Nonstandard Models and Analytic Equivalence Relations

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Abstract. We improve a result of Hjorth[93] concerning the nature of thin analytic equivalence relations. The key lemma uses a weakly compact cardinal to construct certain nonstandard models, which Hjorth obtained using #'s.

The purpose of this note is to improve the following result of Hjorth [93].

Theorem. (Hjorth) Suppose that for every real $x, x^{\#}$ exists. Let E be an analytic equivalence relation, Σ^1_1 in parameter x_0. Then either there exists a perfect set of pairwise E-inequivalent reals or every E-equivalence class has a representative in a set-generic extension of $L[x_0]$.

Hjorth's proof makes use of his analysis of nonstandard Ehrenfeucht-Mostowski models built from #'s. Instead, we construct the necessary nonstandard models using infinitary model theory, assuming only the existence of weak compacts.

Theorem 1. Suppose that for every real x there is a countable ordinal which is weakly compact in $L[x]$. Then the conclusion of the Theorem still holds.

The main lemma is the following.

Lemma 2. Suppose that there is a weakly compact cardinal κ in $L[x], x$ a real, such that κ^+ of $L[x]$ is countable. Then there is a countable nonstandard ω-model M_x of ZF such that $x \in M_x$ and $L(M_x) = (L$ in the sense of $M_x)$ has an isomorphic copy in a set-generic extension of $L[x_0]$, for any real x_0.

It is not known if the conclusion of Lemma 2 holds in ZFC alone, for arbitrary x (with ZF replaced by an arbitrary finite subtheory).

Proof of Theorem 1 from Lemma 2. Suppose that E is an analytic equivalence relation, Σ^1_1 in the parameter x_0 and choose an x_0-recursive tree T on $\omega\times\omega\times\omega^\omega$ that $xEy \longleftrightarrow T(x,y)$ has a branch. For each countable ordinal α we define $xE_\alpha y \longleftrightarrow \text{rank}(T(x,y))$ is at least α; then E_α is Borel in (x_0, c) where c is any real coding α and E is the intersection of the E_α's. We may assume that each E_α is an equivalence relation. A theorem of Harrington-Silver says that a Π^1_1-equivalence relation has a perfect set of pairwise inequivalent reals or each equivalence class has a representative constructible from the parameter defining the relation. As E_α is Borel in (x_0, c) where c is a real coding α and as we may assume that E and hence each E_α has no perfect set of pairwise inequivalent reals, we know that each E_α-equivalence class has a representative in $L[x_0, c]$ where c is any real coding α.

Now let x be arbitrary and by Lemma 2 choose a countable nonstandard ω-model M_x of ZF containing (x_0, x) such that $L(M_x)$ has an isomorphic copy in a set-generic extension N of $L[x_0]$. Let $a \in \text{ORD}(M_x)$ be nonstandard and let c be a code for a, generic over M_x; then by applying Harrington-Silver in $M_x[c]$ we conclude that there is y in $L(M_x)[x_0, c]$ such that $yE_a x$. By choosing c to be generic over N as well we get that y belongs to a set-generic extension of $L[x_0]$. Finally, yEx since if not, $yE_\alpha x$ would fail for some α admissible in (y, x) and hence for some (standard) $\alpha < a$. $\dashv$

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Proof of Lemma 2. We use infinitary logic. Fix a real x and assume $V = L[x]$. Let κ be weakly compact and introduce the language $\mathcal{L}$ consisting of the formulas in the language of set theory with constants $\underline{a}$ for $a \in L_\kappa[x]$, closed under conjunctions and disjunctions of size less than κ (however we allow a formula to have only finitely many free variables). Let T be the theory of $\langle L_\kappa[x], a\rangle$, $a \in L_\kappa[x]$ in this language. An *n-type* is a set of formulas Γ with free variables $v_1 \dots v_n$, and such a Γ is *consistent with* T if there is a model M of T and $m_1 \dots m_n$ in M such that $M \models \varphi(m_1 \dots m_n)$ for each $\phi \in \Gamma$, where M exists in a set-generic extension of $V = L[x]$. Γ is *complete* if for every $\varphi(v_1 \dots v_n)$ either φ or $\sim \varphi$ belongs to Φ.

Now work in Levy collapse $L[x, c]$ where c is a real coding κ^+ of $L[x]$. Let $d_1, d_2, \dots$ be ω-many new constant symbols and for $D \subseteq \{d_1, d_2, \dots\}$ let the language $\mathcal{L}_D$ be defined like $\mathcal{L}$ but with the new constant symbols from D. Define $T_0 = T \subseteq T_1 \subseteq \dots$ and $D_0 = \phi \subseteq D_1 \subseteq D_2 \subseteq \dots$ inductively as follows: if T_n, D_n have been defined select a complete k-type $\Gamma_n(\vec{v})$ in $L[x]$ consistent with T_n, choose $D_n \subseteq D_{n+1}$ so that card $(D_{n+1} - D_n) = k$ and let $T_{n+1} = T_n \cup \Gamma_n(\vec{d})$ where $\vec{d}$ enumerates $D_{n+1} - D_n$. This can be done in such a way that $\bigcup_n T_n = T^*$ is *L[x]-saturated*: if $\Gamma(\vec{v}, \vec{w})$ is an $L[x]$-type, $\vec{d}$ a finite sequence from D, $\Gamma(\vec{d}, \vec{w})$ consistent with T^* then T^* includes $\Gamma(\vec{d}, \vec{e})$ for some $\vec{e}$. And note that each T_n belongs to $L[x]$ (though of course T^* itself makes use of the Lévy collapse c).

Let M_x be the model determined by T^*, whose universe consists of (equivalence classes) of the constants $d_n, n \in \omega$. Note that a set in $L[x]$ of sentences in some $\mathcal{L}_\mathcal{D}$ is consistent iff each subset of $L[x]$-cardinality $< \kappa$ is. An easy consequence is that M_x is nonstandard with standard ordinal κ.

Now consider $L(M_x)$: every n-type in the language $\mathcal{L}_0 =$ (same as $\mathcal{L}$ but restricted to L_κ) that is realized in $L(M_x)$ belongs to L, as each of its initial segments (obtained by restricting to some $L_\alpha, \alpha < \kappa$) belongs to L and κ is weakly compact. Also, just as M_x is saturated for types in $L[x]$, $L(M_x)$ is saturated for types in L, since again by weak compactness any $\mathcal{L}_0$-type in L consistent with T can be extended to a complete $\mathcal{L}$-type consistent with T in $L[x]$.

Now it is clear that $L(M_x)$ has an isomorphic copy in $L[c]$: using c we can do the same construction as we did above in $L[x, c]$, obtaining M_0, a model that is saturated for $\mathcal{L}_0$-types in L and realizing only types in L. Now construct an isomorphism via a back and forth argument in ω steps between M_0 and $L(M_x)$.

Finally note that by the countability of κ^+ of $L[x]$, the desired model M_x exists not only in $L[x, c]$ but also in V. $\dashv$

Remark. Lemma 2 can also be used to establish the following improvement of the Glimm-Effros style dichotomy theorem of Hjorth-Kechris [96]: Let E be a Σ^1_1 equivalence relation. Assume that for every real x there is a countable ordinal which is weakly compact in $L[x]$. Then either E_0 is continuously reducible to E or E is reducible to $2^{<\omega_1}$ by a function Δ^1_2 in the codes.

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