Definable well–orders of $H(\omega_2)$ and GCH

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Abstract

Assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, we build a partial order that forces the existence of a well–order of $H(\omega_2)$ lightface definable over $\langle H(\omega_2), \in\rangle$ and that preserves cardinal exponentiation and cofinalities.

1 Introduction and terminology

In this paper we prove the following theorem.

Theorem 1.1 *Suppose $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. There is then a totally proper partial order of size $\aleph_2$ with the $\aleph_2$–chain condition which forces that there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ by a parameter–free formula.*

Recall that a poset is totally proper if it is proper and does not add new reals. Any poset $\mathcal{P}$ witnessing the theorem obviously preserves all cofinalities and CH. Also from $2^{\aleph_1} = \aleph_2$ it follows, for all uncountable cardinals $\kappa < \lambda$, that $\mathcal{P}$ will force $2^\kappa = \lambda$ if and only if $2^\kappa = \lambda$ holds in V.

The problem of forcing lightface definable well–orders of structures of the form $H(\kappa)$, κ a cardinal, while preserving (instances of) GCH has been addressed before.[1] For example, in [As2] the first author builds, for any given

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[1] [As1] and [As3] contain historical information on the more general problem of forcing lightface definable well–orders of $H(\kappa)$ (focusing on $H(\omega_2)$).

regular cardinal $\kappa \geq \omega_2$, a proper κ–distributive partial order that preserves stationary subsets of κ and forces the existence of a well–order of $H(\kappa^+)$ definable over the structure $\langle H(\kappa^+), \in\rangle$ by a parameter–free formula.[2] Furthermore, this poset has the κ^+–chain condition and preserves GCH below κ^+ if this cardinality assumption holds in V. The proof of this result involves the manipulation by forcing of certain weak club–guessing properties for club–sequences defined on stationary subsets of κ. The reason why the case $\kappa = \omega_1$ is not covered by that forcing construction is that there is only one infinite regular cardinal below ω_1, whereas the definition of the forced well–order of $H(\kappa^+)$ exploits the fact that below κ there is another infinite regular cardinal apart from ω. More specifically, the construction proceeds by performing the following tasks simultaneously through a certain forcing iteration of length κ^+.

(1) Coding every subset B of κ by an ordinal $\delta < \kappa^+$ with respect to a certain fixed pair $(F, \mathcal{S})$ (added at an initial stage of the iteration), where $F : \kappa \longrightarrow \mathcal{P}(\kappa)$ and $\mathcal{S} = \langle S_i \,:\, i < \kappa\rangle$ is a sequence of pairwise disjoint stationary subsets of $\kappa \cap cf(\geq \omega_1)$, and where '$\delta$ codes B with respect to $(F, \mathcal{S})$' means that there is a club $E \subseteq \mathcal{P}_\kappa(\delta)$ such that for every $X \in E$ and every $i < \kappa$, if $X \cap \kappa \in S_i$, then $ot(X) \in F(X \cap \kappa)$ if and only if $i \in B$.

(2) Making the decoding parameter $(F, \mathcal{S})$ definable by ensuring that some $A \subseteq \kappa$ decoding it becomes the set of perfect ordinals τ of countable cofinality for which there is a certain club–sequence on a subset of $\kappa \cap cf(\omega)$ of height τ and with a certain club-guessing property.[3]

There is no interference between these tasks because (1) deals with ordinals of uncountable cofinality whereas (2) deals with ordinals of countable cofinality. This guarantees that the whole construction can be carried out successfully.

In [As2] there is also a similar result for the case $\kappa = \omega_1$ using an inaccessible cardinal: If Ω is inaccessible, then there is an ω_1–distributive proper poset

[2] In fact, letting NS_κ and $\mathcal{L}$ denote, respectively, the non-stationary ideal on κ and the language of $\langle H(\kappa^+), \in, NS_\kappa\rangle$, there is an $\mathcal{L}$–formula $\Phi(x, y)$ of complexity Σ_3 and an $\mathcal{L}$–formula $\Psi(x, y)$ of complexity Π_3, both without parameters, such that in the extension $\Phi(x, y)$ and $\Psi(x, y)$ define over $\langle H(\kappa^+), \in, NS_\kappa\rangle$ the same well–order of $H(\kappa^+)$.

[3] The concepts mentioned or referred to in this sentence will be explained in a while. In fact they will be a key ingredient in the construction in the present paper.

forcing the existence of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ by two formulas without parameters, one of which is Σ_3 and the other Π_3. However, this poset does not preserve cardinals, and in fact Ω becomes ω_2 in the extension. This times one applies methods of Justin Moore (see [M]) for coding subsets of ω_1 by ordinals in ω_2 relative to a given parameter $(\vec{C}, \mathcal{U})$ added at an initial stage of the iteration, where $\vec{C}$ is a ladder system defined on ω_1 and $\mathcal{U}$ is a sequence of pairwise disjoint stationary subsets of ω_1, while at the same time making the parameter $(\vec{C}, \mathcal{U})$ definable as in (2). The need for the inaccessible cardinal Ω stems from the fact that each instance of Moore's forcing for coding a subset of ω_1 by an ordinal collapses the current ω_2 to ω_1. The construction this time is a countable support iteration of length Ω, and a consequence of this is that Ω becomes ω_2 in the end.

In [As-F] we build on the methods of [As2] to construct, under GCH, class–sized partial orders that preserves cofinalities and GCH and that force the existence of a well–order of the universe which is locally definable, in the sense that its restriction to $H(\kappa^+) \times H(\kappa^+)$, for all regular cardinals $\kappa \geq \omega_2$, is a well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in\rangle$ by a formula without parameters. In addition, these forcings preserve various instances of large cardinals from the ground model. Specifically, one of the forcing notions preserves all supercompact cardinals and in fact all regular instances of local supercompactness,[4] and another one preserves many of the n–huge cardinals there may be in the ground model (for all n).[5]

One of the questions left open by the work in [As2] and in [As-F] is the following.

Question 1.1 *Is there, under* GCH*, any partial order preserving* GCH *and cofinalities and forcing the existence of a well–order of $H(\omega_2)$ lightface definable over $\langle H(\omega_2), \in\rangle$?*

[4]That is, whenever $\kappa \leq \lambda$ are regular and κ is λ–supercompact, κ remains λ–supercompact in the extension.

[5]Earlier work along similar lines is that of Andrew Brooke-Taylor in [B], who also builds a class–forcing which adds a lightface definable well–order of the universe and which preserves GCH as well as many large cardinals from the ground model. The well–order constructed in his model does not admit a local definition in the sense of [As-F]. The coding technique in Brooke-Taylor's construction is also different from the one in [As-F]: He encodes a given bit of information at a suitably chosen cardinal κ by making the diamond–principle $\diamondsuit^*$ hold or fail at κ^+.

Friedman's [F] (see also [FH]) contains the construction of a forcing that, given any subset A of ω_2, adds a subset B of ω_1 which codes A relative to canonical functions, in the sense that for every nonzero ordinal $\alpha < \omega_2$ and every surjection $\pi : \omega_1 \longrightarrow \alpha$ there is a club $C \subseteq \omega_1$ of ordinals ν such that $ot(\pi``\nu) \in B$ if and only if $\alpha \in A$. This forcing has an ω_1–closed dense subset, and it has the $\aleph_2$–chain condition if CH holds in V.

One first naïve approach for tackling the above question was to start from a model of GCH and build a countable support forcing iteration of length ω_2 with the $\aleph_2$–chain condition in which one simultaneously

(A) uses Friedman's forcing from [F] for coding the ground model $H(\omega_2)$ (and a well–order of it) by a subset B of ω_1 and

(B) uses Asperó's forcing from [As2] to make B definable.

This would suffice to make the ground model $H(\omega_2)$ (together with a well–order of it) lightface definable over the final $\langle H(\omega_2), \in\rangle$, but of course this is not good enough, as we would like the set $B \subseteq \omega_1$ to somehow encode the generic filter as well.[6] However, any subset of ω_1 added during an iteration of length ω_2 with the $\aleph_2$–chain condition will have appeared at some of its initial stages, which makes it impossible for such a set to encode the entire generic filter.

The forcing we finally build can be (roughly) described as a restricted product rather than a forcing iteration, which nonetheless "mimics" the natural iteration – let us call it $\mathcal{I}$ – for performing the following variation of the above tasks (A) and (B):

(A)' Starting from a stationary and co-stationary $S \subseteq \omega_1$, use Friedman's forcing for coding, *relative to S,* the ground model $H(\omega_2)$ (and a well–order of it) by a subset $B \subseteq \omega_1$. This means: Make sure that there is $A \subseteq [\omega_1, \omega_2)$ coding the ground model $H(\omega_2)$ (and a well–order of it) and that there is $B \subseteq \omega_1$ such that for every $\gamma \in [\omega_1, \omega_2)$ and every canonical function $f : \omega_1 \longrightarrow \omega_1$ for γ there are club–many $\nu < \omega_1$ with the property that if $\nu \in S$, then $f(\nu) \in B$ if and only if $\gamma \in A$.

[6] If all proper initial segments of the iteration have dense subsets of size at most $\aleph_1$, then forcing the generic filter G and a well–order $\mathcal{W}$ of $H(\omega_2)^V$ to become lightface definable is enough to yield a lightface definable well–order of the new $H(\omega_2)$: Given any $x \in H(\omega_2)^{V[G]}$, one looks at the $\mathcal{W}$–first name in $H(\omega_2)^V$ whose interpretation by G is x. This name exists by the $\aleph_2$–chain condition of the iteration.

(B)' Use Asperó's forcing to make B definable, *relative to* $\omega_1 \backslash S$. This means that, letting $(\eta_i)_{i<\omega_1}$ be the increasing enumeration of all infinite perfect ordinals $\eta < \omega_1$, B should become the set of $i < \omega_1$ such that there is a coherent strongly type-guessing club–sequence of height η_i with stationary domain included in $\omega_1 \backslash S$.

(C) Arrange that the class of S in $\mathcal{P}(\omega_1)/NS_{\omega_1}$ is the largest class K in $\mathcal{P}(\omega_1)/NS_{\omega_1}$ consisting of sets T for which there is a ladder system $\langle E_\delta : \delta \in T \rangle$ with the property that for every club $C \subseteq \omega_1$ there are club–many δ such that $|E_\delta \backslash C| < \aleph_0$ if $\delta \in T$.[7]

A condition in the forcing can be roughly described as consisting of the piece of information about a typical condition in $\mathcal{I}$ which is completely determined in V.

The above is actually just an approximation to what we really do: Note that arranging (A)', (B)' and (C) would again suffice to make the ground model $H(\omega_2)$ definable in the extension $V[G]$, but not necessarily $H(\omega_2)^{V[G]}$. For this we actually make sure – in (A)' – that $B \subseteq \omega_1$ codes a set $A^G \subseteq [\omega_1, \omega_2)$ which not only codes the ground model $H(\omega_2)$ and a well–order of it, but also codes the generic filter G itself. The restrictions imposed by the necessary feedback between the coding set $B \subseteq \omega_1$ being added, the generic set $A^G \subseteq [\omega_1, \omega_2)$ decoded by B, and also the generic filter G making B definable[8] are the reason why we cannot do with a forcing iteration in the conventional sense. Theorem 1.1 settles Question 1.1 by means of this forcing construction.

The proof of Theorem 1.1 (and in particular Lemma 2.4) will show that the following variation thereof also holds.

Theorem 1.2 *Suppose $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ and let $T \subseteq \omega_1$ be stationary and co-stationary. There is then a totally proper partial order of size $\aleph_2$ with the $\aleph_2$–chain condition which forces $\diamondsuit(T)$ together with the existence of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in \rangle$ by a parameter–free formula.*

[7]It is not clear whether the versions of tasks (A)' and (B)' not mentioning S or $\omega_1 \backslash S$ (let us call them (A)" and (B)") can be performed simultaneously with our approach. For technical reasons one needs to deal with both tasks on complementary stationary sets while making the classes of these stationary sets definable. The problem with the implementation of tasks (A)" and (B)" was detected by the referee.

[8]Of course B is itself "part of" G.

In fact, given T we can modify the forcing $\mathcal{P}$ from the proof of Theorem 1.1 (relative to $S := Lim(\omega_1) \setminus T$) in such a way that it also adds a sequence $(X_\alpha)_{\alpha \in T}$, with $X_\alpha \subseteq \alpha$, by initial segments. Using the fact that the resulting forcing is T–closed by the corresponding version of Lemma 2.4, it follows by standard arguments that $(X_\alpha)_{\alpha\in T}$ is a $\diamondsuit(T)$–sequence in the extension. Of course the coding $\mathcal{A}(p)$ (see below) has to be suitably modified so as to incorporate the information on the sequence $(X_\alpha)_{\alpha\in T}$ being added.

The proof of Theorem 1.1 will stretch through Sections 2 and 3. In the rest of this first section we give some pieces of terminology that will come up in the proof.

Given a set X of ordinals, $ot(X)$ denotes the order type of X. For an ordinal $\gamma < \omega_2$ and a surjection $\pi : \omega_1 \longrightarrow \gamma$, the function $f_\gamma : \omega_1 \longrightarrow \omega_1$ defined by $f_\gamma(\nu) = ot(\pi``\nu)$ will be called *the canonical function for* γ *corresponding to* π. It represents the ordinal γ in the generic ultrapower $Ult(V, G)$ for every $\mathcal{P}(\omega_1)\setminus NS_{\omega_1}$–generic filter G over V (where NS_{ω_1} denotes the non-stationary ideal on ω_1). The use of the term 'canonical' is justified by the following well–known fact.

Fact 1.3 *Given ordinals $\gamma < \gamma' < \omega_2$ and surjections $\pi_0 : \omega_1 \longrightarrow \gamma$, $\pi_1 : \omega_1 \longrightarrow \gamma$ and $\pi' : \omega_1 \longrightarrow \gamma'$, if f_0 and f_1 are the canonical functions for γ corresponding to, respectively, π_0 and π_1, and f' is the canonical function for γ' corresponding to π', then there is a club $E \subseteq \omega_1$ such that $f_0(\nu) = f_1(\nu) < f'(\nu)$ for all $\nu \in E$.*

We are also going to use the following terminology, borrowed from [As2] (see also [As-F]).

A *club–sequence* will be defined to be a sequence $\vec{C} = \langle C_\delta : \delta \in X\rangle$, where X is a set of limit ordinals, such that each C_δ is a club subset of δ. The set X will be called the *domain of* $\vec{C}$ and will be denoted by $dom(\vec{C})$. Also, $range(\vec{C})$ will denote the set $\bigcup_{\delta\in dom(\vec{C})} C_\delta$. We will use the convention of denoting $\vec{C}(\delta)$ by C_δ and similarly with names for club–sequences involving superscripts.[9]

We will say that $\vec{C}$ is *coherent* in case there is a club–sequence $\vec{D}$ with $dom(\vec{D}) \supseteq dom(\vec{C})$ and $\vec{D} \restriction dom(\vec{C}) = \vec{C}$ and such that $\gamma \in dom(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$ whenever $\delta \in dom(\vec{D})$ and γ is a limit point of D_δ. In that case we will say that $\vec{D}$ *witnesses the coherence of* $\vec{C}$.

[9]So, for example, if $\vec{C}^i$ is a club–sequence, C^i_δ will represent $\vec{C}^i(\delta)$.

The *height of* $\vec{C}$, if defined, is the unique ordinal τ such that $ot(C_\delta) = \tau$ for all $\delta \in dom(\vec{C})$. In this case we will write $ht(\vec{C}) = \tau$. If $ht(\vec{C}) = \omega$, we say that $\vec{C}$ *is a ladder system*.

Given a set X of ordinals and an ordinal δ, the *Cantor–Bendixson rank of* δ *with respect to* X, $rnk_X(\delta)$, is defined by specifying that $rnk_X(\delta) \geq 1$ if and only if δ is a limit point of X and, for each ordinal $\eta \geq 1$, that $rnk_X(\delta) > \eta$ if and only if δ is a limit ordinal and there is a sequence $(\delta_\xi)_{\xi < cof(\delta)}$ converging to δ such that $rnk_X(\delta_\xi) \geq \eta$ for every ξ. An ordinal δ will be said to be *perfect* if $rnk_\delta(\delta) = \delta$.[10] Note that $rnk_\delta(\delta) \leq \delta$ for every ordinal δ and that, given any uncountable cardinal μ, the set of perfect ordinals below μ forms a club of μ of order type μ.

We will make use of the following non-symmetric operation: Given sets of ordinals X and Y, $X \cap^* Y$ denotes the set of all ordinals $\alpha \in X \cap Y$ such that α is not a limit point of X.

We will say that a club–sequence $\vec{C}$ with stationary domain is *strongly type-guessing* if it is the case that for every club $C \subseteq \omega_1$ there is a club $D \subseteq \omega_1$ such that $ot(C_\delta \cap^* C) = ot(C_\delta)$ for every $\delta \in dom(\vec{C}) \cap D$.

Given a set $T \subseteq \omega_1$, we will denote the equivalence class of T in the quotient algebra $\mathcal{P}(\omega_1)/NS_{\omega_1}$ by $[T]$. In other words, $T' \in [T]$ if and only if $T' \Delta T$ is a nonstationary subset of ω_1.

For ordinals α and β, $[\alpha, \beta)$ (resp., (α, β)) represents the interval of ordinals ξ such that $\alpha \leq \xi < \beta$ (resp., such that $\alpha < \xi < \beta$), and $\alpha \cdot \beta$ denotes ordinal multiplication. The closure of a set X of ordinals will be the closure of X with respect to the order topology.

It will be convenient to fix a canonical (and definable) way of coding pairs of ordinals by ordinals. For concreteness we are going to use the Gödel pairing function $\Gamma : Ord \times Ord \longrightarrow Ord$ for this,[11] and consequently we will sometimes identify a pair $\langle \alpha, \beta \rangle$ of ordinals with the ordinal $\Gamma(\alpha, \beta)$. We will refer to Gödel pairing simply by 'pairing'. Finally, given sets X, Y of ordinals, we will let $X \bigoplus Y$ denote $\{2 \cdot \alpha \,:\, \alpha \in X\} \cup \{2 \cdot \alpha + 1 \,:\, \alpha \in Y\}$.[12]

[10] Note that the first perfect ordinal is 0 and the second is $\epsilon_0 = sup\{\omega, \omega^\omega, \omega^{(\omega^\omega)}, \omega^{(\omega^{(\omega^\omega)})}, \ldots\}$.

[11] Where $\Gamma(\alpha, \beta)$ is the order type of $\{\langle \gamma, \delta \rangle \in Ord \times Ord \,:\, \langle \gamma, \delta \rangle < \langle \alpha, \beta \rangle\}$, and where $\langle \gamma, \delta \rangle < \langle \alpha, \beta \rangle$ if and only if either $max\{\gamma, \delta\} < max\{\alpha, \beta\}$, or $max\{\gamma, \delta\} = max\{\alpha, \beta\}$ and $\gamma < \alpha$, or $max\{\gamma, \delta\} = max\{\alpha, \beta\}$, $\gamma = \alpha$ and $\delta < \beta$ (see for example [J], p. 30).

[12] $X \bigoplus Y$ is the result of "putting together" X and Y. Notice that the operation $\bigoplus$ is neither commutative nor associative.

2 The construction

Let us assume $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. By $2^{\aleph_1} = \aleph_2$ we can fix a bookkeeping function $F : \omega_2 \longrightarrow H(\omega_2)$.[13] We also fix a stationary and co-stationary $S \subseteq \omega_1$ consisting of limit ordinals.

Let us also fix surjections $\pi_\gamma : \omega_1 \longrightarrow \gamma$ for all nonzero ordinals $\gamma < \omega_2$ and let $f_\gamma : \omega_1 \longrightarrow \omega_1$ be the canonical function for γ corresponding to π_γ.

Let us fix a subset A of ω_2 such that $L_{\omega_2}[A] = H(\omega_2)$ and such that A codes in some canonical way over $H(\omega_2)$, F, S, π_γ for all nonzero $\gamma < \omega_2$, and also a well–order $\mathcal{W}$ of $H(\omega_2)$. This can be arranged again by $2^{\aleph_1} = \aleph_2$.

Finally, let $(\eta_\xi)_{\xi<\omega_1}$ be the strictly increasing enumeration of all countable perfect ordinals and let $\mathcal{C}$ be the set of all limit ordinals $\alpha \leq \omega_2$ such that $\omega_1 \cdot \alpha' < \alpha$ for all $\alpha' < \alpha$.

We are going to construct by recursion a certain $\subseteq$–increasing sequence of partial orders $\langle \mathcal{P}_\alpha, \leq_\alpha \rangle$ $(\alpha \in \mathcal{C})$. Our forcing $\mathcal{P}$ witnessing Theorem 1.1 is going to be $\langle \mathcal{P}_{\omega_2}, \leq_{\omega_2} \rangle$.

In general, for any given tuple of the form

$$p = \langle b, \vec{E}, (c_\gamma \,:\, \gamma \in a), (\langle \vec{C}^i, \vec{D}^i \rangle \,:\, i < \beta) \rangle$$

we will set $\beta^p = \beta$, $a^p = a$, and similarly for other objects occurring in the definition of p. For any given ordinal α we will denote by $p \restriction \alpha$ the tuple

$$\langle b, \vec{E}, (c_\gamma \,:\, \gamma \in a \cap \alpha), (\langle \vec{C}^i, \vec{D}^i \rangle \,:\, i < \beta) \rangle$$

Also, suppose p is a tuple as above such that

(i) β is a countable ordinal,

(ii) $b \subseteq \omega_1$,

(iii) $\vec{E}$ is a ladder system with $dom(\vec{E}) \subseteq \omega_1$,

(iv) a is a countable subset of $[\omega_1, \omega_2)$ and each c_γ is a subset of ω_1, and

(v) for all $i < \beta$, $\vec{C}^i$ and $\vec{D}^i$ are club–sequences with domain included in ω_1.

[13] That is, for every $a \in H(\omega_2)$, $F^{-1}(a)$ is unbounded in ω_2.

Then we associate to p a certain set $\mathcal{A}(p) \subseteq [\omega_1, \omega_2)$ which canonically codes A and p. For concreteness we are going to take this to mean the following: First we let $\mathcal{A}_0 = \{\omega_1 + \gamma \,:\, \gamma \in b\} \bigoplus \{\omega_1 + \varsigma \,:\, \varsigma \in \mathcal{B}\}$, where $\mathcal{B} = \{\langle \delta, \varrho\rangle \,:\, \delta \in dom(\vec{E}),\ \varrho \in E_\delta\}$, $\mathcal{A}_1 = \mathcal{A}_0 \bigoplus \bigcup_{\xi \in a}\{\omega_1 \cdot \xi + \varsigma \,:\, \varsigma \in c_\xi\}$, and then we let $\mathcal{A}^*(p)$ be $\mathcal{A}_1 \bigoplus \bigcup_{i<\beta}\{\omega_1 \cdot (1+i) + \varsigma \,:\, \varsigma \in \mathcal{B}^i\}$, where $\mathcal{B}^i$ is, for each $i < \beta$, $\{\langle \delta, \varrho\rangle \,:\, \delta \in dom(\vec{C}^i),\ \varrho \in C^i_\delta\} \bigoplus \{\langle \delta, \varrho\rangle \,:\, \delta \in dom(\vec{D}^i),\ \varrho \in D^i_\delta\}$.[14] Finally we let $\mathcal{A}(p) = \mathcal{A}^*(p) \bigoplus \{\omega_1 + \gamma \,:\, \gamma \in A\}$.

Note that an ordinal in $\mathcal{A}(p)$ will be odd if and only if it codes an ordinal in $\{\omega_1 + \gamma \,:\, \gamma \in A\}$, that it will be divisible by 2 but not by 4 if and only if it codes an ordinal in $\bigcup_{i<\beta}\{\omega_1 \cdot (1+i) + \varsigma \,:\, \varsigma \in \mathcal{B}^i\}$, that it will be divisible by 4 but not by 8 if and only if it codes an ordinal in $\bigcup_{\xi \in a}\{\omega_1 \cdot \xi + \varsigma \,:\, \varsigma \in c_\xi\}$, and that it will be divisible by 8 if and only if it codes an ordinal in $\mathcal{A}_0$. The following fact follows easily from the way $\mathcal{A}(p)$ has been set up together with the fact that $\omega_1 \cdot \alpha' < \alpha$ for all $\alpha' < \alpha$ and all α in $\mathcal{C}$.

Fact 2.1 *Let p be a tuple for which $\mathcal{A}(p)$ is defined, and let $\alpha \in \mathcal{C}$. Then, for every $\gamma \in [\omega_1, \alpha)$, $\gamma \in \mathcal{A}(p)$ if and only if $\gamma \in \mathcal{A}(p \restriction \alpha)$.*

Given $\alpha \in \mathcal{C}$, and assuming $\mathcal{P}_{\alpha'}$ has been defined for all $\alpha' < \alpha$ in $\mathcal{C}$, conditions in $\mathcal{P}_\alpha$ are tuples of the form

$$p = \langle b, \vec{E}, (c_\gamma \,:\, \gamma \in a), (\langle \vec{C}^i, \vec{D}^i\rangle \,:\, i < \beta)\rangle$$

satisfying the following conditions (1)–(8).

(1) β is a countable ordinal closed under pairing.

(2) a is a countable subset of $\bigcup_{1 \le \rho < \alpha}[\omega_1 \cdot \rho,\ \omega_1 \cdot \rho + \beta)$ and $\gamma' \in a$ whenever $\gamma' \geq \omega_1$ and $\gamma \in a$ is of the form $\omega_1 \cdot \gamma' + \varsigma$ with $\varsigma < \omega_1$.

(3) $\vec{E}$ is a ladder system of the form $\vec{E} = \langle E_\delta \,:\, \delta \in S \cap (\beta + 1)\rangle$.

(4) b is a countable subset of ω_1 without a maximum and with $ot(b \cap \beta) = \beta$.

(5) for each $\gamma \in a$, c_γ is a closed subset of $\beta + 1$ and for all $\nu \in c_\gamma \cap S$,

(5.1) $f_\gamma(\nu) < sup(b)$, and

[14] Here, and of course also in the above specification of $\mathcal{B}$, we are identifying pairs of ordinals with ordinals via the pairing function.

(5.2) $f_\gamma(\nu) \in b$ if and only if $\gamma \in \mathcal{A}(p)$.

(6) For every $i < \beta$, $\vec{C}^i$ and $\vec{D}^i$ are club–sequences with domain included in $\beta + 1$, $ht(\vec{C}^i)$ is defined and is a perfect ordinal, and $\vec{C}^i$ is coherent as witnessed by $\vec{D}^i$. Moreover,

$$b \cap \beta = \{\xi < \beta \, : \, (\exists i < \beta) \, ht(\vec{C}^i) = \eta_\xi\}$$

(7) For all $i < \beta$, $(dom(\vec{D}^i) \cup range(\vec{D}^i)) \cap S = \emptyset$, $dom(\vec{D}^i) \cap (i+1) = \emptyset$, for all i, $j < \beta$, $dom(\vec{C}^i) \cap range(\vec{C}^j) = \emptyset$ and, if $i \neq j$, $dom(\vec{D}^i) \cap dom(\vec{D}^j) = \emptyset$.

(8) Let $\gamma \in a$ be given and suppose there is a least $\alpha' \in \gamma \cap \mathcal{C}$ such that $F(\gamma)$ is a $\mathcal{P}_{\alpha'}$–name for a club subset of ω_1. Then $p \restriction \alpha'$ is a condition in $\mathcal{P}_{\alpha'}$ and for every $\nu < max(c_\gamma)$, $p \restriction \alpha'$ either forces $\nu \in F(\gamma)$ or forces $\nu \notin F(\gamma)$. Let C_γ be the set of all $\nu < max(c_\gamma)$ such that $p \restriction \alpha' \Vdash_{\mathcal{P}_{\alpha'}} \nu \in F(\gamma)$. Then

(8.1) $ot(C^i_\delta \cap^* C_\gamma) = ht(\vec{C}^i)$ for every $i < \beta$ and every $\delta \in c_\gamma \cap dom(\vec{C}^i)$, and

(8.2) $E_\delta \setminus C_\gamma$ is finite for every $\delta \in c_\gamma \cap S$.

Given conditions

$$p_\epsilon := \langle b^\epsilon, \vec{E}^\epsilon, (c^\epsilon_\gamma \, : \, \gamma \in a^\epsilon), (\langle \vec{C}^{i,\epsilon}, \vec{D}^{i,\epsilon} \rangle \, : \, i < \beta^\epsilon) \rangle$$

for $\epsilon \in \{0, 1\}$, we set $p_1 \leq_\alpha p_0$ if and only if

(a) $\beta^0 \leq \beta^1$, $a^0 \subseteq a^1$ and $b^0 = b^1 \cap sup(b^0)$,

(b) $\vec{E}^0 \subseteq \vec{E}^1$,

(c) for all $\gamma \in a^0$, $c^0_\gamma = c^1_\gamma \cap (\beta^0 + 1)$, and

(d) $\vec{C}^{i,0} = \vec{C}^{i,1} \restriction (\beta^0 + 1)$ and $\vec{D}^{i,0} = \vec{D}^{i,1} \restriction (\beta^0 + 1)$ for all $i < \beta^0$.

Note that the c_γ's play a triple role: They are used for the canonical function coding (5.2), for the strong type-guessing (8.1) and for the ladder system coding of $[S]$ (8.2).

$\leq_\alpha$ is obviously transitive. Also, it is easy to check that $\mathcal{P}_{\alpha'} \subseteq \mathcal{P}_\alpha$ and $\leq_{\alpha'} \subseteq \leq_\alpha$ hold for $\alpha' < \alpha \leq \omega_2$ in $\mathcal{C}$.

Let $\langle \mathcal{P}, \leq \rangle = \langle \mathcal{P}_{\omega_2}, \leq_{\omega_2} \rangle$. By CH it is clear that $\mathcal{P}$ has size $\aleph_2$.

The following result will be of crucial importance.

Lemma 2.2 *For every $\alpha \leq \omega_2$ in $\mathcal{C}$ and every $p \in \mathcal{P}_\alpha$, $p \restriction \alpha'$ is a condition in $\mathcal{P}_{\alpha'}$ for all $\alpha' \leq \alpha$ in $\mathcal{C}$. Furthermore, $\mathcal{P}_{\alpha'}$ is a complete suborder of $\mathcal{P}_\alpha$ for all $\alpha' \leq \alpha$ in $\mathcal{C}$.*

Proof: Let $p = \langle b, \vec{E}, (c_\gamma : \gamma \in a), (\langle \vec{C}^i, \vec{D}^i \rangle : i < \beta) \rangle$. Let us fix $\alpha' < \alpha$ in $\mathcal{C}$. The verification of conditions (1)–(4) and (6)–(8) in the definition of $\mathcal{P}_{\alpha'}$ for $p \restriction \alpha'$ is completely routine. For condition (5) one just needs to check that if $\gamma \in a \cap \alpha'$ and $\nu \in c_\gamma \cap S$, then $f_\gamma(\nu) \in b$ if and only if $\gamma \in \mathcal{A}(p \restriction \alpha')$, but this follows from Fact 2.1 since $\gamma < \alpha'$ and since $f_\gamma(\nu) \in b$ if and only if $\alpha \in \mathcal{A}(p)$.

From the fact that $p \restriction \alpha' \in \mathcal{P}_{\alpha'}$ for all $p \in \mathcal{P}_\alpha$, together with the definition of $\leq_{\alpha'}$, it easily follows that incompatible conditions in $\mathcal{P}_{\alpha'}$ are also incompatible as conditions in $\mathcal{P}_\alpha$.

To see that $\mathcal{P}_{\alpha'}$ is a complete suborder of $\mathcal{P}_\alpha$, let B be a maximal antichain of $\mathcal{P}_{\alpha'}$ and let $q = \langle b, \vec{E}, (c_\gamma : \gamma \in a), (\langle \vec{C}^i, \vec{D}^i \rangle : i < \beta) \rangle \in \mathcal{P}_\alpha$. Let $p = \langle b^0, \vec{E}^0, (c^0_\gamma : \gamma \in a^0), (\langle \vec{C}^{i,0}, \vec{D}^{i,0} \rangle : i < \beta^0) \rangle$ be a condition in $\mathcal{P}_{\alpha'}$ extending both a condition in B and $q \restriction \alpha'$.

Let c^*_γ be c^0_γ if $\gamma \in a^0$ and let it be c_γ if $\gamma \in a \backslash \alpha'$. It suffices to see that the tuple $q^* = \langle b^0, \vec{E}^0, (c^*_\gamma : \gamma \in a^0 \cup a), (\langle \vec{C}^{i,0}, \vec{D}^{i,0} \rangle : i < \beta^0) \rangle$ is a condition in $\mathcal{P}_\alpha$ extending q and p. For this, we are going to prove by induction on $\tilde{\alpha} \leq \alpha$ that if $\tilde{\alpha} \in \mathcal{C}$, then $q^* \restriction \tilde{\alpha}$ is a condition in $\mathcal{P}_{\tilde{\alpha}}$. (It will then obviously follow that $q^* \restriction \tilde{\alpha}$ extends $p \restriction \tilde{\alpha}$ and $q \restriction \tilde{\alpha}$.) So let us fix some $\tilde{\alpha} \leq \alpha$ in $\mathcal{C}$ and let us start by showing that $q^* \restriction \tilde{\alpha} \in \mathcal{P}_{\tilde{\alpha}}$. The verification of conditions (1)–(4), (6) and (7) for $q^* \restriction \tilde{\alpha}$ is immediate.

For condition (5), note that if $\gamma \in a^0$ and $\nu \in c^0_\gamma \cap S$, then $f_\gamma(\nu) \in b^0$ if and only if $\gamma \in \mathcal{A}(p)$, and that if $\gamma \in a \setminus a^0$ and $\nu \in c_\gamma \cap S$, then $\gamma \in \mathcal{A}(q)$ if and only if $f_\gamma(\nu) \in b$ if and only if (since $f_\gamma``(c_\gamma \cap S) \subseteq sup(b)$ and b^0 is an end–extension of b) $f_\gamma(\nu) \in b^0$. Hence, it suffices to show that

(a) if $\gamma \in [\omega_1, \alpha')$, then $\gamma \in \mathcal{A}(p)$ if and only if $\gamma \in \mathcal{A}(q^* \restriction \tilde{\alpha})$, and

(b) if $\gamma \in \omega_2 \setminus \alpha'$, then $\gamma \in \mathcal{A}(q)$ if and only if $\gamma \in \mathcal{A}(q^* \restriction \tilde{\alpha})$.

If γ is as in (a), then $\gamma \in \mathcal{A}(q^* \restriction \tilde{\alpha})$ if and only if $\gamma \in \mathcal{A}(q^* \restriction \alpha')$ (by Fact 2.1). But $q^* \restriction \alpha' = p$ by the way we have defined q^*. If γ is as in (b), we have that γ is in $\mathcal{A}(q^* \restriction \tilde{\alpha})$ if and only if γ is either in $\mathcal{A}(q^* \restriction \tilde{\alpha}) \setminus \mathcal{A}(q^* \restriction \alpha')$ or else codes an ordinal in $\{\omega_1 + \varrho \,:\, \varrho \in A\}$ (with $\varrho \geq \alpha'$, in fact). If $\gamma \in \mathcal{A}(q^* \restriction \tilde{\alpha}) \setminus \mathcal{A}(q^* \restriction \alpha')$, then γ codes an ordinal in $\bigcup_{\xi \in a^0 \cup a}\{\omega_1 \cdot \xi + \zeta \,:\, \zeta \in c^*_\xi\}$ with $\xi \in a \setminus \alpha'$, which means that it codes an ordinal in $\bigcup_{\xi \in a}\{\omega_1 \cdot \xi + \zeta \,:\, \zeta \in c_\xi\}$. It follows that $\gamma \in \mathcal{A}(q^* \restriction \tilde{\alpha}) \setminus \mathcal{A}(q^* \restriction \alpha')$ if and only if $\gamma \in \mathcal{A}(q) \setminus \mathcal{A}(q \restriction \alpha')$, and hence in this case we have that $\gamma \in \mathcal{A}(q^* \restriction \tilde{\alpha})$ if and only if $\gamma \in \mathcal{A}(q)$.

Let us finish with the verification of condition (8) for $q^* \restriction \tilde{\alpha}$. Let $\gamma \in a^0 \cup a$, $\gamma < \tilde{\alpha}$, and suppose there is a least $\overline{\alpha} < \gamma$ in $\mathcal{C}$ such that $F(\gamma)$ is a $\mathcal{P}_{\overline{\alpha}}$–name for a club subset of ω_1. It will suffice to show, first, that for every $\nu < max(c^*_\gamma)$ either $q^* \restriction \overline{\alpha} \Vdash_{\mathcal{P}_{\overline{\alpha}}} \nu \in F(\gamma)$ or $q^* \restriction \overline{\alpha} \Vdash_{\mathcal{P}_{\overline{\alpha}}} \nu \notin F(\gamma)$ and, second, that letting C_γ be the set of all $\nu < max(c^*_\gamma)$ such that $q^* \restriction \overline{\alpha} \Vdash_{\mathcal{P}_{\overline{\alpha}}} \nu \in F(\gamma)$,

(i) $ot(C^{i,0}_\delta \cap^* C_\gamma) = ht(\vec{C}^{i,0})$ for all $i < \beta^0$ and all $\delta \in c^*_\gamma \cap dom(\vec{C}^{i,0})$, and

(ii) $E^0_\delta \setminus C_\gamma$ is finite for every $\delta \in dom(\vec{E}^0) \cap c^*_\gamma$.

If $\gamma \in a^0$ this is immediate, so we may assume that $\gamma \in a \setminus \alpha'$. But then, by (8) for q, the first conclusion holds for all $\nu < max(c_\gamma)$ (since $q^* \restriction \overline{\alpha} \leq_{\overline{\alpha}} q \restriction \overline{\alpha}$ by induction hypothesis and since $q \restriction \overline{\alpha}$ decides the statement '$\nu \in F(\gamma)$' for every such ν). As to the second conclusion, notice that part (i) holds for every $i < \beta^0$ and every $\delta \in c^*_\gamma \cap dom(\vec{C}^{i,0})$ since $\delta \in c^*_\gamma \cap dom(\vec{C}^{i,0}) = c_\gamma \cap dom(\vec{C}^{i,0})$ implies $i < max(c_\gamma) \leq \beta$ (as $dom(\vec{C}^{i,0}) \cap (i+1) = \emptyset$ by condition (7) in the definition of $\mathcal{P}_{\alpha'}$) and therefore $C^{i,0}_\delta = C^i_\delta$ and $ot(C^{i,0}_\delta \cap^* C_\gamma) = ot(C^i_\delta \cap^* C_\gamma) = ht(\vec{C}^i) = ht(\vec{C}^{i,0})$. The verification of part (ii) of the second conclusion is similar. □

A condition q^* obtained as in the proof of Lemma 2.2 from p and q will be denoted by $p \wedge q$. This notation will come up in the proof of Lemma 3.2.

Lemmas 2.3, 2.4 and 2.6 prove part of Theorem 1.1.

Lemma 2.3 *$\mathcal{P}$ has the $\aleph_2$–chain condition.*

Proof: Let $\{p_\epsilon \,:\, \epsilon < \omega_2\}$ be a set of $\mathcal{P}$–conditions. Let $p_\epsilon = \langle b^\epsilon, \vec{E}^\epsilon, (c^\epsilon_\gamma \,:\, \gamma \in a^\epsilon), (\langle \vec{C}^{i,\epsilon}, \vec{D}^{i,\epsilon}\rangle \,:\, i < \beta^\epsilon)\rangle$ for each ϵ. By a standard Δ–system argument using CH, together with the fact that all objects occurring in the description of any given condition in $\mathcal{P}$ are countable, we may assume that there are objects a, b, $\vec{E}$, $(c_\gamma : \gamma \in a)$ and $(\langle \vec{C}^i, \vec{D}^i \rangle : i < \beta)$ such that, for all distinct ϵ, $\epsilon' < \omega_2$,

(i) $b^\epsilon = b$, $\vec{E}^\epsilon = \vec{E}$, $(\langle \vec{C}^{i,\epsilon}, \vec{D}^{i,\epsilon} \rangle : i < \beta^\epsilon) = (\langle \vec{C}^i, \vec{D}^i \rangle : i < \beta)$, and

(ii) $a^\epsilon \cap a^{\epsilon'} = a$ and $c^\epsilon_\gamma = c_\gamma$ for all $\gamma \in a$.

Let $p_{\epsilon,\epsilon'}$ be the tuple defined by b, $\vec{E}$, $(c^{\bar{\epsilon}}_\gamma : \bar{\epsilon} \in \{\epsilon, \epsilon'\}, \gamma \in a^{\bar{\epsilon}})$ and $(\langle \vec{C}^i, \vec{D}^i \rangle : i < \beta)$, and let us prove by induction on $\alpha \leq \omega_2$ that if $\alpha \in \mathcal{C}$, then $p_{\epsilon,\epsilon'} \restriction \alpha$ is a condition in $\mathcal{P}_\alpha$ $\leq_\alpha$–extending $p_\epsilon \restriction \alpha$ and $p_{\epsilon'} \restriction \alpha$. It is easy to check that conditions (1)–(4), (6) and (7) hold for $p_{\epsilon,\epsilon'} \restriction \alpha$.

For (5), notice that if $\gamma \in a^\epsilon$ is such that c^ϵ_γ (together with f_γ and b) codes a bit of information of the form '$\xi \in c_{\tilde{\gamma}}$' or '$\xi \notin c_{\tilde{\gamma}}$', for $\tilde{\gamma} < \gamma$, then the way $\mathcal{A}^*(p^\epsilon)$ has been set up, together with the closure property of a^ϵ given by condition (2) for p_ϵ, entails that this bit of information cannot conflict with the information corresponding to $p^{\epsilon'}$. The reason is that, in case of conflict, $\tilde{\gamma}$ would be in $a^\epsilon \cap a^{\epsilon'} = a$, but then $c^\epsilon_{\tilde{\gamma}} = c^{\epsilon'}_{\tilde{\gamma}}$, so there would be no conflict after all.

As to condition (8) for $p_{\epsilon,\epsilon'} \restriction \alpha$, suppose $\gamma \in a^\epsilon \cap \alpha$ is such that there is a minimal $\alpha' < \gamma$ in $\mathcal{C}$ such that $F(\gamma)$ is a $\mathcal{P}_{\alpha'}$–name for a club of ω_1.[15] We want to see that for all $\nu < max(c^\epsilon_\gamma)$, $p_{\epsilon,\epsilon'} \restriction \alpha'$ decides the statement '$\nu \in F(\gamma)$' and that, letting C_γ be the set of $\nu < max(c^\epsilon_\gamma)$ for which $p_{\epsilon,\epsilon'} \restriction \alpha'$ forces $\nu \in F(\gamma)$,

(i) the equality $ot(C^i_\delta \cap^* C_\gamma) = ht(\vec{C}^i)$ holds for every $i < \beta$ and every $\delta \in c^\epsilon_\gamma \cap dom(\vec{C}^i)$, and

(ii) $E_\delta \backslash C_\gamma$ is finite for every $\delta \in dom(\vec{E}) \cap c^\epsilon_\gamma$.

We know by (8) for p_ϵ that $p_\epsilon \restriction \alpha'$ decides '$\nu \in F(\gamma)$'. Since $p_{\epsilon,\epsilon} \restriction \alpha' \leq_{\alpha'} p_\epsilon \restriction \alpha'$ by induction hypothesis, we get the first conclusion. Parts (i) and (ii) of the second conclusion follow by a similar argument again using the induction hypothesis for α'.

Hence, $p_{\epsilon,\epsilon'} \restriction \alpha$ is a condition in $\mathcal{P}_\alpha$, and obviously it extends both $p_\epsilon \restriction \alpha$ and $p_{\epsilon'} \restriction \alpha$. $\square$

[15] The argument when $\gamma \in a^{\epsilon'} \cap \alpha$ is of course the same as neither $\epsilon < \epsilon'$ nor $\epsilon' < \epsilon$ is being assumed here.

Lemma 2.4 *For every $\leq$–decreasing sequence $(p_n)_{n<\omega}$ of $\mathcal{P}$–conditions, if $sup_n \beta^{p_n} \in \omega_1 \backslash S$, then there is a $\mathcal{P}$–condition extending each p_n.*

Proof: Let $p_n = \langle b^n, \vec{E}^n, (c^n_\gamma \,:\, \gamma \in a^n), (\langle \vec{C}^{i,n}, \vec{D}^{i,n}\rangle \,:\, i < \beta^n)\rangle$ for all n. Let $\beta = sup_n \beta^n$, $a = \bigcup_n a^n$, $b = \bigcup_n b^n$ and $\vec{E} = \bigcup_n \vec{E}^n$.

Suppose first that $(\beta_n)_n$ is not eventually constant. As we are about to see, the proof in this case relies on the fact that we are not obliged to put β in the domain of any club–sequence nor obliged to control $f_\gamma(\beta)$ for $\gamma \in a$, as $\beta \notin S$.

Let n be given. For any $i < \beta_n$ let $\vec{C}^i$ and $\vec{D}^i$ be, respectively, the club–sequence $\bigcup_{m \geq n} \vec{C}^{i,m}$ and the club–sequence $\bigcup_{m \geq n} \vec{D}^{i,m}$, and for $\gamma \in a^n$ let $c_\gamma = \bigcup_{m \geq n} c^m_\gamma \cup \{sup_{m \geq n}\, max(c^m_\gamma)\}$. Note that for all n and $\gamma \in a^n$, $sup_{m \geq n}\, max(c^m_\gamma) \notin \bigcup_{m \geq n} c^m_\gamma$ if and only if $(c^m_\gamma)_m$ does not stabilize, and that this happens if and only if $sup_{m \geq n}\, max(c^m_\gamma) = \beta$.

Let $q = \langle b, \vec{E}, (c_\gamma \,:\, \gamma \in a), (\langle \vec{C}^i, \vec{D}^i\rangle \,:\, i < \beta)\rangle$ and let us check by induction on $\alpha \leq \omega_2$, $\alpha \in \mathcal{C}$, that $q \restriction \alpha$ is a condition extending all $p_n \restriction \alpha$: The verification of conditions (1)–(7) for $q \restriction \alpha$ is easy, using the fact that $\beta \notin S$. For condition (5) one uses the way the coding $\mathcal{A}(q \restriction \alpha)$ has been set up (as in the proof of Lemma 2.2).

As to the verification of (8), let $n < \omega$, $\gamma \in a^n \cap \alpha$, and suppose $\alpha' < \gamma$ is minimal such that $F(\gamma)$ is a $\mathcal{P}_{\alpha'}$–name for a club of ω_1. First we want to see, given m and $\nu < max(c^m_\gamma)$, that $q \restriction \alpha'$ decides '$\nu \in F(\gamma)$'. But this is true, by (8) for p_m, since $q \restriction \alpha' \leq_{\alpha'}$–extends $p_m \restriction \alpha'$ by induction hypothesis. Let C_γ be the set of all ordinals $\nu < max(c_\gamma)$ such that $q \restriction \alpha' \Vdash_{\mathcal{P}_{\alpha'}} \nu \in F(\gamma)$. Let us fix $i < \beta$ and $\delta \in c_\gamma \cap dom(\vec{C}^i)$ as in (8.1). To conclude the verification of (8.1), we are going to argue that $ot(C^i_\delta \cap^* C_\gamma) = ht(\vec{C}^i)$.

Case 1: There is an $m \geq n$ such that $\delta \in c^m_\gamma$. In this case, since $C_\gamma \cap\ max(D_\gamma)$ is precisely the set of all $\nu < max(c_\gamma)$ such that $p_m \restriction \alpha' \Vdash_{\mathcal{P}_{\alpha'}}$ $\nu \in F(\gamma)$, by condition (8) for p_m we have that $ot(C^i_\delta \cap^* C_\gamma) = ot(C^{i,m}_\delta \cap^* C_\gamma) = ht(\vec{C}^{i,m}) = ht(\vec{C}^i)$.

Case 2: Otherwise. As we observed, this means that $\delta = \beta$. But then $\delta \notin dom(\vec{C}^i)$ by our construction of $\vec{C}^i$, so this case does not apply.

Condition (8.2) can be checked by similar arguments, using the fact that $\beta \notin dom(\vec{E})$.

We have thus seen that $q \restriction \alpha \in \mathcal{P}_\alpha$, and obviously it extends $p_n \restriction \alpha$ for all n.

The case when $(\beta_n)_n$ has eventually constant value β is easier. For each $\gamma \in a$ let c_γ be now such that $c^n_\gamma = c_\gamma$ for all sufficiently large n (with $\gamma \in a^n$), and let $\langle \vec{C}^i, \vec{D}^i\rangle$ be, for all $i < \beta$, such that $\langle \vec{C}^i_n, \vec{D}^i_n\rangle = \langle \vec{C}^i, \vec{D}^i\rangle$ for all sufficiently large n (with $i < \beta^n$). Note that $\vec{E} = \vec{E}_n$ for all sufficiently large n. Now it is easy to check that the tuple $\langle b, \vec{E}, (c_\gamma : \gamma \in a), (\langle \vec{C}^i, \vec{D}^i\rangle : i < \beta)\rangle$ is a condition q extending all p_n. As before, one uses for this the way the coding $\mathcal{A}(q)$ has been set up. $\square$

We will refer to the property of $\mathcal{P}$ expressed in the above lemma by saying that *$\mathcal{P}$ is $(\omega_1 \backslash S)$–closed.* Also, we will say that a poset $\mathbb{P}$ is *totally S–proper* in case for every large enough θ, every countable $N \preccurlyeq H(\theta)$ containing $\mathbb{P}$ and such that $N \cap \omega_1 \in S$, and every $p \in N \cap \mathbb{P}$ there is a condition extending p and extending, for every dense set $D \subseteq \mathbb{P}$ in N, a condition in $D \cap N$. Before proving that $\mathcal{P}$ is totally S–proper, we will prove the following multi–purpose density lemma.

Lemma 2.5 *Let $p = \langle b, \vec{E}, (c_\gamma : \gamma \in a), (\langle \vec{C}^i, \vec{D}^i\rangle : i < \beta)\rangle$ be a condition in $\mathcal{P}$.*

(a) For every $\beta' < \omega_1$ there is a $\mathcal{P}$–condition p' extending p such that $\beta^{p'} > \beta'$ and such that $(c^{p'}_\gamma : \gamma \in a^{p'}) = (c_\gamma : \gamma \in a)$. Hence, $\mathcal{P}$ is σ–distributive.

(b) For every $\gamma \in [\omega_1, \omega_2)$ there is a $\mathcal{P}$–condition p' extending p such that $\gamma \in a^{p'}$.

(c) For every $\gamma \in a$ and every $\nu < \omega_1$ there is a $\mathcal{P}$–condition p' extending p such that $\nu < max(c^{p'}_\gamma)$ and $c^{p'}_\gamma \cap \nu = c_\gamma \cap \nu$.

Proof: For the first conclusion of (a), simply pick $\beta^* < \omega_1$ such that $\eta_{\beta^*} = \beta^*$ and such that for every $\gamma < \beta^*$, $\beta^* \setminus S$ contains closed subsets of order type $\gamma + 1$, β^* above β' and above $sup(b)$, pick $b^* \subseteq \beta^*$ end-extending b and such that $ot(b^*) = \beta^*$, extend $\vec{E}$ to a ladder system $\vec{E}_*$ on $S \cap (\beta^* + 1)$, and extend $(\langle \vec{C}^i, \vec{D}^i\rangle : i < \beta)$ to $(\langle C^i_*, \vec{D}^i_*\rangle : i < \beta^*)$ in such a way that conditions (6) and (7) in the definition of the forcing hold for $(\langle C^i_*, \vec{D}^i_*\rangle : i < \beta^*)$ and b^*. For this, just let $\langle \vec{C}^i_*, \vec{D}^i_*\rangle = \langle \vec{C}^i, \vec{D}^i\rangle$ if $i < \beta$ and, for $i \in [\beta, \beta^*)$, let

$\vec{C}^i_* = \{(sup(X_i), X_i)\}$ and $\vec{D}^i_* = \{(\gamma, X_i \cap \gamma) : \gamma$ a limit point of $X_i\}$, where X_i is a subset of $\beta^* \setminus (S \cup (i+1))$ of order type η_{ρ_i} closed in $sup(X_i)$, for $(\rho_i)_{i\in[\beta,\beta^*)}$ being the increasing enumeration of $b^* \setminus \beta$ in type $\beta^* \setminus \beta$ (we make also sure that, for $i \neq i'$, $(X_i \cup \{sup(X_i)\}) \cap (X_{i'} \cup \{sup(X_{i'})\}) = \emptyset$). Now it suffices to take $p' = \langle b^*, \vec{E}_*, (c_\gamma : \gamma \in a), (\langle \vec{C}^i_*, \vec{D}^i_*\rangle : i < \beta^*)\rangle$. The second conclusion of (a) follows from the first conclusion together with Lemma 2.4.

For (b), let $\{\gamma_i\}_{i<n}$ (for some $n < \omega$) be the closure of $\{\gamma\}$ under the operation sending an ordinal $\omega_1 \cdot \gamma' + \varsigma$ (with $\gamma' \geq \omega_1$ and $\varsigma < \omega_1$) to γ', let $\beta' < \omega_1$ be such that for all i there is some ρ with $\omega_1 \cdot \rho \leq \gamma_i < \omega_1 \cdot \rho + \beta'$, and let $c_{\gamma'} = \emptyset$ for all $\gamma' \in \{\gamma_i\}_{i<n} \setminus a$.[16] Let $p_* = \langle b^*, \vec{E}_*, (c_\gamma : \gamma \in a), (\langle \vec{C}^i_*, \vec{D}^i_*\rangle : i < \beta^*)\rangle$ be obtained by an application of (a) to p and β'. Now it suffices to set $p' = \langle b^*, \vec{E}_*, (c_\gamma : \gamma \in a \cup \{\gamma_i\}_{i<n}), (\langle \vec{C}^i_*, \vec{D}^i_*\rangle : i < \beta^*)\rangle$.

Finally we prove (c). We start by extending p to a condition p' such that $\beta^{p'} > \nu$. This can be done by conclusion (a). Now we pick a countable elementary substructure N of some $H(\theta)$ containing $\mathcal{P}$ and p' and such that $\nu' := N \cap \omega_1 \notin S$, and we build a decreasing N-generic sequence $(p_n)_{n\in\omega}$ with $p_0 = p'$. Finally we may build a condition q extending all p_n as in the proof of Lemma 2.4, except that now we make sure to set $c^q_\gamma = \bigcup_{n\in\omega} c^{p_n}_\gamma \cup \{\nu'\}$ rather than $c^q_\gamma = \bigcup_{n\in\omega} c^{p_n}_\gamma \cup \{sup_{n\in\omega}\, max(c^{p_n}_\gamma)\}$. In order to argue that q is a condition we only need to show that if $\alpha \in \gamma \cap \mathcal{C}$ is minimal such that $F(\gamma)$ is a $\mathcal{P}_\alpha$–name for a club of ω_1 and $\nu'' < \nu'$, then there is n such that $p_n \restriction \alpha$ decides whether or not ν'' is in $F(\gamma)$. This conclusion follows from the fact that $(p_n \restriction \alpha)_{n\in\omega}$ meets all dense subsets of $\mathcal{P}_\alpha$ in N, together with the fact that $\mathcal{P}_\alpha$ is σ–distributive by (a), and therefore for each $\overline{\nu} < \omega$, the set of conditions in $\mathcal{P}_\alpha$ deciding $F(\gamma) \cap \overline{\nu}$ is dense in $\mathcal{P}_\alpha$. $\square$

Lemma 2.6 *$\mathcal{P}$ is totally S–proper.*

Proof: Suppose N is a countable elementary substructure of some large enough $H(\theta)$ containing $\mathcal{P}$ and such that $\beta := N \cap \omega_1 \in S$ and let $p \in N \cap \mathcal{P}$. We build a decreasing $(N, \mathcal{P})$–generic sequence $(p_n)_{n<\omega}$ of conditions in N extending p, together with a strictly increasing sequence $(\beta_n)_{n<\omega}$ converging to β in such a way that for every n and every $\gamma \in a^{p_n}$, if $\alpha' \in \gamma \cap \mathcal{C}$ is least such that $F(\gamma)$ is a $\mathcal{P}_{\alpha'}$–name for a club subset of ω_1, then for sufficiently large $m < \omega$, $p_{m+1} \restriction \alpha' \Vdash_{\mathcal{P}_{\alpha'}} \beta_m \in F(\gamma)$.

[16]Note that $\{\gamma_i\}_{i<n}$ is indeed finite since $\gamma' < \omega_1 \cdot \gamma' + \varsigma$ for all γ' and ς as above.

These two sequences can be easily built, as follows: Without loss of generality we may assume that N is the union of an $\in$–increasing chain $(N_n)_{n<\omega}$ of elementary substructures of $H(\theta)$ containing p and with $N_n \cap \omega_1 \notin S$. We let $\beta_n = N_n \cap \omega_1$ for each n, and we set $p_0 = p$ and let p_{m+1} be a lower bound in N_{m+1} of an $(N_m, \mathcal{P})$–generic ω–sequence of conditions in N_m extending p_m. Such a p_{m+1} exists by Lemma 2.4 together with the fact that, by Lemma 2.5 (a), $sup_n \beta^{r_n} = \beta_m$ whenever $(r_n)_{n<\omega}$ is an $(N_m, \mathcal{P})$–generic sequence of conditions in N_m.

Let $\vec{E} = (\bigcup_n \vec{E}^{p_n}) \cup \{\langle \beta, (\beta_n)_{n<\omega}\rangle\}$. Also, let b be a countable set end–extending $\bigcup_n b^{p_n}$ and such that for every $\gamma \in \bigcup_n a^{p_n}$, $f_\gamma(\beta) \in b$ if and only if $\gamma \in \bigcup_n \mathcal{A}(p_n)$. This set b exists because for all distinct γ, γ' in $\bigcup_n a^{p_n}$, $f_\gamma(\beta) \geq \beta$ and $f_\gamma(\beta) \neq f_{\gamma'}(\beta)$, which is true since N contains a club of ordinals $\xi < \omega_1$ such that $f_\gamma(\xi) \geq \xi$ and $f_\gamma(\xi) \neq f_{\gamma'}(\xi)$ (by Fact 1.3).

Finally, by arguments as in the proof of Lemma 2.4 (using Lemma 2.5 (c)) it is easy to see that $q = \langle b, \vec{E}, (c_\gamma \,:\, \gamma \in a), (\langle \vec{C}^i, \vec{D}^i\rangle \,:\, i < \beta)\rangle$ is a condition in $\mathcal{P}$ extending all p_n, where $(c_\gamma \,:\, \gamma \in a)$ and $(\langle \vec{C}^i, \vec{D}^i\rangle \,:\, i < \beta)$ are defined from $(p_n)_{n<\omega}$ as in that proof. $\square$

It follows from Lemmas 2.4 and 2.6 that $\mathcal{P}$ is totally proper.

Now let G be $\mathcal{P}$–generic over V and let $b^G = \bigcup\{b^p \,:\, p \in G\}$. For much of the rest of the proof of Theorem 1.1 we are going to work in $V[G]$.

Let $C^G_\gamma = \bigcup\{c^p_\gamma \,:\, p \in G, \gamma \in a^p\}$ for all uncountable $\gamma < \omega_2$. By Lemma 2.4 (or Lemma 2.6) we know that $\omega_1^V = \omega_1^{V[G]}$, and therefore each C^G_γ is a club of ω_1 (by Lemma 2.5 (c) and by the definition of $\leq$). Let also $\vec{E}^G = \bigcup\{\vec{E}^p \,:\, p \in G\}$, and let $\vec{C}^{i,G} = \bigcup\{\vec{C}^{i,p} \,:\, p \in G, i < \beta^p\}$ and $\vec{D}^{i,G} = \bigcup\{\vec{D}^{i,p} \,:\, p \in G, i < \beta^p\}$ for all $i < \omega_1$. By the definitions of $\mathcal{P}$ and $\leq$ (and by Lemma 2.5 (a)) we also have that $\vec{E}^G$ is a ladder system with domain S, that each $\vec{C}^{i,G}$ is a coherent club–sequence with nonempty domain disjoint from S, and that b^G is the set of all $\xi < \omega_1$ for which there is some i with $ht(\vec{C}^{i,G}) = \eta_\xi$. Finally, let $\mathcal{A}^G = \bigcup\{\mathcal{A}(p) \,:\, p \in G\}$.

Lemma 2.7 *In $V[G]$ there is a well–order R of $H(\omega_2)^{V[G]}$ definable over the structure $\langle H(\omega_2), \in, [S]\rangle^{V[G]}$ by a formula (in the corresponding language) with b^G as parameter.*

Proof: To start with notice that S is stationary in $V[G]$ by Lemma 2.6. Let $\gamma < \omega_2$ be uncountable. By Lemma 2.5 (b) we know that there is some

$p(\gamma) \in G$ such that $\gamma \in a^{p(\gamma)}$ and such that (by the definition of the coding $\mathcal{A}(p)$) $\gamma \in \mathcal{A}(p(\gamma))$ if and only if $\gamma \in \mathcal{A}(p')$ for all (equivalently, for some) condition p' extending $p(\gamma)$. We thus have that $\mathcal{A}^G$ is definable from b^G and $[S]$ as the set of uncountable $\gamma < \omega_2$ for which there are $S' \in [S]$ and a canonical function g for γ such that $\{\nu < \omega_1 : \nu \in S' \to g(\nu) \in b^G\}$ contains a club.[17] The existence of this club (for f_γ) will be witnessed by C^G_γ.

Hence, we have that the original set $A \subseteq \omega_2$ is definable in $H(\omega_2)^{V[G]}$ from b^G and $[S]$, and therefore so is the well–order $\mathcal{W}$ of $H(\omega_2)^V$, as well as S, the sequence $(\pi_\gamma : 0 < \gamma < \omega_2)$, and the bookkeeping function F.[18] From this we also have that $\mathcal{P}$ is definable from b^G over $\langle H(\omega_2)^{V[G]}, \in, [S]^{V[G]}\rangle$.

Also, by the $\aleph_2$–chain condition of $\mathcal{P}$ we may look, for every $x \in H(\omega_2)^{V[G]}$, at the $\mathcal{W}$–least $\mathcal{P}$–name $\dot{x}$ such that $\dot{x}_G = x$. Identifying x with the $\mathcal{W}$–rank of $\dot{x}$ gives then a well–order R of $H(\omega_2)^{V[G]}$ (this way of defining a well–order of $H(\omega_2)^{V[G]}$ was anticipated in the introduction). Hence, the proof of Lemma 2.7 will be finished if we can show that G is definable from b^G and $[S]$, and for this it is enough to show that G is definable from $\mathcal{A}^G$.

To see that G is definable from $\mathcal{A}^G$, note that all the objects b^G, $\vec{E}^G$, $\langle \vec{C}^{i,G} : i < \omega_1\rangle$, $\langle \vec{D}^{i,G} : i < \omega_1\rangle$ and $\langle C^G_\gamma : \omega_1 \leq \gamma < \omega_2\rangle$ are definable from $\mathcal{A}^G$, and that if $p = \langle b, \vec{E}, (c_\gamma : \gamma \in a), (\langle \vec{C}^i, \vec{D}^i\rangle : i < \beta)\rangle \in \mathcal{P}$, then $p \in G$ if and only if

(i) $b^G \cap sup(b) = b$ and $\vec{E} = \vec{E}^G \restriction (\beta + 1)$,

(ii) for all $\gamma \in a$, $C^G_\gamma \cap (\beta + 1) = c_\gamma$, and

(iii) $\vec{C}^{i,G} \restriction \beta + 1 = \vec{C}^i$ and $\vec{D}^{i,G} \restriction \beta + 1 = \vec{D}^i$ for all $i < \beta$.

□

It is worth observing that, as a consequence of the $\aleph_2$–chain condition of $\mathcal{P}$, the subset b^G of ω_1 is actually added by $G \cap \mathcal{P}_\alpha$ for some $\alpha < \omega_2$ in $\mathcal{C}$. As we have seen, G can be read off from b^G and $[S]$ in $V[G]$. But this does not mean that knowledge of the restriction of the generic filter to $\mathcal{P}_\alpha$ implies complete knowledge of the whole generic filter. The reason is that, although it is true that the decoding set b^G has already been given by $G \cap \mathcal{P}_\alpha$, it is in general

[17]By the definition of $\mathcal{P}$ we actually get that if $\gamma \in [\omega_1, \omega_2) \backslash \mathcal{A}^G$ and g is a canonical function for γ, then $\{\nu < \omega_1 : \nu \in S' \to g(\nu) \notin b^G\}$ contains a club for any such S'.

[18]Since these objects can be decoded in $H(\omega_2)^V = L_{\omega_2}[A]$ from A.

not the case, for a given $\gamma \in [\alpha, \omega_2)$, that $G \cap \mathcal{P}_\alpha$ decides whether or not b^G codes "$\gamma \in \mathcal{A}^{\dot{G}}$". More precisely, the fact whether there is a club witnessing "$\gamma \in \mathcal{A}^{\dot{G}}$" or whether there is a club witnessing "$\gamma \notin \mathcal{A}^{\dot{G}}$" typically will not be decided by $G \cap \mathcal{P}_\alpha$. This means that the quotient forcing $\mathcal{P}/(G \cap \mathcal{P}_\alpha)$[19] will be responsible for shooting clubs through the relevant sets, in such a way that in the end b^G (together with $[S]$) decodes the generic filter in the intended way.[20]

3 Defining b^G and $[S]^{V[G]}$

The purpose of this final section is to show that b^G and $[S]^{V[G]}$ are lightface definable in $\langle H(\omega_2), \in\rangle^{V[G]}$. By Lemma 2.7 this will conclude the proof of Theorem 1.1.

If it exists, let K_0 be the unique $K \in \mathcal{P}(\omega_1)/NS_{\omega_1}$ such that

(a) for every (equivalently, for some) $T \in K$ there is a ladder system of the form $\langle E_\delta : \delta \in Lim(\omega_1) \cap T\rangle$ with the property that for every club $C \subseteq \omega_1$ there are club–many δ such that $|E_\delta \backslash C| < \aleph_0$ if $\delta \in T$, and

(b) for every stationary $T \subseteq \omega_1$ and every ladder system $\langle E_\delta : \delta \in T\rangle$, if $T \backslash T'$ is stationary for some (equivalently, for any) $T' \in K$, then there is a club $C \subseteq \omega_1$ such that $|E_\delta \backslash C| = \aleph_0$ for stationary many $\delta \in T$.

If K_0 exists, then it is obviously lightface definable in $\langle H(\omega_2), \in\rangle$.

Also, if K_0 exists, let I be the set of all $\xi < \omega_1$ with the property that there is a coherent strongly type-guessing club–sequence $\vec{C}$ with stationary domain disjoint from T for some $T \in K_0$ and such that $ht(\vec{C}) = \eta_\xi$. This set is also definable in $\langle H(\omega_2), \in\rangle$ by a formula without parameters (if K_0 exists).

We are going to prove that K_0 exists, that $K_0 = [S]^{V[G]}$, and that $I = b^G$. The proof will be an adaptation to the present construction of an analogous proof in [As2]. We start with the following lemma.

Lemma 3.1 *Every $\vec{C}^{i^*,G}$ is a coherent strongly type-guessing club–sequence with stationary domain disjoint from S.*

[19]Where $\mathcal{P}/(G \cap \mathcal{P}_\alpha)$ is the suborder of $\mathcal{P}$ consisting of those $q \in \mathcal{P}$ such that $q \restriction \alpha \in G \cap \mathcal{P}_\alpha$ (see also the proof of Lemma 3.2).

[20]These clubs will in fact be the C^G_γ's for $\gamma \geq \alpha$.

Proof: Every $\vec{C}^{i^*,p}$ (for $p \in \mathcal{P}$ and $i^* < \beta^p$) is a coherent club–sequence with domain disjoint from S, and therefore so is $\vec{C}^{i^*,G}$.

To see that $\vec{C}^{i^*,G}$ has stationary domain, let $\dot{C}$ be a $\mathcal{P}$–name for a club of ω_1 and let $p \in G$ with $i^* < \beta^p$. Stepping back to V, it will suffice to see that there is a condition $q \in \mathcal{P}$ extending p and forcing $dom(\vec{C}^{i^*,q}) \cap \dot{C} \neq \emptyset$.

Let $\eta = ht(\vec{C}^{i^*,p})$ and let us fix a $\subseteq$–continuous chain $(N_\sigma)_{\sigma\leq\eta+1}$ of countable elementary substructures of $\langle H(\theta), \in, \Delta\rangle$, for some large enough θ and some well–order Δ of $H(\theta)$, containing $\dot{C}$ and p, and with $\delta_\sigma := N_\sigma \cap \omega_1 \notin S$ and $(N_{\sigma'})_{\sigma'\leq\sigma} \in N_{\sigma+1}$ for all $\sigma \leq \eta$. Such a chain $(N_\sigma)_{\sigma\leq\eta+1}$ can be found since for any club $D \subseteq \omega_1$, $D \setminus S$ is stationary and therefore contains arbitrarily long closed countable subsets. We aim to build a decreasing sequence $(q_\sigma)_{\sigma\leq\eta}$ of conditions extending p in such a way that, for all σ,

(i) $q_\sigma \in N_{\sigma+1}$ is an $(N_\sigma, \mathcal{P})$–generic condition and $\beta^{q_\sigma} = \delta_\sigma$,

(ii) if $\sigma < \eta$ is a limit ordinal, then $\delta_\sigma \in dom(\vec{D}^{i^*,q_\sigma})$ and $D^{i^*,q_\sigma}_{\delta_\sigma} = \{\delta_{\sigma'} : \sigma' < \sigma\}$, and

(iii) $\delta_\eta \in dom(\vec{C}^{i^*,q_\eta})$ and $C^{i^*,q_\eta}_{\delta_\eta} = \{\delta_\sigma : \sigma < \eta\}$.

This is enough: Since q_η is $(N_\eta, \mathcal{P})$–generic, it forces $\delta_\eta \in \dot{C}$. Hence, q_η is a condition extending p and forces $\dot{C} \cap dom(\vec{C}^{i^*,q_\eta}) \neq \emptyset$.

The construction of $(q_\sigma)_{\sigma\leq\eta}$ is quite standard. Given σ and assuming $q_{\sigma'}$ has been built for all $\sigma' < \sigma$, we can find q_σ in the following way.

Suppose σ is 0 or a successor ordinal. Let $(p_n)_{n<\omega}$ be the Δ–least $(N_\sigma, \mathcal{P})$–generic sequence of conditions extending p (if $\sigma = 0$) or q_{σ_0} (if $\sigma = \sigma_0+1$) and let q_σ be obtained from $(p_n)_{n<\omega}$ by an application of the proof of Lemma 2.4 (using Lemma 2.5 (a)). Certainly this q_σ is in $N_{\sigma+1}$ and is an $(N_\sigma, \mathcal{P})$–generic condition. Further, by Lemma 2.5 (a) we have that $\beta^{q_\sigma} = \delta_\sigma$.

If $\sigma < \eta$ is a limit ordinal, we can let q_σ be obtained from $(q_{\sigma'})_{\sigma'<\sigma}$ as in the proof of Lemma 2.4 by putting δ_σ in $dom(\vec{D}^{i^*,q_\sigma})$, but not in $dom(\vec{C}^{i^*,q_\sigma})$, and making $D^{i^*,q_\sigma}_{\delta_\sigma} = \{\delta_{\sigma'} : \sigma' < \sigma\}$. Again, $\beta^{q_\sigma} = \delta_\sigma$. In this case, the verification that q_σ is a condition in $\mathcal{P}$ and that it extends $q_{\sigma'}$ for all $\sigma' < \sigma$ is exactly as in the proof of Lemma 2.4. As in that proof, we use the fact that $\delta_\sigma \notin dom(\vec{C}^{i,q_\sigma})$ for all $i < \delta_\sigma$. Also, clearly q_σ is in $N_{\sigma+1}$, and it is $(N_\sigma, \mathcal{P})$–generic because $N_\sigma = \bigcup_{\sigma'<\sigma} N_{\sigma'}$ and because each $q_{\sigma'}$ is $(N_{\sigma'}, \mathcal{P})$–generic.

If $\sigma = \eta$, we build q_η from $(q_\sigma)_{\sigma<\eta}$ again as in the proof of Lemma 2.4, but this time putting δ_η in $dom(\vec{C}^{i^*,q_\eta})$ and making $C^{i^*,q_\eta}_{\delta_\eta} = \{\delta_\sigma : \sigma < \eta\}$.

As usual we prove by induction on $\alpha \leq \omega_2$, $\alpha \in \mathcal{C}$, that $q_\eta \restriction \alpha$ is a $\mathcal{P}_\alpha$–condition extending all $q_\sigma \restriction \alpha$ $(\sigma < \eta)$. The proof proceeds as in the proof of Lemma 2.4. The only problem could come up in the verification of property (8) for q_η, and more specifically of (8.1) for some γ, i and δ. Let γ, α', C_γ, i and δ be as in the verification of (8.1) in the proof of Lemma 2.4. By arguing as in that proof, the case when either there is some $\sigma < \eta$ such that $\delta \in c_\gamma^{q_\sigma}$ or $i \neq i^*$ goes through easily.

The only nontrivial case is when $i = i^*$ and $\delta = \delta_\eta$. We want to prove that $ot(C_{\delta_\eta}^{i^*,q_\eta} \cap^* C_\gamma) = \eta$. In that case we argue that, since $F(\gamma)$ is a $\mathcal{P}_{\alpha'}$–name for a club of ω_1 and each q_σ $(\sigma < \eta)$ is $(N_\sigma, \mathcal{P})$–generic (so in particular $q_\sigma \restriction \alpha'$ is $(N_\sigma, \mathcal{P}_{\alpha'})$–generic if $\gamma \in a^{q_\sigma}$ by the fact that $\mathcal{P}_{\alpha'}$ is a complete suborder of $\mathcal{P}$ and that $q_\sigma \restriction \alpha'$ is the reduction of q_σ to $\mathcal{P}_{\alpha'}$), $q_\eta \restriction \alpha'$ forces $\delta_\sigma \in F(\gamma)$ for all $\sigma < \eta$ such that $\gamma \in a^{q_\sigma}$ (note that $q_\eta \restriction \alpha'$ extends $q_\sigma \restriction \alpha'$ by induction hypothesis). Hence, we have in fact that a final segment of $C_{\delta_\eta}^{i^*,q_\eta}$ is contained in C_γ.

Now that we have that q_η is a condition in $\mathcal{P}$, checking that it extends all q_σ, for $\sigma < \eta$, is easy.

We still need to check that $\vec{C}^{i^*,G}$ is strongly type-guessing. For this, let us remain in V, let $\dot{C}$ be a $\mathcal{P}$–name there for a club and let $p \in \mathcal{P}$. By the $\aleph_2$–chain condition of $\mathcal{P}$ we know that there is some $\alpha < \omega_2$ in $\mathcal{C}$ such that $\dot{C}$ is in fact a $\mathcal{P}_\alpha$–name for a club, and of course we may assume $\dot{C} \in H(\omega_2)$. Suppose also that α is minimal with the above property. Since F is a bookkeeping function, we can find $\gamma > \alpha$ such that $F(\gamma) = \dot{C}$.

By Lemma 2.5 (b) we may extend p to a condition p^* such that $\gamma \in a^{p^*}$. But by condition (8) in the definition of $\mathcal{P}$ we know that every $p' \in \mathcal{P}$ extending p^* is such that $p' \restriction \alpha$ decides, for every $\nu < max(c_\gamma^{p'})$, whether or not ν is in $\dot{C}$. Furthermore, for every such p', letting C_γ be the set of $\nu < max(c_\gamma^{p'})$ such that $p' \restriction \alpha \Vdash_{\mathcal{P}_\alpha} \nu \in \dot{C}$, we know that every $\delta \in c_\gamma^{p'} \cap dom(\vec{C}^{i^*,p'})$ is such that $ot(C_\delta^{i^*,p'} \cap^* C_\gamma) = \eta$. That is, p' forces $ot(C_\delta^{i^*,\dot{G}} \cap^* \dot{C}) = \eta$ for every such δ. This shows that, in $V[G]$, $ot(C_\delta^{i^*,G} \cap^* \dot{C}_G) = \eta$ for all $\delta \in C_\gamma^G \cap dom(\vec{C}^{i^*,G})$. Hence, C_γ^G is a witness for $\dot{C}_G$ to the fact that $\vec{C}^{i^*,G}$ is strongly type-guessing.
$\square$

It remains to see that if $\eta < \omega_1$ is either ω or an infinite perfect ordinal such that $\eta \notin \{\eta_\xi : \xi \in b^G\}$, then in $V[G]$ there is no coherent strongly type-guessing club–sequence with stationary domain disjoint from S, and of height η. This will finish the proof of Theorem 1.1 as the case $\eta = \omega$ ensures the definability of $[S]$ via conditions (a), (b) stated at the start of this section, and the case $\eta \neq \omega$ ensures the definability of b^G.

Lemma 3.2 *In $V[G]$, let $\vec{C}$ be a coherent club–sequence with stationary domain, suppose $dom(\vec{C}) \cap S = \emptyset$, and suppose $\eta = ht(\vec{C})$ is either ω or a nonzero perfect ordinal such that $\eta \notin \{\eta_\xi \,:\, \xi \in b^G\}$. Then there is some $\alpha < \omega_2$ such that $\{\delta \in dom(\vec{C}) \,:\, ot(C_\delta \cap^* C^G_\gamma) < \eta\}$ is stationary for all $\gamma \in (\alpha, \omega_2)$.*[21]

Proof: Modulo notational changes, most of the proof of this lemma (but not all of it) will be an almost exact copy of a corresponding proof in [As2]. As in the proof of Lemma 3.1, using the $\aleph_2$–chain condition of $\mathcal{P}$ in V we may fix some $\alpha < \omega_2$ in $\mathcal{C}$ such that $\vec{C} = \tau_G$, where $\tau \in H(\omega_2)^V$ is a $\mathcal{P}_\alpha$–name for a coherent club–sequence of height η, and some $\bar{p} \in \mathcal{P}_\alpha \cap G$ forcing (in $\mathcal{P}$) that $\eta \notin \{\eta_\xi \,:\, \xi \in b^{\dot{G}}\}$[22] (if η is perfect) and that $dom(\tau)$ is a stationary set disjoint from S.

Let $\gamma > \alpha$, $\gamma < \omega_2$, and let $\dot{C}$ be a $\mathcal{P}$–name for a club of ω_1. Let also p' be a condition extending $\bar{p}$. By Lemma 2.5 (b), we may assume that $\gamma \in a^{p'}$. It will suffice to find a condition $q \leq p'$ and some $\delta \in c^q_\gamma$ such that $q \Vdash_{\mathcal{P}} \delta \in \dot{C} \cap dom(\tau)$ and such that $q \Vdash_{\mathcal{P}} ot(\tau_\delta \cap^* c^q_\gamma) < \eta$ (where of course τ_δ is a name for $\tau(\delta)$).

Let G^* be $\mathcal{P}_\alpha$–generic over V with $p' \restriction \alpha \in G^*$. Note that, since $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}$, every generic filter G' for $\mathcal{P}/G^*$ over $V[G^*]$[23] is such that $G' \cap \mathcal{P}_\alpha = G^*$ and is $\mathcal{P}$–generic over V as a filter of $\mathcal{P}$, and that, conversely, every $\mathcal{P}$–generic filter G' over V with $G' \cap \mathcal{P}_\alpha = G^*$ is $\mathcal{P}/G^*$–generic over $V[G^*]$.

We will temporarily work in $V[G^*]$. Let $\vec{C}^* = \tau_{G^*}$ and let $\vec{C}^{*\,i} = \bigcup\{\vec{C}^{i,p} \,:\, p \in G^*,\, i < \beta^p\}$ for all $i < \omega_1$. Let θ be a sufficiently large regular cardinal and let Δ be a well–order of $H(\theta)^{V[G^*]}$. Let $(N_\sigma)_{\sigma<\omega_1}$ be a $\subseteq$–continuous chain of countable elementary substructures of $\langle H(\theta)^{V[G^*]}, \in, \Delta\rangle$ containing everything relevant and such that $(N_{\sigma'})_{\sigma'\leq\sigma} \in N_{\sigma+1}$ for all $\sigma < \omega_1$. Let $\delta_\sigma = N_\sigma \cap \omega_1$ for all $\sigma < \omega_1$ and let $D_0 = \{\delta_\sigma \,:\, \sigma < \omega_1\}$.

[21] The proof also shows, for $\vec{C}$ as in the hypothesis, that if the set $dom(\vec{C}) \setminus \bigcup\{dom(\vec{C}^{i,G}) \,:\, ht(\vec{C}^{i,G}) < \eta\}$ is stationary, then in fact $\{\delta \in dom(\vec{C}) \,:\, sup(C_\delta \cap^* C_\gamma) < \eta\}$ is stationary for co-finally many γ in ω_2.

[22] That is, with $ht(\vec{C}^{i,\bar{p}}) > \eta$ for some $i < \beta^{\bar{p}}$ and such that $\eta \notin \{ht(\vec{C}^{i,\bar{p}}) \,:\, i < \beta^{\bar{p}}\}$.

[23] Where $\mathcal{P}/G^*$ is the suborder of $\mathcal{P}$ consisting of those conditions q such that $q \restriction \alpha \in G^*$.

Claim 3.2.1 *There is a limit ordinal $\overline{\sigma} < \omega_1$ with $\delta_{\overline{\sigma}} \in dom(\vec{C}^*)$, $\eta < \delta_{\overline{\sigma}}$, with $(D_0 \cap \delta_{\overline{\sigma}}) \backslash (C^*_{\delta_{\overline{\sigma}}} \cup \bigcup_{i<\omega_1} dom(\vec{C}^{*\,i}) \cup S)$ unbounded in $\delta_{\overline{\sigma}}$, and such that $ot(C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) = ot(C^{*\,i}_{\delta_{\overline{\sigma}}})$ in case $i < \omega_1$ is such that $\delta_{\overline{\sigma}} \in dom(\vec{C}^{*\,i})$.*

Proof: Note that, by Lemma 3.1,

$$\mathcal{X} := \{\delta < \omega_1 \,:\, (\forall i)\,(\delta \in dom(\vec{C}^{*\,i}) \rightarrow ot(C^{*\,i}_\delta \cap^* D_0) = ot(C^{*\,i}_\delta))\}$$

is forced by $\mathcal{P}/G^*$ to contain a club.[24] Also, $X := D_0 \backslash (\bigcup_{i<\omega_1} dom(\vec{C}^{*\,i}) \cup S)$ is clearly unbounded in ω_1 (by an argument as in the proof of Lemma 2.4, for example), and therefore the set D_1 of $\delta \in D_0$ such that $rnk_X(\delta) > \eta$ is a club. Since $dom(\vec{C}^*)$ is forced by $(\mathcal{P}/G^*) \restriction p'$ to be stationary, it must have stationary intersection with $\mathcal{X} \cap D_1$. We may take $\overline{\sigma}$ so that $\delta_{\overline{\sigma}} > \eta$ is in this intersection. This is enough since then $(D_0 \cap \delta_{\overline{\sigma}}) \backslash (C^*_{\delta_{\overline{\sigma}}} \cup \bigcup_{i<\omega_1} dom(\vec{C}^{*\,i}) \cup S)$ must be unbounded in $\delta_{\overline{\sigma}}$ as $rnk_X(\delta_{\overline{\sigma}}) > \eta$ and $ot(C^*_{\delta_{\overline{\sigma}}}) = \eta$.[25] $\square$

Let $\overline{\sigma}$ be as given by Claim 3.2.1. We will eventually find, in V, a condition q extending p' and forcing both $\delta_{\overline{\sigma}} \in \dot{C} \cap dom(\tau)$ and $ot(\tau_{\delta_{\overline{\sigma}}} \cap^* c^q_\gamma) < \eta$. Let us move back to $V[G^*]$.

Case 1: There is a (unique) i such that $\delta_{\overline{\sigma}} \in dom(\vec{C}^{*\,i})$ and $\eta < ht(\vec{C}^{*\,i})$.

Let x be the set of ordinals in $C^*_{\delta_{\overline{\sigma}}}$ above $min(C^*_{\delta_{\overline{\sigma}}})$ which are not limit points of $C^*_{\delta_{\overline{\sigma}}}$ and let $(t_k)_{k<\omega}$ be an increasing sequence converging to the height of $\vec{C}^{*\,i}$. Note that, since $ht(\vec{C}^{*\,i})$ is a perfect ordinal above η, for every $k < \omega$ there are unboundedly many ordinals δ in x such that $ot((C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) \cap J) \geq t_k$, where J is the interval $(max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ \delta)$. Otherwise $ht(\vec{C}^{*\,i})$ would be bounded by $t_k \cdot \eta$ for some k, which would contradict the fact that $ht(\vec{C}^{*\,i})$ is perfect and that t_k and η are less than $ht(\vec{C}^{*\,i})$. Since every ordinal in $C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0$ is of the form δ_σ for some $\sigma < \overline{\sigma}$, it follows that we may find a strictly increasing sequence $(\sigma_k)_{k<\omega}$ converging to $\overline{\sigma}$ such that $\delta_{\sigma_0} > i$ and such that

$$ot((C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) \cap (max(C^*_{\delta_{\overline{\sigma}}} \cap \overline{\delta}_k),\ \overline{\delta}_k)) > t_{k+1}$$

[24] This is because $dom(\vec{C}^{*\,i}) \cap (i+1) = \emptyset$ for all i. For every i there is, in $V^{\mathcal{P}/G^*}$, a club $C_i \subseteq \{\delta < \omega_1 \,:\, \delta \in dom(\vec{C}^{*\,i}) \rightarrow ot(C^{*\,i}_\delta \cap^* D_0) = ot(C^{*\,i}_\delta)\}$. The required club can be taken to be the diagonal intersection $\Delta_{i<\omega_1} C_i$.

[25] Note that $Z \backslash Y$ is unbounded in $sup(Z)$ whenever Z and Y are sets of ordinals with $rnk_Z(sup(Z)) > ot(Y)$.

for $\overline{\delta}_k := min(x\backslash\delta_{\sigma_k})$ (for all k). It follows that there is a function h defined on x such that $max(C^*_{\delta_{\overline{\sigma}}} \cap \delta) \leq h(\delta) < \delta$ for every $\delta \in x$ and such that

$$ot((C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) \cap \bigcup_{\delta \in x \cap \delta'} (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ h(\delta))) \geq t_k$$

whenever $k \geq 1$, $\delta' \in x$ and $\delta_{\sigma_k} \leq \delta'$. We may assume that h is defined by letting $h(\delta)$ be, for every $k < \omega$ and every $\delta \in x \cap \delta_{\sigma_k}$, the least ϵ in $(max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ \delta)$ such that $ot((C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) \cap (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ \epsilon)) \geq \overline{t}$, where $\overline{t}$ is the maximal member t of the set $\{0\} \cup \{t_{k'}\ :\ k' \leq k\}$ for which there is some ϵ, $max(C^*_{\delta_{\overline{\sigma}}} \cap \delta) < \epsilon < \delta$, such that $ot((C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) \cap (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ \epsilon)) \geq t$. Note that, since $\vec{C}^{*\,i}$ and $\vec{C}^*$ are coherent sequences, $h \restriction x \cap \delta_\sigma \in N_{\sigma+1}$ for every $\sigma < \overline{\sigma}$. The reason is that $N_{\sigma+1}$ contains all initial segments of $\vec{C}^{*\,i}$ and of $\vec{C}^*$ of length less than $\delta_{\sigma+1}$, the sequence $(N_{\sigma'})_{\sigma' \leq \sigma}$, and the finite set $\{0\} \cup \{t_{k'}\ :\ k' \leq k\}$ (for a relevant k).

Let

$$\Sigma = \{\sigma < \overline{\sigma}\ :\ \delta_\sigma \in (C^{*\,i}_{\delta_{\overline{\sigma}}} \cap^* D_0) \cap \bigcup_{\delta \in x} (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ h(\delta))\}$$

and let $\overline{\Sigma}$ be the closure of Σ. Note that $ot(\Sigma) = ht(\vec{C}^{*\,i})$ and that Σ does not contain any of its accumulation points. In fact, if $\sigma \in \Sigma$, then δ_σ is a member of $C^{*\,i}_{\delta_{\overline{\sigma}}}$ which is not a limit point of $C^{*\,i}_{\delta_{\overline{\sigma}}}$ (by the definition of $\cap^*$).

Now we can build by recursion a decreasing sequence $(p_\sigma)_{\sigma \in \overline{\Sigma} \cap \overline{\sigma}}$ of conditions in $\mathcal{P}/G^*$ extending p' such that the following conditions hold for each $\sigma \in \overline{\Sigma}$.

(i) $p_\sigma \in N_{\sigma+1}$.

(ii) if $\sigma \in \Sigma$, then p_σ is a lower bound of an ω–sequence $(q^\sigma_n)_{n<\omega}$ of conditions in N_σ and forces $\delta_\sigma \in \dot{C}$.

(iii) Given any two $\sigma_0 < \sigma_1$ in $\overline{\Sigma}$ and any $\gamma' \in a^{p_{\sigma_0}}$, if there is a minimal $\overline{\alpha} < \gamma'$ in $\mathcal{C}$ such that $F(\gamma')$ is a $\mathcal{P}_{\overline{\alpha}}$–name for a club of ω_1, then $p_{\sigma_1} \restriction \overline{\alpha}$ forces $\delta_{\sigma_1} \in F(\gamma')$.

(iv) If $\sigma \in \Sigma$ and $\delta \in x$ is such that $\delta_\sigma \in (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ h(\delta))$, then $max(C^*_{\delta_{\overline{\sigma}}} \cap \delta) < min(c^{p_\sigma}_\gamma \backslash (c^{p'}_\gamma \cup (sup\{\delta_{\sigma'}\ :\ \sigma' \in \Sigma \cap \sigma\} + 1)))$.

We want to show first, given any $\sigma \in \Sigma$ and assuming $p_{\sigma'}$ has been built for all $\sigma' \in \overline{\Sigma} \cap \sigma$, how to find p_σ in $N_{\sigma+1}$ so that (ii) and (iv) hold about p_σ, and so that (iii) holds about $\sigma < \sigma^\circ$ with $\sigma^\circ := min(\Sigma \backslash (\sigma+1))$. Moreover we want to show how to perform the construction in a uniform definable way.

The proof of the following claim is quite standard. It appears in [A-S] in a different but similar context.

Claim 3.2.2 *For every dense set $\mathcal{D} \subseteq \mathcal{P}/G^*$, $q \in \mathcal{P}/G^*$, and every $\rho \in a^q \backslash \alpha$ there is a club $C \subseteq \omega_1$ with the property that for every $\delta \in C$ and every $\delta' < \delta$ there is a condition $q' \in \mathcal{D}$ extending q with $\beta^{q'} < \delta$ and such that $c_\rho^{q'} \backslash c_\rho^q \subseteq (\delta',\ \delta)$.*

Proof: By Lemma 2.5 (c) we may take this club to be $\{M_j \cap \omega_1 \ :\ j < \omega_1\}$ for an $\in$–chain $(M_j)_{j<\omega_1}$ of elementary substructures of $H(\chi)$ (for some large enough χ) containing $\mathcal{D}$, q and G^*: Fix such a j and fix $\delta' < M_j \cap \omega_1$ above $max(c_\rho)$. By Lemma 2.5 (c), there is a condition $\overline{q} \in N_j$ in $\mathcal{P}/G^*$ such that $c_\rho^{\overline{q}} \cap \delta' = c_\rho$ and $max(c_\rho^{\overline{q}}) > \delta'$. (Given any condition $r \in \mathcal{P}_\alpha \cap M_j$ extending $q \restriction \alpha$, extend $r \wedge q$ to a condition $\overline{q}$ as above using Lemma 2.5 (c). The fact that this is always possible implies by density that there is $\overline{r} \in G^* \cap M_j$ with the property that there is some $\overline{q}$ as above and such that $\overline{q} \restriction \alpha = \overline{r}$.) Now we may extend $\overline{q}$ to a condition q' in $\mathcal{D} \cap N_j$ $\square$

Now, p_σ can be built as a lower bound in $N_{\sigma+1} \cap \mathcal{P}/G^*$ of a decreasing sequence $(q_n^\sigma)_{n<\omega}$ of $\mathcal{P}/G^*$–conditions in N_σ extending $p_{max(\overline{\Sigma}\cap\sigma)}$ (if $\overline{\Sigma} \cap \sigma \neq \emptyset$) or extending p' (if σ is the first member of Σ) such that, for a suitable sequence $(\mathcal{D}_n)_{n<\omega}$ of dense subsets of $\mathcal{P}/G^*$, all of them belonging to N_σ,

(a) $q_n^\sigma \in \mathcal{D}_n$ for all n,

(b) $sup_{n \geq m} max(c_\rho^{q_n^\sigma}) = \delta_\sigma$ for every $m < \omega$ and every $\rho \in a^{q_m^\sigma}$, and

(c) if $\delta \in x$ is such that $\delta_\sigma \in (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta),\ h(\delta))$, then q_0^σ puts some ordinal above $max(C^*_{\delta_{\overline{\sigma}}} \cap \delta)$, but no new ordinal below $max(C^*_{\delta_{\overline{\sigma}}} \cap \delta)+1$, inside $c_\gamma^{q_0^\sigma}$.[26]

[26] This means that $c_\gamma^{q_0^\sigma} \backslash (c_\gamma^r \cup (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta) + 1)) \neq \emptyset$ and $c_\gamma^{q_0^\sigma} \cap (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta) + 1) = c_\gamma^r \cap (max(C^*_{\delta_{\overline{\sigma}}} \cap \delta) + 1)$, where $r = p_{max(\overline{\Sigma}\cap\sigma)}$ if $\overline{\Sigma} \cap \sigma \neq \emptyset$ and $r = p'$ if σ is the first member of Σ.

Let $\tilde{S} = \omega_1 \backslash (S \cup \bigcup_{i<\omega_1} dom(\vec{C}^{*\,i}))$. The sequence $(q^\sigma_n)_{n<\omega}$ has a lower bound because $\delta_\sigma \in \tilde{S}$ (by condition (7) in the definition of $\mathcal{P}$, since $\delta_\sigma \in range(\vec{C}^{*\,i})$) and because $\mathcal{P}/G^*$ is *sufficiently* $\tilde{S}$*–closed* (in the natural sense that if $(r_n)_{n<\omega}$ is a decreasing sequence of conditions for which there is some $\delta \in \tilde{S}$ such that $sup_{n\in\omega}\beta^{r_n} = \delta$ and $sup_{n\geq m} max(c^{r_n}_\gamma) = \delta$ for every $m < \omega$ and every $\gamma \in a^{r_m}$, then $(r_n)_{n<\omega}$ has a lower bound in $\mathcal{P}/G^*$), and that $\mathcal{P}/G^*$ is sufficiently $\tilde{S}$–closed in the above sense can be seen as in the proof of Lemma 2.4. Specifically, we let r be obtained from $(r_n)_n$ as in the proof of Lemma 2.4.[27] By the proof of Lemma 2.4, r is a $\mathcal{P}$–condition extending all r_n, so it remains to see $r \restriction \alpha \in G^*$. Suppose towards a contradiction that $s \in G^*$ forces that $r \restriction \alpha \notin G^*$, and assume without loss of generality that $s \in G^*$ is $\mathcal{P}_\alpha$–incompatible with the $\mathcal{P}_\alpha$–condition $r \restriction \alpha$. By the construction of $r \restriction \alpha$[28] it follows then that s is $\mathcal{P}_\alpha$–incompatible with some $r_n \restriction \alpha$, which is impossible since both $r_n \restriction \alpha$ and s are in G^*.

Conditions (a)–(c) can be met simultaneously once $\mathcal{D}_n$ has been fixed since, by correctness, N_σ contains a club as given by Claim 3.2.2 for $\mathcal{D} = \mathcal{D}_n$ and for q being either $p_{max(\overline{\Sigma}\cap\sigma)}$, p', or q^σ_{n-1}.

Let $(\epsilon_n)_{n<\omega}$ be the Δ–first sequence of ordinals in N_σ converging to δ_σ. As to the choice of $(\mathcal{D}_n)_n$, we take each $\mathcal{D}_n$ to be the set of conditions q' forcing some ordinal above ϵ_n to be in $F(\gamma') \cap \dot{C} \cap c^{q'}_{\gamma'}$ whenever $\gamma' \in a^q$ (for the right choice of q) and there exists an $\overline{\alpha} \in \gamma' \cap \mathcal{C}$ such that $F(\gamma')$ is a $\mathcal{P}_{\overline{\alpha}}$–name for a club of ω_1. The fact that a^q, for q as above, is countable and that $\mathcal{P}/G^*$ is sufficiently $\tilde{S}$–closed guarantees that such a dense set $\mathcal{D}_n$ exists. Moreover, all choices can be made in a uniform way using Δ to pick all relevant objects to be the Δ–least possible with the desired properties.

The sequence of conditions can now be built, again by the usual argument involving the well–order Δ. At the limit stages σ of the construction we extend all conditions built up to that point by considering a lower bound p_σ of the sequence with $p_\sigma \restriction \alpha \in G^*$. The fact that $h \restriction x \cap \delta_\sigma \in N_{\sigma+1}$ for every σ ensures that the choice of p_σ takes place inside $N_{\sigma+1}$.

Note that, given any $\sigma \in \overline{\Sigma}$, $sup\{\delta_{\sigma'} : \sigma' \in \Sigma \cap \sigma\}$ can be a member of $C^*_{\delta_{\overline{\sigma}}} \cap c^{p_\sigma}_\gamma$ only if it is a limit point of $C^*_{\delta_{\overline{\sigma}}}$. Hence, by condition (iv) in the construction there is some $\overline{\delta} < \delta_{\overline{\sigma}}$ such that

[27] Note that, since $\mathcal{P}_\alpha$ is ω_1–distributive, the sequence $(r_n)_n$ is in V.

[28] More precisely, by the fact that each component of $r \restriction \alpha$ is the $\subseteq$–minimal object extending all corresponding components of all $r_n \restriction \alpha$ for all n for which this makes sense (and such that this object is a club, if required).

($\star$) $C^*_{\delta_{\overline{\sigma}}} \cap^* c^{p_\sigma}_\gamma \subseteq \overline{\delta}$ for each σ.

Also, given any $\gamma' \in [\gamma, \omega_2)$ and any $\sigma_0 \in \Sigma$, if $\gamma' \in a^{p_{\sigma_0}}$, then $\gamma' \in N_{\sigma_0+1}$. Hence, if $\overline{\alpha} \in \gamma' \cap \mathcal{C}$ is minimal such that $F(\gamma')$ is a $\mathcal{P}_{\overline{\alpha}}$–name for a club of ω_1, then each δ_σ (for $\sigma \in \Sigma$, $\sigma > \sigma_0$) is a member of $C^{*\,i}_{\delta_{\overline{\sigma}}}$ which is not a limit point of $C^{*\,i}_{\delta_{\overline{\sigma}}}$ and which is forced by p_σ to be in $F(\gamma')$. Using this we can prove, again by an argument as for Lemma 2.4 (since $\delta_{\overline{\sigma}} \in dom(\vec{C}^*)$ and $dom(\vec{C}^*) \cap S = \emptyset$), that there is a condition $q \in \mathcal{P}/G^*$ extending all p_σ with $C^{i,q}_{\delta_{\overline{\sigma}}} = C^{*\,i}_{\delta_{\overline{\sigma}}}$ and such that $q \Vdash_{\mathcal{P}} \tau_{\delta_{\overline{\sigma}}} = C^*_{\delta_{\overline{\sigma}}}$. This condition q can be built by first considering a lower bound $\tilde{q}$ of all p_σ such that $\tilde{q} \restriction \alpha \in G^*$ and such that $C^{i,\tilde{q}}_{\delta_{\overline{\sigma}}} = C^{*\,i}_{\delta_{\overline{\sigma}}}$ and then taking q to be $q^* \wedge \tilde{q}$, where $q^* \in G^*$ is a condition extending $\tilde{q} \restriction \alpha$ and forcing $\tau_{\delta_{\overline{\sigma}}} = C^*_{\delta_{\overline{\sigma}}}$.

This finishes the proof in this case, since $q \Vdash_{\mathcal{P}} \delta_{\overline{\sigma}} \in \dot{C}$ and since $q \Vdash_{\mathcal{P}} \tau_{\delta_{\overline{\sigma}}} \cap^* c^q_\gamma \subseteq \overline{\delta}$ (by ($\star$)).

Case 2: There is a (unique) i such that $\delta_{\overline{\sigma}} \in dom(\vec{C}^{*\,i})$ and $\eta > ht(\vec{C}^{*\,i})$.

Let $\Sigma = \{\sigma < \overline{\sigma} \,:\, \delta_\sigma \in C^{*\,i}_{\delta_{\overline{\sigma}}},\, \eta < \delta_\sigma\}$. Let $(\sigma_j)_{j<ht(\vec{C}^{*\,i})}$ be the strictly increasing enumeration of Σ. This time we build a decreasing sequence $(p_j)_{j<ht(\vec{C}^{*\,i})}$ of conditions in $\mathcal{P}/G^*$ extending p' and satisfying the following conditions.

(i) $p_j \in N_{\sigma_j+1}$ for every j.

(ii) For every infinite limit ordinal $j < ht(\vec{C}^{*\,i})$, p_j is a lower bound of $(p_k)_{k<j}$.

(iii) Given any $j < ht(\vec{C}^{*\,i})$, p_{j+1} is a lower bound of an ω–sequence $(q^j_n)_{n<\omega}$ of conditions in $N_{\sigma_{j+1}}$ and forces $\delta_{\sigma_{j+1}} \in \dot{C}$.

(iv) Given any $j < j^* < ht(\vec{C}^{*\,i})$ and any $\gamma' \in a^{p_j}$, if $\overline{\alpha} \in \gamma' \cap \mathcal{C}$ is minimal such that $F(\gamma')$ is a $\mathcal{P}_{\overline{\alpha}}$–name for a club of ω_1, then p_{j^*} forces $\delta_{\sigma_{j^*}} \in F(\gamma')$.

(v) Given any $j < ht(\vec{C}^{*\,i})$, $c^{p_{j+1}}_\gamma \cap (\delta_{\sigma_j},\ \delta_{\sigma_{j+1}}) \cap C^*_{\delta_{\overline{\sigma}}} = \emptyset$.

Assuming p_j has been defined, the choice of $p_{j+1} \in N_{\sigma_{j+1}+1}$ can be made as in the previous case: p_{j+1} can be taken as a lower bound of a decreasing sequence $(q^j_n)_{n<\omega}$ of conditions in $N_{\sigma_{j+1}}$ meeting the members of a suitably chosen sequence $(\mathcal{D}_n)_{n<\omega}$ of dense subsets of $\mathcal{P}/G^*$ in $N_{\sigma_{j+1}}$.

This time we pick the conditions q^j_n in such a way that, for all n,

(a) $q^j_n \in \mathcal{D}_n$,

(b) $sup_{n\geq m} max(c^{q^j_n}_\gamma) = \delta_{\sigma_{j+1}}$ for every $m < \omega$ and every $\gamma \in a^{q^j_m}$, and

(c) q^j_n does not put any ordinal in $C^*_{\delta_{\overline{\sigma}}} \setminus (\delta_{\sigma_j} + 1)$ inside $C^{\dot{G}}_\gamma$.

Conditions (a)–(c) can be met, once $\mathcal{D}_n$ has been fixed, since $ht(\vec{C}^*) < \delta_{\sigma_{j+1}} = N_{\sigma_{j+1}} \cap \omega_1$ and since $N_{\sigma_{j+1}}$ contains a club as given by Claim 3.2.2 for $\mathcal{D} = \mathcal{D}_n$ and for q being p_j or q^j_{n-1}.

This is enough, since then $\{p_j\}_{j<ht(\vec{C}^{*\,i})}$ has a lower bound $\tilde{q}$ in $\mathcal{P}/G^*$[29] such that $(c^{\tilde{q}}_\gamma \cap C^*_{\delta_{\overline{\sigma}}}) \setminus (\delta_{\sigma_0} + 1) \subseteq \{\delta_{\sigma_j} \,:\, 0 < j < ht(\vec{C}^{*\,i})\}$ (by (v)) and forcing $\delta_{\overline{\sigma}} \in \dot{C}$ (by (iii)). As in the previous case, we can extend $\tilde{q}$ to a condition q, with $q \restriction \alpha \in G^*$, forcing $\tau_{\delta_{\overline{\sigma}}} = C^*_{\delta_{\overline{\sigma}}}$. This is enough, since then q forces $ot(\tau_{\delta_{\overline{\sigma}}} \cap^* c^q_\gamma) \leq ht(\vec{C}^{*\,i}) < \eta$.[30]

Case 3: $\delta_{\overline{\sigma}} \notin \bigcup_{i<\omega_1} dom(\vec{C}^{*\,i})$.

The proof is now easier than in the previous two cases. Let $(\sigma_j)_{j<\omega}$ be a strictly increasing sequence converging to $\overline{\sigma}$ and with $\{\delta_{\sigma_j}\}_{j<\omega}$ disjoint from $C^*_{\delta_{\overline{\sigma}}} \cup \bigcup_{i<\omega_1} dom(\vec{C}^{*\,i}) \cup S$. We can build by recursion a decreasing sequence $(p_j)_{j<\omega}$ of conditions in $\mathcal{P}/G^*$ extending p' such that, for each j,

(i) $p_j \in N_{\sigma_j+1}$,

(ii) p_j is a lower bound of an $(N_{\sigma_j}, \mathcal{P}/G^*)$–generic sequence of conditions in N_{σ_j}, and

(iii) if $j > 0$, $min(c^{p_j}_\gamma \setminus (\delta_{\sigma_{j-1}} + 1)) > max(C^*_{\delta_{\overline{\sigma}}} \cap \delta_{\sigma_j})$.

Finally, since $\delta_{\overline{\sigma}} \notin dom(\vec{C}^{*\,i})$ for all i, $(p_j)_{j<\omega}$ has a lower bound $\tilde{q}$ with $\tilde{q} \restriction \alpha \in G^*$ and forcing that $\delta_{\overline{\sigma}}$ is in $\dot{C}$. Again, we can extend $\tilde{q}$ to a condition

[29] As in the previous case, the fact that, for some γ' with $F(\gamma')$ a $\mathcal{P}_{\overline{\alpha}}$–name for a club (for some $\overline{\alpha} < \gamma'$ in $\mathcal{C}$), $\delta_{\overline{\sigma}}$ is necessarily put in $c^{\dot{G}}_{\gamma'}$ by every condition extending all p_j is not an obstacle, since a final segment of $\{\delta_\sigma \,:\, \sigma \in \Sigma\}$ is then put in $F(\gamma')$ (by (iv)). As before, we also make use of the ω_1–distributivity of $\mathcal{P}_\alpha$ to ensure that the relevant set of conditions is an actual member of V.

[30] In fact, q forces $ot((\tau_{\delta_{\overline{\sigma}}} \setminus (\delta_{\sigma_0} + 1)) \cap c^q_\gamma) \leq ht(\vec{C}^{*\,i}) < \eta$.

forcing $\tau_{\delta_{\overline{\sigma}}} = C^*_{\delta_{\overline{\sigma}}}$. It follows that q forces that $\tau_{\delta_{\overline{\sigma}}} \cap^* C^{\dot{G}}_\gamma$ (and in fact $\tau_{\delta_{\overline{\sigma}}} \cap C^{\dot{G}}_\gamma$) is bounded in $\delta_{\overline{\sigma}}$.

The construction in this last case finishes the proof of Lemma 3.2. □

Lemma 3.2 concludes the proof of Theorem 1.1.

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