MULTIVERSE CONCEPTIONS AND THE HYPERUNIVERSE PROGRAMME

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Abstract. We review some conceptions of the set-theoretic multiverse and evaluate their strength. In §1, we introduce the *universe/multiverse* dichotomy and discuss its significance. In §2, we discuss three alternative conceptions. Finally, in §3, we present our own conception as integral to the Hyperuniverse Programme launched by Friedman and Arrigoni in [10]. We believe that ours strongly differentiates itself from those examined in §2 (and, more generally, from any other merely *descriptive* multiverse conception), insofar as it is primarily concerned with the search for new axioms.

1. The Set-theoretic Multiverse

The current situation in the foundations of set theory sees the *multiverse view* and the *universe view* in confrontation with each other. The latter conception is easily characterised: its supporters think that there is a definite and unique set-theoretic structure which captures all true properties of sets. They are aware that the first-order axioms of sets do not have a fixed *reference*, but from this fact they only infer that the currently known axioms are not sufficient to express in full the intuitive pre-axiomatic notion of 'set'. As a consequence, they think that set-theoretic indeterminacy may be significantly reduced by adding new axioms which will provide us with a more determinate picture of the universe. In turn, axioms should reflect our intuitive understanding of the realm of sets.[1]

Usually, supporters of this view are mathematical *realists*, or, to be more precise, *platonists*. It is generally assumed that platonists ought not to be strongly concerned about the axioms' failure to provide an adequate description of a reality they think pre-exists and may also significantly outstrip our understanding. However, platonists might still see certain strands of set-theoretic practice as relevant to the confirmation of their views. For instance, they might see the success of the axioms of ZFC and their apparent consistency as evidence of the fundamental correctness of our intuitive grasp of the notion of set. If pressed to explain why, then, those axioms are arguably less successful in referring to a

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[1]However, advocates of this stance may not agree about what the universe is like, whether the cumulative hierarchy represented by V, or portions of it, such as L or $H(\kappa)$ (where κ is, for instance, an inaccessible cardinal). Furthermore, they might not agree about whether the universe contains *large cardinals* or not, and, if so, then which.

unique structure, they might invoke a sort of *incompleteness* in our understanding. A very typical universe-view supporter, thus, holds the same views as presented in the following passage, due to Gödel:

It is to be noted, however, that on the basis of the point of view here adopted [that is, the 'platonistic conception', *our note*], a proof of the undecidability of Cantor's conjecture from the accepted axioms of set theory (in contradistinction, e.g., to the proof of the transcendency of π) would by no means solve the problem. For if the meaning of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some *well-determined reality*,[2] in which Cantor's conjecture must be either true or false.[3] ([13], in [14], p. 260)

On the other hand, the multiverse view supporter can be characterised as someone believing that the absence of a fixed reference of the first-order axioms of set theory is an inevitable phenomenon and that one should make one's peace with it.

The multiversist front, as it were, is internally divided. There is an extreme view which maintains that there are as many *set concepts* as *models* of the axioms. Its advocates believe that we could not have hoped to fix a reference of the axioms from the beginning, insofar as we already had many set concepts at hand, which differ from each other in some respects.

Adherents of this radical view often hold that there is no set-theoretic reality beyond models. As a consequence of this, either one is a supporter of platonism in the same way as Gödel and, hence, inclined to accept some sorts of *theological* arguments in favour of the universe-view or one should simply take the *multiverse* as being an indisputable and permanent matter of fact.

This alternative seems to stand out neatly, for instance, in Putnam's [22]. In that work, Putnam examines the relativity of set-theoretic concepts as a consequence of Löwenheim-Skolem results for first-order logic and then extends the range of 'Skolemian relativity' to other set-theoretic notions, such as 'constructibility' or 'power of the continuum'. As a consequence of such a conceptual relativity, he argues that the *moderate realist* position, which asserts that we have means to describe a unique universe of sets, is untenable. However, he does not follow Skolem on the point of absolute relativity: in his view, we can still make sense of set-theoretic notions, by invoking our use and understanding of language: for instance, to understand what the concept 'uncountable' means consists in showing what it means *to prove* that a set is uncountable.[4] As a consequence,

[2] Our italics.

[3] However, supporters of the universe view may not necessarily be platonist, at least, not in the way described in Gödel's quote. For instance, a realist in *truth-value* but not in *ontology* (see, e.g., Shapiro, [23], p. 29) could also be viewed as being a universe-view supporter, in the sense that, although this person would not assert that there is only one structure meeting our concept of set, he would still claim that there are *objective* (unique) answers to set-theoretic problems.

[4] A natural consequence of Putnam's arguments in that work would be that of being able to debunk the typical claim often made by set-theorists that something is true in 'the picture of V given by a model', but not in the 'real V'. In his view, any such claim is meaningless, as any model of the axioms must be 'real'

if we are told, "axiomatic set theory does not capture the intuitive notion of set", then it is natural to think that *something else* - our "understanding" - does capture it. But what can our "understanding" come to, at least for a naturalistically minded philosopher, which is more than *the way we use our language*? ([22], p. 466)

Indeed, the typical multiverse-view supporter may share Putnam's views, and, accordingly, maintain that one is able to grasp set-theoretic notions, but, at the same time, live with different models.

The typical supporter of the multiverse view seems to fit more aptly into the label of 'anti-realist', and such are undoubtedly the proponents of the insolubility (or meaninglessness) of such questions as the Continuum Problem.[5] However, as we shall see, there are some supporters of the multiverse who have platonistic leanings. The kind of platonism involved therein, however, is different from *standard* platonism.

The supporter of the multiverse view may ascribe a form of *objectivity* to the global set-theoretic undertaking also in a different way: there might be some universes which are *preferable*, in view of certain particular purposes, or in view of certain predominant *philosophical maxims*, to use Maddian slang. It will be our task to show that this position, as integral to the Hyperuniverse Programme, makes perfect sense.

2. Multiverse Conceptions

2.1. **Three Alternative Views.** In the preceding section, we have briefly hinted at some features of the multiverse view. In this section, we want to examine three alternative conceptions, which, we think, will help us illustrate this position in more detail. These three alternative conceptions have been proposed by working set-theorists.

The first one is the already presented *radical view*, which will be examined here more thoroughly.

The second one is a form of *pluralism*, which, not unlike the radical view, is quietist on the existence of concurrent and competing axiomatisations (in particular, competing extensions of ZFC).

The last conception we will discuss results from an unwarranted restriction of the multiverse to a certain selection of *models* (initial segments of all set-generic extensions of a given model).

However, as we shall set forth very clearly, we are also not content with the *indifferentism* of the radical view, since we believe that a philosophically justified internal selection of preferred universes represents a fully legitimate and, to some extent, unavoidable move.

We wish to argue that all three conceptions are insufficient in some way, and we present relevant strands of criticism.

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in some sense, and, hence, there cannot be a 'real V', which our intuition would be able to capture (see further, p. 8, for an alternative assessment of this view, as connected to Hamkins' position).

[5]Consider, e.g., staunch supporters of *predicativism*, such as Feferman, who have expressed many times the view that some set-theoretic problems may be hopelessly vague.

2.2. **The Radical Multiverse View.** As we said, proponents of the radical view think that there is not a unique *set concept* (or, alternatively, that there is no unique *instantiation* of a set concept). Accordingly, in this view, each universe instantiaties one among the many possible concepts of set.

An inevitable consequence of the existence of many models of the axioms of ZFC would, therefore, be the shift from a notion of 'truth' *tout court* to that of 'truth in a model': given any two models M_1 and M_2, a particular φ may be true in one and false in the other. The reason would be, in M_1 the axioms refer to a set concept which is different from that referred to in M_2.

A standard expression of the radical view can be found, for instance, in Balaguer's [2] and [3].

The Balaguerian conception is glued to what its author has launched as a reform of classical Platonism, that is, *full-blooded platonism* (FBP).[6] Balaguer characterises his conception of set-theoretic truth in this way:

> According to FBP, both ZFC and ZF+ not-C[7] truly describe parts of the mathematical realm; but there is nothing wrong with this, because they describe *different* parts of that realm. This might be expressed by saying that ZFC describes the universe of sets_1, while ZF+not-C describes sets_2, where sets_1 and sets_2 are different kinds of things. ([2], p. 315)

Contrary to what one would maybe expect, according to Balaguer, his own notion of *mathematical truth* is not revisionary. We can still hold that a certain φ has a *definite* truth-value (that is, we can still commit ourselves to *logical bivalence*), so long as we specify what is the universe wherein φ lives. What we cannot hold, in some cases, is that φ has a *unique* truth-value.

> It is worth noting that what FBP does *not* advocate is a *shift* in our conception of mathematical truth. Now, it *does* imply (when coupled with a corresponding theory of truth) that the consistency of a mathematical sentence is sufficient for its truth. [...] What mathematicians *ordinarily* mean when they say that some set-theoretic claim is true is that it is true of the *actual* universe of sets. Now, as we have seen, according to FBP, there is no *one* universe of sets. There are many, but nonetheless, a set-theoretic claim is true just in case it is true of *actual* sets. What FBP says is that there are so many different kinds of sets that every consistent theory is true of an *actual* universe of sets. ([2], p. 315)

In other terms, Balaguer rejects the alternative 'truth is *absolute* or there is *no* truth': truth can be *pluralistic*, so long as it is relativised in the way indicated.

The existence of an intuitive framework for the universe of sets would seem to imply the aforementioned alternative, and what Balaguer suggests is that our prior belief that this was the case is simply misguided: we have an intuitive notion of a unique universe, no doubt, but this simply belies the existence of a *plurality* of universes. By no means

[6]FBP is equivalent to the claim that there exist as many *platonistic* universes as universes (models) of any set of axioms.

[7]ZF plus the negation of the Axiom of Choice.

are we forced to accept *absoluteness of truth* as a result of the existence of an intuitive pre-theoretical picture of the universe of sets.

Within the multiverse debate among set-theorists, the radical view has recently been advocated by Hamkins in a similar fashion.[8] In particular, Hamkins avows a form of platonism along the same lines as the FBP-ist's.

The multiverse view is one of higher-order realism - Platonism about universes - and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. ([15], p. 417)

However, Hamkins' stance differs from Balaguer's in that he does not think that *any model* is an existing universe of sets. The view he commits himself to is that we have different concepts of set and that the best way to study a specific concept of set is to find a model which instantiates it.

Often, the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise. ([15], p. 417)

Therefore, while there is a direct implication between a set concept and a model which instantiates it, one should not assume that the converse direction also holds: the mere *existence* (or construction) of a model is not sufficient to relate it to a *new* set concept.

Another point on which maybe Hamkins would distance himself from Balaguer is the issue of *multiverse-membership*. In the radical view supported by Balaguer, any model is on a par with any other model, and the multiverse consists of all *conceivable universes*. This implies that the multiverse is composed of all models of all given sets of axioms. There is no restriction on models and axioms, and, as a consequence, the multiverse must be *maximal*, in a sense.

Hamkins' conception seems to be more restrictive than this. He adapts Maddy's maxim 'MAXIMIZE' (which asserts that there are no limitations on what sets might exist) for his purposes, and uses it as a general underlying criterion for multiverse-membership. However, he does not seem to commit himself to the idea that *any* model of *any* set of axioms in the language of set theory is a member of the multiverse. On the one hand, he says:

Here, we may follow a multiverse analogue of MAXIMIZE, by placing no undue limitations on what universes might exist in the multiverse. ([15], p. 437)

Furthermore:

The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP and so on, perhaps even down to second order number theory, as this is set-theoretic in a sense. ([15], p. 436)

[8] The discussion here is essentially based on [15].

On the other hand, he specifies that:

> At the same time, there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. The point is that there is little need to draw sharp boundaries as to what counts as a set-theoretic universe, and we may easily regard some universes as more set-theoretic than others. ([15], p. 436-437)

Hamkins' claim that some universes might look more set-theoretic than others is not substantiated further. However, it would seem that what he has in mind is set-theoretic practice. Although, *ontologically*, all universes are on a par, for mathematical purposes, it might be more convenient to identify certain universes as more suitable to our investigations. Furthermore, certain model-theoretic constructions might be *irrelevant* or *redundant*. In any case, what he emphasises is the fact that we do not have complete knowledge of multiverse-membership and that, accordingly, there is no reason to set *a priori* criteria of inclusion.

So far, Hamkins' conception seems to be almost entirely modelled upon Balaguer's. A novelty is maybe represented by his conception of truth.

In Balaguer's view, 'truth' is reduced to 'truth in a model', hence independent set-theoretic statements may be true in one universe and false in another. In Hamkins' conception, this view is retained, but also somewhat altered. Statements such as CH may be true or false, but the answer to the Continuum Problem is not the value of the continuum (and, hence, the truth-value of CH) in a specific universe. Rather, the answer to an independent statement such as CH may consist in a careful explanation of the facts concerning its holding or failing across the multiverse.

> On the multiverse view, consequently, the continuum hypothesis is a settled question; it is *incorrect*[9] to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties. Of course, there are and will always remain questions about whether one can achieve CH or its negation with this or that hypothesis, but the point is that the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question. ([15], p. 429)

In other terms, on Hamkins' conception of the multiverse, while, *ontologically*, one cannot but surrender to the fact that CH has different truth-values in different universes, *epistemologically* one will get the significant result that the Continuum Problem is solved, since the answer to it consists in our detailed knowledge of where CH holds and where it does not. And the same, presumably, will apply to all other set-theoretic statements which are not decided by ZFC.

He also adds that:

[9] Our italics.

On the multiverse view, set theory remains a foundation for the classical mathematical enterprise. The difference is that when a mathematical issue is revealed to have a set-theoretic dependence, as in the independence results, then the multiverse response is a careful explanation that the mathematical fact of the matter depends on which concept of set is used. [...] ([15], p. 419)

The radical multiverse conception embraced by Hamkins seems to be unsatisfactory in many respects. Some of our objections are directed at the features of the FBP-ist's conception, and others at Hamkins' use of this conception for his purposes.

We do not want to provide a systematic refutation of FBP here, but rather mention and expand on existing criticism.

Colyvan and Zalta, in [5], argue that the FBP-ist's notion of truth may be implausible. In the radical view, truth needs to be constantly relativised to universes. This would imply that, whenever we think of any two different universes U_1 and U_2, then we automatically think of two entirely different sets of objects (and truths about them) living within them. But this is hardly the case. For instance, all forcing extensions of a model of the axioms of ZFC leave the truth-value of arithmetical sentences unchanged. This would suggest that those models display a high level of *ontological stability*, at least relative to their finite stages. What, then, is the reason to conjecture that the natural numbers referred to by M_1 are not the same as the natural numbers referred to by M_2, where M_1 and M_2 are two different forcing extensions? This kind of maximal ontological proliferation does not seem to be plausible to us either.

The second objection we wish to mention questions FBP's pretension to represent platonism correctly. For, platonists believe that the reality they are talking about puts some mentally efficacious constraints on our mathematical thought, whereas in the FBP-ist's view, there are no such constraints, as *any* of our set-theoretic conceptualisations is acceptable. The point is developed at length by Potter:

...for a view to count as realist, [...], it must hold the truth of the sentences in question to be metaphysically constrained by their subject matter more substantially than Balaguer can allow. A realist conception of a domain is something we win through to when we have gained an understanding of the nature of the objects the domain contains and the relations that hold between them. For the view that bare consistency entails existence to count as realist, therefore, it would be necessary for us to have a quite general conception of the whole of logical space as a domain populated by objects. But it seems quite clear to me that we simply have no such conception. ([21], p. 11)

In his paper on undecidable statements, also Field seems to acknowledge this interpretative difficulty:

Ontologically speaking, then, plenitudinous platonism is highly platonistic, indeed more platonistic than standard platonism: roughly, it postulates multiple mathematical universes where standard platonism (especially Quinean platonism) postulates only one. But methodologically speaking, plenitudinous platonism is quite anti-platonistic (or as I prefer to say, anti-objectivistic). ([7], p. 333)

But if Field's suggestion is correct, and FBP is methodologically anti-platonistic, how can one reliably use it to achieve the result Hamkins wants to achieve, that is, to secure the *existence* of multiple universes?

Now, let us turn to consider more specifically Hamkins' point of view.

First of all, it should be noticed that Hamkins strongly needs FBP in order to support his multiverse conception. The reason is, how can one otherwise legitimately conceive of a forcing extension as being an existent *universe of sets* without commitment to the actual instantiation of a set concept within a universe of sets? FBP purposefully allows one to state that any consistent theory, corresponding to one single *set concept*, *truly* describes a realm of mathematical objects. Hamkins can, then, use this fact to assert that any forcing extension of the universe, corresponding to a single set concept (for instance, a set concept which implies that $2^{\aleph_0} = \aleph_2$) truly describes an existing realm of mathematical objects.

In a recent paper, Koellner has expressed numerous concerns with the legitimacy of Hamkins' assumption that models obtained through forcing can be seen as such definite ontological constructs. After briefly reviewing various model-theoretic constructions in Hamkins' paper, Koellner says:

> In summary, on the face of it, all three methods provide us with models that are either sets in V or inner models (possibly non-standard) of V or class models that are not two-valued. In each case one sees by construction that (just as in the case of arithmetic) the model is non-standard. One can by an act of imagination treat the new model as the "real" universe. The broad multiverse position[10] is a *consistent* position. But we have been given no reason for taking that imaginative leap. We have given no reason for embracing the broad multiverse. ([18], p. 22)

Indeed, the reasons for Hamkins to take what Koellner calls 'imaginative leap' are provided by just what we are we are pointing out, that is, Hamkins' adherence to FBP.[11]

The third objection we want to bring forward is about multiverse-membership. It seems to us that multiverse-membership, as arising from Hamkins' conception, is not adequately calibrated to suit our mathematical purposes. For instance, on the radical view, ill-founded models belong to the multiverse. While this is perfectly consistent with the radical multiversist's presuppositions, one does not see how any progress in the understanding of set-theoretic truth will be enhanced by this stance. Indeed, in section 3, we will make it clear that we will consider *well-founded models* as the only legitimate members of our multiverse construct.

In conclusion, while Hamkins' conception is certainly consistent, it seems to us that it is unconvincing. Our major qualms are about his attempt at building a definite *multiverse ontology*, and his use of FBP as a necessary background conception.

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[10]What Koellner calls the *broad multiverse position* is equivalent to what we call the *radical multiverse position.*

[11]Hamkins declares his interpretation of forcing to be *naturalistic.* He takes his *naturalism* to imply that, insofar as set-theorists refer to forcing extensions as universes of sets, we should follow suit. However, as we have seen, he also maintains that his position is one of *higher-order realism*, something which is not necessarily implied by or connected to naturalism.

2.3. **Pluralism.** The second conception we will be reviewing is a slightly different form of *pluralism*, which has been expressed by Shelah. In his [24], he says:

> My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality". ([24], p. 211)

Shelah's view has no ontological flavour, and understandably so. Pluralism is the view that there are no facts of the matter about undecidable mathematical statements and, in general, about mathematics as a whole. Such a position is sometimes defined *if-thenist*, as the person advocating it is only interested in seeing what consequences (theorems) derive *if* one assumes certain axioms.

As a consequence, Shelah's lack of emphasis on what universes count as relevant to our purposes is accordingly equally understandable: on the pluralist view, each set of axioms is interpreted as referring to a collection of models, none of which has more reason to be seen as the correct one.

At the same time, Shelah does not seem to be conceding that mathematics is entirely *meaningless*, a position which would qualify him as an extreme formalist.[12]

Although Shelah seems to be quite keen on the radical view, he sets forth some important *caveats*. In principle, all universes are equally preferable, but at least some of them may still be differentiated (and preferred) according as whether they are more or less set-theoretically 'typical':

> ...a typical citizen will not satisfy $(\forall\alpha)[2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7}]$ but will probably satisfy $(\exists\alpha)[2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7}]$. However, some statements do not seem to me clearly classified as typical or atypical. You may think "Does CH, i.e., $2^{\aleph_0} = \aleph_1$ hold?" is like "Can a typical American be Catholic?"

In the radical view, there is no need to prefer any universe to another one. However, one could always set forth some criteria in view of particular purposes or, as it were, tastes.[13]

[12]Or, maybe, a 'radical pluralist', if one adopts the point of view expressed in Koellner, [17] (reprinted in [4], pp. 80-116).

[13]See, for instance, Field's take on this in [6], p. 300: '...we can still advance aesthetic criteria for preferring certain values of the continuum over others; we must now view these not as *evidence that* the continuum has a certain value, but rather as *reason for refining our concepts so as to give* the continuum that value, [...]'. Some sorts of correctives to the indifferentism of the radical view are also suggested by Balaguer in [2], p. 317: 'There are at least two ways in which the FBP-ist can salvage the objective bite of mathematical disputes. The first has to do with the notion of *inclusiveness* or *broadness*: the dispute over CH, for instance, might be construed as a dispute about whether ZF+CH or ZF+not-CH characterizes a broader notion of set. And a second way in which FBP-ists can salvage objective bite is by pointing out that certain mathematical disputes are disputes about whether some sentence is true in a *standard model.*' The second solution hinted at by Balaguer seems to us inapplicable: given that all universes are on a par, it does not even make sense to talk about *standard* models. In general, it does not seem to us that 'any objective bite' can be salvaged by FBP-ists, nor should this be relevant to their purposes.

Likewise, Shelah thinks that all new axiom candidates are on a par, in a sense. Since they are far less intuitive than the ZFC axioms, he calls them *semi-axioms*. Each semi-axiom is selected in virtue of some, mostly pragmatic, reasons, among which the ability to solve certain problems is possibly the most relevant. He mentions, for instance, GCH as a typical example of a semi-axiom.

> Clearly, even after forcing was found, it seems better to prove that something follows from GCH than just proving it is consistent; statements which we treat like this we shall call semi-axioms. [...] Of course, the extent to which we consider a statement a semi-axiom is open to opinion and may change in time. ([24], p. 210)

He does not furnish any criterion with regard to the selection of semi-axioms. He only maintains that any semi-axiom which decides statements which are undecidable in ZFC has a certain appeal and can, thus, be seen as legitimate as any other, although there would hardly be any room to view it as a 'true' axiom.

> Generally, I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt), is a sufficient reason to say that it is a "true" axiom. In particular, I do not find it compelling at all to see it as true. ([24], p. 212)

Shelah's pluralism, although unrelated to the radical multiverse conception, fosters the same conclusions. Therefore, Shelah's position does not do any better than Hamkins'. In particular, if one is interested, as we are, in expanding the range of set-theoretic truths in some way, there is nothing in this conception which can be of any help.

To be fair, it would seem that Shelah would be keen on identifying certain mathematical criteria whereby one could, at least, get a clearer picture of the options at hand. This is what originally leads him to introduce the notion of *semi-axiom*. But precisely when one would expect Shelah to analyse more carefully what a semi-axiom is, he remains silent, and does not provide us with any further relevant details.

2.4. **A 'Restrictive' Conception.** The multiverse, as it appears, either explicitly or implicitly, in the work of Balaguer, Hamkins and Shelah discussed so far, is not construed by its respective authors as an instrument to find new candidates for axioms. Their main motivation is to provide an interpretive framework for the inevitable phenomenon of incompleteness of ZFC in a mathematically useful way.[14] In this section we will briefly discuss Woodin's conception which, insofar as its author seems to be a supporter of the *universe view*, is naturally more concerned with the search for new axioms.[15]

[14] In fact, Balaguer's conception, upon which Hamkins' significantly relies, aimed to present a viable alternative to *standard* platonism in order to overcome the epistemological issues with which this latter is fraught, see [2]. In this respect, it strongly differentiates itself from Hamkins', Shelah's and Woodin's conceptions.

[15] Woodin's adherence to the universe view and also to standard platonism is particularly manifest in [28], which is partly devoted to showing that there may *exist* a *non-physical realm* of objects (mathematical entities).

Woodin does not attempt to define the multiverse in a general way; he is satisfied with taking the multiverse to mean the collection of all set-generic extensions of a given model:[16]

> Is the generic-multiverse position a reasonable one? The refinements of Cohen's method of forcing in the decades since his initial discovery of the method and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt the generic-multiverse position. This has been reinforced by some rather unexpected consequences of large cardinal axioms which I shall discuss later in this section. ([28], p. 17)

His motivation for taking this restrictive view of the multiverse is that he can then apply powerful mathematical tools, often developed by him, to derive certain mathematical results. However, the narrowing of focus to set-forcings is ill-motivated both philosophically and mathematically in that there is a priori no reason to identify a method (forcing) with the goal (the investigation of truth of first-order statements across models).

Woodin partly acknowledges this deficiency and makes a remark concerning class forcing as a possible candidate for enlarging the multiverse:

> At present there is no reasonable candidate for the definition of an expanded version of the generic-multiverse which allows *class forcing* extensions and yet which preserves the existence of large cardinals across the multiverse.[17] ([28], p. 18)

Despite this, some of Woodin's results are still being interpreted as showing that the very notion of 'multiverse' is inherently flawed. Woodin has certainly contributed to this confusion by implicitly conflating 'multiverse' and 'set-generic multiverse' in some of his other work (see for instance the discussion of Ω-logic in [27]).[18]

In surveying Woodin's position, we also attempt to clear up this confusion and comment on Woodin's work from the perspective of the Hyperuniverse Programme. In particular, we wish to stress that Woodin's result that the set-generic multiverse (and multiverses included in this one) is too restrictive can be seen in a positive light, subsuming it into our programme: as will be set forth in this paper (and was already stipulated in [10]), our initial choice of the multiverse is the collection of all countable transitive models of ZFC (the *hyperuniverse*). It is our philosophical and mathematical position that it is

[16]More precisely, he closes the given V under set-generic extensions and set-generic ground models, and considers rank initial segments V_α^W, of the resulting models W.

[17]In our view, even the addition of class forcing, hyperclass forcing, etc. would be insufficient; these more general forcings are, as said, just instances of one method. Woodin appears to make *utilitarian* choices here: he would consider adding a new type of model to the multiverse only if he could apply his large cardinal techniques to them. The obvious question is why the preservation of large cardinals should limit our choices. We briefly discuss Woodin's position on large cardinals later in this section.

[18]This confusion is further fuelled by work of Bagaria, Todorčević, Woodin, and many others, which identifies the concept of *absoluteness* with *set-generic absoluteness* (see, for instance, Bagaria, [1]). This leads some to propose *set-generic absoluteness* principles as candidates for new axioms of set theory. But as we argue in this paper, the fact that the truth of a statement cannot be changed by *set-forcing* does not guarantee its absoluteness, as it only means that its truth is invariant over the *set-generic multiverse*, a small fragment of the full multiverse as we regard this concept.

unwarranted to consider any smaller subfamily of models as representative enough. In particular, we see no reason to limit our attention to set-generic extensions for the sole reason that we happen to have good tools at hand for the investigation of a single model. Woodin's result, reviewed briefly in the next paragraph, complements this position by providing a mathematical argument which shows that under some starting assumptions, the subfamily of all set-generic extensions of a given model is provably too restrictive.

What is the crux of Woodin's refutation of the set-generic multiverse? The main idea is wrapped up in a complex theory built around Woodin cardinals. Woodin proved that if there is a proper class of Woodin cardinals in a countable transitive model M which is minimal in some sense, then the set-generic multiverse built around this M fails to have some desirable properties.[19]

Notice that the conclusion of the theorem depends on the assumption of a proper class of Woodin cardinals.[20] For Woodin, the result seems in essence unconditional because he believes that large cardinals (in particular, a proper class of Woodin cardinals) exist. This is a subtle point as not everyone takes consistency, let alone existence of large cardinals, for granted. In [28], Woodin does his best to convince the reader that there are strong structural reasons for the adoption of large cardinals. He claims that set-theoretic results are not merely a formalistic exercise, but that they have a physical meaning as well, in the sense that the prediction concerning the consistency of certain theories is a statement which can be disproved by an experiment. He further argues that once we accept the existence of large cardinals, many other set-theoretic statements, such as AD (Axiom of Determinacy), become meaningful and refer to existing entities. Importantly for our discussion of the multiverse, the inadequacy of the set-generic multiverse becomes, in Woodin's reading, an unconditional mathematical fact.[21]

It would seem that the initial motivation of Woodin and others in investigating the set-generic multiverse was to derive *absoluteness results*.[22] However, if the set-generic multiverse is too restrictive, as is the starting position of the present authors, and Woodin concluded as well, then the importance of such absoluteness is questionable. Clearly, one needs to consider a larger collection of models to retain the claim that the invariance of the truth of a statement across these models should compel us to consider adopting this statement as a new axiom.

3. The Hyperuniverse Programme

[19]'Desirable' here means that 'truth across the multiverse' should not have a syntactically simple definition. This condition is expressed in [28], p. 19, by means of two principles which Woodin calls Multiverse Laws. See, again, [28] for the statement of the theorem, and Kanamori, [16], p. 360, for the definition of Woodin cardinals.

[20]It is here that the restriction to *set forcings* is relevant, as a set forcing cannot destroy a proper class of Woodin cardinals.

[21]The exposition in [28] is rather convoluted. At times, Woodin appears to argue that the existence of large cardinals follows from some other mathematical or physical facts. He however refrains from stating such arguments explicitly, inasmuch as they might appear circular.

[22]See above, footnote 17.

3.1. **The Search for New Axioms.** The conceptions we have reviewed thus far present certain inadequacies which, in our view, make them partly unsuitable to make sense of the notion of *set-theoretic multiverse*. Hamkins' conception seemed to us to be dependent upon Balaguer's FBP, and, consequently, it carried over the shortcomings of the FBP-ist's conception. Shelah's conception, although related in some way to the existence of multiple universes, ultimately contents itself with re-stating the formalist's indifference to questions of truth. Woodin introduces an *ad hoc* version of the multiverse, that he, afterwards, rejects as flawed. However, the crux of his proof rests upon a misrepresentation of the multiverse concept.

Altogether considered, these conceptions share one major defect, that is, they do not lead us to view the multiverse as a theoretical tool to discover new set-theoretic truths. To be fair, these conceptions are deliberately *static* and *quietist* on the purpose of establishing the truth-value of undecidable statements, or, as in the case of Woodin's, may lead to an ultimately static view, that of the single 'real' universe.

In contrast to all the aforementioned conceptions, within the Hyperuniverse Programme, as we shall see, we foster the search for criteria for selecting certain universes from the *hyperuniverse*, a process whereby some set-theoretic statements – those holding in the selected universes – can be declared as *new* set-theoretic axioms.[23] However, we also emphasise that the choice of mathematical criteria to select universes is not fixed once and for all and, hence, that the statements which we declare as new axioms may vary. This fact leads to a *dynamic* view of set-theoretic truth.

Another key point which needs emphasising very clearly is the following. In introducing the hyperuniverse programme, we want to present procedures for the acceptance of new axioms which are based on *intrinsic evidence*. In our view, intrinsic evidence has not been addressed properly in the debate on the foundations of set theory, and we aim to fill this gap. As will be clearer from what follows (see §3.2), we claim that philosophically justified principles provide the desired intrinsic evidence. Consequently, one of the main tasks of the programme is to show how the choice of philosophically justified principles leads, as its final outcome, to the identification of new axioms.

3.2. **The Hyperuniverse and a Multi-level Process.** Our multiverse conception can be described as a *multi-level* process, which starts with the collection of all transitive countable models of ZFC and ends with the newly identified set-theoretic axioms. We define the collection of all transitive countable models of ZFC as the *hyperuniverse*. The hyperuniverse occupies the top level in the process.

This multiverse concept reduces the ontological messiness of the radical conception to the simplicity and clarity of one single model-theoretic construct. At the same time, the hyperuniverse is sufficiently rich to allow all kinds of universe constructions, most prominently *set-forcing*, *class-forcing* and *hyperclass-forcing*, and, therefore, it avoids the concerns raised by Woodin with regard to a purely set-generic multiverse.

[23]The hyperuniverse programme emerged from foundational work done by Friedman (see, for instance, [8] and [9]) and was first exposed by Friedman and Arrigoni in [10]. In the present work, we expand on and clarify some of the notions presented there.

However, it should be noticed preliminarily, and will be further clarified in the next subsection, that we do not commit ourselves to any notion of *existence* of the universes within the hyperuniverse. Granted, we define an ontological framework, but this should not be taken to be a description of any *mathematical reality* in any strong sense. The main reason for our quite deflationary ontological view is that we are more interested in studying the properties of the members of the hyperuniverse rather than in formulating a precise ontological doctrine about them. We believe that the features of models and the knowledge they give us are more important than their mere existence.

The second level consists of philosophically justified principles, such as *maximality* or *omniscience* (see [10], p. 88 and p. 91, respectively). Shelah referred to *typicality* as one further principle and in the next section Gödel's *principle of uniformity* will be briefly introduced. One of the key tasks of the hyperuniverse programme is to investigate such philosophical principles in light of the role they play ast underlying motivations for defining or accepting certain mathematical criteria. It will be equally interesting to address the more general problem of what kind of *justification* we can provide for our starting philosophical principles.

On the third level, we find mathematical criteria, which are the counterpart of the philosophical principles in the level above. These criteria are usually expressed by *second-* or *higher-order* set-theoretic statements applied to the members of the hyperuniverse. In [10], some instances of such higher-order set-theoretic statements have been mentioned: the criteria IMH, IMH* and SIMH corresponding to the principle of *maximality* and the criterion of *mathematical omniscience* (based on work done by M. Stanley, [25]) corresponding to the principle of *omniscience*,[24] and more may surface in future research.

After having chosen these mathematical criteria, one then explores the collection of members of the hyperuniverse which satisfy them. The fourth and last level consists, therefore, in isolating *first-order* set-theoretic statements of interest which hold in the universes which satisfy the criteria. For instance, the Singular Cardinal Hypothesis (SCH) holds in all universes in which IMH holds.[25] However, there might be *alternative* mathematical criteria deriving from the same philosophical principle which lead to mutually contradictory statements. This fact accounts for the tree-like character of the process, wherein the final results (first-order statements) depend on the *path* chosen in the tree.

We stress that we do not commit ourselves to the position that the axioms identified on the fourth level of the process are *unrevisable* and *definitive*. The tree-like structure of the process presented above allows one to follow different paths and, accordingly, find alternative axioms. In our view, *methodologically* and *epistemologically*, the *dialectical* process described is tantamount to explaining what axioms derive from what choices of underlying motivating principles, when they are formulated as mathematical criteria.

One may hope for this dynamic search for set-theoretic truth to converge to a consistent set of new axioms. This would indeed be a very exciting outcome. It is, however, too early

[24]$\sharp$-generation is now also currently being investigated as a criterion of maximality. See [11], and 3.6 below.

[25]See 3.6 for more mathematical details and references.

to say if this will come about, and it is only through further work on the Hyperuniverse Programme that an answer to this question will emerge.

3.3. **The Ontology of the Hyperuniverse.** Our multiverse conception within the Hyperuniverse Programme is tied to *epistemological* concerns. As already said, we are more concerned about *what* new properties about sets we can *discover* and in *what way*, using a multiverse framework, than in what the multiverse *is like* and, in what sense, if any, it *exists*.

However, a few words on this issue might be necessary. The problem of the existence of model-theoretic constructs such as, for instance, 'forcing extensions' is the problem of the *ontological status* of universes of set theory. As we saw, in order to state that certain model-theoretic constructs could legitimately be seen as definite ontological constructs, that is universes, Hamkins had to embrace a form of plentiful realism, whereby one can automatically instantiate any set concept (such as that arising through forcing) into an existing universe.

In opposition to the Balaguer-Hamkins point of view and conception, we wish to argue that the problem of the ontological status of such universes may be addressed in a different way. In order to advocate the multiverse view, what we only need do is to look at set-theoretic practice. We know that the axioms of first-order set theory are *non-categorical*, and that set theory has many models. Set-theorists deal with models by ascribing them a sufficient degree of ontological distinctness, at least as much distinctness as required by the postulation of the existence of different universes.

The issue of what is, then, the role left for V, if one simply takes the axioms to refer to the multiverse consisting of all models producible through several model-theoretic techniques, no doubt, is a relevant one. However, as we said, one simple solution consists in seeing a countable transitive model of ZFC as a specific picture of V. In this sense, V is a *partially defined* construct, which continually needs to be instantiated through a member of the hyperuniverse.

But for the time being and in view of our purposes in this paper, the thick philosophical question whether the members of the hyperuniverse really exist is to be left aside, as it clearly exceeds the boundaries of our theoretical perspective.

3.4. **Re-structuring the notion of 'truth in V'.** An issue parallel to the previous one is that of the meaning of 'true in V' in our conception. A desirable consequence of the programme would be to re-interpret such notion as the outcome of a reliable and justifiable process of selection of universes: all set-theoretic sentences holding across universes selected as preferred on the basis of justifiable criteria will be considered 'truths in V'.

First of all, it should be noticed that, for us, *truth* reflects a state of affairs generated by selections across the hyperuniverse. This view has been set forth very clearly by Friedman and Arrigoni:

> ...in formulating the Hyperuniverse Program the expression "*true* in V" is not used to reflect an ontological state of affairs concerning the universe of all sets as a reality to which existence can be ascribed independently of set-theoretic practice. Instead "*true* in V" is meant as a *façon de parler*

that only conveys information about set-theorists' epistemic attitudes, as a description of the status that certain statements have or are expected to have in set-theorists' eyes. ([10], p. 80)

In [10], it is also explicitly said that we distinguish between *de facto* and *de jure* truths. However, the expression '*de facto* truths' only conveys the idea that there are first-order statements that set-theorists have come to see as *non-revisable*. These include, at least, the axioms of ZFC; we do not wish to argue in favour of the non-revisability of these axioms. We only take their widespread acceptance as the standard axiomatic framework as a matter of fact, which is motivated by partly historical and partly theoretical reasons.

Accordingly, what Friedman and Arrigoni called *de jure* truths in [10], are nothing but the statements that set-theorists may come to recognise as most 'compelling', once the reasons for accepting them and their role in the whole set-theoretic undertaking have been exactly calibrated.

...formulating de jure set-theoretic truths, which lies at the core of the Hyperuniverse Program, may be understood as the *active* response of a non-Platonistically minded mathematician, who believes that it makes sense to search for new truths in V beyond *de facto* truths. ([10], p. 81)

Instead of relying upon mathematical practice, that is, *extrinsic evidence*, we wish to propose mathematical criteria which can be justified through the study of *philosophically motivated* principles.

We believe that the mere postulation of the existence of many universes is philosophically *sterile*, insofar as it does not lead to the identification of new truths. It is only the definition of procedures by which more truths are selected that arguably represents a way to expand our knowledge of truth in set theory.

Granted, the existence of the multiverse may, by itself, dissolve the problem of whether undecidable statements have a truth-value. But here we are taking a more active approach to the question. In our view, it is not sufficient to say, as Hamkins does, that the unsolved problems should, on the epistemological side, considered to be solved through the acceptance of a multiverse framework. As said, we believe that there are well-defined logical procedures to tie the selection of certain truths to intrinsic evidence (motivating principles) in a successful manner, thus overtaking the epistemological indefferentism implicit in Hamkins' and Shelah's conception. It is an important task of the programme to show in full detail what these logical procedures are and to provide convincing justifications for them.

3.5. **Principles and Criteria.** As said, within the programme, we are pre-eminently interested in examining *philosophical principles* which might possibly be turned into *mathematical criteria*.

We want, at this point, to be a bit clearer on what counts as a *philosophical principle* through some examples belonging to the history of set theory.

The story of the emergence of set theory can be construed as the gradual acceptance of *philosophical principles* which legitimise the notion of *actual infinite* in mathematics. However, from the point of view of post-Cantorian set theory, the emergence of set theory can be construed in terms of the accumulation of mathematical evidence in support of certain axioms. The mediation between philosophical principles and mathematical practice was provided by the emerging first-order axiomatisation of set theory, which included the Axiom of Infinity.

One might describe this process as a route from a philosophical principle, actual infinite, to a criterion (Axiom of Infinity in ZFC).[26]

One further historical example is famously furnished by Gödel. While arguing in favour of the intuitive implausibility of $V = L$ as a new axiom candidate, he says:

> ...from an axiom in some sense opposite to this one [$V = L$], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets, whereas axiom A [$V = L$] states some minimum property. Note that only a maximum property would seem to harmonize with the concept of set mentioned in footnote 11.[27]([13], pp. 478-9)

We can read this quote as stating that $V = L$ can be construed as a mathematical criterion based on a philosophical *principle of minimality*, that he rejected as implausible. In other cases, the philosophical principle invoked might be deeper and less obvious.

According to what Wang says in [26], Gödel might have argued in favour of the acceptance of a different philosophical principle, that is, the 'principle of uniformity':

> 8.7.5 *Uniformity of the universe of sets* (analogous to the uniformity of nature). The universe of sets does not change its character substantially as one goes from smaller to larger sets or cardinals. In some cases, it may be difficult to see what the analogous situations or properties are. But in case of simple and, in some sense, "meaningful" properties it is pretty clear that there is no analogue except the property itself. ([26], p. 281)

In this quote, there is no mention of any mathematical criterion which should correspond to this principle, and we do not intend to force any unwarranted interpretation of Gödel's words. However, it is reasonable to assume that Gödel might have thought that his principle of uniformity might be turned into a strong form of *mathematical reflection*.[28]

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[26]Another example is the mathematisation of the complementary *notion of finitude*: it is known that, without the Axiom of Choice, there are non-equivalent formulations (criteria) of finiteness (*Dedekind finiteness*, *Tarski finiteness*, etc.).

[27]That is, a concept of 'set' construed as the operation of 'set of' ranging over an indefinite multitude of objects. The operation of 'set of', therefore, would have to be considered a *conceptual primitive*.

[28]In any case, a mathematical criterion corresponding to Gödel's *principle of uniformity* should be formulated with some care, because the principle may suggest analogies which are not really warranted: not all sets share the same properties – there is a big difference between being *countable* and *uncountable*, for instance (the closed unbounded filter has no analogue at ω). This example suggests that the path from a reasonably sounding principle to a concrete criterion is far from obvious.

We believe that the aforementioned cases exemplify the way philosophical principles can be transformed into mathematical criteria. In our future work, we aim to develop and expand on these considerations.[29]

3.6. **Mathematical Remarks.** We conclude this work with some mathematical remarks. The above mentioned principle of *maximality* was formulated as a certain mathematical criterion of maximality in [8].[30] Here, a model M of set theory is called *internally maximal* if any statement which holds in an inner model of an outer model of M holds already in an inner model of M, and we say that M satisfies IMH (Inner Model Hypothesis). A priori, it is not even obvious that such a model M exists. However, in [8] models of IMH were constructed from PD (Projective Determinacy), and in [12] it was shown that existence of an internally maximal model implies that there exists an inner model with measurable cardinals of arbitrarily high Mitchell order. So IMH has a large cardinal consistency strength. Surprisingly, it was also shown in [8] that if M satisfies IMH, then in M there exists a real R such that $L_\alpha[R]$ does not satisfy ZFC for all ordinals α; in particular, there can be no inaccessible cardinals! This does not contradict the fruitful large cardinal techniques used in current set theory: while there can be no inaccessible cardinals in M, there can be *inner models* of M with *arbitrarily strong* large cardinals.[31] Such results suggest that the connection between large cardinals and maximality principles is not as clear as one might think. Is there a version of maximality such as IMH which does allow for the existence of large cardinals in the universe? This was recently solved positively in [11]: one adds to internal maximality a form of vertical maximality – a property formulated in terms of a generalized reflection principle for ordinals – and argues that if a model satisfies this combined form of maximality, it allows for the existence of arbitrarily large cardinals.

There are more principles which can be turned into mathematical criteria. Instead of maximising size, one can consider richness of expressive power: a model M is rich in this sense if it can define the collection of sentences which hold in arbitrary outer models of M. Again, it is not obvious that such models exist, but an unpublished work of Stanley

[29]As far as the issue of the role of extrinsic evidence is concerned, in [10], Friedman and Arrigoni say: "When declaring the intention of extending ZFC so as to settle independent questions, one also requires that one be as *unbiased* as possible as to the way such questions should be settled and as to which principles and criteria for preferred universes one should formulate. In particular, the latter must not be chosen at the outset so as to be apt for settling questions independent of ZFC, or for meeting the needs of some particular area existing in set-theoretic practice." ([10], p. 81). The attitude the authors deprecate has been the prevalent one. In particular, some of the most relevant literature (see, especially, Maddy, [19], [20] or, again, Koellner, [17]) has essentially focused on local areas of set theory and on a defence of maxims (and corresponding axioms) which are acceptable in view of their consequences in those areas. This is one important reason for not considering extrinsic evidence in the Hyperuniverse Programme.

[30]We provide here a brief review of the mathematical results relevant for the Hyperuniverse Programme. For more details, please consult [10], [8], and [12].

[31]Recall how large cardinals are often used – a given large cardinal κ is collapsed to some small cardinal, such as $\aleph_2$, and one further argues that this small cardinal still satisfies some combinatorial property previously holding for the large cardinal κ; to give a specific example, if the tree property – a local maximality property – should hold at every regular cardinal $\kappa > \aleph_1$ (a goal now seriously pursued), then in this model there can be no inaccessible cardinals while inner models for very large cardinals exist.

[25] suggests they may (this criterion is called in [10] *omniscience*). One can consider generalized forms of Levy absoluteness with parameters, or some sort of *saturation* which could be the mathematical counterpart of Shelah's notion (principle) of *typicality*. In all cases, however, there is a major mathematical challenge involved: one wishes to not only construct a model for the given criterion, but also to find an interesting mathematical statement which holds across such models. For instance, for IMH, this was SCH. For other criteria, this is still an open question.

4. Conclusion

We have reviewed some conceptions of the set-theoretic multiverse and argued that their shortcomings make them inadequate. First and foremost, all these conceptions do not view the multiverse as in connection with the goal of defining new set-theoretic truths and proposing new set-theoretic axioms, whereas our Hyperuniverse Programme is essentially concerned with this goal. We have tried to make it clear that our main interest lies in describing *philosophical principles* which might be turned into higher-order mathematical criteria leading to the selection of universes wherein new set-theoretic axioms, formulated as first-order statements, hold. However, we see the process of finding new axioms as *dynamic* and open to the wealth of possible choices within a complex *dialectical* process. We believe that the way we deal with this and other issues marks the strong *epistemological* character of our conception.

We also think that what has been said is sufficient to characterise our conception of the set-theoretic multiverse and, in particular, strongly differentiate it from the others reviewed here.

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