{H_\delta \mid \delta < \lambda\} \quad \text{for limit } \lambda,$$

$$\gamma_\lambda = \sup\{\gamma_\delta \mid \delta < \lambda\}.$$

Finally set $H = \bigcup \{H_\delta \mid \delta < \aleph^L_m\}$, $\beta'_2 = \sup\{\gamma_\delta \mid \delta < \aleph^L_m\}$. Let $\pi: H \simeq L_{\alpha'}$ and define $y' = \pi(y'')$. Note that $\pi(\aleph^L_{k+2}) = \beta'_2$ and $L_{\alpha'} \vDash \phi(y', \aleph^L_{k+1}, \beta'_2)$ since π is the identity on L_{β_2}. And most importantly $L_{\alpha'} \vDash \beta'_2$ is regular but there is a $\Sigma_2(L_{\alpha'})$ cofinal increasing $f: \aleph^L_m \to \beta'_2$, given by $f(\delta) = \gamma_\delta$. So $\hat{\beta}'_2 = \alpha'$.

Now we repeat the above with an $\aleph^L_n$-iteration of Σ_2 Skolem hulls inside $L_{\alpha'}$. Thus:

$$K_0 = \Sigma_2 \text{ Skolem hull of } \aleph^L_k \cup \{y', \beta'_2, \aleph^L_{k+1}\} \text{ in } L_{\alpha'},$$

$$\mu_0 = K_0 \cap \aleph^L_{k+1},$$

$$K_{\delta+1} = \Sigma_2 \text{ Skolem hull of } \aleph^L_k \cup \{y', \beta'_2, \mu_\delta, \aleph^L_{k+1}\} \text{ in } L_{\alpha'},$$

$$\mu_{\delta+1} = K_{\delta+1} \cap \aleph^L_{k+1},$$

$$K_\lambda = \bigcup \{K_\delta \mid \delta < \lambda\}, \quad \lambda \text{ limit},$$

$$\mu_\lambda = K_\lambda \cap \aleph^L_{k+1}, \quad \lambda \text{ limit}.$$

Finally set $K = \bigcup \{K_\delta \mid \delta < \aleph^L_n\}$, $\beta_1 = \sup\{\mu_\delta \mid \delta < \aleph^L_n\}$. Let $\sigma: K \simeq L_\alpha$, $y = \sigma(y')$, $\beta_2 = \sigma(\beta'_2)$. Note that $\sigma(\aleph^L_{k+1}) = \beta_1$ and $L_\alpha \vDash \phi(y, \beta_1, \beta_2)$. As $\aleph^L_{k+1}$, β'_2 are regular in $L_{\alpha'}$, we have that β_1, β_2 are regular in L_α. But there is a $\Sigma_3(L_\alpha)$ cofinal increasing $g: \aleph^L_n \to \beta_1$, namely $g(\delta) = \mu_\delta$. And if $f: \aleph^L_m \to \beta'_2$ is the cofinal, increasing $\Sigma_2(L_{\alpha'})$ function mentioned earlier, we have that $\sigma \circ f \cdot \aleph^L_m \to \beta_2$ is cofinal, increasing and $\Sigma_2(L_\alpha)$ since K is a Σ_2 elementary submodel of $L_{\alpha'}$ containing the defining parameters for f (namely y', β'_2, $\aleph^L_m$). So $\hat{\beta}_1 = \hat{\beta}_2 = \alpha$. As $\beta_1 \in S(\aleph^L_n)$, $\beta_2 \in S(\aleph^L_m)$ we have proved:

Theorem 2.1. *$S(\aleph^L_m)$, $S(\aleph^L_n)$ do not form a minimal pair for any $m, n < \omega$.*

The same proof shows:

Corollary 2.2 (to proof). *$S(\aleph^L_{n_1}), \ldots, S(\aleph^L_{n_k})$ have a common nonzero lower bound for any $n_1, \ldots, n_k < \omega$.*

A refinement of the proof shows: The set

$$\{(\beta_1, \ldots, \beta_k) \mid \beta_i \in S(\aleph^L_{n_i}) \text{ all } i, \hat{\beta}_1 = \hat{\beta}_2 = \cdots = \hat{\beta}_k\}$$

is a non $\aleph^L_\omega$-recursive lower bound for $S(\aleph_n)$ if and only if $n = n_i$ for some i. This can be used to produce an order-reversing embedding of the tree $\omega^{<\omega}$ into the $\aleph^L_\omega$-RE degrees (and without use of the priority method).

3. Characters of $\aleph^L_\omega$-RE sets

We continue to fix $\alpha = \aleph^L_\omega$. The set $S(\omega)$, as was shown in Friedman [1], has the property that

$$\langle L_{\aleph^L_n}, S(\omega) \cap \aleph^L_n \rangle <_1 \langle L_\alpha, S(\omega) \rangle$$

for each $n > 1$ ($<_1$ denotes 'Σ_1-elementary substructure'). Not every α-r.e. set has this property. In particular, let C be of complete degree. Then C is not hyperregular and so there is a function $f : \omega \to \alpha$ such that $f \leqslant_{\omega\alpha} C$ and $\sup\{f(n) \mid n < \omega\} = \alpha$. By introducing parameters if necessary, one may assume that $|f(n)| < |f(n+1)|$ for all n. This imples that for all sufficiently large n, $\langle L_{\aleph^L_n}, C \cap \aleph^L_n \rangle$ is not a Σ_1-elementary substructure of $\langle L_\alpha, C \rangle$. We are thus naturally led to the following definition.

Definition 3.1. Let A be a subset of α. The *character* of A, denoted char(A), is the set of all $n < \omega$ such that

$$\langle L_{\aleph^L_n}, A \cap \aleph^L_n \rangle <_1 \langle L_\alpha, A \rangle.$$

Proposition 3.2. *Let A and B in L be such that $A \leqslant_\alpha B$. Then* char$(A) \supseteq^*$ char(B), *where $\supseteq^*$ means containment module finite sets.*

Proof. Let $\exists y\, \phi$ and $\exists y\, \bar{\phi}$ be $\Sigma_1(B)$ sentences such that for all α-finite K:

$$K \subseteq A \Leftrightarrow \langle L_\alpha, B \rangle \models \exists y\, \phi(y, K)$$

and

$$K \subseteq \bar{A} \Leftrightarrow \langle L_\alpha, B \rangle \models \exists y\, \bar{\phi}(y, K).$$

Let n_0 be chosen so that $\aleph^L_{n_0}$ is greater than the parameters occurring in ϕ and $\bar{\phi}$. Let $n > n_0$ and $n \in \text{char}(B)$. Let $\exists x\, \psi(x)$ be $\Sigma_1(A)$ with parameters less than $\aleph^L_n$. Suppose that

$$\langle L_\alpha, A \rangle \models \exists x\, \psi(x).$$

Then

$$\langle L_\alpha, B \rangle \models \exists K_1 K_2\, \exists y_1 y_2\, \exists \gamma\, (\phi(y_1, K_1)\ \&\ \bar{\phi}(y_2, K_2)$$
$$\&\ \langle L_\gamma, K_1, K_2 \rangle \models \exists x\ \psi'(x)) \qquad (*)$$

where ψ' is obtained from ψ by replacing '$z \in A$' by $z \in K_1$, '$z \notin A$' by $z \in K_2$. By the $\Sigma_1(B)$-stability of $\aleph^L_n$, the statement $(*)$ is true in the structure $\langle L_{\aleph^L_n}, B \cap \aleph^L_n \rangle$.

In other words,

$$\langle L_{\aleph_n^L}, B \cap \aleph_n^L \rangle \models \exists K_1 K_2 \exists y_1 y_2 \exists \gamma\, (\phi(y_1, K_1) \;\&\; \bar{\phi}(y_2, K_2) \;\&\; \langle L_\gamma, K_1, K_2 \rangle \models \exists x\, \psi'(x)).$$

Hence $\langle L_{\aleph_n^L}, A \cap \aleph_n^L \rangle \models \exists x\, \psi(x)$, so that $n \in \mathrm{Char}(A)$.

Remark. It follows from Proposition 3.2 that if $A \equiv_\alpha B$, then $\mathrm{char}(A) = \mathrm{char}(B)$ modulo finite sets (written $\mathrm{char}(A) =^* \mathrm{char}(B)$). The converse is not true. Indeed since characters are degree-theoretically invariant (modulo finite sets), and there are only $\aleph_1^L$ possibilities for characters while there are $\alpha = \aleph_\omega^L$ many α-r.e. degrees, there certainly exist many different α-r.e. degrees whose characters are only finitely different.

In the Sacks–Simpson [3] construction the incomparable α-r.e. degrees have characters $=^* \omega$, as with the $S(n)$, for $n = \aleph_0, \aleph_1^L, \ldots, \aleph_n^L, \ldots$.

We now prove a theorem which is a weak form of the density theorem. The theorem says that for each constructible $K \subset \mathrm{char}(A)$, there is a B such that $A \leqslant_\alpha B$ and $\mathrm{char}(B) =^* K$. This result has the following consequences: (i) (Representability) Every constructible $K \subset \omega$ is the character of an α-r.e. set A_K; (ii) (Friedberg–Muchnik–Sacks–Simpson theorem) There exist $\aleph_1^L$-many pairwise incomparable α-r.e. degrees; (iii) (Upward Density Theorem) If $\mathrm{char}(A) \neq^* \emptyset$, then there exist $\aleph_1^L$ many α-r.e. degrees d such that $\deg(A) <_\alpha d <_\alpha 0'$. While (ii) and (iii) are known results (in particular, Shore [6] has proved the full Density theorem), our method of proof is different and *does not use a priority argument.* Furthermore, (i) provides a classification of hyperregular α-r.e. degrees according to the measure of stability (in terms of characters) of the sets sitting in them. This may provide a useful tool in the study of the fine structure of hyperregular α-r.e. degrees.

Theorem 3.3. *Let A be α-r.e. and let $K \subset \mathrm{char}(A)$ be constructible. There exists an α-r.e. set $B \geqslant_\alpha A$ such that $\mathrm{char}(B) =^* K$.*

Proof. If $\mathrm{char}(A) =^* \emptyset$, then we let $B = A$. Hence we may assume that $\mathrm{char}(A) \neq^* \emptyset$ and let $K \subset \mathrm{char}(A)$ be constructible.

The set B will consist of pairs $\langle x, y \rangle$ such that

$$\forall x (\langle x, 0 \rangle \in B \leftrightarrow x \in A).$$

This ensures that $A \leqslant_\alpha B$. We let $B_0 = \{x \mid \langle x, 0 \rangle \in B\}$.

Let $B_1 = \{y \mid \langle 0, y \rangle \in B\}$. We construct B so that for each $n \notin K$, $B_1 \supseteq (\aleph_{n-1}^L, \aleph_n^L)$. Without loss of generality we may assume $n \geqslant 1$. Furthermore, it is safe to assume that $\omega - K \neq^* \emptyset$ since otherwise we may again take $B = A$ to prove the theorem.

The objective is to construct a B such that $B \geqslant_\alpha A$ and for all but finitely many $n \in \omega$,

$$\langle L_{\aleph_n^L}, B \cap \aleph_n^L \rangle <_1 \langle L_\alpha, B \rangle$$

if and only if $n \in K$.

To kill the B-stability of $\aleph_n^L$, $n \notin K$, note that if $B_1 \supseteq (\aleph_{n-1}^L, \aleph_n^L)$, then

$$\langle L_\alpha, B\rangle \models \exists x (x \notin B_1 \wedge x > \aleph_{n-1}^L) \qquad (**)$$

but $\langle L_{\aleph_n^L}, B \cap \aleph_n^L\rangle$ fails to satify the same sentence.

At stage σ, let $A^{<\sigma}$ be the set of ordinals which have been enumerated in A before σ, and define $B^{<\sigma}$, $B_i^{<\sigma}$ $(i=0,1)$ similarly. Let σ' be the least $\tau > \sigma$ such that:

(i) $L_\tau \models \exists\, n < \omega(|\sigma| = \aleph_n)$,

(ii) let τ_n denote the nth infinite cardinal of L_τ. Suppose that $|\sigma|^{L_\tau} = \tau_m$. Then for each $n \leq m$ such that $n \in K$,

$$\langle L_{\tau_n}, A^{<\tau} \cap \tau_n\rangle <_1 \langle L_\tau, A^{<\tau}\rangle.$$

Note that $i \in \mathrm{char}(A)$, $\sigma < \aleph_i^L$ implies that $\tau = \aleph_i^L$ obeys (i), (ii). So σ' exists.

Let $n(\sigma)$ be the integer m such that $|\sigma| = \sigma'_m$ in $L_{\sigma'}$. If $n(\sigma)+1 \in K$, let

$$B^\sigma = B^{<\sigma} \cup \{\langle x, 0\rangle \mid x \in A^{\sigma'}\}.$$

If $n(\sigma)+1 \notin K$, then let

$$B^\sigma = B^{<\sigma} \cup \{\langle x, 0\rangle \mid x \in A^{\sigma'}\} \cup \{\langle 0, y\rangle \mid y \in (\sigma'_{n(\sigma)}, \sigma)\}.$$

Finally let

$$B = \bigcup_{\sigma<\alpha} B^\sigma.$$

Lemma 3.4. *If $n \notin K$, then $B_1 \supseteq (\aleph_{n-1}^L, \aleph_n^L)$.*

Proof. This follows easily from our construction. Let σ be chosen so that $\sigma'_{n(\sigma)} = \aleph_{n-1}^L$. Then $(\aleph_{n-1}^L, \sigma) \subseteq B_1^\sigma$. As such σ's occur unboundedly often in $\aleph_n^L$, we have the lemma.

Lemma 3.5. *Suppose $n \in K$ and $L_{\aleph_{n-1}^L}$ contains the parameters which define A and B. Then*

$$\langle L_{\aleph_n^L}, B \cap \aleph_n^L\rangle <_1 \langle L_\alpha, B\rangle.$$

Proof. Let $n \in K$ be as given and let $\exists x\, \phi(x, a_1, \ldots, a_r, B)$ be a $\Sigma_1(B)$-sentence with parameters $a_1, \ldots, a_r$ in $L_{\aleph_n^L}$. Suppose that this sentence is true in $\langle L_\alpha, B\rangle$. Then

$$\langle L_\alpha, A\rangle \models \exists \sigma\, \exists x\, [B_\sigma^{<\sigma} = A \cap \sigma = A^{<\sigma} \ \&\ \phi(x, a_1, \ldots, a_r, B^{<\sigma})].$$

As $K \subset \mathrm{char}(A)$, we have $n \in \mathrm{char}(A)$ and so $\langle L_{\aleph_n^L}, A \cap \aleph_n^L\rangle$ satisfies this sentence which we denote by ψ.

Choose $\hat\sigma$ to be the least $\sigma \geq \aleph_{n-1}^L$ for which ψ is true in $\langle L_\sigma, A \cap \sigma\rangle$. Our goal is to show that $B^{<\hat\sigma} = B \cap \hat\sigma$ and therefore that $\exists x\, \phi$ is true in $\langle L_{\hat\sigma}, B \cap \hat\sigma\rangle$.

Claim. $n(\hat\sigma)+1 = n$.

Clearly $n(\hat{\sigma})+1 \geqslant n$.

Suppose $n(\hat{\sigma})>n-1$. In the notation of our construction, let $\hat{\sigma}'_{n(\hat{\sigma})}$ be the cardinality of $\hat{\sigma}$ in $L_{\hat{\sigma}'}$. Then by (ii) in the definition of $\hat{\sigma}'$, we have $\hat{\sigma}'_n \leqslant \hat{\sigma}'_{n(\hat{\sigma})}$ and

$$\langle L_{\hat{\sigma}'_n}, A^{<\hat{\sigma}'} \cap \hat{\sigma}'_n \rangle <_1 \langle L_{\hat{\sigma}'}, A^{<\hat{\sigma}'} \rangle.$$

In particular $\langle L_{\hat{\sigma}'_n}, A^{<\hat{\sigma}'} \cap \hat{\sigma}'_n \rangle = \langle L_{\hat{\sigma}'_n}, A \cap \hat{\sigma}'_n \rangle$ satisfies ψ. Hence there is a $\nu < \hat{\sigma}'_n$ for which ψ is true in $\langle L_\nu, A \cap \nu \rangle = \langle L_\nu, A^{<\hat{\sigma}'} \cap \nu \rangle$. But this contradicts the choice of $\hat{\sigma}$, and the claim is proved.

It follows from our construction that $B^{<\hat{\sigma}} = B \cap \hat{\sigma}$, since $A^{<\hat{\sigma}} = A \cap \hat{\sigma}$ and no L_τ for $\tau > \hat{\sigma}'$ will tell us that $\hat{\sigma}$ has smaller cardinality than it has in $L_{\hat{\sigma}'}$. Thus $\langle L_{\hat{\sigma}}, B \cap \hat{\sigma} \rangle \models \exists x\, \phi$ and so $\langle L_{\aleph_n^L}, B \cap \aleph_n^L \rangle <_1 \langle L_\alpha, B \rangle$. This proves Theorem 3.3, as it was observed earlier that for $n \notin K$, $(\aleph_{n-1}^L, \aleph_n^L) \subseteq B$.

Corollary 3.6. *Let $K \subset \omega$ be constructible. Then there is an α-r.e. set A such that* $\mathrm{char}(A) =^* K$.

Proof. Let A be α-recursive. Then $\mathrm{char}(A) =^* \omega$. As $K \subset^* \omega$, by Theorem 3.3, we have an α-r.e. set A_K such that $\mathrm{char}(A_K) =^* K$.

Corollary 3.7. *There exist $\aleph_1^L$-many pairwise incomparable α-r.e. sets.*

Proof. There exist $\aleph_1^L$ many almost disjoint constructible reals K. For each such K, let A_K be α-r.e. such that $\mathrm{char}(A_K) =^* K$. By Proposition 3.2, these sets are pairwise α-incomparable.

Corollary 3.8. *Let* $\mathrm{char}(A) \neq^* \emptyset$. *Then there exist $\aleph_1^L$-many α-r.e. degrees which lie strictly between* $\deg(A)$ *and* $0'$.

Proof. There exist $\aleph_1^L$ many almost disjoint constructible reals $K \subseteq \mathrm{char}(A)$ such that $(\mathrm{char}(A) - K) \neq^* \emptyset$ and $K \neq^* \emptyset$. Then by Proposition 3.2 each $\deg(A_K)$ where $\mathrm{char}(A_K) =^* K$ and K is as above lies strictly between $\deg(A)$ and $0'$.

7. Some open problems

(a) Prove the following Character Density theorem.

If $\alpha = \aleph_\omega^L$ and $A \leqslant_\alpha B$ are α-RE, $\mathrm{char}(A) \supseteq K \supseteq \mathrm{char}(B)$, $K \in L$, *then there is an α-RE C such that $A \leqslant_\alpha C \leqslant_\alpha B$ and* $\mathrm{char}(C) = K$.

(b) Let $C = \{(\beta_1, \beta_2) \mid \beta_1 \in S(\aleph_n^L), \beta_2 \in S(\aleph_m^L) \text{ and } \hat{\beta}_1 = \hat{\beta}_2\}$, as in Section 2. Is C a greatest lower bound for $S(\aleph_n^L)$, $S(\aleph_m^L)$ in the sense of $\aleph_\omega^L$-degree?

(c) Find a non $\aleph_\omega^L$-recursive A which is $\aleph_\omega^L$-recursive in each of the sets $S(\aleph_n^L)$, $n \in \omega$.

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