Hyperclass Forcing in Morse-Kelley Class Theory

Carolin Antos, Sy-David Friedman*
Kurt Gödel Research Center for Mathematical Logic
University of Vienna

Abstract

In this article we introduce and study hyperclass-forcing (where the conditions of the forcing notion are themselves classes) in the context of an extension of Morse-Kelley class theory, called MK**. We define this forcing by using a symmetry between MK** models and models of ZFC⁻ plus there exists a strongly inaccessible cardinal (called SetMK**). We develop a coding between β-models $\mathcal{M}$ of MK** and transitive models M^+ of SetMK** which will allow us to go from $\mathcal{M}$ to M^+ and vice versa. So instead of forcing with a hyperclass in MK** we can force over the corresponding SetMK** model with a class of conditions. For class-forcing to work in the context of ZFC⁻ we show that the SetMK** model M^+ can be forced to look like $L_{\kappa^*}[X]$, where κ^* is the height of M^+, κ strongly inaccessible in M^+ and $X \subseteq \kappa$. Over such a model we can apply definable class forcing and we arrive at an extension of M^+ from which we can go back to the corresponding β-model of MK**, which will in turn be an extension of the original $\mathcal{M}$. Our main result combines hyperclass forcing with coding methods of [BJW82] and [Fri00] to show that every β-model of MK** can be extended to a minimal such model of MK** with the same ordinals. A simpler version of the proof also provides a new and analogous minimality result for models of second-order arithmetic.

1 Introduction

When considering forcing notions with respect to their size, there are three different types: the original version of forcing, where the forcing notion is a set, called set forcing; forcing in ZFC, where the forcing notion is a class, called definable class forcing and class forcing in Morse-Kelley class theory (MK). In this article we consider a fourth type which we call definable hyperclass forcing and give applications for this forcing in the context of

Morse-Kelley class theory, where hyperclass forcing denotes a forcing with class conditions. We will define hyperclass forcing indirectly by using a correspondence between certain models of MK and models of a version of ZFC$^-$ (minus PowerSet) and show that we can define definable hyperclass forcing by going to the related ZFC$^-$ model and using definable class forcing there.

Two problems arise when considering definable class forcing in ZFC: the forcing relation might not be definable in the ground model and the extension might not preserve the axioms. As an example consider $Col(\omega, ORD)$ with conditions $p : n \to Ord$ for $n \in \omega$ which adds a cofinal sequence of length ω in the ordinals. Here Replacement fails[1]. These problems were addressed in a general way by the second author in [Fri00]) where class forcings are presented which are definable (with parameters) over a model $\langle M, A \rangle$ where M is a transitive model of ZFC, $A \subseteq M$ and Replacement holds in M for formulas mentioning A as a unary predicate. Two properties of the forcing notion are introduced, pretameness and tameness and it is shown that for a pretame forcing notion the Definability Lemma holds and Replacement is preserved and that tameness (which is a strengthening of pretameness) is equivalent to the preservation of the Power Set axiom. In this article we will adjust this approach to definable class forcing in ZFC$^-$. Pretameness is defined as follows:

Definition 1. *A forcing notion P is pretame iff whenever $\langle D_i | i \in a \rangle$, $a \in M$, is an $\langle M, A \rangle$-definable sequence of dense classes and $p \in P$ then there is $q \leq p$ and $\langle d_i | i \in a \rangle \in M$ such that $d_i \subseteq D_i$ and d_i is predense $\leq q$ for each i.*

For definable hyperclass forcing we will work in the context of Morse-Kelley class theory, by which we mean a theory with a two-sorted language, i.e. the object are sets and classes and we have corresponding quantifiers for each type of object. We denote the classes by upper case letters and sets by lower case letters, the same will hold for class-names and set-names and so on. Hence atomic formulas for the $\in$-relation are of the form "$x \in X$" where x is a set-variable and X is a set- or class-variable. The models $\mathcal{M}$ of MK are of the form $\langle M, \in, \mathcal{C} \rangle$, where M is a transitive model of ZFC, $\mathcal{C}$ the family of classes of $\mathcal{M}$ (i.e. every element of $\mathcal{C}$ is a subset of M) and $\in$ is the standard $\in$ relation (from now on we will omit mentioning this relation). We use the following axiomatization of MK:

A) Set Axioms:

    1. Extensionality for sets: $\forall x \forall y (\forall z \, (z \in x \leftrightarrow z \in y) \to x = y)$.

    2. Pairing: For any sets x and y there is a set $\{x, y\}$.

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[1] A detailed analyses on how even the Definability Lemma for class forcings can fail can be found in [HKL$^+$].

3. Infinity: There is an infinite set.

4. Union: For every set x the set $\bigcup x$ exists.

5. Power set: For every set x the power set $P(x)$ of x exists.

B) Class Axioms:

1. Foundation: Every nonempty class has an $\in$-minimal element.

2. Extensionality for classes: $\forall z\, (z \in X \leftrightarrow z \in Y) \rightarrow X = Y$.

3. Replacement: If a class F is a function and x is a set, then $\{F(z) : z \in x\}$ is a set.

4. Class-Comprehension:

$$\forall X_1 \ldots \forall X_n \exists Y\ Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

where φ is a formula containing class parameters in which quantification over both sets and classes are allowed.

5. Global Choice: There exists a global class well-ordering of the universe of sets.

Class forcing in MK was defined by the first author in [Ant]. Here the Definability Lemma holds for unrestricted forcing notions, but for the preservation of the axioms we still need pretameness and tameness.

The structure of this article will be as follows: First, we define the correspondence between certain models of a version of MK and ZFC$^-$ and show that this correspondence is indeed a coding between a variant of MK and certain models of the ZFC$^-$ which allows us to go back and forth between them. Then we define definable hyperclass forcing and show how the problems of definable class forcing in the setting of ZFC$^-$ can be handled. We conclude the chapter by giving an example of definable hyperclass forcing by showing that every β-model of a variant of MK can be extended to a minimal β-model of the same variant of MK with the same ordinals.

2   Coding between MK* and SetMK*

In the context of ZFC we can talk about definable class forcings as done in [Fri00], where we deal directly with the class forcing notion as it is definable from a class predicate. Here we want to develop a way of defining definable hyperclass forcings in MK, i.e. forcings with class conditions, but we will choose an indirect approach, which will allow us to reduce the technical problems as much as possible to the context of definable class forcing. So instead of talking directly about hyperclasses, we will use a correspondence between models of a variant of MK (called MK*) and models of a variant of ZFC$^-$ (called SetMK*). We get an idea of how such a model of SetMK* looks

by considering the following model of MK: $\langle V_\kappa, V_{\kappa+1}\rangle$ where κ is strongly inaccessible. Similar to this model we will show how to define a model of SetMK* with a strongly inaccessible cardinal κ which is the largest cardinal such that the sets of the MK* model are elements of V_κ and the classes are elements of V_{κ^*}, where κ^* is the height of the SetMK* model. We will then force over such a model with a definable class forcing which will give us an extension of the SetMK* model. From this extension we can then go back to a model of MK* and this is the definable hyperclass-generic extensions of the original MK* model.

$$\text{In SetMK}^*\text{: } M^+ \xrightarrow[\text{forcing}]{\text{def.class}} M^+[G]$$

$$\text{In MK}^*\text{: } \quad \mathcal{M} \xrightarrow[\text{forcing}]{\text{def.hyperclass}} \mathcal{M}[G]$$

In the following we will describe how we can go from MK* to SetMK* and vice versa and show that the basic properties of class forcing over a model of SetMK* hold. Then we give an application of definable hyperclass forcing regarding minimal models of MK**.

But before we develop the relation between these models further we will impose a restriction on the models we are considering.

Definition 2. *A model $\mathcal{M}$ of Morse-Kelley class theory is a β-model of MK if a class is well-founded in $\mathcal{M}$ if and only if it is true that the class is well-founded.*

We introduce this restriction for two reasons: First, we will define a coding which allows us to go from a β-model of MK* to a transitive model of SetMK* and this coding only works in the intended way if we know that every well-founded class in the model is really well-founded (see section 2). Secondly we will prove a theorem about minimal models and such a notion only makes sense if we work with minimal β-models. So from now on, we will always talk about β-models of (variants of) MK.

The associated model of set theory will be a model of ZFC$^-$ (i.e. minus the Power Set Axiom) where we understand such a model to include the Collection (or Bounding) Principle[2]. To ensure this we have to add the Class-Bounding Principle, a "class version" of the Bounding Principle, and we call the resulting axiomatic system MK*:

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[2]Note that in ZFC minus Power Set the Bounding Principle does not follow from Replacement. This is used in [Zar82], where he showed that in ZF$^-$ the different formulations of the Axiom of Choice are not equivalent. As for MK, work done in [GH] shows that for example ultrapower constructions don't work without first adding a version of Class-Bounding. For more information see [GHJ16]

Definition 3. *The axioms of MK* consist of the axioms of MK plus the Class-Bounding Axiom*

$$\forall x \, \exists A \, \varphi(x, A) \rightarrow \exists B \, \forall x \, \exists y \, \varphi(x, (B)_y)$$

where $(B)_y = \{z \mid (y, z) \in B\}$.

Note that as we have Global Choice, this is equivalent to AC_∞:

$$\forall x \, \exists A \, \varphi(x, A) \rightarrow \exists B \, \forall x \, \varphi(x, (B)_x).$$

Equivalently, SetMK* will include the set version of Bounding (here called Set-Bounding):

$$\forall x \in a \, \exists y \, \varphi(x, y) \rightarrow \exists b \, \forall x \in a \, \exists y \in b \, \varphi(x, y)$$

As we will show in the proof of Theorem 8 and the proof of Theorem 13, Set-Bounding in SetMK* follows from Class-Bounding in MK* and vice versa.

We are now going to show how to translate the theory of MK* to a first-order set theory SetMK*. The axioms of SetMK* are:

1. ZFC^- (including Set-Bounding).

2. There is a strongly inaccessible cardinal κ.

3. Every set can be mapped injectively into κ.

We can construct a transitive model M^+ of SetMK* out of any β-model $(M, \mathcal{C})$ of MK* by taking all sets which are coded by a pair (M_0, R), where M_0 belongs to $\mathcal{C}$ and R is a binary relation within $\mathcal{C}$. We will show that M^+ is the unique model of SetMK* with largest cardinal κ such that $M = V_\kappa^{M^+}$ and the elements of $\mathcal{C}$ are the subsets of M in M^+.

To describe the coding between SetMK* and MK* we will define what a coding pair (M_0, R) is and what it means for a coding pair (M_0, R) to code a set x in a model of SetMK*. In the coding below we work with relations which are classes, i.e. objects of rank $ORD\S$ which provide isomorphic copies of the membership relation on the transitive closure of x for a set x which may have rank greater than ORD (see Definitions 5 and 7).

Definition 4. *A pair (M_0, R) is a coding pair in the β-model $\mathcal{M} = (M, \mathcal{C})$ if M_0 is an element of $\mathcal{C}$ with a distinguished element a, $R \in \mathcal{C}$ and R is a binary relation on M_0 with the following properties:*

a) *$\forall z \in M_0 \, \exists! n$ such that z has R-distance n from a, i.e. there is an R-chain $(zRz_{n-1}R \ldots Rz_1Ra)$,*

   b) *if $x, y, z \in M_0$ with $y \neq z$, yRx, zRx then $(M_0, R) \upharpoonright y$ is not isomorphic to $(M_0, R) \upharpoonright z$, where $(M_0, R) \upharpoonright y$ denotes the R-transitive closure below y (i.e. y together with all elements which are connected to y via an R-chain), respectively for z,*

   c) *if $y, z \in M_0$ are on level n (i.e. have the same R-distance n from a) and $y \neq z$ then $vRy \rightarrow \neg(vRz)$,*

   d) *R is well-founded.*

Note that in the definition of the codes in $(M, \mathcal{C})$ we need the assumption that $(M, \mathcal{C})$ is a β-model as for a class to code a set in M^+ it has to be well-founded not only in the MK model but "in the real world".

The meaning of the definition becomes clearer when we view the coding pair as a tree T whose nodes are exactly the distinct elements of M_0, the top node is a and R is the extension relation of the tree. A tree T' with top node a' is a subtree of T if a' is a node of T and T' contains all T-nodes (not only immediately) below a'. If T' is a subtree of T such that a' lies directly below a then T' is called a direct subtree of T. Then property $b)$ states that for every node x distinct direct subtrees are not isomorphic and property $c)$ implies that the trees below two distinct points on the same level are disjoint (and not only on the next level).

The idea behind the coding pairs is, that every coding pair will define a unique set x in the SetMK* model. Note that at the same time every x in M^+ can correspond to different coding pairs in $\mathcal{M}$.

In the following we will give some intuition on what such a correspondence between coding pairs in $\mathcal{M}$ and sets in M^+ should look like: Every $x \in M^+$ is coded by a tree T_x where x is associated to the top node a_x of T_x, the elements $y \in x$ are associated to the nodes on the first level below a_x so that every node on this level gives rise to a subtree T_y which codes y so that the elements of y are associated to the nodes on the second level below a_x and so on:
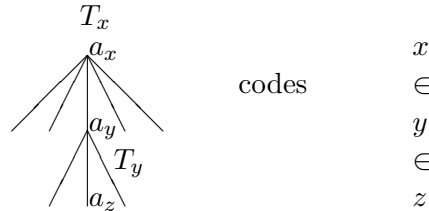

Note that there are only countably many levels but a level can have class many elements. If two elements a_y, a_z have the same R_x predecessor (i.e. are connected to the same node on the previous level) their subtrees T_y, T_z will never be isomorphic and therefore don't code the same element of M^+

(by property b) of Definition 4). But it can happen that there are isomorphic subtrees on different levels or on the same level but not connected to the same node on the level above. This can be made clear in the following two examples: First let $y \in x$, $v \in y$ and $w \in y$ and $v \in w$. Then there are two isomorphic trees T_v and T_v' both coding v but on different levels:
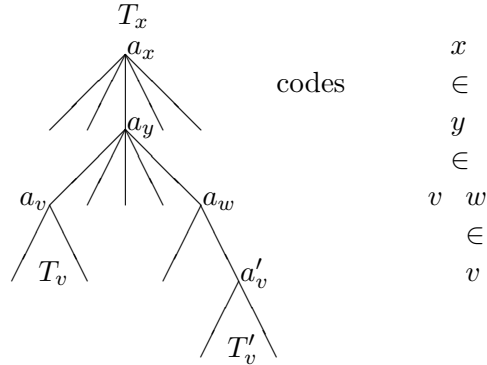
Secondly let $v \in w$, $v \in y$ and $w, y \in x$. Again there are two isomorphic trees T_v and T_v' coding v but this time on the same level:
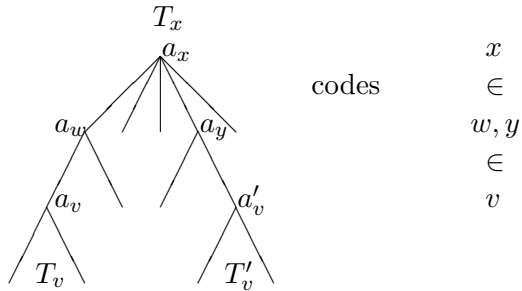

To show that Definition 4 indeed defines a coding, we have to show that there is a correspondence between x and its coding pair. As we want to include non-transitive sets we will work with $(TC(\{x\}), \in)$ (note that we used the transitive closure of $\{x\}$ rather than the transitive closure of x as the transitive closure of two different sets could be the same). As we have seen, the coding tree will have a lot of isomorphic subtrees, for example many different pairs $(a_i, \{\})$ coding the empty set. So the tree T_x itself will not be isomorphic to $(TC(\{x\}), \in)$ and we will have to collapse (M_x, R_x) to a structure $(M_x, R_x)/\approx$ in which we have identified all these isomorphic subtrees. We define this quotient of the coding pair in the following way:

Definition 5. *For a coding pair (M_0, R), let $[a] = \{b \in M_0 \,|\, (M_0, R) \restriction b$ isomorphic to $(M_0, R) \restriction a\}$ be the equivalence class of all the top nodes of subtrees of the coding tree T which are isomorphic to the subtree T_a (here $(M_0, R) \restriction b$ denotes the "sub-coding pair" which is the subtree T_b as detailed*

in Definition 4). By Global Choice let $\tilde{a}$ be a fixed representative of this class. Then let $\tilde{M}_0 = \{\tilde{a} \mid a \in M_0\}$ and define the relation $\tilde{R}$ as follows: $\tilde{a}\tilde{R}\tilde{b}$ iff $\exists a_0, b_0$ such that $a_0 \in [a]$ and $b_0 \in [b]$ and $a_0 R b_0$.

Note that if $a_0 \approx a_1$ and $b_0 R a_0$ then there is b_1 with $b_1 R a_1$ such that $b_0 \approx b_1$ as the isomorphism between T_{a_0} and T_{a_1} will restrict to the trees T_{b_0} and T_{b_1}.

The following example shows how this quotient structure looks for a possible coding tree of the set 3:
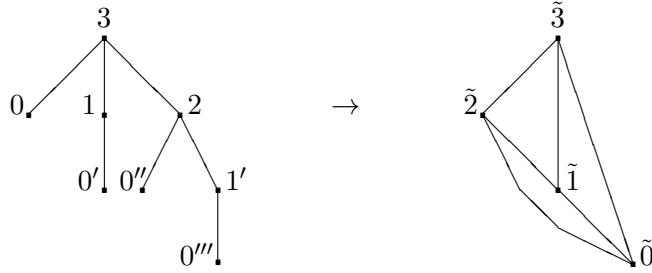
As one can see, the resulting structure $(\tilde{M}_3, \tilde{R}_3)$ is then isomorphic to $(TC(\{3\}), \in)$. In the following we will show that this construction works in general:

Lemma 6. *Let (M_0, R) be a coding pair. Then the quotient structure $(\tilde{M}_0, \tilde{R})$ as defined in Definition 5 is extensional and well-founded.*

Proof. By Class-Comprehension $\tilde{R} \in \mathcal{C}$ and $\tilde{R}$ is well-founded as we can always find an R-minimal element a, build the equivalence class $[a]$ and find its representative $\tilde{a}$. Then $\tilde{a}$ is $\tilde{R}$ minimal as otherwise there exists $\tilde{a}'$ such that $\tilde{a}'\tilde{R}\tilde{a}$ and therefore there is $a_0' \in [a']$ such that $a_0' R a$.

To show that $\tilde{R}$ is extensional, let $\tilde{y}, \tilde{z} \in \tilde{M}_0$ with $\tilde{y} \neq \tilde{z}$ and assume that they have the same extension $\{\tilde{x} \mid \tilde{x}\tilde{R}\tilde{y}\} = \{\tilde{x} \mid \tilde{x}\tilde{R}\tilde{z}\}$. Going back to (M_0, R) this means that the elements of the related equivalence classes $[y]$, $[z]$ have the same isomorphism types of children, i.e. for every $x_0, y_0, z_0 \in M_0$ with $x_0 R y_0$, $y_0 \in [y]$ and $z_0 \in [z]$ we can find x_1 with $x_1 R z_0$ such that $x_0, x_1 \in [x]$. By using property $b)$ of Definition 4 it follows that the $[y] = [z]$, because we do not have multiplicities in (M_0, R), i.e. isomorphic subtrees that are connected to the same R-predecessor. It follows that $\tilde{y} = \tilde{z}$. $\square$

Note that the quotient structure always has a fixed top node which is the representative of the equivalence class of the distinguished node of (M_0, R), which has the distinguished node as its only element.

It follows from Mostowski's Theorem that there is a unique transitive structure with the $\in$-relation that is isomorphic to $(\tilde{M}_0, \tilde{R})$. This structure then has the form $(TC(\{x\}), \in)$ for a unique set x.

Definition 7. *A coding pair (M_x, R_x) is called a coding pair for x, if x is the unique set such that $(\tilde{M}_x, \tilde{R}_x)$ is isomorphic to $(TC(\{x\}), \in)$.*

In the following we will use this coding to associate a transitive model of SetMK* to each β-model of MK* and vice versa.

Theorem 8. *Let $\mathcal{M} = (M, \mathcal{C})$ be a β-model of MK* and*

$$M^+ = \{x \mid \text{there is a coding pair } (M_x, R_x) \text{ for } x\}$$

*Then M^+ is the unique, transitive set that obeys the following properties:*

a) *$M^+ \models$ SetMK*,*

b) *$\mathcal{C} = P(M) \cap M^+$,*

c) *$M = V_\kappa^{M^+}$, κ is the largest cardinal in M^+ and strongly inaccessible in M^+.*

The coding between $\mathcal{M}$ and M^+ is the key to prove the theorem. So before proving this theorem we will prove two useful fact about the coding.

As we have seen there can be more than one coding pair for an $x \in M^+$. Of course these coding pairs are isomorphic because they are all built according to Definition 4 but we also would like to know that they are isomorphic in $\mathcal{M}$. For elements of M^+ that can be coded by sets in $\mathcal{M}$ this is trivial but for elements that are coded by proper classes we have to show the following:

Lemma 9 (Coding Lemma 1)**.** *Let $\mathcal{M} = (M, \mathcal{C})$ be a transitive β-model of MK^*. Let $N_1, N_2 \in \mathcal{C}$ and R_1, R_2 be well-founded binary relations in $\mathcal{C}$ such that (N_1, R_1) and (N_2, R_2) are coding pairs as described in Definition 4. Then if there is an isomorphism between (N_1, R_1) and (N_2, R_2) there is such an isomorphism in $\mathcal{C}$.*

Proof. Let T_1, T_2 be the coding trees associated to the coding pairs (N_1, R_1), (N_2, R_2). Assume to the contrary that there is an isomorphism between T_1 and T_2 but not one in $\mathcal{C}$. It follows that the tree below the top node of T_1 is isomorphic to the tree below the top node of T_2, but there is no such isomorphism in $\mathcal{C}$. Then, as T_1 and T_2 are well-founded we can choose a T_1-minimal node a_1 of T_1 such that for some node a_2 of T_2 the tree U_1 (the tree T_1 below and including a_1) is isomorphic to U_2 (the tree T_2 below and including a_2) but there is no isomorphism in $\mathcal{C}$. Because of the minimality of a_1 we know that for every node $a_{1,i}$ of U_1 just below a_1 and every node $a_{2,j}$ of U_2 just below a_2, if $U_{1,i}$ is isomorphic to $U_{2,j}$ then there is an isomorphism in $\mathcal{C}$. Moreover the property "$U_{1,i}, U_{2,j}$ are ismorophic" is expressible in $(M, \mathcal{C})$.
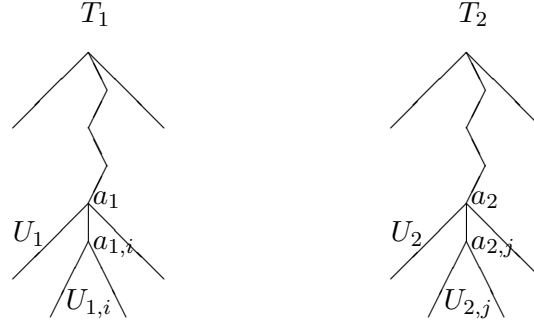
Now we can apply the Class Bounding Principle of MK* to get a class B so that for each $a_{1,i}$, $a_{2,j}$ for which $U_{1,i}$, $U_{2,j}$ are isomorphic, $(B)_c$ is such an isomorphism for some set c. Using the global well-order of M we can choose a unique $c(a_{1,i}, a_{2,j})$ for each relevant pair $\langle a_{1,i}, a_{2,j} \rangle$ and combine the isomorphisms $(B)_{c(a_{1,i}, a_{2,j})}$ to get an isomorphism between U_1 and U_2 in $\mathcal{C}$, which is a contradiction.

<div align="right">□</div>

So all coding trees of the same element of M^+ are isomorphic in $\mathcal{C}$. For the converse it is obvious that two isomorphic coding trees code the same element in M^+ as they give rise to the same $(\tilde{M}_x, \tilde{R}_x)$.

The next lemma shows that we are able to see something of the coding in M^+:

Lemma 10 (Coding Lemma 2)**.** *For all $x \in M^+$ there is a one-to-one function $f \in M^+$ such that $f : x \to M_x$, where (M_x, R_x) is a coding pair for x.*

Proof. Let T_x be a coding tree for x and for each $y \in x$ let T_y is the subtree of T_x with top node a_y lying just below the top node of T_x such that T_y codes y. Note that the choice of a_y is unique after having fixed the tree T_x.

To show that $f = \{ \langle y, a_y \rangle \mid y \in x \}$ belongs to M$^+$, we have to find a coding tree for f. Firstly we construct a coding tree $T_{\langle y, a_y \rangle}$ for every $\langle y, a_y \rangle$ with $y \in x$. As a_y is a set in M, it is a set in M^+ and therefore coded by some T_{a_y}. So we can build $T_{\langle y, a_y \rangle}$ by connecting the trees T_y and T_{a_y}. To make sure that the relation $R_{\langle y, a_y \rangle}$ on the new tree is well-defined we can relabel the nodes of the tree T_{a_y} and so we get the following picture:
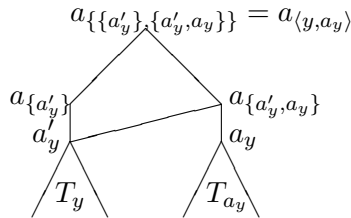
In this way we code every pair $\langle y, a_y \rangle$ with $y \in x$ and we can now join all the codes to code f.

Let (M_f, R_f) be the following pair: $M_f = \bigcup_{z \in x} M_{\langle z, a_z \rangle} \cup \{a_f\}$ where $a_f \in M$ and $a_f \notin M_{\langle z, a_z \rangle}$ for every $z \in x$. Then R_f is the binary relation which is defined using R_x as parameter:

$$R_f = \{\langle v, w \rangle \mid \text{ for some } y \in x \text{ either } \langle v, w \rangle \in R_{\langle y, a_y \rangle} \text{ or}$$
$$v = a_{\langle y, a_y \rangle} \text{ and } w = a_f\}$$

M_f and R_f are well-defined because of Class Comprehension in MK^* and so f is coded by the tree T_f which is ordered by R_z below every a_z and by putting $a_{\langle y, a_y \rangle}$ below a_f otherwise.

$\square$

Now we give the proof of Theorem 8.

Proof. a) We show that if $\mathcal{M}$ is a β-model of MK^* then $M^+ \models SetMK^*$. The first step is proving that M^+ satisfies ZFC^- with Set-Bounding.

Observe that M^+ is transitive: Let $x \in M^+$. Then for every $y \in x$ there is a coding tree for y (namely the corresponding subtree of T_x). Therefore $y \in M^+$ and so $x \subseteq M^+$. From transitivity it follows that Extensionality and Foundation hold in M^+; Infinity follows as $\omega \in M^+$.

Pairing: Let x, y be coded by T_x, T_y respectively. Then $\{x, y\}$ is coded by the tree:
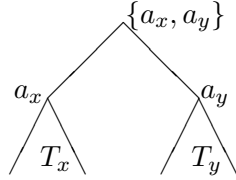
Union: Let x be coded by T_x:
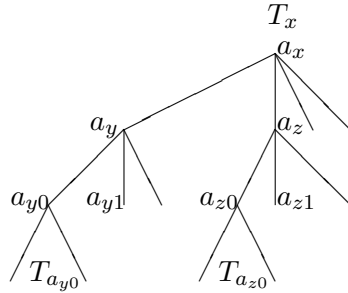
The obvious way to code $\bigcup x$ would be to join the $a_{y0}, a_{y1}, \ldots, a_{z0}, a_{z1}, \ldots$ together by one top node $a_{\bigcup x}$. But in general this is not a coding tree by reasons of isomorphism: Our coding trees have the property that subtrees which are connected to the same node on the next level above are all pairwise non-isomorphic. In this case that means that the trees $T_{a_y}, T_{a_z}, \ldots$

are pairwise non-isomorphic, as are the trees $T_{a_{y0}}, T_{a_{y1}}, \ldots$ and the trees $T_{a_{z0}}, T_{a_{z1}}, \ldots$ and so on. But, as we explained before, it can happen that some of the $T_{a_{yi}}$ are isomorphic to, for example, some of the $T_{a_{zj}}$. So if we connect these trees by a top node the resulting tree would have isomorphic subtrees connected by the same node on the next level and therefore would not be a coding tree. This problem can easily be resolved by taking equivalence classes of the subtrees of T_x from the second level below a_x (where two trees are equivalent if the are isomorphic). Then we take a representative from each equivalence class and connect them to the top node $a_{\bigcup x}$ (as before, this is possible by Class Comprehension in MK^* and Coding Lemma 1).

To prove Comprehension and Bounding we need to take a closer look at how formulas in M^+ translate to formulas in $\mathcal{M}$:

Lemma 11. *For each first-order formula φ there is a formula ψ of second-order class theory such that for all $x_1, \ldots, x_n \in M^+$, $M^+ \models \varphi(x_1, \ldots, x_n)$ if and only if $\mathcal{M} \models \psi(c_1, \ldots, c_n)$ for any choice of codes $c_1, \ldots, c_n$ for $x_1, \ldots, x_n$.*

Proof. The proof is by induction over the complexity of the formula φ. For the first atomic case assume that $M^+ \models y \in x$. Let c_x and c_y be codes for x and y respectively and let T_x, T_y be the associated coding trees. As we know that $y \in x$ it follows that there is a direct subtree $T_{y'}$ of T_x such that $T_{y'}$ is a coding tree for y ("direct subtree" means a subtree whose top node lies just below the top node of the original tree). As $T_{y'}$ and T_y are both codes for y they are isomorphic and by Coding Lemma 1 we know that they are isomorphic in $\mathcal{M}$. So $\mathcal{M} \models$ "c_y is isomorphic to a direct subtree of c_x" and this therefore is the desired ψ.

For the second atomic case assume that $M^+ \models y = x$. Let c_x and c_y be codes for x and y respectively. As $y = x$, c_y is also a code for x and again by Coding Lemma 1 we know that the codes are isomorphic in $\mathcal{M}$ thus giving us the desired ψ.

The cases of $\neg\varphi$, $\varphi_1 \wedge \varphi_2$ follow easily by using the induction hypothesis. For the quantifier case consider $M^+ \models \forall x \varphi$. By induction hypothesis let ψ be the second-order formula associated to φ. Then $M^+ \models \forall x \varphi$ translates to $\mathcal{M} \models \forall c$, if c is a code then $\psi(c)$. $\square$

Comprehension: Let $a, x_1, \ldots, x_n \in M^+$ and let $\varphi(x, x_1, \ldots, x_n, a)$ be any first-order formula. We will show that $b = \{x \in a : M^+ \models \varphi(x, x_1, \ldots, x_n, a)\}$ in an element of M^+ by using Class Comprehension in $\mathcal{M}$ to find the corresponding $B \in \mathcal{C}$ and build from it a coding tree for b.

Let $T_{x_1}, \ldots, T_{x_n}, T_a$ be codes for the corresponding elements of M^+ and let ψ be the formula corresponding to φ provided by Lemma 11. Assume that b is non-empty, i.e. that there is x_0 in a such that φ holds. Therefore there is a c_0 such that $\psi(c_0, T_{x_1}, \ldots, T_{x_n}, T_a)$ holds. Let c be a variable that

varies over the level directly below the top level of T_a so that each $T_{a(c)}$ denotes a direct subtree of T_a. Then by Class Comprehension there is a class B such that if $\psi(T_{a(c)}, T_{x_1}, \ldots, T_{x_n}, T_a)$ holds then $(B)_c$ is the direct subtree $T_{a(c)}$ of T_a and if not then $(B)_c$ is T_{c_0}.

So let T_b be the coding tree with top node a_b and whose direct subtrees are all of the $(B)_c$:

$$T_b \qquad \overset{a_b}{\bigwedge} \qquad \qquad \text{ordered by } R_{a(c)}$$

with $a(c)$ and $T_{a(c)}$ labelling the subtree
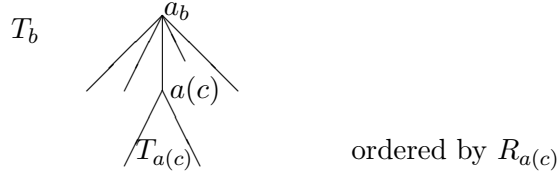
Then T_b codes $b \in M^+$ with $b = \{x \in a : \varphi(x, x_1, \ldots, x_n, a)\}$.

Bounding: We have to show that for $a \in M^+$ and φ a first-order formula

$$M^+ \models \forall x \in a \, \exists y \, \varphi(x, y) \to \exists b \, \forall x \in a \, \exists y \in b \, \varphi(x, y).$$

So assume that $\forall x \in a \, \exists b \, \varphi(x, y)$. Let T_y, T_a be coding trees for y and a respectively and let ψ be the second-order formula corresponding to φ provided by Lemma 11. By Class-Bounding in MK* we know that

$$\exists B \, \forall T_x \text{ direct subtree of } T_a \exists y' \, \psi(T_x, (B)_{y'}),$$

where $(B)_{y'} = \{z \mid (y', z) \in B\}$. By Class Comprehension we can join together all the section $(B)_{y'}$ which are coding trees $T_{(B)_{y'}}$ to obtain a tree T_b with top node a_b such that the $T_{(B)_{y'}}$ are the direct subtrees of T_b. It follows that in $\mathcal{M}$ there is a tree T_b such that for every tree T_x subtree of T_a there is a $T_{(B)_{y'}}$ direct subtree of T_b such that $\psi(T_x, T_{(B)_{y'}})$ and the tree T_b gives us the desired b in M^+.

Replacement: Follows from Comprehension and Bounding.

Choice: We have to show that every element of M^+ can be well-ordered (we aim for the strongest version of the axiom of Choice in a set-theory without Power Set (see [Zar82]). So let $x \in M^+$ and let T_x be a coding tree for x with top node a_x. We know that the direct subtrees T_y of T_x code the elements y of x and their top nodes a_y are elements of M. As we have a well-order of M we can well-order the class $B = \{a_y \mid a_y$ is the top node of a direct subtree T_y of $T_x\}$. We call this well-order W. Now we can build a tree for every pair $\langle a_y, a_z \rangle \in W$ by using the trees T_y, T_z analogous as we did in the proof of Coding Lemma 2:

$$\{\{a_y\}, \{a_y, a_z\}\} = a_{\langle y,z \rangle}$$

$$\{a_y\} \qquad \{a_y, a_z\}$$

$$a_y \qquad a_z$$

$$T_y \qquad T_z$$
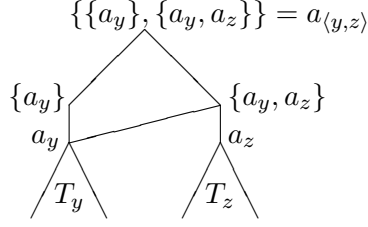
So for every $\langle a_y, a_z \rangle \in W$ we get a coding tree for the pair $\langle y, z \rangle$ with $y, z \in x$. As we have shown in the proof of Coding Lemma 2 we can now join together the trees by a single top node a_w using Class Comprehension. We now get a tree T_w which is a coding tree for an element w of M^+ and w is a well-order of x.

Remark 12. *The next two results below (b and c) will show, that there even is a global choice function for the sets in $V_\kappa^{M^+}$ for κ an inaccessible cardinal, as there is a class which well-orders M and we will show that every class in $\mathcal{C}$ is an element of M^+.*

b) We have to show that $\mathcal{C} = P(M) \cap M^+$. So assume that $X \in \mathcal{C}$ and $y \in X$. Then $y \in M$ and so can be coded by the following tree: y is the top node of the tree T_y. On the first level below the top node there are nodes for every element of y which are named by pairwise different elements z_i of $M \setminus \{y\}$. On the first level below such an z_i there are nodes for every element in z_i named by pairwise different elements v_j of $M \setminus \{y, z_i\}$ and so on. So T_y is a coding tree for y and therefore $y \in M^+$. This can be done for all $y \in X$ and by Class Comprehension the trees T_y can be connected to a tree T_X with top node a_X. Then the pair (M_X, R_X) gives a code for X with $M_X = \bigcup_{y \in X} M_y \cup \{a_X\}$ and

$$R_X = \{\langle v, w \rangle \mid \text{for some } y \in X \text{ either } \langle v, w \rangle \in R_y \text{ or } v = a_y \text{ and } w = a_X\}$$

Therefore $X \in M^+$.

For the converse, let $x \in M^+$ and $x \subseteq M$. Then there exists a coding pair (M_x, R_x) of x such that $(\tilde{M}_x, \tilde{R}_x) \cong (TC\{x\}, \in)$ (see Lemma 6). As $(\tilde{M}_x, \tilde{R}_x)$ is in $\mathcal{C}$, has rank $\mathrm{Ord}(M)$ and we can build $TC(\{x\})$ by transfinite induction from $(\tilde{M}_x, \tilde{R}_x)$, we can decode x in $\mathcal{C}$ and so $x \in \mathcal{C}$.

c) Now we will show that there is a strongly inaccessible cardinal κ in M^+ which is the largest cardinal in M^+ and the elements of M (the sets in $\mathcal{M}$) are exactly the elements of $V_\kappa^{M^+}$.

Let κ be $\mathrm{Ord}(M)$. Then as $\kappa \subseteq M$ and $\kappa \in M^+$ it follows from b) that κ is a class in $\mathcal{C}$. Let $f : \beta \to \kappa$ with β is a ordinal less than κ be a function in $\mathcal{C}$. From the Class Bounding Principle it follows that f is bounded in κ.

So κ is regular in $\mathcal{M}$ and therefore regular in M^+. Moreover, again by b), any subset of an ordinal β of M which belongs to M^+ is a class in $\mathcal{C}$ and indeed a set in M, so the power set of β in M^+ equals the power set of β in M and so κ is strongly inaccessible. It follows that if $x \in M$ then $x \in V_\kappa^{M^+}$. For the converse let $x \in V_\kappa^{M^+}$ and let (M_x, R_x) be a coding pair and T_x the associate coding tree for x . By Coding Lemma 2 any coding tree of a set is a set, so T_x is an element of M. Clause 3 of the axioms of SetMk* follows directly from Coding Lemma 2 and so κ is the largest cardinal in M^+.

That M^+ is unique follows from its construction: Let M^{++} be another such model of SetMK* (i.e. it is transitive, $\mathcal{C} = P(M) \cap M^{++}$ and $M = V_\kappa^{M^{++}}$ with κ largest cardinal in M^{++} and strongly inaccessible cardinal in M^{++}). Then M^+ and M^{++} have the same largest cardinal κ, they have the same subsets of κ and as every set in both models can be coded by a subset of κ they are the same.

This concludes the proof of Theorem 8. $\hfill\square$

The converse of Theorem 8 follows by the corresponding axioms in the SetMK* model:

Theorem 13. *Let N be a transitive model of SetMK* that has a strongly inaccessible cardinal κ that is the largest cardinal, let $\mathcal{C} = P(M) \cap N$ and M is defined to be V_κ^N. Then $\mathcal{M} = (M, \mathcal{C})$ is a β-model of MK* and the model M^+ derived from $\mathcal{M}$ by Theorem 8 equals N.*

Proof. We have to show that $(M, \mathcal{C})$ fulfills the axioms of MK*: Extensionality, Pairing, Infinity, Union, Power Set, and Foundation follow directly by the corresponding axioms of SetMK*. By the definition of M and $\mathcal{C}$ it follows that every set is a class and elements of classes are sets.

For the remaining axioms, note that there is an easy converse for Lemma 11: For each formula φ of second-order class theory there is a first-order formula ψ such that for all $x_1, \ldots, x_n \in \mathcal{M}$, $\mathcal{M} \models \varphi(x_1, \ldots, x_n)$ if and only if $N \models \psi(x_1, \ldots, x_n)$. This holds because by assumption all elements of $\mathcal{M}$ are elements of $\mathcal{C}$ or M and therefore elements of N and so φ and ψ are the same where the statement that x is a set in $\mathcal{M}$ translates to $x \in V_\kappa^N$ and the statement that X is a class in $\mathcal{M}$ translates to $X \in P(M) \cap N$. So for Class Comprehension we have to show that the following holds:

$$\forall X_1 \ldots \forall X_n \exists Y \ Y = \{x : \varphi(x, X_1, \ldots, X_n)\}$$

where φ is a formula containing class parameters in which quantification over both sets and classes is allowed. By the definition of M and $\mathcal{C}$ this statement is exactly the Comprehension Axiom of N where ψ is the first-order formula corresponding to φ: $y = \{x \in V_\kappa^N : N \models \psi(x, x_1, \ldots, x_n, V_\kappa^N)\}$.

For Class Bounding we have to show:

$$\forall x \, \exists A \, \varphi(x, A) \to \exists B \, \forall x \, \exists y \, \varphi(x, (B)_y)$$

where $(B)_y = \{z \mid (y, z) \in B\}$. So assume that $\forall x \,\exists A \, \varphi(x, A)$ holds in $\mathcal{M}$. Then translating this to N we know by Set-Bounding that

$$\forall x \in V_\kappa^{M^+} \, \exists A \in P(M) \cap M^+ \, \psi(x, A) \to \exists b \, \forall x \in V_\kappa^{M^+} \, \exists y \in b \, \psi(x, y)$$

where ψ is the first-order formula corresponding to φ. By Set-Comprehension we can form a set b_0 from b such that $b_0 = \{y \mid y \in b \wedge y \subseteq V_\kappa^N\}$. Then there is a function $f \in N$ from V_κ^N onto b_0 (as b_0 has size less or equal κ) and so f is also an element of $\mathcal{M}$. The we can define the class $(B)_z = \{w \mid w \in f(z)\}$ and therefore also $B = \{(z, w) \mid z \in V_\kappa^N \wedge w \in f(z)\}$. So Class-Bounding holds.

For Global Choice we have to show that there is a well-ordering of M. We know that every element of N can be well-ordered and so V_κ^N can be well-ordered. The well-order is therefore an element of $\mathcal{C}$.

$(M, \mathcal{C})$ has to be a β-model: Any well-founded relation in $(M, \mathcal{C})$ corresponds to a well-founded relation in N and because N is a transitive model of ZF^-, well-foundedness is absolute (we can define a rank function into the "real" ordinals which witnesses the well-foundedness in V).

Finally when we build the M^+ of $\mathcal{M}$ according to Theorem 8, M^+ and N are both transitive, have the same largest cardinal κ and the same subsets of κ and are therefore equal. $\qquad\square$

Remark 14. *We can also use this switching between models of MK* and SetMK* for class-forcing: Instead of doing class-forcing over MK* we go to SetMK* and do a set-forcing there. Note that by doing this indirect version of class-forcing we don't lose the tameness requirement for the forcing: Assume the class-forcing is not tame (as for example a forcing which collapses the universe to ω). Then we go to $M^+ \models SetMK^*$ and force with the associated set-forcing. But such a forcing destroys the inaccessibility of κ and therefore the preservation of PowerSet in the MK* extension $\mathcal{M}[G]$.*

Corollary 15.
$$M^+ = \bigcup_{C \in \mathcal{C}} L_{\kappa^*}(C).$$

where κ^* is the height of M^+ and

$$\begin{aligned} L_0(C) &= TC(\{C\}) \\ L_{\beta+1}(C) &= Def(L_\beta(C)) \\ L_\lambda(C) &= \bigcup_{\beta < \lambda} L_\beta(C), \ \lambda \ limit. \end{aligned}$$

Proof. Let $x \in M^+$. Then there is a coding pair (M_x, R_x) for x such that $(\tilde{M}_x, \tilde{R}_x)$ is isomorphic to $(TC(\{x\}), \in)$. As $\tilde{M}_x$ and $\tilde{R}_x$ are elements of $\mathcal{C}$ we can code the pair $(\tilde{M}_x, \tilde{R}_x)$ by a class $C_x \in \mathcal{C}$. As C_x is an element of M^+, $L_{\kappa^*}(C_x)$ is an inner model in M^+. But now we can decode x in

$L_{\kappa^*}(C_x)$ as we can build $(TC(\{x\})$ by transfinite induction from $(\tilde{M}_x, \tilde{R}_x)$. So $x \in L_{\kappa^*}(C_x)$.

For the converse, let $x \in \bigcup_{C \in \mathcal{C}} L_{\kappa^*}(C)$, i.e. there is an $C_x \in \mathcal{C}$ such that $x \in L_{\kappa^*}(C_x)$. As $L_{\kappa^*}(C_x)$ is an inner model of $\mathcal{M}$, x is an element of $\mathcal{C}$ and by Theorem 8 b) it is an element of M^+. $\qquad\square$

3 Hyperclass Forcing and Forcing in SetMK∗∗

In the last section we have seen how to move back and forth between a model of MK* and its associated SetMK* model. Now we will use this relation between a model of class theory and a model of set theory to define hyperclass forcing. A hyperclass is a collection whose elements are classes. The key idea is that instead of trying to formalize forcing for a definable hyperclass forcing notion, we can go to the associated model of SetMK* where the forcing notion is now a class and so we force with a definable class forcing there and then go back to a new MK* model. First let us define the relevant notions:

Definition 16. *Let $\mathcal{M} = (M, \mathcal{C})$ be a model of MK* and for $\mathbb{P} \subseteq \mathcal{C}$ let $(\mathbb{P}, \leq) = \mathbb{P}$ be an $\mathcal{M}$-definable partial ordering with a greatest element $1^{\mathbb{P}}$. $P, Q \in \mathbb{P}$ are compatible if for some R, $R \leq P$ and $R \leq Q$. A definable hyperclass $D \subseteq \mathbb{P}$ is dense if $\forall P \exists Q(Q \leq P$ and $Q \in D)$. Then a $\mathbb{G} \subseteq \mathcal{C}$ is called a $\mathbb{P}$-generic hyperclass over $\mathcal{M}$ iff $\mathbb{G}$ is a pairwise compatible, upward-closed subcollection of $\mathbb{P}$ which meets every dense subcollection of $\mathbb{P}$ which is definable over $\mathcal{M}$.*

We will assume that for each $P \in \mathbb{P}$ there exists $\mathbb{G}$ such that $P \in \mathbb{G}$ and $\mathbb{G}$ is $\mathbb{P}$-generic over $\mathcal{M}$ (this is always possible if the model $\mathcal{M}$ is countable).

To define the structure $(M, \mathcal{C})[\mathbb{G}]$ where $\mathbb{G}$ is a $\mathbb{P}$-generic hyperclass over $(M, \mathcal{C})$ we will use Theorem 8 and Proposition 13. By Theorem 8 we go to the model $M^+ \models SetMK^*$. As $\mathbb{P}$ is a subcollection of $\mathcal{C}$ in $\mathcal{M}$ it becomes a subclass of $P(M) \cap M^+$ and is an M^+-definable class, $\mathbb{G}$ remains a pairwise compatible, upward-closed subclass of $\mathbb{P}$ which meets every dense subclass of $\mathbb{P}$ which is definable over M^+ and therefore is definable class-generic over M^+. Then we define names, their interpretation and the extension of M^+ as usual: A $\mathbb{P}$-name in M^+ is a set in M^+ consisting of pairs (τ, p) where τ is a $\mathbb{P}$-name in M^+ and p belongs to $\mathbb{P}$ (as we are in the set model we now denote the elements of $\mathbb{P}$ with lower-case letters). Then $\mathcal{N} = \cup\{\mathcal{N}_\alpha \mid \alpha \in Ord(M^+)\}$ is the collection of all names where $\mathcal{N}_0 = \emptyset$, $\mathcal{N}_{\alpha+1} = \{\sigma \mid \sigma$ is a subset of $\mathcal{N} \times P$ in $M^+\}$ and $\mathcal{N}_\lambda = \cup\{\mathcal{N}_\alpha \mid \alpha < \lambda\}$ for a limit ordinal λ. For a $\mathbb{P}$-name σ its interpretation is $\sigma^{\mathbb{G}} = \{\tau^{\mathbb{G}} \mid p \in \mathbb{G}$ for some $(\tau, p) \in \sigma\}$. Then $M^+[\mathbb{G}]$ is the set of all such $\tau^{\mathbb{G}}$. Finally we can define the extension of $\mathcal{M}$:

Definition 17. *Let $\mathcal{M} = (M, \mathcal{C})$ be a β-model of MK^*, $\mathbb{P}$ be a definable hyperclass forcing and $\mathbb{G} \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic hyperclass over $\mathcal{M}$. Let M^+*

be the model of SetMK* associated to $\mathcal{M}$ by Theorem 8 and assume that $M^+[\mathbb{G}] \models SetMK^*$ with largest cardinal κ with $M^+[\mathbb{G}]$ transitive. Then $\mathcal{M}[\mathbb{G}] = (M, \mathcal{C})[\mathbb{G}]$ is the β-model of MK* derived from $M^+[\mathbb{G}]$ by Theorem 13, whose sets are the elements of $V_\kappa^{M^+[\mathbb{G}]}$ and whose classes are the subsets of $V_\kappa^{M^+[\mathbb{G}]}$ in $M^+[\mathbb{G}]$, where κ is the largest cardinal of $M^+[\mathbb{G}]$ and is strongly inaccessible. Such a model is called a definable hyperclass-generic outer model of $\mathcal{M}$.

This definition assumes that the definable class-forcing $\mathbb{P}$ again produces a model of SetMK* with the same largest cardinal κ where κ is strongly inaccessible (we say in short that $\mathbb{P}$ does not change κ). Unfortunately the assumption that SetMK* is preserved is not as straightforward as it might seem. Definable class-forcing was developed by [Fri00]. There the concept of pretameness and tameness of a forcing notion is introduced and it is shown that such a forcing has a definable forcing relation and preserves the axioms. In the case of SetMK* we now have the added problem that we are not forcing over a model of full ZFC but rather over ZFC$^-$, i.e. without the Power Set Axiom. This can cause problems when we use concepts like the hierarchy of the V_α, for example to prove that pretame class-forcings preserve the Replacement (or in our case the Set-Bounding) Axiom. So we cannot simply transfer the results of [Fri00] but have to prove the Definability Lemma and the preservation of the axioms again without making use of the Power Set Axiom.

To define definable class-forcing in SetMK* first note that the following still holds: Let M^+ be a transitive model of SetMK*, P be a M^+-definable forcing notion and G P-generic over M^+. Then $M^+[G]$ is transitive and $Ord(M^+[G]) = Ord(M^+)$. It follows from the definition of the interpretation of names and the definition of $M^+[G]$ that if $y \in \sigma^G$ then $y = \tau^G$ for some $\tau \in TC(\sigma)$ and therefore $M^+[G]$ is transitive. Furthermore for every $x \in Ord(M^+)$ there exists a name σ for x (i.e. $x = \sigma^G$ as defined above) with name-rank of σ = the least $\alpha \in Ord(M)$ such that $\sigma \in \mathcal{N}_{\alpha+1}$ and by induction the von Neumann rank of σ^G is at most the name rank of σ. So we know that if "new" sets are added by the forcing they have size at most the "old" sets from M^+ and so $Ord(M^+[G]) \subseteq Ord(M^+)$.

We will first treat the case where we already assume that the forcing relation is definable and P is a pretame class-forcing and then show how we can ensure that in general pretame class-forcings preserve the axioms and the Definability Lemma holds.

Proposition 18. *Let M^+ be a model of SetMK* and let P be a pretame definable class-forcing over M^+ that does not change κ and whose forcing relation is definable. Let $G \subseteq P$ be definable class-generic over M^+. Then $M^+[G]$ is a model of SetMK*.*

Proof. Extensionality, Pairing, Comprehension, Infinity, Foundation and Choice still hold by the proof for definable class-forcing over full ZFC. We have to show that Set-Bounding holds in $M^+[G]$, i.e.

$$M^+[G] \models \forall x \in a \, \exists y \, \varphi(x,y) \to \exists b \, \forall x \in a \, \exists y \in b \, \varphi(x,y)$$

Let σ be a name for a. We can extend any p for which $p \Vdash \forall x \in \sigma \, \exists y \, \varphi(x,y)$ to force that there is an isomorphism between σ and an ordinal α (by using AC) and so we can assume without loss of generality that σ is $\check{\alpha}$ where $\alpha \in Ord$ and therefore $p \Vdash \forall x < \alpha \, \exists y \, \varphi(x,y)$. Then for such a fixed p and for each $x < \alpha$ we can define by the Definability of the forcing relation $D_x = \{q \leq p \,|\, \exists \tau \, q \Vdash \varphi(x,\tau)\}$ where D_x is dense below p. By pretameness there is a $q \leq p$ and $\langle d_x \,|\, x < \alpha \rangle \in M^+$ such that for all $x < \alpha$, d_x is pretense $\leq q$ and by genericity there is such a q in G. Then we know that for all pairs $\langle x, r \rangle$ where $x < \alpha$ and $r \in d_x$ there is τ such that $r \Vdash \varphi(x,\tau)$. By the Set-Bounding principle in M^+ we get a set $T \in M^+$ such that $\forall(x,r)$ with $r \in d_x$ $\exists \tau \in T$ such that $r \Vdash \varphi(x,\tau)$. Finally let π be a name for $\{\tau^G \,|\, \tau \in T\}$, i.e. $\pi = \{\langle \tau, 1^{\mathbb{P}} \rangle \,|\, \tau \in T\}$. Then, because the generic below q hits every d_x, $\varphi(x,\tau)$ will hold for some $\tau \in T$. It follows that $q \Vdash \forall x < \alpha \, \exists y \in \pi \, \varphi(x,y)$. Then Union follows with the use of Set-Bounding. □

With this proposition we have shown that in a model of MK* we can force with a definable hyperclass-forcing $\mathbb{P}$ and preserve MK*, provided $\mathbb{P}$ translates to a pretame class-forcing in SetMK* which preserves the inaccessibility of κ and whose forcing relation is definable. But in practice we don't usually know if the forcing relation is definable, even if we know that P is pretame due to the absence of a suitable hierarchy (like the V-hierarchy which suffices when forcing over ZF-models). So we will introduce a preparatory forcing which does not add any new sets but converts the SetMK* model M^+ into a model of the form $L_\alpha[A]$ for some generic class predicate $A \subseteq ORD$ preserving SetMK* (relative to A). This will allow us to use the relativized L hierarchy and therefore adapt the proof of the Definability Lemma for a pretame class-forcing and the fact that it preserves the axioms.

With this proposition we have shown that in a model of MK* we can force with a definable hyperclass-forcing $\mathbb{P}$ and preserve MK*, provided $\mathbb{P}$ translates to a pretame class-forcing in SetMK* which preserves the inaccessibility of κ and whose forcing relation is definable. But in the setting of SetMK* we don't usually know if the forcing relation is definable, even if we know that $\mathbb{P}$ is pretame, due to the absence of a suitable hierarchy. In the standard case of proving the Definability Lemma for pretame class forcing in models (M, A) of ZF we define a function by induction using the fact that we have the V-hierarchy to refer to the "least possible stage" where something occurs (see [Fri00], page 34ff)). Similarly, in the proof that the forcing preserves the axioms we construct predense sets associated to certain stages in the V-hierarchy. To be able to adapt these proofs to the case of ZF$^-$ we

will introduce a preparatory forcing which does not add any new sets but converts the SetMK* model M^+ into a model of the form $L_\alpha[A]$ for some generic class predicate $A \subseteq ORD$ preserving SetMK* (relative to A). This will allow us to use the relativized L-hierarchy instead of the V-hierarchy to prove the Definability Lemma and verify that the axioms are preserved.

Such a preparatory forcing presents us with two difficulties: first we have to show that its forcing relation is definable and the forcing is pretame, so that we can infer from Proposition 18 that it preserves the axioms. Secondly we have to show that the predicate A, that was added by the forcing, can be coded into a subset of κ so as to avoid problems when going back to the MK* model.

To prove the pretameness of such a forcing we have to add a new axiom to SetMK*, namely a variant of Dependent Choice. To ensure that this axiom holds in M^+, we will add its class version to MK* and show that it is transformed to the appropriate set version using the coding introduced in the last section.

Definition 19. *Let MK** consist of the axioms of MK* plus Dependent Choice for Classes (we denote this with DC_∞):*

$$\forall \vec{X} \exists Y \varphi(\vec{X}, Y) \rightarrow \forall X \exists \vec{Z} \, (Z_0 = X \wedge \forall i \in ORD \, \varphi(\vec{Z} \upharpoonright i, Z_i))$$

where $\vec{X}$ is an α-length sequence of classes for some $\alpha \in ORD$, $\vec{Z}$ is an ORD-length sequence of classes and $Z \upharpoonright i$ is the sequence of the "previously chosen" Z_j, $j < i$.

In the resulting SetMK** model M^+, DC_∞ becomes a form of κ-Dependent Choice:

$$\forall \vec{x} \exists y \varphi(\vec{x}, y) \rightarrow \forall x \exists \vec{z} \, (z_0 = x \wedge \forall i < \kappa \, \varphi(\vec{z} \upharpoonright i, z_i))$$

where $\vec{x}$ is a $< \kappa$-length sequence of sets, $\vec{z}$ is a κ-length sequences of sets and $z \upharpoonright i$ is the sequence of the "previously chosen" z_j, $j < i$.

The coding between MK** and SetMK** works exactly as in the MK* case, we only have to prove that it transforms DC_∞ into DC_κ and vice versa.

Proposition 20.   *1. Let $\mathcal{M} = (M, \mathcal{C})$ be a β-model of MK**. Then we can define a model*

$$M^+ = \{x \, | \, there \ is \ a \ coding \ pair \ (M_x, R_x) \ that \ codes \ x\}$$

Then M^+ is the unique, transitive set that obeys the following properties:

    a) $M^+ \models SetMK^{**}$,
    b) $\mathcal{C} = P(M) \cap M^+$,

c) $M = V_\kappa^{M^+}$, κ is the largest cardinal in M^+ and strongly inaccessible in M^+.

2. Let M^+ be a model of SetMK** that has a strongly inaccessible cardinal κ, let $\mathcal{C} = P(M) \cap M^+$ and $M = V_\kappa^{M^+}$. Then $\mathcal{M} = (M, \mathcal{C})$ is a model of MK**.

Proof. For 1.: Using the proof of Theorem 8 it only remains to show that M^+ is a model of κ-Dependent Choice, where κ is strongly inaccessible in M^+: $M^+ \models \forall \vec{x} \exists y\, \varphi(\vec{x}, y) \to \forall x \exists \vec{z}(z_0 = x \wedge \forall i < \kappa\, \varphi(\vec{z} \upharpoonright i, z_i))$ where $\vec{x}, \vec{z}$ are κ-length sequences. So assume that $M^+ \models \forall \vec{x} \exists y\, \varphi(\vec{x}, y)$. From what we have show above, we know that $\vec{x}$ is an ordinal length sequence of elements in $\mathcal{M}$ and also y is an element of $\mathcal{M}$ (as these can be classes we will write them with upper case letters in $\mathcal{M}$). Let ψ be the second-order formula associated to φ, i.e. ψ is the formula that says exactly the same as φ only that its variables can be classes. Then by DC_∞ we have that $\forall \vec{X} \exists Y\, \psi(\vec{X}, Y) \to \forall X \exists \vec{Z}\,(Z_0 = X \wedge \forall i \in ORD\, \psi(\vec{Z} \upharpoonright i, Z_i))$ where $\vec{X}, \vec{Z}$ are sequences of classes with ordinal length and $Z \upharpoonright i$ is the sequence of the previously "chosen" Z_j, $j < i$. As before all the classes mentioned here are elements of M^+ where $\vec{Z}$ is a κ-length sequence and so we have proven the κ-Dependent Choice.

For 2.: Again we only have to proof the case of DC_∞ and this is an direct analog to the proof of the Comprehension Axiom in the proof of Proposition 13. $\qquad\square$

Lemma 21. *Let M^+ be a model of SetMK** with largest cardinal κ and P be an M^+-definable class forcing notion. Then if P is $\leq \kappa$-closed it is $\leq \kappa$-distributive.*

Proof. Let $p \in P$ and $\langle D_i \,|\, i < \beta \rangle$ is an M^+ definable sequence of dense classes, $\beta \leq \kappa$, and we want to show that there is a $q \leq p$ meeting each D_i (q meets D_i if $q \leq q_i \in D_i$ for some q_i). As we have shown that P is $\leq \kappa$-closed we want to construct a descending sequence $p_0 \geq p_1 \geq \ldots \geq p_i \geq \ldots$ ($i < \beta$) with $p_i \in D_i$ for all $i < \beta$. Here we need the SetMK** version of the Dependent Choice Axiom we added to MK*: Recall that κ-Dependent Choice says that $\forall \vec{x} \exists y \varphi(\vec{x}, y) \to \forall x \exists \vec{z}\,(z_0 = x \wedge \forall i < \kappa\, \varphi(\vec{z} \upharpoonright i, z_i))$ where $\vec{x}$ is a $< \kappa$-length sequence of sets, $\vec{z}$ is a κ-length sequences of sets and $z \upharpoonright i$ is the sequence of the previously "chosen" z_j, $j < i$. If we take $\varphi(\vec{x}, y)$ to mean that "$\vec{x}$ is a descending sequence of conditions, $x_i \in D_i$ for $i <$ length $\vec{x}$, y is a lower bound for $\vec{x}$ and $y \in D_{length_{\vec{x}}}$" then we know that we can find a descending sequence $p_0 \geq p_1 \geq \ldots \geq p_i \geq \ldots$ ($i < \beta$) with $p_i \in D_i$ for all $i < \beta$ such that there is an $q \in P$ with $q \leq p$ and $q \leq p_i$ for all $i < \beta$ and so q meets all D_i. $\qquad\square$

Theorem 22. *Let M^+ be a model of SetMK** with largest cardinal κ and let κ^* denote the height of M^+. Then there is an M^+-definable forcing P such that the Definability Lemma holds and P is pretame, which adds a class predicate $A \subseteq \kappa^*$ such that $M^+ = L_{\kappa^*}[A]$ and $(M^+, A) \models SetMK^{**}$ relativized to A.*

Proof. Let $P = \{p : \beta \to 2 \,|\, \beta < \kappa^*, p \in M^+\}$ and let G be P-generic over M^+. Let $\bigcup G = g : \kappa^* \to 2$ and $A = \{\gamma < \kappa^* \,|\, g(\gamma) = 1\}$. Note that G is an amenable predicate, i.e. $G \cap a$ belongs to M^+ for every $a \in M^+$ and P is $\leq \kappa$-closed, as for every $\lambda \leq \kappa$ and every descending sequence $p_0 \geq p_1 \geq \ldots \geq p_i \geq \ldots \; (i < \lambda)$ there is $q = \bigcup_{i<\lambda} p_i \in P$ such that $\forall i < \lambda \; q \leq p_i$.

To show that the forcing relation is definable in the ground model, we will concentrate on the atomic cases "$p \Vdash \sigma \in \tau$" and "$p \Vdash \sigma = \tau$". Then the other cases follow by induction. For $p \Vdash \sigma \in \tau$ first consider the case where the length of p is larger then the ranks of σ and τ (i.e. there is an γ such that $rank\,\sigma, rank\,\tau < \gamma$ and $Dom(p) > \gamma$). Then the question if $\sigma^G \in \tau^G$ is already decided by p, meaning that $\sigma^G \in \tau^G$ exactly when $\sigma^p \in \tau^p$ with $\tau^p = \{\pi^p \,|\, \langle \pi, q \rangle \in \tau, p \leq q\}$ as p "has no holes" and therefore a condition that extends p will never change the decisions made below the length of p. This now defines the forcing relation because P doesn't add any new sets and therefore σ^p and τ^p are already elements of the ground model. If p is not large enough to decide if σ^G is an element of τ^G, then we have to check that every q that extends p decides that this is the case so we get the definition "$p \Vdash \sigma \in \tau \leftrightarrow \forall q \leq p \,(\,|q| \,> rank\,\sigma, rank\,\tau \to \sigma^q \in \tau^q)$". The definitions for the "$=$" case can be given the same way and so the forcing is definable. The Truth Lemma then follows from Definability by the usual arguments.

Next we want to show that P is pretame: As P is $\leq \kappa$-closed, we know by Lemma 21 that P is $\leq \kappa$-distributive. Then P is also pretame for sequences of dense classes of length $\leq \kappa$ and therefore P is pretame.

We have shown that P doesn't add any new sets to the extension but a subclass $A \subset \kappa^*$. So the forcing just reorganizes M^+ and adds A as a predicate. Then every set of ordinals from M^+ is copied into an interval of the generic and so every set of ordinals and therefore also every set is coded by A. Also as A adds no new sets it holds that $L_{\kappa^*}[A] \subseteq M^+$. It follows that $M^+[G] = L_{\kappa^*}[A]$ and therefore already $M^+ = L_{\kappa^*}[A]$.

It remains to show that $(M^+, A) \models (SetMK^{**})^A$, i.e. $SetMK^{**}$ holds for formulas which can mention A as a predicate. As P preserves the strongly inaccessibility of κ it follows by Proposition 18 that $M^+[G] \models SetMK^*$ and that means that $(M^+, A) \models SetMK^*$. But as the Comprehension and Bounding can mention the generic this implies that $(M^+, A) \models (SetMK^*)^A$. For the DC_κ note that by adding A we now have a global well-order of the extension. That means that if we have a $< -\kappa$ sequence $\vec{x}$ in $M^+[G]$ such that $\forall \vec{x} \exists y \, \varphi(\vec{x}, y)$ and we want to find a κ-length sequence $\vec{z}$ such that

$\forall x \exists \vec{z} (z_0 = x \wedge \forall i < \kappa \, \varphi(\vec{z} \restriction i, z_i))$ we can just take z_i to be least so that $\varphi(\vec{z} \restriction i, z_i)$ for each i. $\qquad \square$

As our ultimate goal is to go back to an MK** model, we want to show that the predicate A can be coded into a subset of κ:

Theorem 23. *Let (M^+, A) be a model of SetMK** relativized to a predicate A, with largest cardinal κ and let κ^* denote the height of (M^+, A), where A is the generic predicate added by the forcing P in Theorem 22 and $M^+ = L_{\kappa^*}[A]$. Then we can force that there is a $X \subseteq \kappa$ such that $L_{\kappa^*}[A] \subseteq L_{\kappa^*}[X]$, SetMK** is preserved and κ remains strongly inaccessible.*

Proof. To get A definable in $M^+[X]$, for some $X \subseteq \kappa$, we want to use an almost disjoint forcing which codes the predicate A into such an X. The forcing will be along the following lines: we will need to define a family S of almost disjoint sets (i.e. for $x, y \subseteq \kappa$, x and y are almost disjoint if $x \cap y$ is bounded in κ) A_β which we will use to code the predicate $A \subseteq \kappa^*$ into an X. We will define A_β to be the least subset of κ (i.e. least in the canonical well-order of $L_{\kappa^*}[A \cap \beta]$) in $L_{\kappa^*}[A \cap \beta]$ which is distinct from the $A_{\bar{\beta}}$ for $\bar{\beta} < \beta$. The idea is that we can decode A in $L_{\kappa^*}[X]$ if we know the A_β's. But as A is a proper class we don't know that we can always find such distinct A_β's. So we will have to assume that the cardinality of β is at most κ not only in $L_{\kappa^*}[A]$ but also in $L_{\kappa^*}[A \cap \beta]$ because now to find an A_β distinct from each $A_{\bar{\beta}}$, $\bar{\beta} < \beta$, we can list these $A_{\bar{\beta}}$'s as $\langle A_i \,|\, i < \kappa \rangle$ and obtain A_β by diagonalization. To fulfill that assumption however we have to "reshape" A into a predicate A' that has the property that if $\beta < \kappa^*$ then the cardinality of β is $\leq \kappa$ in $L_{\kappa^*}[A' \cap \beta]$. Then we can code A as the even part of A' to get $(M^+, A') \models (SetMK**)^{A'}$ and finally code A' by a subset of κ.

So the proof consists of two steps: First we have to show that we can reshape A and then we have to force with an almost disjoint forcing to show that the reshaped predicate A' can be coded into a subset of κ, preserving SetMK** in each step.

Step 1: We add a reshaped predicate A' over $(L_{\kappa^*}[A], A)$ by the following forcing:

$$P = \{p : \beta \to 2 \,|\, \kappa \leq \beta < \kappa^*, \forall \gamma \leq \beta \, (L_{\kappa^*}[A \cap \gamma, p \restriction \gamma] \models |\gamma| \leq \kappa)\}$$

The main obstacle is to show that P is definably-distributive, i.e. we have to show that for a $p \in P$ and (M^+, A)-definable sequences of dense classes of set-length $\langle D_i \,|\, i < \alpha \rangle$ for all $\alpha \leq \kappa$, there is a $q \leq p$ meeting each D_i with $q \in P$.

Claim 24. *P is definably-distributive.*

Proof. Note that it suffices to show definable-distributivity for κ; so we consider an (M^+, A)-definable sequence of dense classes $\langle D_i \,|\, i < \kappa \rangle$. We

want to define a descending sequence of conditions $p \geq p_0 \geq p_1 \geq \ldots$ where $p_i \geq q$, $q \in P$ and $p_{i+1} \in D_i$ for each $i < \kappa$. To show that the p_i are indeed conditions we have to show that $L_{\kappa^*}[A \cap \gamma, p_i \restriction \gamma] \models |\gamma| \leq \kappa$ for every $\gamma \leq |p_i|$. In the following we will use the fact that a condition is always extendible to any length $< \kappa^*$: $\forall p \forall \beta < \kappa^* \exists q \leq p, |q| \geq \beta, q \in P$. This holds because there is an $x \subseteq \kappa$ such that β is coded by x and $p * x \in P$ and has length $|p| + \kappa$. If this is still below β we can lengthen p further by a sequence of 0's: $q = p * x * \vec{0}$. This will again be an element of P as we know from the information in the code x of β that the ordinals will collapse.

First, we assume that the sequence of dense classes is Σ_1-definable, i.e. $\{(q, i) \mid q \in D_i\}$ is Σ_1-definable with parameter.

As we have seen that every condition is extendible, we can extend p to catch a parameter $x \in L_{|p|}[A]$ such that the sequence of the D_i is Σ_1-definable with parameter x. Let p_0 be this extension of p. Then, as we have Global Choice, we can consider the $<_{(M^+,A)}$-least pair (q_0, w_0) such that $q_0 \leq p_0$ and w_0 witnesses "$q_0 \in D_0$". Then we choose p_1 such that p_1 is a condition which extends q_0 such that $w_0 \in L_{|p_1|}[A \cap |p_1|]$. Now we define p_2 in the same way: Choose (q_1, w_1) such that $q_1 \leq p_1$ and w_1 witnesses "$q_1 \in D_1$". Then let $p_2 \leq q_1$ such that $w_1 \in L_{|p_2|}[A \cap |p_2|]$. Define the rest of the successor cases (p_{n+1}, w_{n+1}) similarly.

For the first of the limit cases, let $p_\omega = \bigcup_{n < \omega} p_n$ and we claim that $p_\omega \in P$. So we have to show that $\forall \gamma \leq |p_\omega|$, γ collapses to κ using only $A \cap \gamma$ and $p_\omega \restriction \gamma$. We know that if $\gamma < |p_\omega|$ then $\gamma < |p_n|$ for some n. So we only have to consider the case where $\gamma = |p_\omega|$. It follows from the construction of the p_n's that the sequence $\langle p_n \mid n < \omega \rangle$ is definable over $L_{|p_\omega|}[A \cap |p_\omega|, p_\omega]$ and is a cofinal sequence in p_ω, i.e. it converges to p_ω. Then also the sequence of the lengths of the p_n's, $\langle |p_n| \mid n < \omega \rangle$ is definable over $L_{|p_\omega|}[A \cap |p_\omega|, p_\omega]$ and converges to $|p_\omega|$. As we know that $|p_n|$ collapses to κ for every $n < \omega$, we know that in $L_{|p_\omega|}[A \cap |p_\omega|, p_\omega]$ $|p_\omega|$ definably collapses to κ. So $L_{|p_\omega|+1}[A \cap |p_\omega|, p_\omega] \models |p_\omega|$ is collapsed to κ. The other limit cases can be handled in the same way.

Now we go to the Σ_2-definable case. Note that we cannot simply copy the construction of the p_n-sequence because the witness q_{n+1} we need for the definition of the next p_{n+1} will now be a solution to a Π_1-statement and will therefore not be absolute in the other models. But we know that for $V = L_{\kappa^*}[A]$ it holds that $\forall \alpha < \kappa^* \exists \beta \leq \kappa^*$, $\alpha < \beta$ such that $L_\beta[A]$ is Σ_n-elementary in $L_{\kappa^*}[A]$. This holds because for a pair α, n we can take the Σ_n-Skolem Hull N of α in $L_{\kappa^*}[A]$. Then in M we have a solution for every Σ_n-property with parameters $< \alpha$, M is transitive and bounded by Class-Bounding. Then there is a $\beta \leq \kappa^*$ such that M is equal to $L_\beta[A]$.

So we can always find models that are Σ_1-elementary submodels of (M^+, A) in which we can carry out the definition of the sequence of conditions: As before we choose for every $n < \omega$ a pair (q_n, w_n) such that $q_n \leq p_n$ such that w_n witnesses "$q_n \in D_n$" and then let $p_{n+1} \leq q_n$ such

that $w_n \in L_{|p_{n+1}|}[A \cap |p_{n+1}|, p_{n+1}]$ and $L_{|p_{n+1}|}[A \cap |p_{n+1}|, p_{n+1}]$ is an Σ_1-elementary submodel of $L_{\kappa^*}[A]$. This also holds in the limit case by using the same construction we did for the Σ_1 case where again the model $L_{|p_\omega|+1}[A \cap |p_\omega|, p_\omega]$ is an Σ_1-elementary submodel of $L_{\kappa^*}[A]$. The same can be done for all the Σ_m-definable cases. $\qquad\square$

Now that we know that P is $\leq \kappa$-distributive, we know that P is $\leq \kappa$-pretame and therefore $(M^+, A, A') \models (SetMK^{**})^{A,A'}$ (similar to proof of Theorem 22 by using Proposition 18 and the fact that there is a global well-order of the extension). Then we can code A to be the even part of A' and we get a model $(M^+, A') \models (SetMK^{**})^{A'}$. It remains to show that A' can be coded into a subset of κ.

Step 2: Code A' into $X \subseteq \kappa$. As we know that A' is reshaped we can define a collection of sets $\mathcal{S} = \langle A_\beta \,|\, \beta < \kappa^* \rangle$ in the following way: let A_β be the least $B \subseteq \kappa$ in $L_{\kappa^*}[A' \cap \beta]$ such that $B \notin \{A_{\bar\beta} \,|\, \bar\beta < \beta\}$. $\mathcal{S}$ can be turned into a collection $\mathcal{S}' = \langle A'_\beta \,|\, \beta < \kappa^* \rangle$ of almost disjoint sets A'_β by mapping every set to the set of codes of its proper initial segments: $B \subseteq \kappa$ is mapped to $B' = \{\mathrm{Code}\,(B \cap \alpha) \,|\, \alpha < \kappa\} \subseteq \kappa$. Then for two distinct subsets B and C of κ, $|\,B' \cap C'\,| < \kappa$ and therefore they are almost disjoint. We want to show that we can code A' by a subset X of κ by showing that $X \cap A'_\beta$ is bounded if and only if $\beta \in A'$. This can be done by a forcing Q with the conditions (g, S) where $S \subseteq A'$, $|S| < \kappa$ and g is an element of $^{<\kappa}2$. Extension is defined by: $(g, S) \geq (h, T)$ iff h extends g, $S \subseteq T$ and if $\beta \in S$ and $h(\gamma) = 1$ for a $\gamma \in A'_\beta$ then $g(\gamma) = 1$. Note that two conditions with the same first component $\langle g, S \rangle$ and $\langle g, T \rangle$ are compatible because we can always find a common extension $\langle g, S \cup T \rangle$. Thus a function which maps every element of a definable antichain into its first component is injective (as otherwise the conditions would be compatible). So we have injectively mapped a definable class to a set as there are only κ many first components. By Bounding such a function exists as a set and so Q is set-c.c., i.e. every definable antichain is only set-sized. Then Q is pretame, as every definable dense class can be seen as an antichain. Now let G be a Q-generic, $G_0 = \bigcup \{g \,|\, (g, S) \in G\}$ and $X = \{\gamma \,|\, G_0(\gamma) = 1\}$. we argue that we can find the almost disjoint sets in $L_{\kappa^*}[X]$ because A' is reshaped and therefore it holds for any β that $|\beta| \leq \kappa$ in $L_{\kappa^*}[A' \cap \beta]$. So after X has decoded $A' \cap \beta$ it can find A'_β and then continue the decoding in the following way: $\beta \in A'$ if there is an $(g, S) \in G$ with $\beta \in S$ and by the definition of extension if $G_0(\gamma) = 1$ for a $\gamma \in A'_\beta$ then $g(\gamma) = 1$. So $X \cap A'_\beta = \{\gamma \,|\, g(\gamma) = 1\} \cap A'_\beta$ and that is bounded and therefore we have a code of A' by X via

$$X \cap A'_\beta \text{ is bounded if and only if } \beta \in A'.$$

As this forcing is κ-closed (i.e. closed for $< \kappa$ sequences), κ stays regular and therefore strongly inaccessible and by Proposition 18 SetMK* is preserved and by Proposition 18 SetMK* is preserved. $\qquad\square$

We have seen how definable hyperclass-forcing can be carried out over a model $\mathcal{M}$ of MK**: First we go to the related SetMK** model M^+ (Theorem 8). Then in order to be able to force over this model, we change M^+ to a model $L_{\kappa^*}[A]$ for a generic predicate A (Theorem 22). Finally we showed how to code A into a subset $X \subseteq \kappa$ to avoid having an undefinable predicate once we go back to the extension of the original MK** model (Theorem 23). At this point we can force with any desirable pretame definable class-forcing over $L_{\kappa^*}[X]$, go back to MK** and get the desired definable hyperclass-forcing over MK**.

So we have given a template which allows us to do definable hyperclass-forcing over MK**. In the following we will show how to use this template to produce minimal β-models of MK**.

4 Minimal β-Models of MK**

As an application of definable hyperclass forcing we will show that every β-model of MK** can be extended to a minimal β-model of MK** via the use of SetMK** models. Here a minimal model $M(S)$ of SetMK** is the least transitive model of SetMK** containing a real S and equivalently a minimal β-model $\mathcal{M}(S)$ of MK** is the least β-model of MK** containing a real S.[3] For that we will use and modify the template developed in the last section: We start with an arbitrary β-model $\mathcal{M} = (M, \mathcal{C})$ of MK** and from that we get the corresponding model M^+ of SetMK** (by Theorem 8) with $M = V_\kappa^{M^+}$ and $\mathcal{C} = P(M) \cap M^+$ where κ is strongly inaccessible in M^+. Let κ^* denote the height of M^+ and apply Theorem 22 to arrive at $M^+ = L_{\kappa^*}[A]$ where $A \subseteq \kappa^*$ and (M^+, A) satisfies SetMK$^{(}**)$ relative to A. We now show that we can extend M^+ to a minimal model of SetMK** and then go back to an MK** model, which will be a minimal β-model of MK**.

Theorem 25. *Every β-model of MK** can be extended to a minimal β-model of MK** with the same ordinals.*

Proof. First we use the template described above to arrive at the model $L_{\kappa^*}[A]$ and then we will code the predicate A into a subset of κ by using Theorem 23 with a small modification in the "reshaping" forcing. Instead of forcing that each $\gamma < \kappa^*$ collapses in $L_{\kappa^*}[A \cap \gamma, p \restriction \gamma]$, we will force it to already collapse instantly in the next level, i.e. in $L_{\gamma+1}[A \cap \gamma, p \restriction \gamma]$. So the forcing will be:

$$P = \{p : \beta \to 2 \,|\, \kappa \le \beta < \kappa^*, \forall \gamma \le \beta \,(L_{\gamma+1}[A \cap \gamma, p \restriction \gamma] \vDash \,|\gamma| \le \kappa)\}$$

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[3]We can see here that it is vital to restrict ourselves to β-models in order to talk about minimal models of MK by comparing this to the situation in ZFC: There it also only makes sense to talk about minimal models containing a real for well-founded models (and not for ill-founded models). So by making the transformation from MK to SetMK we have to restrict ourselves to β-models.

The proof that P is definably-distributive then works in exactly the same way. As in Theorem 23 we can code A to be the even part of the predicate A' added by the reshaping forcing which in turn can be coded into an $X \subseteq \kappa$ by an almost disjoint forcing. This gives us that there are no SetMK** models containing X of height between κ and κ^*: In the reshaping forcing we destroyed the Replacement axiom level by level relative to A and in the almost disjoint coding we can now choose the codes instantly level-by-level (i.e. every code for γ appears in $L_{\gamma+1}[X]$). So A' can be recovered level-by-level from X and therefore Replacement is also destroyed level-by-level relative to X. We arrive at a SetMK** model $L_{\kappa^*}[X]$, with $X \subseteq \kappa$, which is the least transitive ZFC^- model containing X (again κ remains regular and indeed strongly inaccessible, because the almost disjoint coding is κ-closed).

We will extend this to a minimal model of SetMK** in two steps: First we extend $L_{\kappa^*}[X]$ to a model $L_{\kappa^*}[Y]$ such that no cardinal $\bar{\kappa} < \kappa^*$ can serve as a "source" for a SetMK** model (i.e. is the largest cardinal of a SetMK** model containing $Y \cap \bar{\kappa}$) and second we show that we can add a real S such that in $L_{\kappa^*}[S]$ there are no SetMK** models containing S below κ^*. Then it only remains to show that from $L_{\kappa^*}[S]$ we can go back to a minimal β-model of MK**.

Step 1: With the modification of Theorem 23, we have shown that there are no SetMK** models containing X between κ and κ^*. But it could still be that there exist cardinals below κ which are sources for SetMK** models. We will destroy these cardinals by shooting a club through a "fat-stationary" set which has no such cardinals and then force all limit cardinals to belong to this club.

So let $S = \{\bar{\kappa} < \kappa \,|\, \bar{\kappa}$ is a limit cardinal and for all $\bar{\beta} > \bar{\kappa}$, if $L_{\bar{\beta}}[X \cap \bar{\kappa}] \vDash ZFC^-$ then $L_{\bar{\beta}}[X \cap \bar{\kappa}] \nvDash \bar{\kappa}$ is strongly inaccessible$\}$.

Definition 26. *S is* fat-stationary *if for every club C in $L_{\kappa^*}[X]$, $S \cap C$ contains closed subsets of any order type less than κ.*

We prove the following:

Lemma 27. *S is fat-stationary and there is a κ-distributive (i.e. $< \kappa$ distributive) forcing of size κ that adds a club $C \subseteq S$.*

Proof. First we will show that S is stationary with respect to clubs in $L_{\kappa^*}[X]$. So suppose C is a club in $L_\alpha[X]$ for an $\alpha < \kappa^*$. We build an increasing sequence $\langle M_n \,|\, n < \omega \rangle$ of sufficiently elementary submodels of $L_\alpha[X]$ in the following way: Let M_0 be the Σ_1-Skolem Hull of $\omega \cup \{X, C\}$ in $L_\alpha[X]$. Then $C \in M_0$ and $\kappa_0 = sup(M_0 \cap \kappa)$ is a cardinal. Next, let M_1 be the Σ_1-Skolem Hull of $\kappa_0 + 1 \cup \{X, C\}$ in $L_\alpha[X]$ and $\kappa_1 = sup(M_1 \cap \kappa)$. Repeat this construction for all $n < \omega$. Then this sequence of elementary submodels is definable over $M_\omega = \bigcup_{n<\omega} M_n$ and $\kappa_\omega = sup_{n<\omega}\kappa_n < \kappa$ is a cardinal in C as C is closed, unbounded in κ. Also κ_ω is an element of S because if

$L_{\bar{\alpha}}[X \cap \kappa_\omega]$ is the transitive collapse of M_ω then there are no ZFC$^-$ models containing $X \cap \kappa_\omega$ of height $< \bar{\alpha}$ (by elementarity), of height $= \bar{\alpha}$ because $\langle \kappa_n \,|\, n < \omega \rangle$ is definable over it (and so κ_ω becomes definably singular) and any ZFC$^-$ model containing $X \cap \kappa_\omega$ of height $> \bar{\alpha}$ sees that κ_ω has cofinality ω (as the κ_n-sequence is an element of it).

To show that S is fat-stationary we can use the same proof as for stationarity except one uses a longer δ-sequence of elementary submodels, for δ a limit cardinal less than κ.

Now for the second part of the Lemma we can force with a set-forcing to add a club. Here we will closely follow the proof of the ZFC version of this claim, as proven in [AS83] (see there for more details). Let $Q = \{p \,|\, p \text{ is a closed, bounded subset of } S\}$ be a forcing notion ordered by end-extensions: $q \leq p$ iff $p = q \cap (\sup(p) + 1)$. For G Q-generic over $L_{\kappa^*}[X]$ let $C = \bigcup G$. Then C is closed and unbounded and a subset of S. To show that Q is κ-distributive we have to show that for every $\tau < \kappa$ and sequence $\mathcal{D} = \langle D_i \,|\, i \in \tau \rangle$ of open, dense subsets of Q, $\bigcap_{i < \tau} D_i$ is dense in Q. Now we can define a sequence of elementary substructures $\langle M_\alpha \,|\, \alpha < \kappa \rangle$ of $L_{\kappa^*}[X]$ such that $c_\alpha = M_\alpha \cap \kappa$ is an ordinal and $\langle c_\alpha \,|\, \alpha < \kappa \rangle$ is an increasing and continuous sequence cofinal in κ. Let E be the collection of the c_α, $\alpha < \kappa$. Because S is fat-stationary, $S \cap E$ contains a closed subset A of order-type $\tau + 1$. Then in the model M_α, with $\alpha = \sup(A)$, we can define an increasing sequence $\langle p_i \,|\, i < \tau \rangle$, such that $p_i \in Q$ and $p_{i+1} \in D_i \cap M_\alpha$. We can define $p_\tau = \bigcup_{i < \tau} p_i \cup \{\alpha\}$ and this will be in $\bigcap_{i < \tau} D_i$. Note that this (set-) forcing is an element of $L_{\kappa^*}[X]$ and therefore preserves ZFC$^-$. Furthermore, as this forcing doesn't add sets of size $< \kappa$, κ stays strongly inaccessible and SetMK* is preserved because of Proposition 18.   □

Let X' be the join of X with the club we added. Then $X' \subseteq \kappa$ and the resulting model is $L_{\kappa^*}[X']$.

Lemma 28. *We can force all limit cardinals to belong to C with a forcing of size κ such that κ remains strongly inaccessible.*

Proof. Enumerate C as follows: $C = \langle \bar{\kappa}_i \,|\, i < \kappa \rangle$. We may assume that each $\bar{\kappa}_i$ is a strong limit cardinal (as κ is strongly inaccessible we can thin out C). Then we can build an Easton product of collapses, where we collapse every $\bar{\kappa}_{i+1}$ to the successor of $\bar{\kappa}_i$ and therefore ensure that all limit cardinals below κ are limits of cardinals in C and therefore are themselves in C.

So for $i < \kappa$ consider $Col_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$, where the conditions are functions p with $dom(p) \subset \bar{\kappa}_i^+$, $|dom(p)| < \bar{\kappa}_i^+$ and $range(p) \subset \bar{\kappa}_{1+1}$. Cardinals below $\bar{\kappa}_i^+$ and above $\bar{\kappa}_{i+1}^{\bar{\kappa}_i}$ are preserved (the size of the forcing is $\bar{\kappa}_{i+1}^{\bar{\kappa}_i}$) and in the extension we have a function which maps $\bar{\kappa}_i^+$ onto $\bar{\kappa}_{i+1}$.

Now we can build the Easton product (product with Easton support) of these collapses for every $i < \kappa$: A condition p in this forcing is a function such that $p = \langle p_i \,|\, i < \kappa \rangle \in \Pi_{1 < \kappa} Col_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$ and the forcing is ordered by

end-extension. p has Easton support, i.e. for every inaccessible cardinal λ, $|\{\alpha < \lambda \mid p(\alpha) \neq \emptyset\}| < \lambda$. As usual with Easton Products the forcing notion P can be split into two parts $P(\leq \lambda) = \Pi_{1 \leq \lambda} \mathrm{Col}_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$ and $P(> \lambda) = \Pi_{\lambda < 1 < \kappa} \mathrm{Col}_i(\bar{\kappa}_i^+, \bar{\kappa}_{i+1})$ for every regular cardinal λ. For this reason and as each $\bar{\kappa}_i$ is a strong limit, each collapse from $\bar{\kappa}_{i+1}$ to $\bar{\kappa}_i^+$ will not be affected by the other collapses and κ remains regular and strong limit. Furthermore, as this forcing is in $L_{\kappa^*}[X']$ (it is of size κ) it preserves SetMK**.

Because of the unboundedness of C, every limit cardinal is also a limit of cardinals in C and therefore, as C is closed, it is an element of C. $\qquad\square$

We conclude Step 1 by choosing X'' to be the join of X' and the above Easton product. Then we arrive at a model $L_{\kappa^*}[X'']$ with $X'' \subseteq \kappa$ such that for every cardinal $\bar{\kappa} < \kappa^*$ there is no model of ZFC$^-$ containing $X'' \cap \bar{\kappa}$ in which $\bar{\kappa}$ is inaccessible and therefore $\bar{\kappa}$ is not a source for a SetMK** model.

Step 2: We want to extend the results from the last step to hold for all ordinals, i.e. for all ordinals $\alpha < \kappa^*$ there is no SetMK** model of height $< \kappa^*$ containing a real S in which α is strongly inaccessible. This makes use of Jensen coding and a result about admissibility spectra which is connected to it. We will use these results as black boxes and will only state the main definitions and theorems here:

Theorem 29 (Jensen Coding)**.** *Suppose that $\langle M, A \rangle$ is a transitive model of ZFC, i.e. M is a transitive model of ZFC, $A \subseteq M$ and Replacement holds in M for formulas mentioning A as a unary predicate. Then there is an $\langle M, A \rangle$-definable class forcing P such that if $G \subseteq P$ is P-generic over $\langle M, A \rangle$, then:*

a) *$\langle M[G], A, G \rangle \models ZFC$.*

b) *For some $R \subseteq \omega$, $M[G] \models V = L[R]$ and $\langle M[G], A, G \rangle \models A, G$ are definable from the parameter R.*

The very elaborate proof of this result uses Jensen's fine structure theory and, very roughly, the forcing involved consists of three components: an almost disjoint coding at successor cardinals, a variation thereof at limit cardinals and a reshaping forcing.[4]

Definition 30. *Let T be the theory of ZF without Power Set and with Replacement restricted to Σ_1 formulas. Then $\Lambda(R)$ for a real R denotes the admissibility spectrum of R and is defined as the class of all ordinals α such that $L_\alpha[R] \models T$, i.e. the class of all R-admissible ordinals.*

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[4]An detailed account of this can be found in [BJW82], a simplified version of the proof can be found in [Fri00].

Theorem 31 (S.-D. Friedman). [5] *Suppose φ is Σ_1 and $L \models \varphi(\kappa)$ whenever κ is an L-cardinal. Then there exists a real $R <_L 0^\sharp$ such that $\Lambda(R) \subseteq \{\alpha \mid L \models \varphi(\alpha)\}$ and R is cardinal preserving over L.*

We will use these theorems to prove the following lemma:

Lemma 32. *We can extend the model $L_{\kappa^*}[X'']$ to be of the form $L_{\kappa^*}[S]$ for a real S such that $L_{\kappa^*}[S] \models SetMK^{**}$ and whenever $\bar\alpha < \kappa^*$ is an ordinal there is no model of $SetMK^{**}$ of height $< \kappa^*$ containing S in which $\bar\alpha$ is strongly inaccessible.*

Proof. First we add a real R to the resulting model of Step 1 and get a model $L_{\kappa^*}[R] \models SetMK^{**}$. This can be done by using Jensen coding over the model $L_{\kappa^*}[X'']$. Although we start from a model of ZFC$^-$ rather than ZFC our model is of the form $L_{\kappa^*}[X'']$ and therefore we can use the standard pretameness argument for Jensen coding to show that ZFC$^-$ is preserved[6]. Also, κ will still be inaccessible in the extension because Jensen coding preserves inaccessibles.[7] Note that the result from Step 1 still holds: In $L_{\kappa^*}[R]$ we have that if $\bar\kappa < \kappa^*$ is a cardinal then there is no transitive model of SetMK** containing R in which $\bar\kappa$ is inaccessible as otherwise there would have been such a model containing $X'' \cap \bar\kappa$ as the latter is coded by R in $L_{\bar\kappa}[R]$.

Now we use Theorem 31 relativized to the real R to produce a new real S such that this holds for ordinals $\bar\kappa$. Theorem 31 works in the context of ZFC$^-$ for the same reasons as for Jensen coding. Note that $L_{\kappa^*}[R] \models \varphi(\bar\kappa)$ for every $L_{\kappa^*}[R]$-cardinal $\bar\kappa$ where $\varphi(\alpha)$ is the following Σ_1 property with parameter R: "Either $L_\alpha[R] \models$ there is a largest cardinal or there is $\beta > \alpha$ such that $L_\beta[R] \models \alpha$ is singular and for all γ with $\alpha < \gamma < \beta$, $L_\gamma[R] \nvDash ZFC^-$". This property says that either α is a successor or we can "see" the singularity of α before we see a ZFC$^-$ model for which it could be a source. Then by Theorem 31 there exists a real S generic over $L_{\kappa^*}[R]$ such that $L_{\kappa^*}[S] \models SetMK^{**}$ and $\Lambda(S) \subseteq \{\alpha \mid L[R] \models \varphi(\alpha)\}$. As α which is inaccessible in a model of ZFC$^-$ containing S is S-admissible, we get the desired property for all ordinals. $\square$

We now have a minimal model $L_{\kappa^*}[S]$ of SetMK**, i.e. the least transitive model of SetMK** containing S. It only remains to show that by going back to MK** we arrive at a minimal β-model of MK^{**}. To see that consider the model $(L_\kappa[S], \mathcal{C})$ where $\mathcal{C}$ consists of the subsets of $L_\kappa[S]$ in $L_{\kappa^*}[S]$. This is a β-model of MK** by Proposition 13 and it is the least such model containing S because otherwise there exists a β-model $(N, \mathcal{C}') \subset (L_\kappa[S], \mathcal{C})$, $(N, \mathcal{C}') \models MK^{**}$ containing S that would give rise to a model N^+ of SetMK**. If

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[5]See [Fri00], Theorem 7.5, p. 142.

[6]See [Fri00], Chapter 4.

[7]This follows from an property called diagonal distributivity (see [Fri00], p. 37).

we then go to the $L[S]$ of N^+ we arrive at a model $L_\alpha[S]$ for some $\alpha < \kappa^*$ which is a model of SetMK**. This is a contradiction to the minimality of $L_{\kappa^*}[S]$.   □

5   Further Work and Open Questions

These results opens up a wider area of further research and related open questions.

In the definition of definable hyperclass forcing we used the restriction to β-models of MK* to make the coding of a transitive SetMK* model work. It would be interesting to investigate what happens if we drop this restriction:

Question: How can definable hyperclass forcing be defined for an arbitrary model of MK**?

Dropping the β-model assumption for the coding would mean to work only internally in the MK** model and restricting ourselves to just coding pairs. We are confident that this can be done, but there are many details to be worked out.

In Theorem 22 we introduced a preparatory forcing to convert the SetMK* model M$^+$ into a model of the form $L_\alpha[A]$ for some generic class predicate $A \subseteq ORD$. This entails a string of further modifications to our original setup: We add the Dependent Choice axiom, arriving at the theory SetMK**, and extend the model M$^+$ to a model where the predicate A is coded into a subset of κ (see Theorem 23). We can ask if and how these modifications could be circumvented:

Question: Can we avoid the use of the preparatory forcing by restricting the original MK* model and/or the hyperclass forcing $\mathbb{P}$?

At the moment, let us just remark that if the classes already carry a "good" wellorder, i.e. a wellorder "$\prec$" such that for each X there is a class $\prec_X$ such that the predecessors of X in $\prec$ are the $(\prec_X)_i$, $i < Ord$, and $X \to \prec_X$ is 2nd order definable, then there is no need for the preparation (or for MK**, MK* is enough) for then M$^+$ will already have the necessary definable hierarchy for class forcing. This will happen if the given MK* model is already minimal, or more generally, if M$^+$ satisfies $V = L[X]$ for some subset X of κ.

In this paper we consider three variants of the axioms of Morse-Kelley; the standard form MK, the extension via Class-Bounding, here called MK* and the additional extension with Dependent Choice, called MK**. The obvious question presents itself, which is how they are related:

Question: Assuming just the consistency of MK, are there models of MK that don't satisfy MK* and models of MK* that don't satisfy MK**?

Another fruitful topic is the analogy between Morse-Kelley and second-order arithmetic.

Question: What results and questions can be transferred from the context of Morse-Kelley class theory to second-order arithmetic and vice versa?

As an example for this transfer, let us consider the question of minimal β-models of MK**. It can be translated to minimal β-models of second-order arithmetic (plus Dependent Choice) in the following way:

Theorem 33. *Every β-model of second-order arithmetic with Dependent Choice can be extended to a minimal β-model of second-order arithmetic with Dependent Choice with the same ordinals.*

Proof outline. Starting with a β-model of second-order arithmetic we can go to a related model of ZFC$^-$ where the inaccessible cardinal κ is now simply $\aleph_0$. Then the question about models below the largest cardinal becomes trivial and we can concentrate on the case of eliminating models of ZFC$^-$ between $\aleph_0$ and the height of the model α^*. First we change the ZFC$^-$ model to a model $L_{\alpha^*}[A]$ in a way analogous to Theorem 22. Then we can adapt the proof of Theorem 23 in a similar way as we did in the beginning of the proof of Theorem 25: we code the predicate A into $X \subseteq \aleph_0$ with an almost disjoint forcing, where we first reshape A to a predicate A'. In this reshaping forcing we destroy the Replacement axiom level-by-level relative to A and therefore it is also destroyed level-by-level relative to X. We arrive at a model $L_{\alpha^*}[X]$, with $X \subseteq \aleph_0$, which is the least transitive ZFC$^-$ model containing X and from this we can go back to a minimal β-model of second-order arithmetic. $\qquad\square$

Of course, definable hyperclass forcing is not the last step in considering a hierarchy of forcing notions via their size. One could ask further:

Question: What would a general hyperclass forcing look like and in which context can it be developed (a hypercass theory)? What would a hyperhyperclass forcings look like, i.e. a forcing where conditions are hyperclasses?

Here we developed a further step in this hierarchy after set forcing, definable class forcing and class-forcing in MK. We hope that it will serve as a basis for further fruitful research.

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