The Effective Theory of Borel Equivalence Relations[*]

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Abstract

The study of Borel equivalence relations under Borel reducibility has developed into an important area of descriptive set theory. The dichotomies of Silver ([19]) and Harrington-Kechris-Louveau ([5]) show that with respect to Borel reducibility, any Borel equivalence relation strictly above equality on ω is above equality on $\mathcal{P}(\omega)$, the power set of ω, and any Borel equivalence relation strictly above equality on the reals is above equality modulo finite on $\mathcal{P}(\omega)$. In this article we examine the effective content of these and related results by studying effectively Borel equivalence relations under effectively Borel reducibility. The resulting structure is complex, even for equivalence relations with finitely many equivalence classes. However use of Kleene's O as a parameter is sufficient to restore the picture from the noneffective setting. A key lemma is the existence of two effectively Borel sets of reals, neither of which contains the range of the other under any effectively Borel function; the proof of this result applies Barwise compactness to a deep theorem of Harrington (see [6]) establishing for any recursive ordinal α the existence of Π_1^0 singletons whose α-jumps are Turing incomparable.

Keywords

Hyperarithmetic equivalence relations; Borel reducibility; Dichotomy theorems.

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1 Introduction

If E and F are Borel equivalence relations on Polish spaces X and Y respectively, then E is *Borel reducible* to F if and only if there is a Borel function $f : X \to Y$ such that xEy if and only if $f(x)Ff(y)$. The study of Borel equivalence relations under Borel reducibility has developed into a rich area of descriptive set theory. Surveys of some of this work may be found in [2, 3, 7, 8, 11, 14]. In the noneffective setting, Borel equivalence relations with countably many equivalence classes are equivalent (i.e. bi-reducible) exactly if they have the same number of equivalence classes. For Borel equivalence relations with uncountably many equivalence classes there are two fundamental dichotomies:

The Silver Dichotomy ([19]). *If E is a Borel equivalence relation with uncountably many equivalence classes then equality on $\mathcal{P}(\omega)$, the power set of ω, is Borel reducible to E.*

The Harrington-Kechris-Louveau Dichotomy ([5]). *If E is a Borel equivalence relation not Borel reducible to equality on $\mathcal{P}(\omega)$ then E_0 is Borel reducible to E, where E_0 is equality modulo finite on $\mathcal{P}(\omega)$.*

In this article we introduce the effective version of this theory. If E and F are effectively Borel (i.e., Δ_1^1) equivalence relations on effectively presented Polish spaces[1] spaces X and Y, respectively, then we say that E is *effectively Borel reducible* to F if there is an effectively Borel function $f : X \to Y$ such that xEy if and only if $f(x)Ff(y)$. The resulting effective theory reveals an unexpectedly rich new structure, even for equivalence relations with finitely many classes. For $n \leq \omega$, let $=_n$ denote equality on n, let $=_{\mathcal{P}(\omega)}$ denote equality on the power set of ω and let E_0 denote equality modulo finite on $\mathcal{P}(\omega)$. The notion of effectively Borel reducibility on effectively Borel equivalence relations naturally gives rise to a degree structure, which we denote by $\mathcal{H}$.

We show the following:

Theorem A. *For any finite n, the partial order of Δ_1^1 subsets of ω under inclusion can be order-preservingly embedded into $\mathcal{H}$ between the degrees of*

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[1]In the sense of Moschovakis, [15, 3B]. In this paper we will deal almost exclusively with the spaces ω, $\mathcal{P}(\omega)$ and $\mathcal{N} = \omega^\omega$.

$=_n$ and $=_{n+1}$. The same holds between the degrees of $=_\omega$ and $=_{\mathcal{P}(\omega)}$, and between $=_{\mathcal{P}(\omega)}$ and E_0.

A basic tool in the proof of Theorem A is the following result, which may be of independent interest:

($*$) *There are effectively Borel sets A and B such that for no effectively Borel function f does one have $f[A] \subseteq B$ or $f[B] \subseteq A$.*

($*$) is proved via a Barwise compactness argument applied to a deep result of Harrington (see [6]) establishing for any recursive ordinal α the existence of Π_1^0 singletons whose α-jumps are Turing incomparable.

We also examine the effectivity of the Silver and Harrington-Kechris-Louveau dichotomies. Harrington's proof of the Silver dichotomy (see [3] or [10]) and the original proof of the Harrington-Kechris-Louveau dichotomy in [5] respectively show that if an effectively Borel equivalence relation has countably many equivalence classes then it is effectively Borel reducible to $=_\omega$ and if it is Borel reducible to $=_{\mathcal{P}(\omega)}$ then it is in fact effectively Borel reducible to $=_{\mathcal{P}(\omega)}$. We complete the picture by showing:

Theorem B. *Let O denote Kleene's O. If an effectively Borel equivalence relation E has uncountable many equivalence classes then there is a $\Delta_1^1(O)$ function reducing $=_{\mathcal{P}(\omega)}$ to E, and this parameter is best possible. If an effectively Borel equivalence relation E is not Borel reducible to $=_{\mathcal{P}(\omega)}$ then there is a $\Delta_1^1(O)$ function reducing E_0 to E, and this parameter is best possible.*

In other words, while Theorem A rules out that the dichotomy Theorems of Silver and Harrington-Kechris-Louveau are effective, Theorem B shows that the Borel reductions obtained in the dichotomy Theorems can in fact be witnessed by $\Delta_1^1(O)$ functions, and that Kleene's O is the best possible parameter we can hope for in general. The proof of Theorem B is based on a detailed analysis of the effectiveness of category notions in the Gandy-Harrington topology, due to the third author.

There remain many open questions in the effective theory. We mention a few of them at the end of the article.

Organization. The paper is organized into 6 sections. In §2 we introduce some basic notation used in the paper, and recall some well-known theorems and facts that our proofs rely on. In §3 we prove ($*$), which serves as

a basic tool throughout the paper. The proof of Theorem A and several extensions of Theorem A is found in §4. In §5 we give a detailed analysis of the effectiveness of category notions in the Gandy-Harrington topology. Finally, Theorem B is proved in §6.

2 Background

Throughout this paper, Hyp stands for Δ^1_1, both for subsets of ω and for subsets of Baire space $\mathcal{N} = \omega^\omega$. Elements of $\mathcal{N}$ are called "reals". We state without proofs some well-known results that we will need in this paper. For further details the reader may consult the provided references.

For a linear ordering $<$ denote by $\mathcal{W}f(<)$ the largest well-ordered initial segment of $<$. We can identify $\mathcal{W}f(<)$ with an ordinal without danger of confusion.

Theorem 1 (Barwise, see [1])**.** *Let $\mathcal{L}$ be a recursive language, $\mathcal{A} = L_{\omega_1^{\mathrm{CK}}}$, and let $\mathcal{L}_{\mathcal{A}}$ be $\mathcal{L}_{\omega_1\omega}$ restricted to $\varphi \in \mathcal{A}$. Suppose $\Phi \subseteq \mathcal{L}_{\mathcal{A}}$ is a $\Sigma_1(\mathcal{A})$ set of sentences and every $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \in \mathcal{A}$ has a model. Then Φ has a model. Moreover, if $< \in \mathcal{L}$ and for all $\alpha < \omega_1^{\mathrm{CK}}$ and $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \in \mathcal{A}$ there is a model of Φ_0 in which $<$ is a linear ordering of length at least α, then Φ has a model in which $<$ is a linear ordering satisfying $\mathcal{W}f(<) = \omega_1^{\mathrm{CK}} \neq Field(<)$.*

Definition 1. *Let Γ be a point-class (in the sense of Moschovakis [15]) and let A be a set of reals. We call A a Γ singleton iff A has exactly one element and A belongs to Γ.*

In this paper Γ will usually be Π^0_1 or Δ^1_1 (i.e. Hyp).

Fact 1 (see [17, 18])**.**   *1. Every Hyp real is a Hyp singleton;*

  2. A countable Hyp set of reals contains only Hyp reals;

  3. For every Hyp real X there is a Π^0_1 singleton Y, such that $X \leq_T Y$.

Theorem 2 (Uniformization, see e.g. Chapter II of [18])**.** *Every Π^1_1 binary relation on $\mathcal{N} \times \mathcal{N}$ contains a Π^1_1 function with the same domain.*

Theorem 3 (Dependent Choice, see Chapter II of [18]). *If P is a Hyp binary relation and for all Hyp reals X there exists a Hyp real Y such that $P(X, Y)$, then for all Hyp reals X, there is a Hyp ω-sequence $X = X_0, X_1, \ldots$ such that $P(X_n, X_{n+1})$, for all n.*

Recall that E_0 is the equivalence relation on 2^ω defined by

$$xE_0y \iff (\exists n)(\forall m \geq n)x(m) = y(m),$$

equivalently, E_0 is equality modulo finite in $\mathcal{P}(\omega)$. The next result is folklore (see e.g. [7, Theorem 3.2]):

Fact 2. *If $h : 2^\omega \to 2^\omega$ is Baire measurable and constant on E_0 classes then h is constant on a comeagre set.*

The following result will be used several times:

Fact 3 (Kechris, [12]). *If $B \subset 2^\omega \times 2^\omega$ is Hyp then $\{x : \{y : (x, y) \in B\}$ is non-meagre$\}$ is Σ_1^1.*

Finally, we will use the following result from [4]. For a sketch of the proof see also [6].

Theorem 4. *For any recursive ordinal α there is a sequence of reals $\langle a_n | n < \omega \rangle$ such that for some recursive sequence $\langle \varphi_n | n < \omega \rangle$ of Π_1^0 formulas, a_n is the unique solution to φ_n for each n and no a_n is recursive in the α-jump of $\langle a_m | m \neq n \rangle$.*

Remark. We will also use the following weaker form of Theorem 4. For every recursive ordinal α there are two Π_1^0 singletons a, b such that a is not recursive in the α-jump of b and b is not recursive in the α-jump of a.

Notation. If a is a real and $\alpha < \omega_1^{\text{CK}}$ then we denote by a^α the α-jump of a.

3 The basic tool: Hyp incomparable Hyp sets of reals

The theorem which we prove in this section will be used repeatedly to obtain the results of this paper.

Theorem 5. *There exist two nonempty Π_1^0 sets $A, B \subseteq \mathcal{N}$, such that for no Hyp function $F : \mathcal{N} \to \mathcal{N}$ do we have $F[A] \subseteq B$ or $F[B] \subseteq A$.*

Remark. If A and B are as in Theorem 5 then neither A nor B contains a Hyp real: Suppose A contains a Hyp real y; then the constant function with value y maps B into A, contradiction. In particular, it follows that there is no Hyp F such that $F[\sim A] \subseteq B$ or $F[\sim B] \subseteq A$.

Proof of Theorem 5. Let $\mathcal{A} = L_{\omega_1^{\mathrm{CK}}}$, $\mathcal{L} \supseteq \{\in, <, \underline{x_0}, \underline{x_1}\} \cup \{\underline{\alpha} : \alpha \in \mathcal{A}\}$, where $\underline{x_0}, \underline{x_1}$ and $\underline{\alpha}$ are constant symbols. Consider the set of sentences Φ consisting of:

(1) ZF^-

(2) $(\forall x)(x \in \underline{\omega} \leftrightarrow \bigwedge_n x = \underline{n})$

(3) $< = \in \upharpoonright$ Ordinals

(4) $\underline{x_0}, \underline{x_1} \subseteq \underline{\omega}$

(5) $\bigvee_{\varphi \in \Pi_1^0} \left[(\exists! v)\varphi(v) \wedge \varphi(\underline{x_i}) \right]$ ($i = 0, 1$, φ ranges over all Π_1^0 formulas.)

(6) $\underline{x_0} \not\leq_T \underline{x_1}^{\underline{\alpha}}, \underline{x_1} \not\leq_T \underline{x_0}^{\underline{\alpha}}$, for all $\alpha \in \omega_1^{\mathrm{CK}}$.

The set Φ is a Σ_1 set of sentences. By the remark following Theorem 4, for every recursive ordinal α there exist Π_1^0 singletons a_α, b_α, such that a_α is not recursive in the αth Turing jump of b_α and b_α is not recursive in the αth Turing jump of a_α. Thus, we can apply Theorem 1. We get a model $\langle M, E, <, x_0, x_1 \rangle \models \Phi$ such that $L_{\omega_1^{\mathrm{CK}}} \subseteq M$, M has nonstandard ordinals and every standard ordinal of M is recursive, i.e., the standard part of $<^M$ is ω_1^{CK}. Then in M there must be Π_1^0 singletons a and b such that $a \not\leq_T b^\alpha, b \not\leq_T a^\alpha$ for $\alpha < \omega_1^{\mathrm{CK}}$ and since $\omega_1^a = \omega_1^b = \omega_1^{\mathrm{CK}}$, a and b are Hyp-incomparable.

Choose Π_1^0 formulas φ_a and φ_b, such that in M, $\varphi_a(x) \leftrightarrow x = a$ and $\varphi_b(x) \leftrightarrow x = b$. Note that a and b are the unique solutions of φ_a and φ_b in M, respectively. Then the formulas φ_a and φ_b define Π_1^0 sets (not singletons) in V. Let $A = \{x : \varphi_a(x)\}$ and $B = \{x : \varphi_b(x)\}$.

Claim 1. *There is no Hyp function F such that $F[A] \subseteq B$.*

Proof. Suppose F were such a function. Consider $F(a) \in M$. It is Hyp in a. On the other hand, $F(a) = t \in B$. Therefore by definition of B, $\varphi_b(t)$ holds in M, and so $t = b$. Thus, b is Hyp in a, contradicting the properties of a and b. $\square$

This completes the proof of the theorem. $\square$

Theorem 6. *There exists a uniform sequence $A_0, A_1, \ldots$ of nonempty Π_1^0 sets such that for each n there is no Hyp function F such that $F[A_n] \subseteq \bigcup_{m \neq n} A_m$.*

Proof. The proof is analogous to the previous proof using Theorems 4 and 1. We consider $\mathcal{L} \supseteq \{\in, <, \underline{x_0}, \underline{x_1}, \ldots\} \cup \{\underline{\alpha} : \alpha \in \mathcal{A}\}$ and the following set of sentences Φ:

(1) ZF^-

(2) $(\forall x)(x \in \underline{\omega} \leftrightarrow \bigwedge_n x = \underline{n})$

(3) $< = \in \upharpoonright \text{Ordinals}$

(4) $\bigwedge_n \underline{x_n} \subseteq \underline{\omega}$

(5) $\bigwedge_n \bigvee_{\varphi \in \Pi_1^0} \left[(\exists! v)\varphi(v) \wedge \varphi(\underline{x_n}) \right]$, ($\varphi$ ranges over all Π_1^0 formulas)

(6) $\bigwedge_{m \neq n} \underline{x_m} \not\leq_T \underline{x_n}^{\underline{\alpha}}$ for all $\alpha \in \omega_1^{\text{CK}}$.

By the properties of sequences $\langle a_n : n < \omega \rangle$ from Theorem 4, we get that the resulting sequence $A_0, A_1, \ldots$ of Π_1^0 sets is uniform and has the required properties exactly as in Theorem 5. $\square$

4 Hyp Equivalence Relations under Hyp Reducibility

Definition 2. *Let E and F be equivalence relations on $\mathcal{N}$. We say that E is* Hyp-reducible *to F if there exists a Hyp function $f : \mathcal{N} \to \mathcal{N}$ such that*

$$xEy \iff f(x)Ff(y),$$

in which case we will write $E \leq_H F$.

This notion induces a natural notion of *Hyp-equivalence* (or *Hyp bi-reducibility*) and *Hyp-degrees*: we let $E \equiv_H F$ if and only if $E \leq_H F$ and $F \leq_H E$.

Definition 3. *For every $n \in \omega, n \geq 1$, let $=_n$ be the Hyp-degree of the following equivalence relation:*

$$x \equiv y \iff x(0) = y(0) \text{ or both } x(0), y(0) \geq n - 1.$$

The Hyp-degree $=_\omega$ is the Hyp-degree of the equivalence relation

$$x \equiv y \iff x(0) = y(0).$$

4.1 Hyp Equivalence Relations with countably many classes

Proposition 1. *Let $1 \leq n \leq \omega$ and let E be a Hyp equivalence relation. Then $=_n \leq_H E$ iff E has at least n classes containing Hyp reals.*

Proof. $(\Rightarrow)$: For every $1 \leq n \leq \omega$, the equivalence relation $=_n$ has exactly n equivalence classes and each of them contains a Hyp real. Under Hyp-reducibility Hyp reals are sent to Hyp reals, equivalent reals are sent to equivalent reals, non-equivalent reals are sent to non-equivalent reals.

$(\Leftarrow)$: If n is finite, pick n Hyp reals $x_0, \ldots, x_{n-1}$ that lie in different equivalence classes of E. The function F that sends the ith equivalence class of $=_n$ to x_i witnesses the reduction. To prove the result for $n = \omega$ we use Theorem 3. Suppose E is an equivalence relations with infinitely many classes containing Hyp reals. We want to prove that $=_\omega$ Hyp-reduces to E. We will find a Hyp sequence of equivalence classes of E with Hyp reals in them. Consider the following relation $P(X, Y)$ on $\omega \times \mathcal{N}^{<\omega}$:

$$P(X, Y) \iff [X = (n, X_0, \ldots, X_n) \wedge \bigwedge_{i \neq j} \neg X_i E X_j] \longrightarrow$$
$$[Y = (n+1, Y_1, \ldots, Y_n, Y_{n+1}) \wedge \bigwedge_i X_i = Y_i \wedge \bigwedge_{i \neq j} \neg Y_i E Y_j]$$

Then P is Hyp. Moreover, as E has infinitely many Hyp classes, for every Hyp X there exists a Hyp Y such that $P(X, Y)$. It follows from Theorem 3 that there exists a uniform sequence of Hyp sets $X_0, X_1, \ldots$ such that

$$\forall i, j (i \neq j \rightarrow \neg X_i E X_j).$$

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Then the function that sends the equivalence class $\{x : x(0) = n\}$ of $=_\omega$ to X_n is Hyp and witnesses the reduction. $\qquad\square$

Corollary 1. *If $=_n \leq_H E$, for all $1 \leq n < \omega$, then $=_\omega \leq_H E$.*

Proposition 2. *Let $1 \leq n \leq \omega$ and let E be a Hyp equivalence relation. Then $E \leq_H =_n$ iff E has at most n classes.*

Proof. The direction ($\Rightarrow$) is obvious since non-equivalent reals are sent to non-equivalent reals under Hyp-reducibility.

To prove ($\Leftarrow$) we need to show that the equivalence classes of a Hyp equivalence relation with at most countably many equivalence classes are uniformly Hyp.

By Harrington's proof of the Silver Dichotomy (see [10, Theorem 32.1] or [3, Theorem 5.3.5]), if E has only countably many classes then every real belongs to a Hyp subset of some equivalence class. Let C be the set of codes for Hyp subsets of an equivalence class; then C is Π_1^1. Consider the relation

$$R = \{(x, c) : c \in C \text{ and } x \in H(c), \text{ the Hyp set coded by } c\}.$$

Then R is Π_1^1 and can be uniformised by a Π_1^1 function F. As the values of F are numbers, F is Hyp and by separation we can choose a Hyp $D \subseteq C$, $D \supseteq \operatorname{ran}(F)$. Now define an equivalence relation E^* on D by:

$$d_0 E^* d_1 \iff (\forall x_0, x_1)(x_0 \in H(d_0) \wedge x_1 \in H(d_1)) \to x_0 E x_1$$
$$\iff (\exists x_0, x_1)(x_0 \in H(d_0) \wedge x_1 \in H(d_1) \wedge x_0 E x_1).$$

i.e. $d_0 E^* d_1$ if and only if $H(d_0)$ and $H(d_1)$ are subsets of the same E-equivalence class. Note that E^* is Hyp. The relation E Hyp-reduces to E^* via $x \mapsto F(x)$. But E^* is just a Hyp relation on a Hyp set of numbers, so E^* is Hyp-reducible to $=_\omega$ (to see this, send c to the least number c^*, $c E^* c^*$).

Thus if E is a Hyp equivalence relation with at most countably many classes then E is Hyp-reducible to $=_\omega$. (In particular, all equivalence classes of E are Hyp.) One can similarly see that if E has at most n classes then E is Hyp-reducible to $=_n$. $\qquad\square$

Obviously, the degree $=_1$ is Hyp-reducible to any other Hyp-degree. But $=_2$, the equivalence relation with the 2 classes $\{x : x(0) = 0\}$ and $\{x : x(0) \geq 1\}$ is *not* the successor to $=_1$. This is the content of the next theorem.

Theorem 7. 1. *There is a Hyp equivalence relation strictly between $=_1$ and $=_2$.*

2. *For every finite n, there is a Hyp equivalence relation strictly between $=_n$ and $=_{n+1}$.*

3. *For every $n_0 < n_1 \leq \omega$, there is a Hyp equivalence relation above $=_{n_0}$, below $=_{n_1}$ and incomparable with $=_n$, for all $n_0 < n < n_1$.*

Proof. The proof is based on the following fact.

Fact 4 ([18])**.** *There is a nonempty Hyp set X which contains no Hyp reals.*

To prove the first statement, take a Hyp equivalence relation E with two equivalence classes X and $\sim X$, where X is from Fact 4. By Proposition 2, E Hyp-reduces to $=_2$. By Proposition 1, $=_2$ does not Hyp-reduce to E.

To prove the second statement, we let E consist of exactly $n+1$ equivalence classes, such that only n of them contain Hyp reals. For each $i < n-1$, we define the ith equivalence class by taking all $x \in \sim X$, such that $x(0) = i$. We take the nth class to contain all $x \in \sim X$ with $x(0) \geq n - 1$. And the $(n+1)$st class is X.

For the proof of the third statement, consider an equivalence relation with n_1 classes such that only n_0 of them contain Hyp reals. $\square$

Theorem 8. *There are incomparable Hyp equivalence relations between $=_1$ and $=_2$.*

Proof. To prove the theorem, we consider the following equivalence relations. Let A and B be as in Theorem 5. We take the equivalence relation E_A with two equivalence classes $A, \sim A$ and E_B with two equivalence classes $B, \sim B$. Then E_A and E_B are Hyp-reducible to $=_2$. By the properties of A and B, the relations E_A and E_B are Hyp-incomparable, as otherwise (using the Remark following Theorem 5) we would have a Hyp function which maps A to B or vice versa. $\square$

Theorem 9. *The partial order of Hyp subsets of ω under inclusion can be order-preservingly embedded into the structure of degrees of Hyp equivalence relations between $=_1$ and $=_2$.*

Proof. Let X be a Hyp subset of ω. Define the corresponding equivalence relation E_X in the following way. We let $xE_X y$ iff both $x, y \in \bigcup_{i \in X} A_i$ or both $x, y \in \sim \bigcup_{i \in X} A_i$, where $A_0, A_1, \dots$ are the sets constructed in Theorem 6. We check that $X \subseteq Y \iff E_X \leq_H E_Y$.

Suppose $X \subseteq Y$. For every $i \in X$ we send A_i into itself. We send $\sim \bigcup_{i \in X} A_i$ into a single Hyp real chosen in $\sim \bigcup_{i \in Y} A_i$. Therefore $E_X \leq_H E_Y$.

Now suppose $X \nsubseteq Y$ but $E_X \leq_H E_Y$ via a Hyp function $F : \mathcal{N} \to \mathcal{N}$. Note that neither $\bigcup_{i \in X} A_i$ nor $\bigcup_{i \in Y} A_i$ contain Hyp reals. Thus F sends $\sim \bigcup_{i \in X} A_i$ to $\sim \bigcup_{i \in Y} A_i$ and $\bigcup_{i \in X} A_i$ to $\bigcup_{i \in Y} A_i$. Choose an $i_0 \in X \setminus Y$. Then $F[A_{i_0}] \subseteq \bigcup_{i \in Y} A_i \subseteq \bigcup_{i \neq i_0} A_i$, contradicting the properties of the sequence $A_0, A_1 \dots$ $\qquad\square$

Corollary 2. *1. There are infinite antichains between $=_1$ and $=_2$.*

2. There are infinite descending chains between $=_1$ and $=_2$.

3. There are infinite ascending chains between $=_1$ and $=_2$.

The same proof shows:

Corollary 3. *For any $1 \leq n_0 < n_1 \leq \omega$ there is an embedding of $\mathcal{P}(\omega) \cap Hyp$ into the structure of degrees of Hyp equivalence relations that are above $=_{n_0}$, below $=_{n_1}$ and incomparable with each $=_n$ for $n_0 < n < n_1$.*

4.2 Hyp Equivalence Relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$

Let $=_{\mathcal{P}(\omega)}$ denote the Hyp-degree of the equivalence relation of $=$ on $\mathcal{P}(\omega)$. By Proposition 2 and Silver's dichotomy [3], every Hyp equivalence relation E is either Hyp reducible to $=_\omega$, or $=_{\mathcal{P}(\omega)}$ is Borel reducible to E. In §6 we will show that "Borel reducible" can be taken to be "Hyp in Kleene's O reducible", and that this is best possible.

Theorem 10. *There exist Hyp-incomparable Hyp equivalence relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Proof. Suppose that A and B are the Π_1^0 sets from Theorem 5: they contain no Hyp reals and there is no Hyp function F such that $F[A] \subseteq B$ or $F[B] \subseteq A$.

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Now consider the equivalence relations E_A and E_B:

$$xE_Ay \iff [(x \in A \land x = y) \lor (x, y \notin A \land x(0) = y(0))]$$

and similarly for E_B with B replacing A.

By sending n to the real $(n, 0, 0, \ldots)$ we get a Hyp reduction $=_\omega$ to E_A and E_B. Also E_A (resp. E_B) Hyp-reduces to $=_{\mathcal{P}(\omega)}$ via the map $G(x) = x$ if x belongs to A (resp. B), $G(x) = (x(0), 0, 0, \ldots)$ for $x \notin A$ (resp. $x \notin B$).

There is no Hyp reduction of E_A to E_B. Indeed, suppose that F were such a reduction and let C be the preimage under F of $\sim B$. As $\sim B$ is Σ_1^0, C is Hyp and therefore $A \cap C$ is also Hyp. But $A \cap C$ must be countable as F is a reduction. So by Fact 1, part 2, if $A \cap C$ were nonempty it would have a Hyp element, contradicting the fact that A has no Hyp elements. Therefore F maps A into B, which is impossible by the choice of A and B. $\square$

Theorem 11. *The partial order of Hyp subsets of ω under inclusion can be embedded into the structure of degrees of Hyp equivalence relations between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Proof. Let $A_0, A_1, \ldots$ be the Π_1^0 sets from Theorem 6. For every Hyp set $X \subseteq \omega$ consider the equivalence relation of the form

$$xE_Xy \iff [(x \in \bigcup_{i \in X} A_i \text{ and } x = y) \text{ or } (x, y \notin \bigcup_{i \in X} A_i \text{ and } x(0) = y(0))].$$

Then $=_\omega \leq_H E_X \leq_H =_{\mathcal{P}(\omega)}$. Suppose $X \subseteq Y$. Then E_X Hyp-reduces to E_Y via the map $G(x) = x$ if $x \in \bigcup_{i \in X} A_i$, $G(x) = (x(0), 0, 0, \ldots)$ for $x \notin \bigcup_{i \in X} A_i$.

Suppose $X \nsubseteq Y$ but $E_X \leq_H E_Y$ via a Hyp function F. Pick $i_0 \in X \setminus Y$. As before, we consider the set

$$A_{i_0} \cap F^{-1}(\sim \bigcup_{j \in Y} A_j).$$

Then this is a countable Hyp set. If it is non-empty then it contains a Hyp real, contradicting the definition of A_{i_0}. Therefore we get $F[A_{i_0}] \subseteq \bigcup_{j \in Y} A_j \subseteq \bigcup_{j \neq i_0} A_j$, contradiction. $\square$

Corollary 4. *There are infinite chains and antichains between $=_\omega$ and $=_{\mathcal{P}(\omega)}$.*

Corollary 5. *For any finite $n_0 \geq 1$, the partial order of Hyp subsets of ω under inclusion can be embedded into the structure of degrees of Hyp equivalence relations between $=_{n_0}$ and $=_{\mathcal{P}(\omega)}$ but incomparable with $=_n$ for $n_0 < n \leq \omega$.*

Proof. For every Hyp $X \subseteq \omega$, consider the equivalence relation of the form

$$
x E_X^{n_0} y \iff x \in \bigcup_{i \in X} A_i \wedge x = y \vee
$$
$$
x, y \notin \bigcup_{i \in X} A_i \wedge (x(0) = y(0) < n_0 - 1 \vee x(0), y(0) \geq n_0 - 1).
$$

Then $E_X^{n_0}$ has exactly n_0 equivalence classes with Hyp reals. Therefore $=_{n_0} \leq_H E_X^{n_0}$ and for $n_0 < n \leq \omega$, the equivalence relation $=_n$ is incomparable with $E_X^{n_0}$. $\qquad \square$

4.3   Hyp Equivalence Relations between $=_{\mathcal{P}(\omega)}$ and E_0

It was shown in [5] that any Hyp equivalence relation is either Hyp reducible to $=_{\mathcal{P}(\omega)}$, or E_0 is Borel reducible to it. In §6 we will show that "Borel" can be taken to be "Hyp in Kleene's O", and that this is best possible.

Theorem 12. *There exist Hyp-incomparable Hyp equivalence relations between $=_{\mathcal{P}(\omega)}$ and E_0.*

Proof. Let A and B be the Hyp sets from Theorem 5, such that for no Hyp function F do we have $F[A] \subseteq B$ nor $F[B] \subseteq A$.

Define two Hyp equivalence relations E_A and E_B on $\mathcal{N} \times 2^\omega$ by

$$
(x, y) E_A (x', y') \iff x = x' \wedge [(x \notin A) \vee (x \in A \wedge y E_0 y')],
$$

and similarly for E_B with B replacing A.

Suppose $F : \mathcal{N} \times 2^\omega \to \mathcal{N} \times 2^\omega$ is a Hyp-reduction of E_A to E_B. Define $F'(x, y) = z \iff (\exists w) F(x, y) = (z, w)$. Note that F' is constant on E_A classes. Define a function $h : \mathcal{N} \to \mathcal{N}$ by

$$
h(x) = z \iff \{y \in 2^\omega : g'(x, y) = z\} \text{ is non-meagre}
$$
$$
(\iff \{y \in 2^\omega : g'(x, y) = z\} \text{ is comeagre.})
$$

By Facts 2 and 3, h is an everywhere defined Hyp function. Suppose $x \in A$. Then for a comeagre set $C \subseteq 2^\omega$ we have $F'(x, y) = h(x)$ for all $y \in C$. We

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claim that $h(x) \in B$. Indeed, otherwise the set $\{x\} \times C$ is mapped by g into a single E_B class, contradicting that all $E_A|\{x\} \times 2^\omega$ classes are meagre in $\{x\} \times 2^\omega$ (in fact, they are countable).

Thus h is a Hyp function with $h[A] \subseteq B$, contradicting the properties of A and B. $\square$

Theorem 13. *The partial order of Hyp subsets of ω can be embedded into the structure of Hyp equivalence relations between $=_{\mathcal{P}(\omega)}$ and E_0.*

Proof. Let $A_0, A_1, \ldots$ be the sequence from Theorem 6. For every Hyp set X we define a Hyp equivalence relation E_X on $\mathcal{N} \times 2^\omega$ in the following way:

$$(x, y)E_X(x', y') \iff x = x' \wedge [(x \notin \bigcup_{i \in X} A_i) \vee (x \in \bigcup_{i \in X} A_i \wedge yE_0y')].$$

Then the theorem follows from an argument similar to that in the proof of Theorems 11 and 12. $\square$

Theorem 14. *For any $n_0 \leq \omega$ the partial order of Hyp subsets of ω can be embedded into the structure of degrees of Hyp equivalence relations between $=_{n_0}$ and E_0, but incomparable with $=_n$ for $n_0 < n \leq \omega$ and incomparable with $=_{\mathcal{P}(\omega)}$.*

Proof. Let $A_0, A_1, \ldots$ be the sequence of Hyp sets from Theorem 6. For a Hyp set $X \subseteq \omega$, define an equivalence relation $E_X^{n_0}$ on $\mathcal{N} \times 2^\omega$ by

$$\begin{aligned}
(x, y)E_X^{n_0}(x', y') \iff &[x, x' \notin \bigcup_{i \in X} A_i \wedge \\
&(x(0) = x'(0) < n_0 - 1 \vee x_0, x_0' \geq n_0 - 1)] \vee \\
&[x, x' \in \bigcup_{i \in X} A_i \wedge x = x' \wedge yE_0y'].
\end{aligned}$$

The relation $E_X^{n_0}$ is Hyp. Clearly it is below E_0. It has only n_0 equivalence classes with Hyp reals, thus it is above $=_{n_0}$ and incomparable with $=_n$ for $n_0 < n < \omega$ and with $=_{\mathcal{P}(\omega)}$. $\square$

5 Category notions in the Gandy-Harrington topology

The spaces under consideration in this section will be of the form $(\omega^\omega)^n$, $1 \leq n \leq \omega$. Baire space, $\mathcal{N} = \omega^\omega$, is a Polish space in the product topology,

and thus so is $\mathscr{N}^n$ for all $n \leq \omega$. We will call this the "usual" topology on $\mathscr{N}^n$. We consider two other topologies on $\mathscr{N}^n$:

(1) The *Gandy-Harrington* topology, which is generated by the (lightface) Σ_1^1 subsets of $\mathscr{N}^n$. This topology will be denoted τ_n if $n > 1$, or simply by τ if $n = 1$.

(2) The product topology τ^n on $\mathscr{N}^n$, when we equip $\mathscr{N}$ with the Gandy-Harrington topology.

These topologies are all different: The usual topology is weaker than τ^n, which again is weaker than τ_n, if $n > 1$.

The purpose of this section is to examine the effectiveness of category notions in the Gandy-Harrington topology. For instance, if we consider a Σ_1^1 set $A \subseteq \mathscr{N}^2$, we would like to know the complexity of the set

$$\{x \in \mathscr{N} : A_x \text{ is not meagre in } \tau\}.$$

We would also like to know how effective we can reasonably expect a winning strategy in the Banach-Mazur game to be, or how effective player II's winning strategy in the Choquet game in $(\mathscr{N}, \tau)$ is. Our analysis is entirely parallel to that found in [12], where the same questions were analyzed for the usual topology on $\mathscr{N}$.

It is important to note that the category of a set may change when changing between these topologies. For instance, a Σ_1^1 singleton $\{x\} \subseteq \mathscr{N}$ is open in the Gandy-Harrington topology, but closed and meagre in the usual topology. The set

$$A = \{(x, x) : \{x\} \text{ is not a } \Sigma_1^1 \text{ singleton}\}$$

is meagre in τ^2 since every section A_x is meagre, but it is open in τ_2 since it is Σ_1^1. However, all $\mathbf{\Sigma}_1^1$ subsets of $\mathscr{N}^n$ have the property of Baire in the topologies τ^n and τ_n. This follows from [13, Theorem 21.8].

5.1 Basic computations

Fix n and let σ be either τ_n or τ^n. Then we form the finite levels of the Borel hierarchy: $\mathbf{\Sigma}_1^0[\sigma]$ consists of the σ-open subsets of $\mathscr{N}^n$, and in general $\mathbf{\Sigma}_{k+1}^0[\sigma]$ consists of countable unions of sets from $\mathbf{\Pi}_k^0[\sigma]$, which itself consists of the complements of sets in $\mathbf{\Sigma}_k^0[\sigma]$.

Let $A \subseteq \omega \times \mathcal{N}^n$ be universal for Σ_1^1. Then

$$G_1 = \{(x, y) \in \mathcal{N} \times \mathcal{N}^n : (\exists k) x(k) > 0 \wedge y \in A_k\}$$

is a Σ_1^1 set which is universal for $\boldsymbol{\Sigma}_1^0[\tau_n]$. The set $\mathcal{N} \times \mathcal{N}^n \setminus G_1$ is Π_1^1 and universal for $\boldsymbol{\Pi}_1^0[\tau_n]$, and

$$G_2 = \{(x, y) \in \mathcal{N}^\omega \times \mathcal{N}^n : (\exists k) y \in (\mathcal{N} \times \mathcal{N}^n \setminus G_1)_{x(k)}\}$$

is Π_1^1 and universal for $\boldsymbol{\Sigma}_2^0[\tau_n]$, and we can continue in this way to find universal sets for $\boldsymbol{\Sigma}_k^0[\tau_n]$ that are Σ_1^1 when k is odd and Π_1^1 when k is even. A similar analysis applies to τ^n.

Proposition 3. *Let σ be either τ_n or τ^n, and let $A \subseteq \mathcal{N} \times \mathcal{N}^n$ be a Σ_1^1 or Π_1^1 universal set for $\boldsymbol{\Sigma}_k^0[\sigma]$, depending on if k is odd or even. Then*

$$\{x \in \mathcal{N} : A_x \text{ is not } \sigma\text{-meagre}\}$$

is $\Delta_1^1(O)$, where O denotes Kleene's O.

Proof. Let $\sigma = \tau_n$. If $A \in \boldsymbol{\Sigma}_1^0[\sigma]$ then there is a sequence T_l of recursive trees on ω^{n+1} such that

$$A = \bigcup_{l \in \omega} p[T_l],$$

where $p[T_l]$ is the projection of the set $[T_l]$ of infinite branches through T_l. Now

$$A \text{ is not meagre} \iff (\exists l)[T_l] \neq \emptyset.$$

is clearly arithmetic in the sequence (T_l) and Kleene's O.

If $A \in \boldsymbol{\Sigma}_{k+1}^0[\sigma]$, find a sequence $B_l \in \boldsymbol{\Pi}_k^0[\sigma]$ such that

$$A = \bigcup_{l \in \omega} B_l.$$

Then

$$\begin{aligned}
A \text{ is not meagre} \iff & (\exists l) B_l \text{ is not meagre} \\
\iff & (\exists l)(\exists T \text{ recursive}) p[T] \setminus B_l \text{ is meagre} \\
\iff & (\exists l)(\exists T \text{ recursive}) \neg(p[T] \setminus B_l \text{ is not meagre})
\end{aligned}$$

which is arithmetic in O and the sequence (B_l) by the inductive hypothesis. The proof of the case $\sigma = \tau^n$ is similar. $\qquad \square$

Our next goal is to prove the following:

Proposition 4. *Let σ be τ_n or τ^n, and let $A \subseteq \mathscr{N} \times \mathscr{N}^n$ be a Σ_1^1 set universal for $\Sigma_1^1(\mathscr{N}^n)$. Then*

$$\{x \in \mathscr{N} : A_x \text{ is not } \sigma\text{-meagre}\}$$

is $\Sigma_1^1(O)$.

Before proving this we need a generalization of Proposition 1.5.2 in [12]. Let (X, σ) be a 2nd countable topological space and let $\mathcal{U}$ be a countable basis for the topology.

A function $f : \mathcal{U} \to \omega^{<\omega}$ is called $\mathcal{U}$-*monotone* if

$$(\forall U, V \in \mathcal{U}) U \subseteq V \implies f(V) \subseteq f(U).$$

For $x \in \mathscr{N}$ we define

$$(\lim_{\mathcal{U}} f)(x) = y \iff (\forall k)(\exists U \in \mathcal{U}) x \in U \wedge \mathrm{lh}(f(U)) \geq k \wedge f(U) \subseteq y.$$

The set

$$\{x \in \mathscr{N} : (\exists y)(\lim_{\mathcal{U}} f)(x) = y\}$$

is G_δ in the topology σ, and $\lim_{\mathcal{U}} f$ defines a function on this set. With these definitions we have the following analogue of [12, Proposition 1.5.2]:

Lemma 1 (Folklore). *Let (X, σ) be a 2nd countable topological space and let $\mathcal{U}$ be a countable basis for the topology σ. Then:*

(1) *If $Y \subseteq X$ is a G_δ set and $\bar{f} : Y \to \omega^\omega$ is continuous w.r.t. the usual topology on ω^ω and σ on Y, then there is a $\mathcal{U}$-monotone function $f : \mathcal{U} \to \omega^{<\omega}$ such that $\bar{f} = \lim_{\mathcal{U}} f$.*

(2) *If $f : \mathcal{U} \to \omega^{<\omega}$ is $\mathcal{U}$-monotone then $\lim_{\mathcal{U}} f$ is continuous on its domain (taking ω^ω with the usual topology and $\mathrm{dom}(\lim_{\mathcal{U}} f)$ with the topology induced by σ), and $\mathrm{dom}(\lim_{\mathcal{U}} f)$ is a G_δ set in the topology σ.*

Proof. (2) is clear from the definition. For (1), let $Y = \bigcap_{n \in \omega} W_n$, where the W_n are open sets. Let (U_n) enumerate $\mathcal{U}$. We can assume that $W_{n+1} \subseteq W_n$. Define $f : \mathcal{U} \to \omega^{<\omega}$ by letting $f(U_k)$ be the longest sequence s such that

$$(\forall l)(U_l \subseteq U_k \implies \bar{f}(U_l) \subseteq N_s) \wedge \mathrm{lh}(s) \leq \min\{k, \max\{n : U_k \subseteq W_n\}\}.$$

(Here N_s denotes the basic open neighborhood determined by s, i.e.

$$N_s = \{x \in \mathcal{N} : s \subseteq x\}.)$$

Since $\bar{f}$ is continuous it follows that if $x \in Y$ then for all $k, n \geq 0$ we can find $U_m \subseteq W_n$ such that $x \in U_m$ and $f(U_m) \subseteq N_s$ for some sequence s of length at least k. Thus $x \in \operatorname{dom}(\lim_{\mathcal{U}} f)$ and clearly $\bar{f}(x) = (\lim_{\mathcal{U}} f)(x)$. On the other hand, if $x \notin Y$ then there is n such that $x \notin W_n$. Thus $\operatorname{lh}(f(U_k)) \leq n$ for all $k \in \omega$, and so $x \notin \operatorname{dom}(\lim_{\mathcal{U}} f)$. $\qquad\square$

We now turn to the proof of Proposition 4. Recall that the set

$$X_{\text{low}} = \{x \in \omega^\omega : \omega_1^x = \omega_1^{\text{CK}}\}$$

is Σ_1^1 and furthermore that it is dense in the Gandy-Harrington topology, see e.g. Appendix A of [3].

Proof of Proposition 4. The proof follows the general lines of [12, Theorem 2.2.5]. For simplicity we consider the case $n = 1$, i.e. $\sigma = \tau$. Let $A \subseteq \mathcal{N}$ be $\boldsymbol{\Sigma}_1^1$ and not meagre, and let T be a tree on $\omega \times \omega$ such that $p[T] = A$. Then by the Jankov-von Neumann uniformization theorem [13, 18.1] we may find a $\mathcal{B}(\boldsymbol{\Sigma}_1^1)$ uniformising function $\bar{f} : A \to \mathcal{N}$ such that

$$(\forall x \in A)(x, \bar{f}(x)) \in [T].$$

(Here $\mathcal{B}(\boldsymbol{\Sigma}_1^1)$ denotes the σ-algebra generated by the $\boldsymbol{\Sigma}_1^1$ sets.) Since every $\boldsymbol{\Sigma}_1^1$ set has the Baire Property in τ it follows that the function $\bar{f}$ is Baire measurable when $\operatorname{dom}(\bar{f})$ is given the topology τ and $\operatorname{codom}(\bar{f})$ is given the usual topology. Since A has the Baire Property in τ we may find a τ-G_δ set $A' \subseteq A$ such that

(a) $A \setminus A'$ is τ-meagre,

(b) $\bar{f}|A'$ is continuous (w.r.t. τ on $\operatorname{dom}(\bar{f})$ and the usual topology in $\operatorname{codom}(\bar{f})$.)

(c) $A' \subseteq X_{\text{low}}$

Now let $B \subseteq \omega \times \mathcal{N}$ be Σ_1^1 such that

$$C \subseteq X_{\text{low}} \text{ is } \Sigma_1^1 \iff (\exists n)C = B_n.$$

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Let $\mathcal{U} = \{B_n : B_n \neq \emptyset\}$. Then by Lemma 1 we can find a monotone $f : \mathcal{U} \to \omega^{<\omega}$ such that $\bar{f}|A' = \lim_{\mathcal{U}} f$ and

$$(\forall s)(\forall n)(s|\operatorname{lh}(f(B_n \cap N_s)), f(B_n \cap N_s)|\operatorname{lh}(s)) \in T.$$

Thus

$$A \text{ is not meagre } \iff$$
$$(\exists f : \omega \to \omega^{<\omega})((\forall n)(B_n = \emptyset \implies f(n) = \emptyset) \wedge$$
$$(\forall m)(\forall n)(B_m \subseteq B_n \wedge B_m \neq \emptyset \implies f(n) \subseteq f(m)) \wedge$$
$$(\forall s \in \omega^{<\omega})(\forall n)(s|\operatorname{lh}(f(n)), f(n)|\operatorname{lh}(s)) \in T \wedge$$
$$\operatorname{dom}(\lim_{\mathcal{U}} f) \text{ is not meagre}).$$

where above, $\lim_{\mathcal{U}} f$ has the natural meaning if we think of f as being defined on $\mathcal{U}$, not on the indices of elements of $\mathcal{U}$.

If $f : \omega \to \omega^{<\omega}$ is (a code for a) monotone function then

$$x \in \operatorname{dom}(\lim_{\mathcal{U}} f) \iff (\forall k)(\exists n) x \in B_n \wedge \operatorname{lh}(f(n)) \geq k,$$

so "$\operatorname{dom}(\lim_{\mathcal{U}} f)$ is not meagre" is $\Delta_1^1(O, f)$ uniformly in f by Proposition 3.

The proof is finished by noting that the statement "$B_m \subseteq B_n$" may be replaced by the statement

$$\neg(B_m \setminus B_n \text{ is not meagre}).$$

To see this, note that by [3, Theorem A.1.6] we have that if $D \subseteq X_{\text{low}}$ is Σ_1^1 then D is τ-clopen in X_{low}. Thus $B_m \setminus B_n = \emptyset$ iff $B_m \setminus B_n$ is meagre. Since by Proposition 3 the statement "$B_m \setminus B_n$ is not meagre" is $\Delta_1^1(O)$, this finishes the proof. $\square$

5.2   The Choquet and Banach-Mazur games

Let $\sigma = \tau_n$ or $\sigma = \tau^n$. Recall the *strong Choquet game* $G_{(\mathcal{N}^n, \sigma)}$:

$$
\begin{array}{ccccc}
\text{I} & x_0, U_0 & & x_1, U_1 & \\
& & & & \cdots \\
\text{II} & & V_0 & & V_1
\end{array}
$$

Players I and II take turns playing. The ith move for Player I consists of a basic open set U_i and a point $x_i \in U_i$. Player II must respond by playing a basic open set $V_i \subseteq U_i$ such that $x_i \in V_i$. Then Player I is required to respond with x_{i+1} and U_{i+1} such that $x_{i+1} \in U_{i+1} \subseteq V_i$. Player II wins iff

$$\bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n \neq \emptyset.$$

It is well-known that II has a winning strategy in the strong Choquet game in $(\mathcal{N}, \tau)$, see e.g. [3, Theorem 4.1.5]. Moreover, the winning strategy for II described in the proof there is Δ_1^1 in the codes. From this we easily get:

Corollary 6. *Let $n \geq 1$ and let $\sigma = \tau_n$ or $\sigma = \tau^n$. Then II has a winning strategy in $G_{(\mathcal{N}^n, \sigma)}$ which is Δ_1^1 in the codes.*

What about the Banach-Mazur game in $(\mathcal{N}^n, \sigma)$? Recall that the Banach-Mazur game $G_\sigma^{**}(A)$, where $A \subseteq \mathcal{N}^n$ is non-empty, is played as follows: Players I and II take turns playing basic open sets U_i and V_i,

$$\text{I} \quad U_0 \qquad\quad U_1$$
$$\cdots$$
$$\text{II} \qquad\quad V_0 \qquad\quad V_1$$

and the players are required to maintain that $U_i \supseteq V_i \supseteq U_{i+1}$ for all $i \geq 0$. II wins iff

$$\bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n \subseteq A.$$

It is well-known (see e.g. [13, 8.33]) that $A \subseteq \mathcal{N}^n$ is comeagre (in σ) if and only if II has a winning strategy in $G_\sigma^{**}(A)$. By (ii) of [13, 8.33], it also follows that A is meagre in a non-empty open set if and only if I has a winning strategy.

In the case $\sigma = \tau_n$ any Δ_1^1 set $A \subseteq \mathcal{N}^n$ is of course σ-clopen, and so if $\mathcal{N}^n \setminus A \neq \emptyset$ then I clearly wins simply by playing $\mathcal{N}^n \setminus A$ in the first move. For $\sigma = \tau^n$ the situation is more complicated:

Proposition 5. *Let $n \geq 2$. If $A \subseteq \mathcal{N}^n$ is Δ_1^1 then there is a winning strategy in $G_{\tau^n}^{**}(A)$ which is $\Delta_1^1(O)$ in the codes.*

Proof. The proof is a variation of [12, Theorem 4.2.1]. For notational simplicity, we deal with the case $n = 2$. Moreover, following [13, Definition 8.25] we will use the notation

$$U \Vdash A,$$

where U is a basic open set and A some subset, to mean that A is comeagre in U, i.e. $U \setminus A$ is meagre. Finally, we fix a Δ_1^1 winning strategy for II in the strong Choquet game in $(\mathcal{N}^2, \tau^2)$. Since there plainly is a danger of confusion here, we will refer to the players of the *strong Choquet game* as I_C and II_C. I and II then refers to the players in the *Banach-Mazur game*.

Without loss of generality, assume that I wins $G_{\tau^2}^{**}(A)$, i.e. $\mathcal{N}^2 \setminus A$ is comeagre in some open set. We will describe a $\Delta_1^1(O)$ winning strategy for I. Player I will be aided by playing (as I_C) a strong Choquet game concurrently with the Banach-Mazur game. Schematically:

$$
\begin{array}{llll}
I_C & : & x_0, B_{k_0} \qquad\qquad\qquad x_2, B_{k_2} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \\
II_C & : & \qquad\quad B_{n_0} \qquad\qquad\qquad\quad B_{n_2} \\
I & : & \qquad\qquad B_{n_0} \qquad\qquad\qquad\qquad B_{n_2} \\
& & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \\
II & : & \qquad\qquad\quad B_{n_1} \qquad\qquad\qquad\qquad B_{n_3}
\end{array}
$$

Fix a Δ_1^1-scale $\{\varphi_m\}_{m \in \omega}$ on $\mathcal{N}^2 \setminus A$. For $x \in \mathcal{N}^2$ let

$$\psi_m(x) = \langle \varphi_0(x), x(0), \cdots, \varphi_m(x), x(m) \rangle,$$

where as in [12], $\langle \gamma_0, \dots \gamma_m \rangle$ is the rank of $(\gamma_0 \dots, \gamma_m)$ in the lexicographic order on $\mathbb{ON}^{<\omega}$. Note that X_{low}^2 is open and dense in $(\mathcal{N}, \tau^2)$. Let $B \subseteq \omega \times \mathcal{N}^2$ be a Σ_1^1 parametrization of

$$\{C_0 \times C_1 : C_0, C_1 \in \Sigma_1^1(\mathcal{N}), C_0, C_1 \subseteq X_{\text{low}}\}.$$

First find B_{k_0} where k_0 is least such that $B_{k_0} \neq \emptyset$ and $B_{k_0} \Vdash \mathcal{N}^2 \setminus A$, and let $x_0 \in B_{k_0}$ be computable in O. Let B_{n_0} be the response of II_C according to the fixed winning strategy in the strong Choquet game when I_C plays x_0, B_{k_0}. I's first move in the Banach-Mazur game is then B_{n_0}. Suppose II responds by playing B_{n_1}. Let $s_1 \in \omega^{<\omega}$ be the least sequence of length 1 such that $B_{n_1} \cap N_{s_1}$ is not meagre. Now

$$
\begin{aligned}
A_2 = \{ & x : x \in \mathcal{N}^2 \setminus A \wedge x \in B_{n_1} \cap N_{s_1} \wedge \\
& \{y : y \in \mathcal{N}^2 \setminus A \wedge y \in B_{n_1} \cap N_{s_1} \wedge \psi_1(x) = \psi_1(y)\} \text{ is not meagre } \wedge \\
& \{y : y \in \mathcal{N}^2 \setminus A \wedge x \in B_{n_1} \cap N_{s_1} \wedge \psi_1(y) < \psi_1(x)\} \text{ is meagre}\}\}
\end{aligned}
$$

is non-meagre, $\Delta_1^1(O)$ and $A_2 \subseteq N_{s_1} \cap \mathcal{N}^2 \setminus A$.

Now let k_2 be least such that $B_{k_2} \neq \emptyset$, $B_{k_2} \subseteq B_{n_1}$ and $B_{k_2} \Vdash A_2$. We may find k_2 in a $\Delta_1^1(O)$ way since, as in the proof of Proposition 4, $B_{k_2} \subseteq B_{n_1}$ may be expressed by saying that $B_{k_2} \setminus B_{n_1}$ is meagre since we work on X_{low}^2. Let $x_2 \in B_{k_2}$ be computable in O and let I_C play x_2, B_{k_2} in the strong Choquet game. II_C responds with B_{n_2}. I plays B_{n_2} in the Banach-Mazur game.

Suppose II responds by playing B_{n_3}. Let s_3 be the least sequence of length 3 such that $s_1 \subseteq s_3$ and $N_{s_3} \cap B_{n_3}$ is not meagre. We let

$$
\begin{aligned}
A_4 = \{&x : x \in A_2 \wedge x \in B_{n_3} \cap N_{s_3} \wedge \{y : y \in A_2 \wedge y \in B_{n_3} \cap N_{s_3} \wedge \\
&\psi_3(x) = \psi_3(y)\} \text{ is not meagre } \wedge \\
&\{y : y \in A_2 \wedge y \in B_{n_3} \cap N_{s_3} \wedge \psi_3(y) < \psi_3(x)\} \text{ is meagre}\}\}.
\end{aligned}
$$

Then $A_4 \subseteq N_{s_3} \cap A_2$ and A_4 is $\Delta_1^1(O)$ and non-meagre, and so we let I_C play x_4, B_{k_4} in the strong Choquet game, where k_4 is least such that $\emptyset \neq B_{k_4} \subseteq B_{n_3}$, $B_{k_4} \Vdash A_4$ and $x_4 \in B_{k_4}$ is computable in O. II_C responds with B_{n_4}, and I plays B_{n_4} in the Banach Mazur game, and so on. At the end of this run of the Banach-Mazur game we have produced a sequence of sets B_{n_i} and a real $\alpha = \bigcup_{i \in \omega} s_{2i+1}$.

Note that since II_C wins the strong Choquet game we have that

$$
\bigcap_{i \in \omega} B_{n_i} \neq \emptyset.
$$

Clearly, it must then be the case that

$$
\bigcap_{i \in \omega} B_{n_i} = \{\alpha\}.
$$

We claim that $\alpha \notin A$. For this, note that by construction we can find a sequence (x_i) such that $x_i \in A_{2i}$ and $s_{2i-1} \subseteq x_i$. For this sequence it holds for all m that $\psi_{2m+1}(x_i)$ is constant for $i \geq m$. Thus $\varphi_m(x_i)$ is eventually constant, and since $x_i \to \alpha$ it follows by the properties of a scale that $\alpha \in \mathcal{N}^2 \setminus A$. Thus I wins this run of the game. $\qquad\square$

Remark. The previous proof relativizes to a parameter in the following way: If $A \subseteq \mathcal{N}^n$ is $\Delta_1^1(z)$ for some real z then one of the players has a $\Delta_1^1(O, z)$ winning strategy in the game $G_{\tau^n}^{**}(A)$. It is easy to see that the same proof also goes through for $G_{\tau_n}^{**}(A)$. Thus we have

Corollary 7. *Let $n \geq 1$ and let $\sigma = \tau^n$ or $\sigma = \tau_n$. If $A \subseteq \mathcal{N}^n$ is $\Delta_1^1(z)$ then there is a winning strategy in $G_\sigma^{**}(A)$ which is $\Delta_1^1(O, z)$ in the codes. In particular, if A is $\Delta_1^1(O)$ then there is a $\Delta_1^1(O)$ winning strategy in $G_\sigma^{**}(A)$.*

From this we get:

Corollary 8. *Let $n \geq 1$ and $\sigma = \tau^n$ or $\sigma = \tau_n$. Suppose $A \subseteq \mathcal{N}^n$ is $\Delta_1^1(O)$ and comeagre. Then there is $C \subseteq \omega \times \mathcal{N}^n$ such that*

(1) *C is $\Sigma_1^1(O)$,*

(2) *For all $n \in \omega$, C_n is σ-open and dense*

(3) *$\bigcap_{n \in \omega} C_n \subseteq A$.*

Proof. As in the proof of Corollary 4.2.4 in [12], we need only note that the construction given in the proof of Theorem 6.1 in [16] produce the desired set C. To see this, fix a Σ_1^1 set $B \subseteq \omega \times \mathcal{N}^n$ such that

$$U \subseteq \mathcal{N}^n \text{ is a basic } \sigma\text{-open set} \iff (\exists n)U = B_n.$$

If we use the strategy described in Proposition 5 above in the proof of Theorem 6.1 in [16], we will obtain a sequence $W_n \subseteq \omega$ of $\Delta_1^1(O)$ sets (uniformly in n) such that

$$C_n = \bigcup_{k \in W_n} B_k$$

is open dense and

$$\bigcap C_n \subseteq A.$$

Thus

$$(n, x) \in C \iff (\exists k)k \in W_n \wedge x \in B_k$$

gives a $\Sigma_1^1(O)$ definition of a set B that is as required. $\square$

6 Parameters in the basic dichotomy theorems

The results of the previous section show that complexity computations involving category notions in the Gandy-Harrington topology can be carried out using Kleene's O as a parameter. In this section we will use this

(specifically, Corollary 8 above) to show that the proofs of the Silver and Harrington-Kechris-Louveau dichotomy Theorems produce reductions that are no worse than $\Delta_1^1(O)$. We also show that this is in some sense the best possible result we can hope for.

We start with Silver's dichotomy:

Theorem 15 (Silver's dichotomy). *Let E be a Hyp equivalence relation on $\mathcal{N}$. Then either*

$$E \leq_H =_\omega$$

or

$$=_{\mathcal{P}(\omega)} \leq_{\Delta_1^1(O)} E.$$

Proof. By Harrington's well-known proof of Silver's dichotomy (see e.g. [10, Theorem 32.1] or [3, Theorem 5.3.5]), either (*i*) every E-equivalence class contains a non-empty Hyp set, or else (*ii*) there is a Σ_1^1 set $H \subseteq \mathcal{N}$ such that $E \cap H \times H$ is τ^2-meagre in $H \times H$. In case (*i*) by Proposition 2 there is a Hyp reduction of E to $=_\omega$. We show that in case (*ii*) there is a $\Delta_1^1(O)$ reduction of $=_{\mathcal{P}(\omega)}$ to E.

By Corollary 8 we may find a $\Sigma_1^1(O)$ set $C \subseteq \omega \times \mathcal{N}^2$ such that C_n is τ^2-open dense in $H \times H$ and

$$\bigcap_{n \in \omega} C_n \subseteq H \times H \setminus E.$$

We may assume that $C_{n+1} \subseteq C_n$ for all n. Harrington's proof (as presented in [3] or [10]) now produces a reduction of $=_{\mathcal{P}(\omega)}$ to E which is Hyp relative to the sequence C_n. To see this, fix a Hyp winning strategy for II in the strong Choquet game $G_{(\mathcal{N}^2,\tau^2)}$. Then we may easily define a scheme consisting of basic open sets $(U_s)_{s \in 2^{<\omega}}$, $(V_s)_{s \in 2^{<\omega}}$ and points $(x_s)_{s \in 2^{<\omega}}$ such that $s \mapsto (U_s, V_s, x_s)$ is $\Delta_1^1(O)$ (in the codes) and the following conditions hold:

(1) $U_\emptyset = V_\emptyset = H$

(2) For each $s \in 2^{<\omega}$ the following is a play according to II's winning strategy:

I	$x_{s\restriction 1}, U_{s\restriction 1}$		$x_{s\restriction 2}, U_{s\restriction 2}$		$\cdots$		x_s, U_s
II		$V_{s\restriction 1}$		$V_{s\restriction 2}$		$\cdots$	V_s

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(3) $\text{diam}(U_s) < 2^{-\text{lh}(s)}$ (with respect to the usual metric on $\mathcal{N}$).

(4) $U_{s^\frown 0} \times U_{s^\frown 1} \subseteq C_{\text{lh}(s)}$.

If we define for $x \in 2^\omega$

$$f(x) = y \iff y \in \bigcap_{n \in \omega} V_n$$

then f is a $\Delta_1^1(O)$ function and is easily seen to be a reduction of $=_{\mathcal{P}(\omega)}$ to E. $\qquad\square$

For the Glimm-Effros dichotomy due to Harrington, Kechris and Louveau we have:

Theorem 16 (Harrington-Kechris-Louveau [5])**.** *Let E be a Hyp equivalence relation on $\mathcal{N}$. Then either*

$$E \leq_H =_{\mathcal{P}(\omega)}$$

or

$$E_0 \leq_{\Delta_1^1(O)} E.$$

Proof. There are again two cases: (1) $E = E^*$, where E^* is the closure of E in the topology τ^2, or (2) $E \neq E^*$.

In the first case, it was observed in [5], p. 922, that there is a Hyp reduction of E to $=_{\mathcal{P}(\omega)}$. So we only have to handle the 2nd case.

We will follow the exposition of the proof of the Harrington-Kechris-Louveau Theorem given in [3, §6.3]. Since $E \neq E^*$ the set

$$X = \{x \in \mathcal{N} : (\exists y)yE^*x \wedge \neg yEx\}.$$

is non-empty and Σ_1^1. By [3, Lemma 6.3.8] E is dense and meagre in $X^2 \cap E^*$. By Corollary 8, we may find $C \subseteq \omega \times \mathcal{N}^2$ such that C_m is τ^2-open dense in X^2, $m \leq n \implies C_m \supseteq C_n$, and

$$\bigcap_{n \in \omega} C_n \subseteq X^2 \setminus E.$$

Define the auxiliary R_k relations, $k \in \omega$, in $2^{<\omega}$ by

$$s R_k t \iff \mathrm{lh}(s) = \mathrm{lh}(t) \wedge (\forall i < k) s(i) = t(i) = 0$$
$$\wedge\, s(k) \neq t(k) \wedge (\forall i < \mathrm{lh}(s))(i > k \implies s(i) = t(i)).$$

We also let

$$R = \bigcup_{k \in \omega} R_k.$$

Fix winning Δ_1^1 strategies for II in the strong Choquet games on $(\mathcal{N}, \tau)$ and $(\mathcal{N}^2, \tau_2)$. Following [3, Lemma 6.3.10], it suffices to construct a scheme consisting of τ-basic open sets $(U_s)_{s \in 2^{<\omega}}$, $(V_s)_{s \in 2^{<\omega}}$ that are subsets of X, points $(x_s)_{s \in 2^{<\omega}}$ in X, and basic τ_2-open sets $(F_{s,t})_{sRt}$, $(E_{s,t})_{sRt}$ that are subsets of $X^2 \cap E$ such that

(i) $U_\emptyset = V_\emptyset = X$

(ii) For each $s \in 2^{<\omega}$ the following is a play according to II's winning strategy:

$$
\begin{array}{cccc}
\text{I} \quad x_{s\upharpoonright 1}, U_{s\upharpoonright 1} & x_{s\upharpoonright 2}, U_{s\upharpoonright 2} & \cdots & x_s, U_s \\[2mm]
\text{II} \quad \qquad V_{s\upharpoonright 1} & V_{s\upharpoonright 2} & \cdots & V_s
\end{array}
$$

(iii) $\mathrm{diam}(U_s) < 2^{-\mathrm{lh}(s)}$ (with respect to the usual metric on $\mathcal{N}$).

(iv) $U_{s^\frown 0} \times U_{s^\frown 1} \subseteq C_{\mathrm{lh}(s)}$.

(v) If $\mathrm{lh}(s) = \mathrm{lh}(t)$ and $s R_k t$ then the following is a play according to II's winning strategy in the Choquet game on $(\mathcal{N}^2, \tau_2)$:

$$
\begin{array}{cccc}
\text{I} \quad (x_{s\upharpoonright 1}, x_{t\upharpoonright 1}), F_{s\upharpoonright 1, t\upharpoonright k} & \cdots & (x_s, x_t), F_{s,t} \\[2mm]
\text{II} \quad \qquad E_{s\upharpoonright 1, t\upharpoonright 1} & \cdots & E_{s,t}
\end{array}
$$

(vi) If sRt then $\mathrm{diam}(F_{s,t}) < 2^{-\mathrm{lh}(s)}$ (with respect to the usual metric on $\mathcal{N}^2$).

The construction of this scheme given in [3] can easily be carried out so that the function $s \mapsto (x_s, U_s, V_s)$ is Δ_1^1 relative to the set C. Thus $s \mapsto (x_s, U_s, V_s)$ is $\Delta_1^1(O)$ and so the function defined by

$$f(x) = y \iff y \in \bigcap_{n \in \omega} V_n$$

is $\Delta_1^1(O)$. Finally, the arguments of [3, p. 146–147] show that f is a reduction of E_0 to E. $\qquad \square$

We will now show that Theorem 15 and 16 are in some sense optimal:

Theorem 17. *Let z be a real in which O is not hyperarithmetic. Then:*

(i) *There is a Hyp equivalence relation E such that $=_{\mathcal{P}(\omega)} \leq_{\Delta_1^1(O)} E$, but $=_{\mathcal{P}(\omega)} \nleq_{\Delta_1^1(z)} E$.*

(ii) *There is a Hyp equivalence relation E such that $E_0 \leq_{\Delta_1^1(O)} E$, but $E_0 \nleq_{\Delta_1^1(z)} E$.*

We need the following Lemma:

Lemma 2. *Suppose z is a real such that every non-empty Π_1^0 set in $\mathcal{N}$ contains a real hyperarithmetic in z. Then O is hyperarithmetic in z.*

Proof. Let $C \subseteq \omega \times \mathcal{N}$ be Π_1^0 and universal for Π_1^0, and let

$$\hat{O} = \{n : C_n = \emptyset\}.$$

By our assumption,

$$\omega \setminus \hat{O} = \{n : (\exists x \in \Delta_1^1(z))x \in C_n\},$$

which is both Σ_1^1 and $\Pi_1^1(z)$. Thus $\hat{O}$ (and therefore also Kleene's O) is $\Delta_1^1(z)$. $\qquad \square$

Proof of Theorem 17. Suppose z is a real in which O is not hyperarithmetic. Then by the previous Lemma there is a non-empty Π_1^0 set $F \subset \mathcal{N}$ which does not contain any elements hyperarithmetic in z, and in particular is uncountable. To prove (i), we let

$$xEy \iff x, y \notin F \vee (x, y \in F \wedge x = y)$$

Then E has uncountably many classes. If $f : 2^\omega \to \mathcal{N}$ were a function witnessing that $=_{\mathcal{P}(\omega)} \leq_{\Delta^1_1(z)} E$ then $f(\bar{0}) \in F$ or $f(\bar{1}) \in F$, which contradicts that F contains no real which is $\Delta^1_1(z)$.

To prove (ii), we instead define E on $\mathcal{N} \times 2^\omega$ by

$$(x_0, y_0)E(x_1, y_1) \iff x_0, x_1 \notin F \lor (x_0, x_1 \in F \land x_0 = x_1 \land y_0 E_0 y_1).$$

Clearly E is not smooth. If $E_0 \leq_{\Delta^1_1(z)} E$ and $f : 2^\omega \to \mathcal{N} \times 2^\omega$ witness this, then the function

$$\pi(x) = x_0 \iff (\exists y_0) f(x) = (x_0, y_0)$$

is constant on E_0 classes. It follows that there is some $x_0 \in F$ such that

$$\{x \in 2^\omega : \pi(x) = x_0\},$$

is comeagre. Since

$$y \in \{x_0\} \iff \{x \in 2^\omega : \pi(x) = y\} \text{ is not meagre}$$

it follows by [12, Theorem 2.2.5] that $\{x_0\}$ is $\Sigma^1_1(z)$, and so $x_0 \in F$ is $\Delta^1_1(z)$, a contradiction. $\qquad\square$

Remark. Let E be a Hyp equivalence relation with $n \leq \omega$ many classes. Then by Proposition 2, $E \leq_H =_n$. On the other hand, since any E-class is Hyp it must contain a real which is hyperarithmetic in O. Thus there is a $\Delta^1_1(O)$ reduction of $=_n$ to E. We have the following dichotomy:

Theorem 18 (The Finite Dichotomy Theorem). *Let E be a Hyp equivalence relation on $\mathcal{N}$. Then:*

(a) If $n < \omega$ then either $E \leq_H =_n$ or $=_{n+1} \leq_{\Delta^1_1(O)} E$.

(b) Either there is $n < \omega$ such that $E \leq_H =_n$, or $=_\omega \leq_{\Delta^1_1(O)} E$.

This Theorem is again optimal by an argument similar to that given for Theorem 17.

We conclude the paper with the following questions. The first question seems related to [9, Question 6.1.B].

Question 1. *Can a Hyp equivalence relation E be Borel reducible to E_0 but not Hyp reducible to E_0?*

Question 2. *Are there any Hyp-degrees of Hyp equivalence relations other than $=_1$ that are comparable with all other Hyp-degrees?*

Question 3. *What is the complexity of the first order theory of the partial order of Hyp-degrees of Hyp equivalence relations?*

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