EXPLAINING MAXIMALITY THROUGH THE HYPERUNIVERSE PROGRAMME

SY-DAVID FRIEDMAN*, CLAUDIO TERNULLO†

ABSTRACT. The (maximal) iterative concept of set is standardly taken to justify ZFC and some of its extensions. In this paper, we show that the maximal iterative concept also lies behind a class of further *maximality principles* expressing the maximality of the universe of sets V in height and width. These principles have been heavily investigated by the first author and his collaborators within the Hyperuniverse Programme. The programme is based on two essential tools: the *hyperuniverse*, consisting of all countable transitive models of ZFC, and V*-logic*, both of which are also fully discussed in the paper.

1. THE MAXIMAL ITERATIVE CONCEPT OF SET

1.1. **Generalities.** In this paper, we will be pre-eminently dealing with maximality principles for the universe of sets, that is, principles which prescribe that the universe is *maximal*. Of course, it is far from obvious what 'maximal' means or implies here, and the next subsections aim to fully clarify what we mean by that.

Maximality principles may be seen as expressing a fundamental feature of the iterative concept of set. It is not too hard to see why, yet it is worth examining this in more detail.

The iterative concept of set consists in the idea that sets are generated in *stages*, starting with ur-elements or, possibly, with the empty set and, then, forming the power-set of the previous levels at stages indexed by successor-ordinals and the union of all previous levels at stages indexed by limit-ordinals. The resulting picture is simply what is standardly acknowledged to be the universe of sets, the union of the V_α for all ordinals α, consisting of all sets formed through all stages.

The history of the progressive development of the axiomatisation of set theory and, in particular, of the emergence of ZFC has shown that all the most widely

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*Kurt Gödel Research Center for Mathematical Logic, 25 Währinger Strasse, 1090 Wien (Austria). The author wishes to thank the FWF for its support through project P 28420.

†Kurt Gödel Research Center for Mathematical Logic, 25 Währinger Strasse, 1090 Wien (Austria).

accepted axioms of set theory are true of the iterative concept and may, in fact, be motivated by it. This fact has gradually evolved into the more robust view that the concept of set is *essentially equivalent* to the iterative concept of set.

There are several issues with the iterative concept and its full justifiability.[1] However, in light of our goals, here we only wish to focus our attention on two prominent features of it: its connection to *maximality* and its closeness to a *platonistic* interpretation of mathematics.

Sometimes it is said that a guiding principle in the 'genetic' approach to sets is that one should form *as many of them as possible*. The principle seems equivalent to the idea that all logical and conceptual constraints on the formation of sets should be removed, and this leads to viewing the iterative concept as a maximum (or maximal) iterative concept. Here is how Wang comments on the principle with respect to the power-set axiom:

> The concept of all subsets is often thought to be opaque because we envisage all possibilities independently of whether we can specify each in words; for example, just as there are 2^{10} subsets of a set with 10 members, we think of 2^a subsets of a set with a members when a is an infinite cardinal number. In particular, we do not concern ourselves over how a set is defined, e.g. whether by an impredicative definition. This is the sense in which the individual steps of iteration are 'maximum'. (in [5], p. 532)

Two main features of the maximal approach are neatly highlighted in the passage above: the fact that 1) the 'infinite should be treated in a way analogous to the finite', a principle which allows us to extend certain set-theoretic operations holding in the finite to the transfinite, and 2) the fact that *impredicative* definitions are seen as entirely legitimate.

As is known, Bernays, in his [6], had construed the aforementioned principles as expressing Platonism in mathematics (set theory). In Bernays' view, central to mathematical (set-theoretic) Platonism would be a *quasi-combinatorial* conception, that is the view that mathematical (set-theoretic) operations, entities and concepts holding in the finite can (and should) be extended to the infinite, even in the absence of any available methods of 'construction'.[2]

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[1]For further details, see the classical Boolos, [7], Parsons, [23] and Wang, [26]. Potter, [24] provides a more recent, but not less accurate, overview of the topic.

[2]Cf. the following two crucial passages of [6]: 'But analysis is not content with this modest variety of platonism [that of arithmetical platonism, *our note*]; it reflects it to a stronger degree with respect to the following notions: set of numbers, sequence of numbers, and function. It abstracts from the possibility of giving definitions of sets, sequences, and functions. These notions are used in a 'quasi-combinatorial' sense, by which I mean: in the sense of an analogy of the infinite to the

For instance, on this view, the existence of the power-set of the integers does not follow from exploring all known methods for constructing subsets of the integers, but rather from an ideal intuitive grasp of the *fully given* collection of subsets of integers.

Now, if the maximal iterative concept really is the expression of a *platonistic* attitude in mathematics, in accordance with this fact, we might, as a consequence, want to (or be forced to) hold that V is *fully given* in an ideal sense, that no possible extension of it is conceivable (a position known as *actualism*) or that all statements of set theory have a unique truth-value, all of which statements do have notable bearings on the availability of the maximality principles we will be introducing. Therefore, all such issues will have to be carefully taken into account throughout the paper.[3]

1.2. **Expanding on the Maximal Concept.** Returning to the the maximal concept of set, its manifestations in set theory are manifold. Bernays, in the previous quote, was mentioning 'methods of collection', such as those permitted, for instance, by the Infinity, the Power-Set or the Replacement Axiom.

One of the most distinctive ways to construe the maximality inherent in the concept of set is the idea that the universe itself, V, be maximal. Again, Wang expounds this further characterisation of the meaning of 'maximal' in the following way:

> In a general way, hypotheses which purport to enrich the content of power sets (say that of integers) or to introduce more ordinals conform to the intuitive model. We believe that the collection of all ordinals is very 'long' and each power set (of an infinite set) is very 'thick'. Hence, any axioms to such effects are in accordance with our intuitive concept. ([5], p. 553)

To rephrase Wang's quote, one could say: the iterative concept of set leads one to realise that there is a rich hierarchy of sets, whose formation is given by the (maximal) procedures associated ('methods of collecting'). Now, it is reasonable to ask whether such methods of collecting (e.g., the Power-Set Axiom) may themselves be maximised in some way. In simpler words, one could say that, according to the maximal iterative concept, the hierarchy of sets should be *as wide as possible* and extend *as far as possible*. However, it is not prima facie clear what 'as long

finite' and later in the text: 'In Cantor's theories, platonistic conceptions extend far beyond those of the theory of real numbers. This is done by iterating the use of the quasi-combinatorial concept of a function and adding methods of collection. This is the well-known method of set theory' (both are reproduced in [5], p. 259-60).

[3]See, in particular, §2.2 and §7.

as possible' and 'as far as possible' mean. It is therefore the task of the study of maximality principles to disclose (or clarify) the meaning of 'maximality'.

1.3. **New Intrinsically Justified Axioms.** The main rationale for exploring maximality principles is to extend ZFC, through, ideally, declaring these principles new set-theoretic *axioms*. One straightforward criterion to evaluate whether a new axiom is acceptable is to checking whether it decides set-theoretic statements which are not decided by ZFC. But there's something else which should guide us in finding new axioms, that is their conceptual 'aptness', measured against the maximal iterative concept.

Since Gödel, [14], it is customary, in the literature, to define these two forms of evidence for new axioms as, respectively, *extrinsic* and *intrinsic*. Intrinsic evidence relates to 'conformity to the intuitive model', as Wang would say, whereas extrinsic evidence to the success of an axiom. Maximality principles, as we formulate them, clearly obey the maximal iterative concept, and this will fully justify our view that the maximality principles described in the next sections are *intrinsically motivated*. Moreover, these principles should ultimately be viewed as new intrinsically justified *axioms*, but, as detailed in the last section, this will require the satisfaction of further epistemic desiderata.[4] It should be noticed that the maximality principles we introduce have also clearly proved to be able to reduce set-theoretic incompleteness, although we will not deal with this in the present paper.

To summarise, a well-established conception of the axioms of set theory holds that ZFC conforms to a maximal iterative concept, and that its extensions should follow suit.[5] Maximality principles are an expression of this attitude and, thus, can be viewed as being intrinsically motivated.

[4]The difference between an axiom's being *intrinsically motivated (plausible)* and *intrinsically justified* consists in the level of definitiveness conveyed by the justificatory process. Thus, an intrinsically motivated axiom (or principle) need not be a definitively accepted axiom (or principle) of set theory. Koellner, [21], p. 207-8, explains the difference as follows: '..the notion of intrinsic justification is intended to be more secure than mere 'intrinsic plausibility' [...], in that whereas the latter merely adds credence, the former is intended to be definitive (modulo the tenability of the conception).'

[5]Incidentally, such a view is already expressed (although very tersely) by Gödel when he discusses the prospects of deciding CH through a new axiom: '..from an axiom in some sense opposite to this one [$V = L$], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas axiom A [i.e. V = L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set...' ([15], p. 262-63, footnote 23).

2. A ZERMELIAN APPROACH TO V

2.1. **Height and Width Maximality.** In this section, we introduce further key concepts concerning the relationship between maximality principles and the maximal iterative concept.

Set theorists have progressively formulated several maximality principles.[6] As has been clarified above, all of these prescribe, in some way, that the universe of sets is a very rich structure, in particular, the richest possible allowed by the set-theoretic axioms.

Now, as hinted at above in Wang's quote, the maximality of V has two dimensions, *maximality in height* and *maximality in width*, which can be characterised as follows:

Height Maximality. The cumulative hierarchy should be as *tall* as possible.

Width Maximality. The cumulative hierarchy should be as *wide* as possible.

Again, making sense of the statements above consists in understanding carefully what it means for V to be as tall as and as wide as possible. Let's start with height maximality.

There is a principle and, in fact, a class of principles, which has attracted set-theorists' attention in the last few decades, which seems to express height maximality very aptly, and this is the Reflection Principle. Very generally, reflection can be described as asserting that the universe cannot be uniquely characterised by any given collection of first-order properties. As is often said, the universe is indescribable (or ineffable).[7] Now, through reflection one can generate new ordinals α's and corresponding new levels V_α's of the hierarchy. Therefore, another way to construe reflection as a maximality principle is viewing it as inducing (maximal) 'lengthenings' of V, and that is precisely our construal of the principle.

Mathematically, in ZFC, reflection is a theorem which asserts that, if V has a first-order property ϕ, then for some ordinal α, there is a V_α which satisfies ϕ^α (that is, the relativisation of ϕ to V_α). A stronger version of reflection, in particular,

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[6] Incurvati, in [18], makes an overview of different forms of maximality in set theory, and also provides a mathematically detailed account of some of the most important maximality principles in use. Among other things, the paper also includes a philosophical examination of the IMH, which is widely discussed in the present paper.

[7] It is widely known that the emergence of the principle is connected to Cantor's idea of the *absolute infinity* of V (for which see Cantor's renowned 1899 letter to Dedekind, in [8], p. 931-5). Gödel was one of the major advocates of reflection, to the point that he seems to have surmised that the axioms of set theory should be essentially reducible to one single *reflection axiom* (see Gödel, [14], Wang, [27] and Ternullo, [25] for this).

implies that, given any arbitrarily high level α, there is always a $\beta > \alpha$ such that $V_\beta \models \phi^\beta$.

Strengthenings of first-order reflection, in particular, *second-order reflection*, including second-order parameters, are able to prove that there exist such large cardinals as *inaccessibles* and *Mahlos*, in fact, that there exist proper classes of them. Therefore, second-order reflection is strong enough to provide new ordinals and, consequently, if one construes height maximality in terms of producing 'lengthenings' of V, then one could say that second-order reflection induces a significant lengthening of V.

As far as width maximality is concerned, things are somewhat less intelligible. The width of the universe is given by the extent of the power-set operation. Now, it is unclear how one could vary the extent of such an operation. As for the height of the universe, one could try to maximise width by also adopting some form of *reflection*. Width reflection has been informally introduced by Koellner in [21],[8] and essentially arises from a construal of the *core model programme* in terms of reaching an approximation of V using L-like models $L[0^\#], L[0^{\#\#}], ..., L[0^\infty], ...$ Each of these, by width reflection, is taken to fail to approximate V by some specified property P. For instance, failure by L to approximate V leads to proving the existence of $0^\#$.[9] So this may overall be construed as a way to produce thickenings of L which fail to fully approximate V from *within* in the same way as V_α must fail to approximate V from *below*. However, it is controversial, to say the least, that this form of width reflection, heavily based on L, is in accordance with the maximum iterative concept.

2.2. **Actualism and Potentialism. Zermelo's Conception.** The aforementioned maximality principles for V, if all fully in line with the maximal iterative concept, are conducive to several issues concerning the correct conceptualisation of V, which need be taken into account.

For instance, it has been argued that if height maximality is essentially expressed by reflection principles construed as prescribing the *indescribability* of V, then one is more naturally inclined to see V as *fully given* in the sense of being inextensible. This is the *actualist* position, which had already been mentioned in §1.1. But if one adopts actualism, then higher-order quantification is less likely to be made sense of.[10]

[8]The idea is also very carefully examined by Incurvati in the mentioned [18] and further explored in [10]. See also p. 17 of the present article.

[9]A brief description of the core model programme is in Jensen, [19].

[10]Again, see [21] for this.

A *potentialist* conception, on the other hand, construes V as non-fixed in height and width and, thus, can make sense of higher-order quantification and, in general, of lengthenings and thickenings more easily. What would be lost for the potentialist, though, is the availability of the full givenness of V, which appears to lie at the root of reflection.

One way to resolve this would be to declare that the Platonism inherent in the maximal iterative concept would automatically imply that one is supposed to be able to intuit V as a *completed* object. By this interpretation, full-fledged actualism would be the only option available and, furthermore, maximality principles referring to widenings and lengthenings of V would inevitably be viewed as meaningless.[11]

Interestingly enough, we will show that the maximality principles formulated within the Hyperuniverse Programme are compatible with both potentialism and actualism. In particular, even a radical form of potentialism can accommodate maximality. And actualism, accompanied by some class theory of the same strength as MK (but, in fact, less robust than that), also fully befits the programme.[12]

All this might lend support to the view that the issues of whether the maximal iterative concept is essentially platonistic in character, and of whether a platonist could only be an actualist about V are less relevant than it might seem at first glance.

In any case, the underlying conception in which the Hyperuniverse Programme is most fruitfully cast was the Zermelian conception, as described in [28]. In that work, Zermelo, after formulating the axioms of set theory (often labelled Z_2), proves that, for those axioms (some of which have a second-order characterisation), the power-set of V is fixed. More specifically, he proves that:

Theorem 1. *Any two models of the axioms of set theory Z_2 are either isomorphic, or one is isomorphic to a proper initial segment of the other.*

This settles things as far as the width of the universe is concerned. Concerning height, Zermelo introduces the concept of a *normal domain*. A normal domain is the least rank initial segment of the hierarchy which satisfies the (second-order) axioms of set theory. The least normal domain which satisfies (second-order) ZFC is, as is known, V_κ, where κ is the least inaccessible cardinal. But then one can iterate this, by considering (second-order) ZFC+'there is one inaccessible'. The

[11]But things are a lot more subtle. Zermelo, for instance, who would seem to have been a platonist was the major proponent of a partly *potentialist* (if width actualist) conception, for which see the next few pages. For Zermelo's ideas on philosophy, set theory and the justifiability of the axioms, see, in particular, [22] and [20].

[12]A full account of this is provided by Antos, Barton and Friedman in [1]. For further details, also see footnote 21.

least normal domain which satisfies (second-order) ZFC+'there is one inaccessible' is V_λ, where $\lambda > \kappa$ is the least inaccessible after κ. Thus, one obtains a vertical multiverse consisting of V_α's, where α is some large cardinal.

The Zermelian picture of the universe has some clear attractions, some of which can be described as follows:

(1) Height potentialism fully befits the form of reflection introduced in the next section. It is very comfortable to define *lengthenings* of the hierarchy required by this principle within the Zermelian picture.
(2) While height actualism seems counter-intuitive to some extent, width actualism would seem to be more easily justifiable insofar as there is no apparent way to address thickenings of V in a way which resembles the ordinal-indexed progress of stages in height.
(3) One can make full sense of *higher-order* quantification more easily within the Zermelian multiverse, insofar as the universe is non-categorical in height. Fully actualist (absolutist) versions of the universe struggle to provide an equally acceptable account of this.

These reasons may be insufficient for a full case in favour of the adoption of Zermelo's picture as the only correct picture of V, but are clearly sufficient to accommodate the maximality principles formulated within the Hyperuniverse Programme. However, although the Zermelian conception may be viewed as the correct conceptualisation of V, we will point the reader, when necessary, to alternative options.

3. Height Maximality: Reflection

In a sense, the principles we will be discussing improve and expand on those already mentioned in §2.1. For instance, our height maximality principle is a form of the reflection principle and, in our view, the strongest possible. In order to see this we have to recall some notions already briefly introduced in §2.1.

As said, height maximality in terms of reflection of the universe V can be intuitively formulated as follows:

(*Reflection*) Any property which holds in V already holds in some rank initial segment V_α of V.

In other words, V cannot be described as the unique initial segment of the universe satisfying a given property. The strength of such reflection depends on what we take the word 'property' to mean[13]. If this just means 'first-order property with

[13]Properties are often formulated using higher-order quantification. Let M be a class. We say that a variable x is 1-st order (or of order 1) if it ranges over elements of M. In general, we say

set parameters' then we obtain Lévy reflection, a form of reflection provable in ZFC.

A priori, there is no need to limit ourselves to first-order properties of V. But to express second-order properties of V we need to move beyond ZFC to Gödel-Bernays class theory GB. The latter has variables ranging over sets and also variables ranging over the larger collection of classes (collections of sets: note that every set is also a class). The $\in$-relation applies between sets and classes and we impose the Comprehension Scheme for formulas with only set-quantifiers (but with both set and class variables). Thus in GB we can quantify over classes but cannot apply Comprehension to formulas containing such quantifiers. We also include *Global Choice* as an axiom, which says that there is a class function F such that $F(x)$ is an element of x for every nonempty set x.

GB is conservative over ZFC. However it can be strengthened by adding second-order reflection axioms to it, such as:

- Π^1_m Reflection If $\varphi(R)$ is a Π^1_m formula with a class variable R, then *reflection for* $\varphi(R)$ is the implication

$$\varphi(R) \to (V_\alpha, V_{\alpha+1}) \vDash \varphi(R \cap V_\alpha)$$

  where on the right-hand-side the set variables range over V_α and the class variables over $V_{\alpha+1}$.

Even Π^1_1 Reflection for sentences (without the class variable R) is rather strong, as it implies the existence of an inaccessible cardinal. That is because the regularity of an ordinal α is equivalent to the truth of a Π^1_1 sentence in $(V_\alpha, V_{\alpha+1})$. By adding parameters we get stronger large cardinals such as Mahlo cardinals and weakly compact cardinals.

But just as ZFC is inadequate for second-order reflection, GB is inadequate for third-order reflection.[14]

that a variable R is $n+1$-st order (or of order $n+1$), $0 < n < \omega$, if it ranges over $\mathcal{P}^n(M)$, where $\mathcal{P}^n(M)$ denotes the result of applying the powerset operation n times to M. A formula φ is Π^n_m if it starts with a block of universal quantifiers of variables of order $n+1$, followed by existential quantification of variables of order $n+1$, and these blocks alternate at most $m-1$ times; the rest of the formula can contain variables of order at most $n+1$, and quantifications over variables of order at most n. Σ^n_m is obtained by switching the words universal and existential.

[14]As an aside, it is worth noting that if formulated with third-order parameters, third-order reflection is in fact inconsistent! For instance, for a third-order parameter $\mathcal{R}$, i.e. a collection of classes, one is tempted by the following natural-looking principle:

- Third-order reflection If $\varphi(\mathcal{R})$ is true in $(V, \mathcal{R})$ then for some α, $\varphi(\bar{\mathcal{R}})$ is true in $(V_\alpha, \bar{\mathcal{R}})$, where $\bar{\mathcal{R}} = \{R \cap V_\alpha \mid R \in \mathcal{R}\}$.

But such a principle will fail if $\mathcal{R}$ consists of all bounded subsets of the ordinals (viewed as a collection of classes) and $\varphi(\mathcal{R})$ simply says that each element of $\mathcal{R}$ is bounded in the ordinals. Therefore when discussing third-order reflection it is customary to only allow *second-order*, and

Of course there is no reason to stop at third-order reflection, and in light of the Zermelian conception, it is meaningful to discuss 'α-th order' reflection for ordinals α in *lengthenings* of V, i.e. in models V^* which have V as a rank initial segment.

This naturally leads to the following form of higher-order reflection:

- Extended Reflection Axiom (ERA) V satisfies the ERA if V has a lengthening V^*, a model of ZFC, such that if φ is first-order and $\varphi(A)$ holds in V^* where A is a subclass of V, then $\varphi(A \cap V_\alpha)$ holds in V_β for some pair of ordinals $\alpha < \beta$ in V.

This allows us to reflect properties (with second-order parameters) that are α-th order, for all ordinals α appearing in the least ZFC model lengthening V. This embodies all of the classical froms of strong reflection and more.

However, clearly the ERA can easily be strengthened further, by requiring the lengthening V^* of V to satisfy more than ZFC, such as ZFC + 'there is ZFC-lengthening of ZFC'. Indeed, it appears that there is no optimal form of reflection which can be described in terms of lengthenings of V, as we can always strengthen such a reflection principle further by requiring a lengthening V^* of V in which the principle holds with reference only to lengthenings of V appearing in V^*.

How are we then to achieve an optimal reflection principle? This problem is fully addressed mathematically in Section 2.2 of [11], where the principle of #-*generation* is introduced. This asserts the existence of a special kind of set called a # *(sharp)* that 'generates' V through iteration. An optimal form of reflection results as this iteration also produces a closed unbounded class of *indiscernibles* for V, adequate for witnessing any conceivable form of reflection. It is crucial that a # generating V cannot be an element of V, otherwise such optimality would not be possible.

We cannot provide the full details of #-generation here, but at least some notions will be briefly discussed.

First, imagine that V can be seen as being the last step in an elementary chain of universes $(V_{\kappa_i} \mid i < \infty)$ and we set $V = V_{\kappa_\infty}$. We can continue the construction of this chain 'beyond' V itself, producing an upwards elementary chain of universes $V = V_{\kappa_\infty} \prec V_{\kappa_{\infty+1}} \prec V_{\kappa_{\infty+2}} \prec \cdots$.

By elementarity, all of these universes will satisfy the same first-order sentences, but we want more. We want that any two pairs of universes 'resemble' each other,

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not third-order parameters. An alternative is to consider *embedding reflection* (see for example the discussion in Section 2.1 of [11], and in [17]) where $\bar{\mathcal{R}}$ results from applying the inverse of an elementary embedding to $\mathcal{R}$. This very strong form of reflection yields supercompact cardinals, however does not appear to be derivable from the maximal iterative conception, as are the forms of reflection consistent with $V = L$.

i.e. satisfy the same first-order sentences, and this can be extended to any pair of n-tuples of universes $W_{\vec{i}}$, where $\vec{i} = i_0 < i_1 < \cdots < i_{n-1}$ and $W_{\vec{j}}$, where $\vec{j} = j_0 < j_1 < \cdots < j_{n-1}$ (to simplify our notation, we use the symbol W_i for $V^*_{\kappa_i}$). But we want to impose an even higher level of resemblance, whereby all n-tuples of models satisfy the same second-order sentences and so on. In the end, the whole process can be seen as the construction of a series of embeddings $\pi_{ij} : V \to V$, leading to an *indiscernibly-generated* V. In more rigorous terms:

Definition 2. *(*[11], *p.6). V is indiscernibly-generated iff: (1) There is a continuous sequence $\kappa_0 < \kappa_1 < \cdots$ of length ∞ such that $\kappa_\infty = \infty$ and there are commuting elementary embeddings $\pi_{ij} : V \to V$, where π_{ij} has critical point κ_i and sends κ_i to κ_j. (2) For any $i \leq j$, any element of V is first-order definable in V from elements of the range of π_{ij} together with κ_k's for k in the interval $[i, j)$.*

Indiscernible-generation has an equivalent but more useful formulation in terms of *#-generation* (for its definition see [11], p. 6). So we will use the term *#-generation* for this strong form of reflection.

Now, one can show that #-generation implies all forms of reflection which are compatible with $V = L$ (again see [11]).

As a consequence of this, we believe that #-generation expresses the strongest possible amount of vertical reflection and therefore can legitimately claim to be the optimal principle expressing the vertical maximality of V.

4. Width Maximality: V-logic, IMH

4.1. **The Strategy.** From the Zermelian perspective, which incorporates height potentialism and width actualism, expressing principles of *width maximality* principles presents a real challenge. Whereas in the case of height maximality we made liberal use of *lengthenings of V*, no analogous notion of *thickening (or outer model) of V* is available.

Now, since [9], the programme has expressed width maximality in terms of the following principle:

- (The Inner Model Hypothesis, IMH) If a first-order sentence holds in an inner model of some outer model of V then it also holds in some inner model of V.

As is clear, the IMH is conceptually problematic for the Zermelian, as it explicitly refers to 'outer models' which are not available in the Zermelian picture. However, if the IMH were referring not to the whole V, but just to some countable transitive model (which we will mostly indicate as 'little-V') of ZFC, then the IMH would make perfect sense even within a Zermelian perspective.[15]

[15]Note that IMH is also known to consistently hold for some choice of little-V. See [12].

However recent developments, discussed in [2], [10] and [4], provide a solution to this problem. The introduction of *V-logic* enables one to express first-order properties of arbitrary outer models (almost) *internally* within V, in the same way as first-order properties of set-forcing extensions of V can be *internalised* using the forcing relation. The word 'almost' occurs because this new 'truth in outer models' relation will not in general be first-order definable over V, but rather over a small *lengthening* (not *thickening*) of V called $Hyp(V)$ (the least 'admissible set' containing V as an element). As lengthenings are available to the Zermelian, this enables her to express principles such as the IMH without loss of content.

Therefore, we shall scrutinise two approaches to width maximality: the first, through the use of V-logic, will allow one to make sense of IMH as if it were referring to the whole V, and the second will construe the IMH as referring to some countable model 'little-V'. The latter approach is particularly convenient, as it entirely befits our goal to reduce the study of the consequences of maximality principles to their consequences in *countable transitive models*.

Let us review the first approach. As we said, the case of IMH is analogous to that of Martin's Axiom (MA), a principle of set-forcing.[16] Several formulations of MA are available, in particular, $MA_{\aleph_1}$ asserts:

- (Outer Model $MA_{\aleph_1}$). Whenever $V[G]$ is a generic extension of V by a partial order $\mathbb{P}$ with the countable chain condition in V, and $\varphi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a Σ_1 formula with a subset of ω_1 as parameter), if in $V[G]$ there is a y such that $\varphi(y)$ holds, then there is also such a y in V.

Note the quantification in this definition over the (generic) outer models $V[G]$ of V. How can the width actualist possibly make sense of this? The answer is of course via the definable forcing relation:

- (Internal $MA_{\aleph_1}$). Whenever $\mathbb{P}$ is a partial order with the countable chain condition in V, and $\varphi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula, if there is a forcing condition p in $\mathbb{P}$ forcing the existence of a y such that $\varphi(y)$ holds, then there is also such a y in V.

These two formulations of $MA_{\aleph_1}$ are equivalent when V is replaced by a countable transitive model little-V of ZFC. When little-V is not countable (and possibly equal to V), we use the latter internal formulation to express $MA_{\aleph_1}$.

Thus we convert a principle that makes reference to outer models of V to one which is internal, expressible within V.

4.2. **V-logic and IMH.** The point of V-logic is that it provides a tool to enable us to do the analogous thing not for just generic outer models, but for outer models in general. *V-logic* has a symbol for $\in$, a predicate symbol $\bar{V}$ to denote V and a

[16] For further on this analogy, see [4].

constant symbol $\bar{x}$ to denote x for each set x. The proof relation $\vdash_V$ of V-logic begins with axioms that assert that $\bar{x}$ belongs to $\bar{V}$ for each set x, together with the usual axioms of first-order logic and all quantifier-free sentences true in V. The rules of inference are modus ponens together with the infinitary rules:

- From $\varphi(\bar{y})$ for all y in x, infer $\forall y \in \bar{x}\varphi(y)$.
- From $\varphi(\bar{x})$ for all x in V, infer $\forall x \in \bar{V}\varphi(x)$.

Proofs are then well-founded trees which can be shown to belong to $Hyp(V)$, the least admissible set containing V as an element. Assuming height potentialism, (which is provided by the Zermelian conception), $Hyp(V)$ makes full sense.

As said, now we proceed in a way fully analogous to what we did above using the forcing relation. Reconsider the IMH:

- (The Inner Model Hypothesis, IMH) If a first-order sentence holds in an inner model of some outer model of V then it also holds in some inner model of V.

We then formulate an internal version of this as follows:

- (The *Internal* Inner Model Hypothesis, IMH) If the theory in V-logic T_φ asserting that the first-order sentence φ holds in an inner model of some outer model of $\bar{V}$ is consistent in V-logic, then there is an inner model of V in which φ holds.

The 'internal' IMH is expressible as a first-order property of $Hyp(V)$, using the fact that the consistency of T_φ in V-logic is equivalent to saying that there is no V-logic proof in $Hyp(V)$ of a contradiction using the axioms of T_φ. And as in the case of $MA_{\aleph_1}$, the two formulations of the IMH, the one using outer models and the internal one, are equivalent when V is replaced by a countable transitive model little-V of ZFC.

Thus V-logic opens the door to expressing a wide range of width maximality principles, even in the Zermelian, width actualist context. With rare exceptions, these principles are formalisable internally in $Hyp(M)$ for arbitrary transitive ZFC models M, and not just for countable ones. In fact, in almost all cases, the study of width maximality principles for V can be reduced to its study for *countable transitive models* of ZFC. We discuss this in the next section.

5. Reduction to $\mathbb{H}$

5.1. **Reduction of IMH.** Our introduction of V-logic was intended to deal with the problem that for an uncountable transitive model of ZFC (such as V itself) there may be no (proper) outer models available and therefore we are required to discuss width maximality in terms of the consistency of V-logic theories.

As promised, we shall now deal with the second approach, where V is taken to be a countable transitive model little-V. Moreover, in this section we show that we can *reduce* our study of width maximality, and to some extent of height maximality, to a study of countable transitive models. As the collection of countable transitive models carries the name *hyperuniverse*, we are led to what is known as the *Hyperuniverse Programme*.

First we illustrate the reduction to the hyperuniverse with the specific example of the IMH. Suppose that we formulate the IMH as above, using V-logic, and want to know what first-order consequences it has.

Fact 3. *Suppose that a first-order sentence φ holds in all countable models of the IMH. Then it holds in all models of the IMH.*

This is for the following reason: Suppose that φ fails in some model M of the IMH, where M may be uncountable. Now notice that the IMH is first-order expressible in $Hyp(M)$, the least admissible lengthening of M. But then apply the downward Löwenheim-Skolem theorem to obtain a countable little-v which satisfies the IMH, as verified in its associated little-$Hyp(V)$, yet fails to satisfy φ. But this is a contradiction, as by hypothesis φ must hold in all *countable* models of the IMH.

So *without loss of generality*, when looking at first-order consequences of width maximality criteria as formulated in V-logic, we can restrict ourselves to countable little-V's. The advantage of this is that, then, we can dispense with the little-V-logic as by the Completeness Theorem for little-V-logic, consistent theories in little-V-logic do have models, thanks to the countability of little-V. Thus for a countable little-V, the IMH simply says:

- (*IMH for little-V's*). Suppose that a first-order sentence holds in an inner model of an outer model of little-V. Then it holds in an inner model of little-V.

But, if V is taken to be 'little-V', then V can really be 'thickened', which means that the Zermelian picture collapses to a *radical potentialist* picture, wherein both height and width of V are not fixed.

As we have seen, the Zermelian and the radical potentialist versions of the IMH coincide on countable models.

5.2. **Reduction of #-generated V. #-generation revisited.** As far as the case of #-generation is concerned, its reduction to the hyperuniverse is not so obvious, and we shall see that the choice of working either within a Zermelian perspective or a radical potentialist perspective makes a big difference.

First, consider the following encouraging analogue for #-generation of our earlier reduction claim for the IMH, which we state here without proof.

Fact 4. *Suppose that a first-order sentence φ holds in all countable models which are $\#$-generated. Then it holds in all models which are $\#$-generated.*

Now the difficulty is this: how do we express $\#$-generation from a width actualist perspective? Recall that to produce a generating $\#$ for V we have to produce a set of rank less than $Ord(V)$ which does not belong to V, in violation of width actualism.

At this point we need to say a bit more about $\#$'s and models generated by them. A pre-$\#$ is a structure (N, U) where U measures the subsets in N of the largest cardinal κ of N, meeting certain first-order conditions; it is a $\#$ if in addition it is *iterable*, i.e. for any ordinal α if we take iterated ultrapowerse of (N, U) for α steps then it remains wellfounded. V is $\#$-generated if it results as the union of the lower parts of the α-iterates of some $\#$ as α ranges over $Ord(V)$.

But notice that to express the iterability of a generating $\#$ for V we are forced to consider theories T_α formulated in $L_\alpha(V)$-logic for *arbitrary* (Gödel-) lengthenings $L_\alpha(V)$ of V: T_α asserts that V is generated by a *pre-$\#$* which is *α-iterable*, i.e. iterable for α-steps. Thus we have no fixed theory that captures $\#$-generation, only a tower of theories T_α (as α ranges over ordinals past the height of V) which capture closer and closer approximations to $\#$-generation.

Therefore, in order to overcome these difficulties, we need to introduce another form of $\#$-generated V, that is, weakly $\#$-generated V.

Definition 5. *V is* weakly $\#$-generated *if for each ordinal α past the height of V, the theory T_α which expresses the existence of an α-iterable pre-$\#$ which generates V is consistent.*

Weak $\#$-generation is meaningful for a width actualist who is also a height potentialist (that is, a Zermelian), as it is expressed entirely in terms of theories internal to lengthenings of V.

A countable little-V is weakly $\#$-generated if it is α-generated for each countable ordinal α (where the witnesssing pre-$\#$ may depend on α). Little-V is $\#$-generated iff it is α-generated when $\alpha = \omega_1$ iff it is α-generated for all ordinals α.

Now we have the following reduction to countable little-V's:

Fact 6. *Suppose that a first-order sentence φ holds in all countable little-V which are weakly $\#$-generated, and this is provable in ZFC. Then φ holds in all models which are weakly $\#$-generated.*

To summarise: as radical potentialists we can comfortably work with full $\#$-generation as our principle of height maximality. But as width actualists, we instead work with weak $\#$-generation, expressed in terms of theories inside Gödel lengthenings $L_\alpha(V)$ of V. Weak $\#$-generation is sufficient to maximise the height of the universe. And properly formulated, the reduction to the hyperuniverse also applies to weak #-generation: to infer that a first-order statement follows from weak #-generation it suffices to show that in ZFC one can prove that it holds in all weakly #-generated countable models.[17]

In what follows we will primarily work with #-generation, as at present the mathematics of weak #-generation is poorly understood. Indeed, as we shall see in the next section, a synthesis of #-generation with the IMH is consistent, but this remains an open problem for weak #-generation.

6. $\mathbb{H}$-AXIOMS: SYNTHESIS OF #-GENERATION WITH IMH-VARIANTS

In light of the reduction to the hyperuniverse ($\mathbb{H}$), we see that maximality features of V such as #-generation and the IMH can be expressed as axioms about countable models, i.e. as properties of members of $\mathbb{H}$ expressed through quantification over $\mathbb{H}$. We refer to these as $\mathbb{H}$-*axioms*.

An important step in the development of the Hyperuniverse Programme is the *synthesis* of the $\mathbb{H}$-axiom of #-generation, expressing vertical maximality, with $\mathbb{H}$-axioms which express horizontal maximality. The first example of such a synthesis is the *IMH#*, which asserts the IMH for vertically-maximal universes:

Definition 7 (IMH#)**.** *M satisfies the IMH# if M is #-generated and whenever a first-order sentence holds in a #-generated outer model of M, it also holds in a definable inner model of M.*

IMH# captures both vertical maximality and aspects of horizontal maximality simultaneously. But the development of $\mathbb{H}$-axioms does not stop here. One may introduce further logical contraints, and derive further principles incorporating them.

An *absolute parameter* is a set p which is uniformly definable over all outer models of V which 'respect' them in the sense that they preserve cardinals up to and including the cardinality of the transitive closure of p. The *SIMH (Strong IMH)* is the IMH for sentences with absolute parameters relative to outer models which respect them:

Definition 8 (SIMH)**.** *If a sentence with absolute parameters holds in an outer model which respects those parameters then it holds in a definable inner model.*

[17]Weak #-generation is indeed strictly weaker than #-generation for countable models: Suppose that $0^\#$ exists and choose α to be least so that α is the α-th Silver indiscernible (α is countable). Now let g be generic over L for Lévy collapsing α to ω. Then by Lévy absoluteness, L_α is weakly #-generated in $L[g]$, but it cannot be #-generated in $L[g]$ as $0^\#$ does not belong to a generic extension of L.

A related principle is the *CPIMH (Cardinal Preserving IMH)*. A *cardinal-absolute parameter* is a set p which is uniformly definable over all cardinal-preserving extensions of V. Then CPIMH asserts the following:

Definition 9 (CPIMH)**.** *If a sentence with cardinal-preserving parameters holds in a cardinal-preserving outer model of V it also holds in a definable inner model of V.*

Restricting SIMH and CPIMH to $\#$-generated universes yields corresponding principles SIMH$\#$ and CPIMH$\#$.[18]

More recent work (see [10]) develops further $\mathbb{H}$-axioms, such as forms of *Cardinal Maximality* (for example: κ^+ of HOD is less than κ^+ for every infinite cardinal κ), *Width Reflection* (for each ordinal α there is an amenable elementary embedding of an inner model into V with critical point greater than α) and its associated analogue of $\#$-generation for width called *Width Indiscernibility* and *Omniscience* (the first-order definability of satisfaction across outer models, see [13]).[19]

7. The Dynamic Search for Truth

We now proceed to review some of the issues we had briefly mentioned at the beginning, relating to the correct interpretation of maximality, and to whether and in what sense the maximal iterative concept should be construed as expressing a *platonistic* conception of mathematics.

It is important to recall once more the way we construe the maximality of V. We said that V can *literally* be maximised, through maximising the ordinals α indexing the V_α and the subsets in $V_\alpha + 1$, for all ordinals α. In turn, this was conceptualised as corresponding to 'lengthening' and 'thickening' the universe. Whenever this was shown not to be possible within the Zermelian picture, we found a way to *internalise* the maximisation through the use of a powerful logic, V-logic.

Now, we have seen that there is an altogether different approach to the maximality of V, that is *full actualism*, whereby such *absolutely infinite* objects as V are viewed as already *maximal*, in a way which cannot be transcended. Full actualism befits *universism*, insofar as it also encourages the idea that there is a *fully determinate* universe of sets.

Universism, although not implausible, is, at least, epistemologically dubious, like the radical form of Platonism which underlies it. The main trouble with this conception is that the associated semantic determinacy (that is, the idea that, for all ϕ, ϕ is *uniquely* decided by a suitable collection of axioms) leaves a considerable portion of set-theoretic practice, dealing with different 'universes', entirely unaccountable. Furthermore, universism, unless it is endowed with a suitably strong

[18]See figure 1 at the end of the paper.

[19]For the consequences of all of these in members of $\mathbb{H}$, see, in particular, [3], [2], [10].

class theory, is inadequate to express the myriad of valuable forms of width maximality that are otherwise available.

It is even more doubtful that universism stems from a correct interpretation of the maximal iterative concept, as proclaimed by its supporters.[20] But even if it were, we have seen that there are ways to incorporate 'thickenings' of the universe even within a width actualist picture and extend this to 'lengthenings' using MK.[21] Therefore, the *platonist absolutist* who believes in the existence of a preferred structure determinately encompassing all truths about sets would not have to abandon her position, even in the case maximality principles should be viewed as more correctly implying the idea of 'thickenings' in height and width (as in the Zermelian or fully potentialist picture).

Moreover, it is not clear what one gains epistemically from holding that universism is the only way to make sense of maximality. It is interesting to briefly take into account the discussion of this issue provided by Hauser. Hauser has, in our view, convincingly, shown that finding objective solutions to such undecidable statements as CH does not depend upon having a pre-formed picture of V, that is, from believing in the full determinacy of V itself. Rather, objective solutions of set-theoretic problems will most likely be the outcome of procedures conforming to particular *evidential* standards of proof. In the author's own words:

> [This position] can be characterized in a nutshell as *objectivity over objects* and involves a twofold inversion of priorities. The first one shifts the attention from ontology to epistemology, i.e., questions about the existence and nature of mathematical items are discussed exclusively in the context of mathematical truth. [...] In the second inversion, evidence is treated as the primary epistemological concept. This reflects the widespread agreement among philosophers (and mathematicians) about what counts as evidence for the truth of a proposition – regardless of their conflicting ideas about the nature of truth. ([16], 265-66)

The author also casts a hypothesis concerning the way the general acceptability of new axioms will be construed in view of such epistemic inversion. In his view, the latter

[20] For a defence of this position see [17].

[21] We come back to the issue briefly discussed in footnote 12. The overall strategy is to formulate the IMH in V-logic, as shown above in §4. Recall that V-logic proofs are carried out in $Hyp(V)$, the least admissible structure containing V as an element. Now, it can be proved that, in a sub-theory of MK, it is also possible to build a class coding $Hyp(V)$, and therefore fully make use of V-logic to handle width maximality.

> ..may be characterized as a gradual convergence towards a *reflective equilibrium* of high-level convictions and their lower level and 'practical' consequences along the lines of the holistic views on theory formation [...]. (*ibid.*, p. 275)

Now, when we evoked 'optimality' with reference to the search for new axioms, we intended to refer precisely to procedures whereby one could select the most suitable $\mathbb{H}$-axioms, by studying their mathematically 'optimal' features, in a way which may plausibly recall the *objectivity over objects* account advocated by Hauser, that is by downplaying the role of ontology (in particular, a *universist* ontology). Only, we do not view 'practical consequences' as crucial to this undertaking (although certainly the consequences of maximality principles are worth examining), nor do we subscribe to a holistic view concerning set-theoretic truth: the idea of 'testing' maximality principles to find optimal $\mathbb{H}$-axioms should not be viewed as subservient to the search for *extrinsic* (that is, 'empirical') evidence for new axioms, but rather to the goal of best expressing the maximality of V.

Within scientific procedures, optimality is provided by the fine-tuning of the general statements of a theory through empirically testing its results. Within set theory, it is hard to say what may count as an analogue of this, unless one takes the study of 'consequences' to play the same role as that of confirmation in physics (which is, to say the least, utterly problematic). For our purposes, though, this can hardly be different from the idea of producing *progressive* refinements and strengthenings of higher-order principles.

The idea of progressive refinements of maximality principles adds an interesting 'dialectical' twist to our search for new axioms: the motivating idea is that different principles should be combined to produce syntheses of their features and better candidates as ultimate maximality principles (as illustrated in Figure 1).

Ideally, then, the study of $\mathbb{H}$-axioms will reach its natural endpoint when optimal maximality principles are found. We believe that the attainment of this might reasonably be described in terms of finding fully *intrinsically justified* new axioms.

At this stage, we cannot say anything definitive, but surely the analysis of the maximality of V conducted within the Hyperuniverse Programme has already led to remarkable findings, that, in conclusion, we may recapitulate as follows:

(1) $\#$-generated V may be viewed as the strongest possible form of reflection construed as 'lengthening' of V
(2) IMH may be viewed as the most natural form of expressing the width maximality of V, insofar as it successfully thrives on a suitable conceptualisation of 'thickenings' (outer models) of V through V-logic

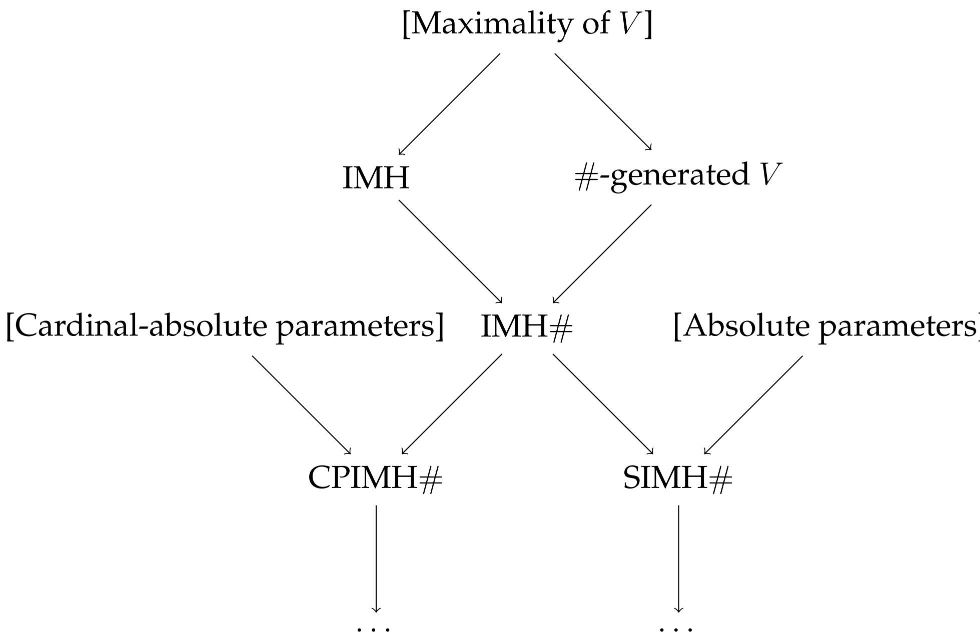

FIGURE 1. Maximality principles

(3) combinations, variants and refinements of these principles construed as quantifying over members of the hyperuniverse ($\mathbb{H}$-axioms) can be shown to have the effect of strongly reducing set-theoretic incompleteness, in such a way as to make it at least plausible to assume that they could be seen as new axioms.

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