SUBCOMPACT CARDINALS, SQUARES, AND STATIONARY REFLECTION

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Abstract. We generalise Jensen's result on the incompatibility of subcompactness with $\square$. We show that α^+-subcompactness of some cardinal less than or equal to α precludes $\square_\alpha$, but also that square may be forced to hold everywhere where this obstruction is not present. The forcing also preserves other strong large cardinals. Similar results are also given for stationary reflection, with a corresponding strengthening of the large cardinal assumption involved. Finally, we refine the analysis by considering Schimmerling's hierarchy of weak squares, showing which cases are precluded by α^+-subcompactness, and again we demonstrate the optimality of our results by forcing the strongest possible squares under these restrictions to hold.

§1. Introduction. A well known result of Solovay [19] is that $\square_\alpha$ must fail for all α greater than or equal to a supercompact cardinal. Jensen refined this result, showing that if κ is subcompact then $\square_\kappa$ fails (see for example [12, Proposition 8]). Jensen's result can be seen to be more or less optimal for $\square_\kappa$ with κ a large cardinal, as Cummings and Schimmerling [7, Section 6] have shown that one can force $\square_\kappa$ to hold for κ 1-extendible, a property just short of subcompactness. However, as is shown below, forcing $\square_\alpha$ at all cardinals which are not subcompact necessarily entails the destruction of stronger large cardinal properties. Moreover, $\square_\kappa$ can hold for κ a Vopěnka cardinal, a consistency-wise stronger assumption which however does not directly imply subcompactness — see [2].

In this article we obtain an optimal result regarding the consistency of square with large cardinals. Specifically, we show that $\square_\alpha$ must fail whenever there is a $\kappa \leq \alpha$ that is α^+-subcompact (appropriately defined). Also, under the GCH, $\square_\alpha$ may be forced to hold at all other cardinals, preserving all β-subcompact cardinals of the ground model for all β, along with other large cardinals, of which we give ω-superstrong cardinals as an example.

Stationary reflection is a combinatorial principle that has more recently come to prominence, and which may be viewed as a strong negation of $\square$. With this strengthening comes a strengthening of the large cardinal needed to imply it: we show that if some κ is α^+-*stationary subcompact* (see Definition 11), then stationary reflection

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holds at α^+. Moreover this is again in some sense optimal: under GCH, we can force to have stationary reflection fail everywhere where stationary subcompactness does not require it to hold (and indeed, having $\square_\alpha$ everywhere possible, as above), whilst preserving the pattern of restrictions due to subcompactness and stationary subcompactness, as well as ω-superstrong cardinals.

In a slightly different vein, there is a hierarchy of weak forms of $\square_\alpha$ introduced by Schimmerling [17]. We show that known results ruling out such weak squares from supercompact cardinals have subcompactness analogues. Moreover, under the GCH these results are again optimal, as we are able to force to obtain a universe in which for every cardinal α the strongest form of $\square_\alpha$ not so precluded holds.

§2. Preliminaries. For any regular carinal α, we denote by H_α the set of all sets of hereditary cardinality strictly less than α. We denote by Lim the class of limit ordinals, and by $\mathrm{Cof}(\alpha)$ the class of ordinals of cofinality α. For any set of ordinals C, we denote by $\mathrm{ot}(C)$ the order type of C and by $\lim(C)$ the set of limit points of C.

Definition 1. *For any cardinal α, a* $\square_\alpha$-sequence *is a sequence $\langle C_\beta \mid \beta \in \alpha^+ \cap Lim\rangle$ such that for every $\beta \in \alpha^+ \cap Lim$,*

- *C_β is a closed unbounded subset of β,*
- $\mathrm{ot}(C_\beta) \leq \alpha$,
- *for any $\gamma \in \lim(C_\beta)$, $C_\gamma = C_\beta \cap \gamma$.*

We say $\square_\alpha$ holds if there exists a $\square_\alpha$-sequence.

The principle $\square_\alpha$ should be viewed as a property of α^+ rather than α: indeed, we shall use below the fact that $\square_\alpha$ can be forced over a model of GCH without changing H_{α^+}. The point is also emphasized by the relationship of $\square$ to *stationary reflection*.

Definition 2. *For regular $\kappa > \lambda$, $\mathrm{SR}(\kappa, \lambda)$ is the statement that for every stationary subset S of $\kappa \cap \mathrm{Cof}(\lambda)$, there is a $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in γ.*

Note that $\square_\alpha$ refutes $\mathrm{SR}(\alpha^+, \lambda)$ for every $\lambda \leq \alpha$: the function $\xi \mapsto \mathrm{ot}(C_\xi)$ from $(\alpha^+ \smallsetminus \alpha + 1) \cap \mathrm{Cof}(\lambda)$ to $\alpha + 1$ is regressive, and so is constant on a stationary set S. But now if $S \cap \gamma$ is stationary in γ, then a pair of distinct elements of $S \cap \lim(C_\gamma)$ can be found, violating coherence.

We now define the large cardinals that we shall be considering.

Definition 3. *For any cardinal α, we say that a cardinal $\kappa < \alpha$ is* α-subcompact *if for every $A \subseteq H_\alpha$, there exist $\bar\kappa < \bar\alpha < \kappa$, $\bar A \subseteq H_{\bar\alpha}$, and an elementary embedding*

$$\pi : (H_{\bar\alpha}, \in, \bar A) \to (H_\alpha, \in, A)$$

with critical point $\bar\kappa$ such that $\pi(\bar\kappa) = \kappa$. We say that such an embedding π witnesses the α-subcompactness of κ for A. *If $\kappa < \alpha$ and*

*κ is β-subcompact for every β strictly between κ and α, we say that κ is $<\alpha$-*subcompact.

Note that κ^+-subcompactness of κ is Jensen's original notion of subcompactness. Also note that since a finite sequences of subsets of H_α may be encoded into a single subset of H_α (for example, with pairs (i, x) for x in the ith subset), we may use structures with any finite number of sets A_i rather than just one. Typical arguments show that if κ is α-subcompact then κ is inaccessible, and indeed we shall show below that it is a very much stronger large cardinal assumption, at the level of supercompactness.

If κ is α-subcompact and $\kappa < \beta < \alpha$, then κ is also β-subcompact. Further, if the GCH holds then H_{α^+} contains all the necessary sets to witness that κ is α-subcompact. Thus, if $\pi : (H_{\bar{\alpha}^+}, \in, \bar{A}) \to (H_{\alpha^+}, \in, A)$ with $\mathrm{cp}(\pi) = \bar{\kappa}$ witnesses α^+-subcompactness of κ for some (arbitrary) $A \subseteq H_{\alpha^+}$, then $\bar{\kappa}$ is $\bar{\alpha}$-subcompact by elementarity. Further, if α is a limit cardinal and $\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)$ with critical point $\bar{\kappa}$ witnessess α-subcompactness of κ with respect to A, then $\bar{\kappa}$ is $<\bar{\alpha}$-subcompact.

The requirement in Definition 3 that $\bar{\alpha}$ be less than κ is a natural one similar to those that are made for a range of other large cardinal axioms: for example, the requirement that $j(\kappa) > \lambda$ for j an embedding witnessing the λ-supercompactness of some κ. As in those cases, this restriction is mostly just a convenience, only ruling out circumstances which are consistency-wise much stronger, as we shall now demonstrate. To this end, let us temporarily define a cardinal κ to be *loosely α-subcompact* if it satisfies the requirements to be α-subcompact except that the cardinals $\bar{\alpha}$ need not be less than κ.

First note that we may usually assume that $\bar{\alpha}$ is strictly less than α.

LEMMA 4. *Suppose κ is loosely α-subcompact,* $\mathrm{cf}(\alpha) > \omega$, *and $A \subseteq H_\alpha$. Then there is an elementary embedding $\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)$ witnessing the α-subcompactness of κ for A such that $\bar{\alpha}$ is strictly less than α.*

PROOF. The proof is just as in Kunen's proof [15] that there can be no nontrivial elementary embedding from $V_{\lambda+2}$ to $V_{\lambda+2}$. Specifically, let $f : [\alpha]^\omega \to \alpha$ be ω-Jónsson for α, that is, for any subset X of α of cardinality α, $f``[X]^\omega = \alpha$; such functions were shown to exist for all α by Erdős and Hajnal [10]. Note that f is a subset of H_α, so there will be an elementary embedding $\pi : (H_{\bar{\alpha}}, \in, \bar{A}, \bar{f}) \to (H_\alpha, \in, A, f)$ witnessing the α-subcompactness of κ for A and f. We claim that this π, when considered as a function from $(H_{\bar{\alpha}}, \in, \bar{A})$ to $(H_\alpha, \in, A)$, satisfies the requirements of the lemma, namely, that $\bar{\alpha} < \alpha$. For suppose $\bar{\alpha}$ were to equal α. Then since $|j``\bar{\alpha}| = \bar{\alpha}$ we would have $f``[j``\bar{\alpha}]^\omega = \alpha$, and so there would be some $s \in [j``\bar{\alpha}]^\omega$ such that $f(s) = \bar{\kappa}$. But now since $\omega < \bar{\kappa}$, s is of the form $j(t)$ for some

$t \in [\bar{\alpha}]^\omega$. By elementarity, $j(\bar{f}(t)) = f(j(t))$, so $\bar{\kappa}$ is in the range of j, a contradiction. $\dashv$

In particular, if κ is the critical point of a rank-plus-one-to-rank-plus-one embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ then it is *not* the case that for all $A \subseteq H_{\lambda^+}$, j witnesses the loose λ^+-subcompactness of κ for A.

PROPOSITION 5. *If κ is loosely α-subcompact, then κ is α-subcompact or κ is the critical point of a rank-to-rank embedding $j : V_\alpha \to V_\alpha$.*

PROOF. From the fact that loose α-subcompactness implies loose β-subcompactness for $\beta < \alpha$, and using Lemma 4, if κ is loosely α-subcompact and there is some $A \subseteq H_\alpha$ such that every $\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)$ witnessing loose α-subcompactness of κ for A has $\bar{\alpha} = \alpha$, then $\alpha = \sup_{n\in\omega}(\pi^n(\kappa))$ for such a π. So α is the limit of inaccessible cardinals, whence $H_\alpha = V_\alpha$. Moreover, as the image of the critical point $\bar{\kappa}$ of $\pi : V_\alpha \to V_\alpha$, κ is the critical point of

$$\pi[\pi] = \bigcup_{\gamma<\alpha} \pi(\pi \restriction V_\gamma)$$

in the left self-distributive system of elementary embeddings from V_α to V_α (see for example [8] for more on such embeddings).

So suppose that for each $A \subseteq H_\alpha$ there is a $\pi_A : (H_{\bar{\alpha}_A}, \in, \bar{A}) \to (H_\alpha, \in, A)$ with critical point $\bar{\kappa}_A$ witnessing the loose α-subcompactness of κ for A such that $\bar{\alpha}_A < \alpha$. For each A take π_A with $\bar{\kappa}_A$ minimal, and with $\bar{\alpha}_A$ minimal amongst those for our fixed $\bar{\kappa}_A$. Then we claim that $\bar{\alpha}_A < \kappa$. For otherwise, we may use the fact that κ is $\bar{\alpha}_A$-subcompact. More specifically, the restriction of

$$\pi_{A,H_{\bar{\alpha}_A},\bar{A},\{\bar{\kappa}\}} : (H_{\bar{\alpha}'}, \in, \bar{A}', H_{\bar{\bar{\alpha}}_A}, \bar{\bar{A}}, \{\bar{\bar{\kappa}}\}) \to (H_\alpha, \in, A, H_{\bar{\alpha}_A}, \bar{A}, \{\bar{\kappa}\})$$

to $H_{\bar{\bar{\alpha}}_A}$ gives an elementary embedding

$$\rho : (H_{\bar{\bar{\alpha}}_A}, \in, \bar{\bar{A}}\ ) \to (H_{\bar{\alpha}_A}, \in, \bar{A}),$$

with critical point at least $\bar{\kappa}$ by the minimality of $\bar{\kappa}$. Since $\rho(\bar{\bar{\kappa}}) = \bar{\kappa}$, $\bar{\bar{\kappa}} = \bar{\kappa}$ and $\mathrm{cp}(\rho)$ is in fact strictly greater than $\bar{\kappa}$. But also $\bar{\bar{\alpha}} < \bar{\alpha}$, so $\pi_A \circ \rho : (H_{\bar{\bar{\alpha}}_A}, \in, \bar{\bar{A}}\ ) \to (H_\alpha, \in, A)$ is an elementary embedding with critical point $\bar{\kappa}$ witnessing the α-subcompactness of κ for A with $\bar{\bar{\alpha}} < \bar{\alpha}$, contradicting the choice of $\bar{\alpha}$. $\dashv$

Now to the matter of the consistency strength of subcompactness itself. It turns out that the levels of subcompactness interleave with the levels of supercompactness in strength. Indeed one gets a result much like Magidor's characterisation of supercompactness [16], just with H_α in place of V_α and the predicate A added.

PROPOSITION 6. 1. *If κ is $2^{<\alpha}$-supercompact, then κ is α-subcompact.*

2. *If κ is $(2^{(\lambda^{<\kappa})})^+$-subcompact, then κ is λ-supercompact.*

In particular, κ is supercompact if and only if κ is α-subcompact for every $\alpha > \kappa$.

We spell out the proof, appropriately modified from [16], for the sake of completeness.

PROOF. 1. Suppose κ is $2^{<\alpha}$-supercompact, and let this be witnessed by $j : V \to M$ with critical point κ, $j(\kappa) > 2^{<\alpha}$, and ${}^{2^{<\alpha}}M \subset M$. For every $A \subseteq H_\alpha$, the restriction of j to H_α is elementary from $(H_\alpha, \in, A)$ to $(H^M_{j(\alpha)}, \in, j(A))$, and since $|H_\alpha| = 2^{<\alpha}$, $j \restriction H_\alpha$ is a member of M. Thus, with α, A and $j \restriction H_\alpha$ as witnesses for the existential quantifications, we have

$$M \models \exists \bar\alpha < j(\kappa)\, \exists \bar A \subseteq H_{\bar\alpha}\, \exists \pi : (H_{\bar\alpha}, \in, \bar A) \to (H_{j(\alpha)}, \in, j(A))$$
$$(\pi \text{ is an elementary embedding } \wedge \pi(\mathrm{cp}(\pi)) = j(\kappa)),$$

whence

$$V \models \exists \bar\alpha < \kappa\, \exists \bar A \subseteq H_{\bar\alpha}\, \exists \pi : (H_{\bar\alpha}, \in, \bar A) \to (H_\alpha, \in, A)$$
$$(\pi \text{ is an elementary embedding } \wedge \pi(\mathrm{cp}(\pi)) = \kappa).$$

2. Suppose κ is $(2^{\lambda^{<\kappa}})^+$-subcompact, and let $\pi : (H_{\bar\alpha}, \in, \{\bar\lambda\}) \to (H_{(2^{\lambda^{<\kappa}})^+}, \in, \{\lambda\})$ witness this for the predicate $A = \{\lambda\}$, with $\mathrm{cp}(\pi) = \bar\kappa$. By elementarity we have that $\bar\alpha = (2^{\bar\lambda^{<\bar\kappa}})^+$, and since $\bar\alpha < \kappa$, we have in particular that $\bar\lambda < \kappa$.

We claim that $\bar\kappa$ is $\bar\lambda$-supercompact. To see this, define an ultrafilter $\mathcal{U}$ on $\mathcal{P}_{\bar\kappa}\bar\lambda$ by

$$X \in \mathcal{U} \leftrightarrow X \subseteq \mathcal{P}_{\bar\kappa}\bar\lambda \wedge \pi(X) \ni \{\pi(\zeta) \mid \zeta \in \bar\lambda\}.$$

It is standard to check that $\mathcal{U}$ so defined is a $\bar\kappa$-complete normal ultrafilter on $\mathcal{P}_{\bar\kappa}\bar\lambda$, noting that $P_{\bar\kappa}\bar\lambda$ belongs to the domain of π and $\pi(\bar\kappa) = \kappa$ is greater than $\bar\lambda$. Now $\mathcal{U} \in H_{(2^{\bar\lambda^{<\bar\kappa}})^+}$, and

$$H_{(2^{\bar\lambda^{<\bar\kappa}})^+} \models \mathcal{U} \text{ is a normal ultrafilter on } \mathcal{P}_{\bar\kappa}\bar\lambda.$$

Therefore by elementarity

$$H_{(2^{\lambda^{<\kappa}})^+} \models \pi(\mathcal{U}) \text{ is a normal ultrafilter on } \mathcal{P}_\kappa\lambda,$$

and $H_{(2^{\lambda^{<\kappa}})^+}$ is clearly correct for this statement. Hence, κ is λ-supercompact. ⊣

Observe that the level of subcompactness required in (2) to imply any supercompactness is at least κ^{++}. Indeed, Cummings and Schimmerling [7, Section 6] note that a κ^+-subcompact cardinal κ need not be measurable, since a measurable κ^+-subcompact cardinal κ must have a normal measure 1 set of ι^+-subcompact cardinals ι below it.

§3. Squares. The general proof of the incompatibility of subcompactness with $\square$ is similar to that for the κ^+ case, due to Jensen.

THEOREM 7. *Suppose κ is α^+-subcompact for some $\kappa \leq \alpha$. Then $\square_\alpha$ fails.*

Proof. Suppose for contradiction that $C = \langle C_\beta \mid \beta \in \alpha^+ \cap \mathrm{Lim}\rangle$ is a $\square_\alpha$-sequence. We can take an α^+-subcompactness embedding

$$\pi : (H_{\bar{\alpha}^+}, \in, \bar{C}) \to (H_{\alpha^+}, \in, C)$$

with critical point some $\bar{\kappa} < \bar{\alpha}^+$, so that $\pi(\bar{\kappa}) = \kappa$. Let λ be the supremum of $\pi``(\bar{\alpha}^+)$, and consider $D = \lim(C_\lambda) \cap \pi``(\bar{\alpha}^+)$. Since $\pi``(\bar{\alpha}^+)$ is countably closed (indeed, it contains all of its limits of cofinality less than $\bar{\kappa}$) and unbounded in λ, D is also unbounded in λ. Therefore, since λ has cofinality $\bar{\alpha}^+$, D is a subset of the range of π which has cardinality at least $\bar{\alpha}^+$, but order type less than α, by the definition of $\square_\alpha$. For $\beta_0 < \beta_1$ in D we have that C_{β_0} is an initial segment of C_{β_1} by coherence, and hence $\mathrm{ot}(C_{\beta_0}) < \mathrm{ot}(C_{\beta_1}) < \alpha$. But then $\{\mathrm{ot}(C_\beta) \mid \beta \in D\}$ must be a subset of the range of $\pi \restriction \bar{\alpha}$, and yet has cardinality $\bar{\alpha}^+$, a contradiction. ⊣

Now to the optimality of this result.

Theorem 8. *Suppose the GCH holds, and let*

$$I = \{\alpha \mid \exists \kappa \leq \alpha (\kappa \textit{ is } \alpha^+\textit{-subcompact})\}.$$

Then there is a cofinality-preserving partial order $\mathbb{P}$ such that for any $\mathbb{P}$-generic G the following hold.

1. *$\square_\alpha$ holds in $V[G]$ for all $\alpha \notin I$.*
2. *If $\kappa < \alpha$ are such that $V \vDash \kappa$ is α-subcompact, then*

$$V[G] \vDash \kappa \textit{ is } \alpha\textit{-subcompact.}$$

In particular, $I^{V[G]} = I$.

Proof. The partial order $\mathbb{P}$ will be a reverse Easton forcing iteration — see for example [5] for an introduction to such forcings. At stage α for α a cardinal not in I, we force with the usual size α^+ (thanks to the GCH), $< \alpha^+$-strategically closed partial order $\mathbb{S}_\alpha$ due to Jensen to obtain $\square_\alpha$, which uses initial segments of the generic $\square_\alpha$ sequence as conditions — see [5, Example 6.3]. At all other stages we take the trivial forcing. Thus, the iteration preserves cofinalities and the GCH, and $\square_\alpha$ holds in $V[G]$ for all $\alpha \notin I^V$. It therefore only remains to show that forcing with $\mathbb{P}$ preserves the α-subcompactness of any κ that is α-subcompact in V.

So suppose κ is α-subcompact in V. By the definition of I, the forcing is trivial on the interval $[\kappa, \alpha)$. Also, the tail of the iteration starting at stage α is $< \alpha^+$-strategically closed since each iterand is — see [5, Proposition 7.8]. Hence, no new subsets of α are added by this part of the forcing. By the GCH, $V \vDash |H_\alpha| = \alpha$, and so the tail of the iteration starting at stage κ adds no new subsets of H_α. Thus, to consider arbitrary subsets of H_α in the generic extension, it suffices to consider those of the form ρ_G for ρ a $\mathbb{P}_\kappa$-name, where $\mathbb{P}_\kappa$ denotes the iteration of length κ that is the initial part of $\mathbb{P}$ up to (but not including) κ. We shall denote by G_κ the generic for $\mathbb{P}_\kappa$

obtained from G, and use corresponding notation for $\bar\kappa$. Note in particular that ρ can be taken to be a subset of H_α.

Applying the α-subcompactness of κ in V, let

$$\pi : (H_{\bar\alpha}, \in, \bar\rho) \to (H_\alpha, \in, \rho)$$

witness the α-subcompactness of κ for ρ, with critical point $\bar\kappa$ taken by π to κ. We wish to lift π to an elementary embedding $\pi' : (H_{\bar\alpha}^{V[G]}, \in, \bar\rho_G) \to (H_\alpha^{V[G]}, \in, \rho_G)$. As noted in Section 2, $\bar\kappa$ is $< \bar\alpha$-subcompact if α is a limit cardinal and $\bar\beta$-subcompact if $\bar\alpha = \bar\beta^+$, so in either case $\mathbb{P}$ is trivial on the interval $[\bar\kappa, \bar\alpha)$. Furthermore, even if the forcing iterand at stage $\bar\alpha$ is non-trivial, it will be $< \bar\alpha^+$-strategically closed, and hence adds no new sets to $H_{\bar\alpha}$. Indeed the tail of the forcing from stage $\bar\alpha$ on is $< \bar\alpha^+$-strategically closed. Therefore $H_{\bar\alpha}^{V[G]} = H_{\bar\alpha}^{V[G_{\bar\kappa}]}$, so combining this with $H_\alpha^{V[G]} = H_\alpha^{V[G_\kappa]}$ our goal becomes to lift π to

$$\pi' : (H_{\bar\alpha}^{V[G_{\bar\kappa}]}, \in, \bar\rho_{G_{\bar\kappa}}) \to (H_\alpha^{V[G_\kappa]}, \in, \rho_{G_\kappa}),$$

for which it suffices by the usual (Silver) argument to show that $\pi``G_{\bar\kappa} \subseteq G_\kappa$. But π is the identity below $\bar\kappa$, so this is immediate. $\dashv$

It should be noted that this lifting argument did not require that the generic contain a non-trivial master condition. Hence, *every* $\mathbb{P}$-generic G over V will preserve all β-subcompacts for all β.

Of course, it is important that our forcing preserve not only α-subcompact cardinals, but stronger large cardinals too. We verify this for a test case near the top of the large cardinal hierarchy, specifically, ω-superstrong cardinals (I2 cardinals in the terminology of [14]). Recall their definition.

Definition 9. *A cardinal κ is* ω-superstrong *if and only if there is an elementary embedding $j : V \to M$ with critical point κ such that, if we let $\lambda = \sup_{n\in\omega}(j^n(\kappa))$, $V_\lambda \subset M$.*

Note that we may take M such that every element of M has the form $j(f)(a)$ for some f with domain V_λ and some $a \in V_\lambda$. Indeed, given any $j : V \to N$ witnessing ω-superstrength, the transitive collapse of the class of elements of this form gives such an M.

Proposition 10. *The forcing iteration $\mathbb{P}$ of Theorem 8 preserves all ω-superstrong cardinals.*

Proof. We again use Silver's method of lifting embeddings. Let κ be ω-superstrong, let $j : V \to M$ witness this, let λ be as in Definition 9, and suppose we have chosen j in such a way that every element of M is of the form $j(f)(a)$ for some a in V_λ and f with domain V_λ. It follows from ω-superstrength that κ is α-subcompact for every $\alpha < \lambda$, that is, $< \lambda$-subcompact. Thus, our forcing $\mathbb{P}$ is trivial between κ and λ. Also, since the definition of $I \cap V_\lambda$ is absolute for models containing V_λ, $j(\mathbb{P}_\lambda^V) = \mathbb{P}_\lambda^M = \mathbb{P}_\lambda^V$ (hence the "non-trivial support" of $\mathbb{P}$ will also be bounded below κ). Below λ, therefore, we may just take the generic for M to be the generic for V, G_λ, and we get a lift j' of j from $V[G_\lambda]$ to $M[G_\lambda]$.

We claim that for the tail of the forcing, the pointwise image of the tail of the generic for V, $j'``G^\lambda$, generates a generic filter for M, by the λ^+-distributivity of this tail forcing. Indeed this is standard for preservation results about ω-superstrongs: compare for example with [12] and [1]. To be explicit: every element of $M[G_\lambda]$ is of the form σ_{G_λ} for some $\sigma = j(f)(a)$ with $a \in V_\lambda$. Suppose D is a dense class in the tail of the forcing iteration, defined in $M[G_\lambda]$ as $\{p \mid \psi(p, d)\}$ for some parameter $d = j(f)(a)_{G_\lambda}$ with $a \in V_\lambda$. Since the tail $\mathbb{P}^\lambda$ of the forcing is $< \lambda^+$-strategically closed and $|V_\lambda| = \lambda$, it is dense for $q \in \mathbb{P}^\lambda$ to extend an element of $D_x = \{p \mid \psi(p, f(x)_{G_\lambda})\}$ whenever $x \in V_\lambda$ and D_x is dense in $\mathbb{P}^\lambda$. We may therefore take such a q lying in G^λ, and by elementarity have that $j(q)$ extends D. That is, $j'``G^\lambda$ indeed generates a generic filter over M for $(\mathbb{P}^\lambda)^M$. ⊣

§4. Stationary reflection. For simplicity, we restrict attention to cofinality ω. This is to ensure that the cofinality of interest is not affected by the embeddings involved — any small enough cofinality would suffice.

As noted after Definition 2, $\mathrm{SR}(\alpha^+, \omega)$ can be seen as a strong failure of square. Correspondingly, we consider a strengthening of subcompactness.

DEFINITION 11. *For any cardinal α, we say that a cardinal $\kappa \leq \alpha^+$ is* (α^+, ω)-stationary subcompact *if for every $A \subseteq H_{\alpha^+}$ and every stationary set $S \subseteq \alpha^+ \cap \mathrm{Cof}(\omega)$, there exist $\bar\kappa < \bar\alpha^+ < \kappa$, $\bar A \subseteq H_{\bar\alpha^+}$, a stationary set $\bar S \subseteq \bar\alpha^+ \cap \mathrm{Cof}(\omega)$ and an elementary embedding*

$$\pi : (H_{\bar\alpha^+}, \in, \bar A, \bar S) \to (H_{\alpha^+}, \in, A, S)$$

with critical point $\bar\kappa$ such that $\pi(\bar\kappa) = \kappa$. We say that such an embedding π witnesses the (α, ω)-stationary subcompactness of κ for A and S.

Thus, (α^+, ω)-stationary subcompactness is α^+-subcompactness with the extra requirement that there be witnessing embeddings respecting the stationarity of any given $S \subseteq \alpha^+ \cap \mathrm{Cof}(\omega)$. As for subcompactness, we can and will freely replace A in the definition by any finite number of subsets of H_{α^+}. Since H_{γ^+} is correct for stationarity of subsets of γ, we have that if $\kappa < \beta^+ < \alpha$ and κ is α-subcompact, then κ is (β^+, ω)-stationary subcompact, and moreover if $\pi : (H_{\bar\alpha}, \in, \bar A) \to (H_\alpha, \in, A)$ is an embedding with critical point $\bar\kappa$ witnessing α-subcompactness of κ for any $A \subseteq H_\alpha$, then for all $\bar\beta^+ < \bar\alpha$, $\bar\kappa$ is $(\bar\beta^+, \omega)$-stationary subcompact.

This strengthened subcompactness notion is sufficient to obtain stationary reflection as a consequence.

PROPOSITION 12. *If there exists some $\kappa \leq \alpha$ such that κ is (α^+, ω)-stationary subcompact, then $\mathrm{SR}(\alpha^+, \omega)$ holds.*

Proof. Suppose $\kappa \leq \alpha$ is (α^+, ω)-stationary subcompact, let S be a stationary subset of $\alpha^+ \cap \mathrm{Cof}(\omega)$, take $A \subseteq H_{\alpha^+}$ arbitrary, and let $\pi : (H_{\bar{\alpha}^+}, \in, \bar{A}, \bar{S}) \to (H_{\alpha^+}, \in, A, S)$ with critical point $\bar{\kappa}$ witness (α^+, ω)-stationary subcompactness of κ for A and S. Let $\lambda = \sup(\pi``\bar{\alpha}^+)$; we claim that $S \cap \lambda$ is stationary in $\lambda$. The pointwise image of $\bar{\alpha}^+$ in $\alpha^+$ is countably closed and unbounded in $\lambda$, so for any club $C \subseteq \lambda$, $C \cap \pi``\bar{\alpha}^+$ is also countably closed and unbounded in λ. Therefore, $\pi^{-1}C$ is countably closed and unbounded in $\bar{\alpha}^+$, and hence has nonempty intersection with $\bar{S}$. But now taking $\xi \in \bar{S} \cap \pi^{-1}C$, we have $\pi(\xi) \in S \cap C$. Hence, $S \cap \lambda$ is stationary. ⊣

Again, we have a complementary result under the GCH.

Theorem 13. *Suppose the GCH holds. Let I be as defined in Theorem 8, and similarly let*

$$J = \{\alpha \mid \exists \kappa \leq \alpha (\kappa \textit{ is } (\alpha^+, \omega)\textit{-stationary subcompact})\} \subseteq I.$$

Then there is a cofinality-preserving partial order $\mathbb{P}$ such that for any $\mathbb{P}$-generic G the following hold.

1. $\mathrm{SR}(\alpha^+, \omega)$ *fails in $V[G]$ for all $\alpha \notin J$.*
2. $\square_\alpha$ *holds in $V[G]$ for all $\alpha \notin I$.*
3. *If $\kappa \leq \alpha$ are such that $V \models \kappa$ is (α^+, ω)-stationary subcompact, then $V[G] \models \kappa$ is (α^+, ω)-stationary subcompact. In particular, $J^{V[G]} = J$.*
4. $I^{V[G]} = I$.

Proof. Again $\mathbb{P}$ will be a reverse Easton iteration. At stage α for $\alpha \in J$, we take the trivial forcing. For cardinals $\alpha \in I \setminus J$, we take the forcing $\mathbb{R}_\alpha$ that adds a non-reflecting stationary set to $\alpha^+ \cap \mathrm{Cof}(\omega)$ by initial segments; this forcing is α^+-strategically closed and (by the GCH) of size α^+ (see [5, Example 6.2]). For cardinals $\alpha \notin I$, we take a three stage iteration, first forcing with $\mathbb{R}_\alpha$. Next, we force with the partial order $\mathbb{C}^{\mathbb{R}}_\alpha$ that makes the generic stationary set from $\mathbb{R}_\alpha$ non-stationary by shooting a club through its complement. Third, we force to make $\square_\alpha$ hold with $\mathbb{S}_\alpha$. The two stage iteration $\mathbb{R}_\alpha * \dot{\mathbb{C}}^{\mathbb{R}}_\alpha$ is $< \alpha^+$-strategically closed (indeed it contains a natural dense suborder that is $< \alpha^+$-closed), and $\mathbb{S}_\alpha$ is also $< \alpha^+$-strategically closed, so $\mathbb{R}_\alpha * \dot{\mathbb{C}}^{\mathbb{R}}_\alpha * \dot{\mathbb{S}}_\alpha$ is $< \alpha^+$-strategically closed. It also has a dense suborder of size α^+. Thus, our reverse Easton iteration will indeed preserve cofinalities, as well as the GCH. The generic extension will also clearly satisfy 1 and 2 of the theorem.

As in the proof of Theorem 8, we will denote by $\mathbb{P}_\kappa$ the iteration up to stage κ and by G_κ the corresponding generic (and similarly for other ordinals); note that since κ is inaccessible, $\mathbb{P}_\kappa$ is a direct limit, so we can and will identify $\mathbb{P}_\kappa$ with $\bigcup_{\gamma<\kappa} \mathbb{P}_\gamma$.

If κ is (α^+, ω)-stationary subcompact, then the forcing is trivial in stages from κ up to (but not necessarily including) α^+, and is $< \alpha^{++}$-strategically closed from stage α^+ onward, so no new elements or subsets of H_{α^+} are added after stage κ. Thus, to show that (α^+,ω)-stationary subcompactness is preserved, it suffices to show that for any $\mathbb{P}_\kappa$-name ρ for a subset of $H_{\alpha^+}^{V[G]}$ and any $\mathbb{P}_\kappa$-name σ for a stationary subset of α^+, there is an embedding from $(H_{\bar\alpha^+}^{V[G]},\in,\bar\rho_G,\bar\sigma_G)$ to $(H_{\alpha^+}^{V[G]},\in,\rho_G,\sigma_G)$ witnessing the α^+-stationary subcompactness of κ for ρ_G and σ_G in $V[G]$.

Because $\mathbb{P}_\kappa$ is only of cardinality κ, there is some $p\in G$ and some $S\in V$ stationary in α^+ such that $p\Vdash\check S\subseteq\sigma$, and H_{α^+} contains all the requisite sets to be correct for this statement. In V let $\pi:(H_{\bar\alpha^+},\in,\bar\rho,\bar\sigma,\bar p,\bar S)\to(H_{\alpha^+},\in,\rho,\sigma,p,S)$ with critical point $\bar\kappa$ and $\bar S$ stationary in $\bar\alpha^+$ witness α^+-stationary subcompactness of κ for ρ,σ,p and S. Then by elementarity, $\bar p\Vdash\check{\bar S}\subseteq\bar\sigma$, and moreover, $\bar p$ is a condition bounded below $\bar\kappa$, so since $\bar\kappa=\mathrm{cp}(\pi)$, $\bar p=\pi(\bar p)=p$. It follows, since $\mathbb{P}_{\bar\kappa}$ is small relative to $\bar\alpha^+$, that $\bar S$ remains stationary under forcing with $\mathbb{P}_{\bar\kappa}$, and so p forces $\bar\sigma$ to be stationary in $\bar\alpha^+$. Now by Silver's lifting of embeddings method again, π lifts to an elementary embedding $\pi':(H_{\bar\alpha^+}^{V[G_{\bar\kappa}]},\in,\bar\rho_{G_{\bar\kappa}},\bar\sigma_{G_{\bar\kappa}})\to(H_{\alpha^+}^{V[G_\kappa]},\in,\rho_{G_\kappa},\sigma_{G_\kappa})$, since $\pi``G_{\bar\kappa}=G_{\bar\kappa}$. That is, we have $\pi':(H_{\bar\alpha^+}^{V[G]},\in,\bar\sigma_G)\to(H_{\alpha^+}^{V[G]},\in,\sigma_G)$ with $\bar\sigma_G$ stationary, as required.

To prove part 4, it now suffices to consider the case when κ is α^+-subcompact but no κ' is (α^+,ω)-stationary subcompact. Furthermore, in order to show that I is preserved, it suffices to only show that α^+-subcompactness of κ is preserved when κ is the *least* α^+-subcompact cardinal. So suppose that κ is the least α^+-subcompact cardinal and no κ' is (α^+,ω)-stationary subcompact.

For any $A\subseteq H_{\alpha^+}$, we claim there is a $\pi:(H_{\bar\alpha^+},\in,\bar A)\to(H_{\alpha^+},\in,A)$ witnessing the α^+-subcompactness of κ for A such that $\bar\alpha\notin I$: no κ' is $\bar\alpha^+$-subcompact. To see this, let B be a subset of $H_{\alpha^+}\times\kappa\subset H_{\alpha^+}$ such that for each cardinal $\gamma<\kappa$, the cross-section $B_\gamma=\{x\in H_{\alpha^+}\mid(x,\gamma)\in B\}$ witnesses the failure of γ to be α^+-subcompact, that is, there is no embedding π' witnessing α^+-subcompactness of γ for B_γ. Let $\pi:(H_{\bar\alpha^+},\in,\bar A,\bar B)\to(H_{\alpha^+},\in,A,B)$ be an embedding witnessing the α^+-subcompactness of κ for A and B with minimal critical point, and given the critical point, minimal $\bar\alpha$. Call the critical point $\bar\kappa$ as always. We claim that π considered as an embedding from $(H_{\bar\alpha^+},\in,\bar A)$ to $(H_{\alpha^+},\in,A)$ is as required. If $\bar\kappa$ itself were $\bar\alpha^+$-subcompact, there would be an elementary embedding $\varphi:(H_{\bar{\bar\alpha}^+},\in,\bar{\bar A},\bar{\bar B})\to(H_{\bar\alpha}^+,\in,\bar A,\bar B)$ witnessing the $\bar\alpha^+$-subcompactness of $\bar\kappa$ for $\bar A$ and $\bar B$, and then $\pi\circ\varphi$ would be an embedding witnessing the α^+-subcompactness of κ for A and B with critical point less than $\bar\kappa$, violating the choice of π. Similarly if some $\kappa'>\bar\kappa$ were $\bar\alpha^+$-subcompact, then there would be an elementary embedding $\varphi:(H_{\bar{\bar\alpha}^+},\in,\bar{\bar A},\bar{\bar B})\to(H_{\bar\alpha}^+,\in,\bar A,\bar B)$ with critical point $\bar\kappa'>\bar\kappa$ witnessing the $\bar\alpha^+$-subcompactness of κ' for $\bar A$ and $\bar B$. In this case, $\pi\circ\varphi$ would be an elementary embedding witnessing the α^+-subcompactness of κ for A and B, with critical point $\bar\kappa$ but with domain $H_{\bar{\bar\alpha}^+}$ for some $\bar{\bar\alpha}^+<\bar\alpha^+$, again violating the choice of π.

Finally, if some $\kappa' < \bar{\kappa}$ were $\bar{\alpha}^+$-subcompact, there would be an elementary embedding $\varphi : (H_{\bar{\bar{\alpha}}^+}, \in, \bar{\bar{A}}, \bar{\bar{B}}\,) \to (H^+_{\bar{\alpha}}, \in, \bar{A}, \bar{B})$ with critical point $\bar{\kappa}'$ witnessing the $\bar{\alpha}^+$-subcompactness of κ' for $\bar{A}$ and $\bar{B}$. But then $\pi \circ \varphi$ witnesses the α^+-subcompactness of $\pi(\kappa') = \kappa'$ for A and B, and in particular, we may view $\pi \circ \varphi$ as an elementary embedding $(H_{\bar{\bar{\alpha}}^+}, \in, \bar{\bar{B}}_{\bar{\kappa}'}) \to (H_{\alpha^+}, \in, B_{\kappa'})$ witnessing the α^+-subcompactness of κ' for $B_{\kappa'}$. This of course violates the choice of B, and so the claim is proven.

Returning to the proof of part 4, we have α and κ such that κ is the least α^+-subcompact cardinal and no κ' is (α^+, ω)-stationary subcompact, and we wish to show that κ remains α-subcompact in the generic extension $V[G]$. The forcing $\mathbb{P}$ is trivial on $[\kappa, \alpha)$, is $\mathbb{R}_\alpha$ at stage α, and is $< \alpha^{++}$-strategically closed thereafter. Thus, $H^{V[G]}_{\alpha^+} = H^{V[G_\kappa]}_{\alpha^+}$, and any subset of $H^{V[G]}_{\alpha^+}$ is named by a $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\kappa * \dot{\mathbb{R}}_\alpha$-name which is a subset of H_{α^+}; of course, any such name is forced by $\mathbb{1}_{\mathbb{P}_\kappa * \dot{\mathbb{R}}_\alpha}$ to be a subset of H_{α^+}.

So suppose σ is such a name; we wish to lift an embedding in V witnessing the α^+-subcompactness of κ for σ to an embedding in $V[G]$ witnessing the α^+-subcompactness of κ for σ_G. Often such lifting arguments simply require one to find an appropriate *master condition* (see for example [5, Section 12]), as there will always be a generic including any condition. However, we wish to lift embeddings for many different names σ, and it is not clear that the corresponding conditions can all lie in a common generic. One fix that is sometimes possible is to use homogeneity of the partial order to argue that the generic can be modified to contain the master condition without altering the genric extension it produces — this is the approach taken in [3], for example. But again this is not appropriate in the present context, as when G is modified to give some G', the interpretation of the name σ may be changed. Instead, we shall show that master conditions for witnessing embeddings are *dense* in the partial order, thus guaranteeing that for each σ there is a corresponding master condition in any given G. A similar technique has been used to demonstrate the indestructibility of Vopěnka's Principle relative to many natural forcing iterations [2].

To this end, let p be an arbitrary element of $\mathbb{P}_{\alpha+1}$, and take π witnessing α^+-subcompactness of κ for σ and $\{p\}$. As shown above, we may assume that no κ' is $\bar{\alpha}^+$-subcompact, but by the comment after Definition 11, $\bar{\kappa}$ will be $\bar{\alpha}$-stationary subcompact. Thus, the forcing is trivial on $[\bar{\kappa}, \bar{\alpha})$, at stage $\bar{\alpha}$ is $\mathbb{R}_{\bar{\alpha}} * \dot{\mathbb{C}}^{\mathbb{R}}_{\bar{\alpha}} * \dot{\mathbb{S}}_{\bar{\alpha}}$, and is $< \bar{\alpha}^{++}$-strategically closed thereafter. In particular, $H_{\bar{\alpha}^+}$ receives no new elements from stage $\bar{\kappa}$ of the forcing onward.

Note that by elementarity, $\bar{\sigma}$ is a $\mathbb{P}_{\bar{\kappa}} * \dot{\mathbb{R}}_{\bar{\alpha}}$-name for a subset of $H_{\bar{\alpha}^+}$, and that p is essentially comprised of $p \restriction \kappa$, a $\mathbb{P}_\kappa$-condition that is thus bounded below κ, and a name $\dot{p}(\alpha)$ for an $\mathbb{R}_\alpha$-condition. Let $\kappa_0 = \mathrm{dom}(p \restriction \kappa)$; then κ_0 is an ordinal less than κ in the range of

π, so the critical point $\bar\kappa$ of π must be greater than κ_0, and $\bar p\restriction\bar\kappa = p\restriction\kappa = p\restriction\kappa_0$. Furthermore, by adding any other singleton $\{\gamma\}$ as a predicate for $\gamma<\kappa$, we can take π with $\bar\kappa$ as large as desired less than κ (without affecting the fact that no κ' is $\bar\alpha^+$-subcompact). Since a direct limit is taken at κ, the set of those conditions in $\mathbb{P}_\kappa$ that extend a $\bar p$ for such an embedding is therefore dense below $p\restriction\kappa$, and so we may assume without loss of generality that if $p\in G$ then $\bar p$ is in the $\mathbb{P}_{\kappa_0} * \dot{\mathbb{R}}_{\bar\alpha}$-generic obtained from G.

For terminological convenience working with $\mathbb{R}_{\bar\alpha}$ we now move to $V[G_{\bar\kappa}]$; note that since π is the identity on $V_{\bar\kappa}$, it lifts in the usual (Silver) way to an embedding $\pi' : H^{V[G_{\bar\kappa}]}_{\bar\alpha^+} \to H^{V[G_\kappa]}_{\alpha^+}$. Let $G_{\mathbb{R}_{\bar\alpha}}$ denote the $\mathbb{R}_{\bar\alpha}$ generic over $V[G_{\bar\kappa}]$ that comes from G. Now, $r=\bigcup\pi'``G_{\mathbb{R}_{\bar\alpha}}$ is a condition in $\mathbb{R}_\alpha^{V[G_\kappa]}$, since the union of the pointwise image of the $\mathbb{C}^{\mathbb{R}}_{\bar\alpha}$-generic component of $G$ is a club in the complement of $r$ in $\sup(\pi'``\bar\alpha^+)$. Since $\bar p(\bar\alpha)\in G_{\mathbb{R}_{\bar\alpha}}$, $r\le p$, and if $r\in G$, π' lifts to an embedding

$$\pi'' : (H^{V[G_{\bar\kappa}*G_{\mathbb{R}_{\bar\alpha}}]}_{\bar\alpha^+}, \in, \bar\sigma_{G_{\bar\kappa}*G_{\mathbb{R}_{\bar\alpha}}}) \to (H^{V[G_\kappa*G_{\mathbb{R}_\alpha}]}_{\alpha^+}, \in, \sigma_{G_\kappa*G_{\mathbb{R}_\alpha}}).$$

But this is the same as

$$\pi'' : (H^{V[G]}_{\bar\alpha^+}, \in, \bar\sigma_G) \to (H^{V[G]}_{\alpha^+}, \in, \sigma_G),$$

and we are done. ⊣

Our forcing for Theorem 13 also preserves stronger large cardinals.

PROPOSITION 14. *The forcing iteration $\mathbb{P}$ of Theorem 13 preserves all ω-superstrong cardinals*

The proof is exactly as for Proposition 10.

§5. Weaker Squares. Schimmerling [17] introduced the following generalisation of $\square_\alpha$.

DEFINITION 15. *For any cardinal α, a $\square_{\alpha,<\mu}$-*sequence *is a sequence $\langle \mathcal{C}_\beta \mid \beta\in\alpha^+\cap Lim\rangle$ such that for every $\beta\in\alpha^+\cap Lim$,*

- *$\mathcal{C}_\beta$ is a set of closed unbounded subsets of β,*
- *$1\le|\mathcal{C}_\beta|<\mu$,*
- *$\mathrm{ot}(C)\le\alpha$ for every $C\in\mathcal{C}_\beta$,*
- *for any $C\in\mathcal{C}_\beta$ and $\gamma\in\lim(C)$, $C\cap\gamma\in\mathcal{C}_\gamma$.*

We say $\square_{\alpha,<\mu}$ holds if there exists a $\square_{\alpha,<\mu}$-sequence, and we write $\square_{\alpha,\nu}$ for $\square_{\alpha,<\nu^+}$.

Of course, $\square_{\alpha,1}$ is simply $\square_\alpha$, and the strength of the statement $\square_{\alpha,<\mu}$ is non-increasing as μ increases; moreover Jensen [13] has shown that $\square_{\alpha,2}$ does not imply $\square_{\alpha,1}$. Jensen's *weak square*, $\square^*_\alpha$, is simply $\square_{\alpha,\alpha}$, and $\square_{\alpha,\alpha^+}$ is provable in ZFC for all α.

It turns out that some of these weaker forms of square are also precluded by α^+-subcompactness of some $\kappa<\alpha$. Indeed, corresponding results are known for κ an α^+-supercompact cardinal, so this should not be surprising.

THEOREM 16. *Suppose κ is α^+-subcompact for some $\kappa \leq \alpha$. Then $\square_{\alpha,<\mathrm{cf}(\alpha)}$ fails.*

PROOF. Suppose for contradiction that $\mathcal{C} = \langle \mathcal{C}_\beta \mid \beta \in \alpha^+ \cap \mathrm{Lim}\rangle$ is a $\square_{\alpha,<\mathrm{cf}(\alpha)}$-sequence. Note that clubs of order type α only occur at ordinals with cofinality $\mathrm{cf}(\alpha)$. We can take an α^+-subcompactness embedding

$$\pi : (H_{\bar{\alpha}^+}, \in, \bar{\mathcal{C}}) \to (H_{\alpha^+}, \in, \mathcal{C})$$

with critical point some $\bar{\kappa} < \bar{\alpha}^+$ such that $\pi(\bar{\kappa}) = \kappa$, and $\bar{\alpha} < \alpha$. Let λ be the supremum of $\pi``(\bar{\alpha}^+)$, let $C$ be an arbitrary member of $\mathcal{C}_\lambda$, and consider the inverse image $\bar{D}$ of $\lim(C)$ under $\pi$. Because $\pi``(\bar{\alpha}^+)$ is $< \bar{\kappa}$-closed and unbounded in λ, $\bar{D}$ is $< \bar{\kappa}$-closed and unbounded in $\bar{\alpha}^+$, so we may take some $\bar{\beta} \in \bar{D}$ of cofinality different from $\mathrm{cf}(\bar{\alpha})$ such that $|\bar{D} \cap \bar{\beta}| = \bar{\alpha}$.

Now, for any $\bar{\gamma} < \bar{\beta}$ in $\bar{D}$, $\pi(\bar{\gamma}) \in C \cap \beta \in \mathcal{C}_\beta$, so by elementarity there is some $\bar{C} \in \bar{\mathcal{C}}_{\bar{\beta}}$ with $\bar{\gamma} \in \bar{C}$. But there are fewer than $\mathrm{cf}(\bar{\alpha})$ elements of $\bar{\mathcal{C}}_{\bar{\beta}}$, each of order type strictly less that $\bar{\alpha}$, so $|\bigcup \bar{\mathcal{C}}_{\bar{\beta}}| < \bar{\alpha}$, and not all $\gamma \in \bar{D} \cap \bar{\beta}$ can be covered in this way. $\dashv$

Note that under the GCH, $\square^*_\alpha$ holds for all regular α (we may take *all* clubs of order type less than α at ordinals of cofinality less than α), making Theorem 16 optimal. For singular α, we leave obtaining a forcing reversal of the result until we have considered obstructions to even weaker variants of $\square$.

Foreman and Magidor [11] observed that if $\square^*_\alpha$ holds then there is a $\square^*_\alpha$ sequence (referred to in [6] as an *improved* square sequence, $\square^{\mathrm{imp}}_{\alpha,\alpha}$) with the added property that for all $\beta < \alpha^+$, there is a $C \in \mathcal{C}_\beta$ with $\mathrm{ot}(C) = \mathrm{cf}(\beta)$. Indeed, if we choose an arbitrary sequence $\langle D_\gamma \mid \gamma \in \mathrm{Lim} \cap \alpha + 1\rangle$ such that D_γ is a club in γ of order type $\mathrm{cf}(\gamma)$, then for any $\square^*_\alpha$-sequence $\mathcal{C}$, we may obtain a $\square^{\mathrm{imp}}_{\alpha,\alpha}$-sequence by adding $\{\delta \in C \mid \mathrm{ot}(C \cap \delta) \in D_\gamma\}$ to $\mathcal{C}_\beta$ for every $C \in \mathcal{C}_\beta$ and γ such that $\mathrm{ot}(C) \in \mathrm{Lim}(D_\gamma) \cup \{\gamma\}$. Using this fact with a trick due to Solovay, we see that if there is some $\kappa > \mathrm{cf}(\alpha)$ that is α^+-subcompact, then even $\square^*_\alpha$ fails.

THEOREM 17. *Suppose κ is α^+-subcompact for some $\kappa \leq \alpha$ with $\kappa > \mathrm{cf}(\alpha)$. Then $\square_{\alpha,\alpha}$ fails.*

PROOF. We essentially follow the proof for the analogous result with κ α^+-supercompact due to Shelah [18] as presented by Cummings [4, Section 6]. Suppose for contradiction that $\mathcal{C}$ is a $\square^{\mathrm{imp}}_{\alpha,\alpha}$ sequence, and let

$$\pi : (H_{\bar{\alpha}^+}, \in, \bar{\mathcal{C}}) \to (H_{\alpha^+}, \in, \mathcal{C})$$

be an embedding witnessing the α^+-subcompactness of κ for $\mathcal{C}$. Since $\mathrm{cf}(\alpha) = \pi(\mathrm{cf}(\bar{\alpha}))$, it is in particular in the range of π, and hence $\mathrm{cf}(\alpha) < \kappa$ implies that in fact $\mathrm{cf}(\alpha) < \bar{\kappa}$. Let $\lambda = \sup(\pi``\bar{\alpha}^+)$, and take $C \in C_\lambda$ with $\mathrm{ot}(C) = \bar{\alpha}^+ = \mathrm{cf}(\lambda)$. Let $\bar{D}$ be the preimage of C

under π; as usual, it is a $<\bar\kappa$-closed unbounded subset of $\bar\alpha^+$. Let ζ be the $\bar\alpha$-th element of $\bar D$. Since $\mathrm{cf}(\bar\alpha)<\bar\kappa$, π is continuous at ζ, and in particular $\pi(\zeta)$ is a limit point of C. Thus, $C\cap\pi(\zeta)\in\mathcal{C}_{\pi(\zeta)}$. Now for every subset X of $\bar D\cap\zeta$ of size less than $\bar\kappa$, $\pi(X)=\pi``X\subset C\cap\pi(\zeta)\in\mathcal{C}_{\pi(\zeta)}$, so by elementarity, there is an element $\bar C_X$ of $\bar{\mathcal{C}}_\zeta$ of order type less than $\bar\kappa$ such that $X\subseteq\bar C_X$. But there are $\bar\alpha^{<\bar\kappa}>\bar\alpha$ such subsets X of $\bar D\cap\zeta$ and at most $\bar\alpha$ such elements of $\bar{\mathcal{C}}_\zeta$, each with at most $2^{<\bar\kappa}=\bar\kappa<\bar\alpha$ subsets, yielding a contradiction. ⊣

Once again we show by forcing that under the GCH, these results are optimal.

THEOREM 18. *Suppose the GCH holds. Let I be as defined in Theorem 8, and similarly let*

$$K=\{\alpha\mid\exists\kappa>\mathrm{cf}(\alpha)(\kappa\text{ is }\alpha^+\text{-subcompact})\}\subseteq I.$$

Then there is a cofinality-preserving partial order $\mathbb{P}$ such that for any $\mathbb{P}$-generic G the following hold.

1. *$\square_\alpha$ holds in $V[G]$ for all $\alpha\notin I$.*
2. *$\square_{\alpha,\mathrm{cf}(\alpha)}$ holds in $V[G]$ for all $\alpha\notin K$.*
3. *$I^{V[G]}=I$.*
4. *$K^{V[G]}=K$.*

PROOF. Once more we use a reverse Easton iteration $\mathbb{P}$. In our iterands, we use the forcing partial order of Cummings, Foreman and Magidor [6, Theorem 16] to force $\square_{\alpha,\mathrm{cf}(\alpha)}$ for singular α. We denote this partial order by $\mathbb{T}_\alpha$; note that it is $<\mathrm{cf}(\alpha)$ directed closed and $<\alpha$-strategically closed, and by the GCH has cardinality α^+.

For regular cardinals $\alpha\notin I$, we force with $\mathbb{S}_\alpha$ at stage α. For singular cardinals $\alpha\in I\smallsetminus K$, we force with $\mathbb{T}_\alpha$. For singular cardinals $\alpha\notin I$, we force with the two-stage iteration $\mathbb{T}_\alpha*\dot{\mathbb{S}}_\alpha$. At all other stages we use the trivial forcing. Clearly this gives a generic extension that satisfies 1 and 2, so we turn to preservation of I and K.

Because of 1, 2, and the fact that cofinalities are preserved, it suffices to lift various embeddings witnessing α^+-subcompactness. Indeed, there are three cases for which we need to check preservation: regular α in I, singular α in K, and singular α in $I\smallsetminus K$. However, for the first two of these, the forcing iteration is trivial at stage α, and the question reduces to lifting embeddings $\pi:(H_{\bar\alpha^+},\in,\bar\sigma)\to(H_{\alpha^+},\in,\sigma)$ for $\mathbb{P}_\alpha$ names σ. We wish to show that it is dense in $\mathbb{P}_\alpha$ to force such a π to lift, so let p be an arbitrary condition in $\mathbb{P}_\alpha$. As in the proof of Theorem 13, the support of p is bounded below κ, and conditions $\bar p$ for embeddings $\pi:(H_{\bar\alpha^+},\in,\{\gamma\},\{\bar p\},\bar\sigma)\to(H_{\alpha^+},\in,\{\gamma\},\{p\},\sigma)$ with $\bar\kappa$ minimal are dense below $p\restriction\kappa$ as γ ranges over ordinals less than κ. Thus we may assume that π is such an embedding with $\bar p\in G$. The structure H_{α^+} correctly computes $I\cap\alpha$ and $K\cap\alpha$, so $\pi(\mathbb{P}_{\bar\alpha})=\mathbb{P}_\alpha$. Let $G_{[\bar\kappa,\bar\alpha)}$ denote the generic over $V[G_{\bar\kappa}]$ for $\mathbb{P}^{[\bar\kappa,\bar\alpha)}$, the part of the iteration from stage $\bar\kappa$ up to but not including stage $\bar\alpha$. Note that the non-trivial iterands in $\mathbb{P}^{[\kappa,\alpha)}$ are all of the form $\mathbb{T}_\beta$ for some singular $\beta \in I \smallsetminus K$, that is, singular β with $\mathrm{cf}(\beta) \geq \kappa$. Since directed closure iterates (see for example [5, Proposition 7.11]), we have that $\mathbb{P}^{[\kappa,\alpha)}$ is $< \kappa$ directed closed. Hence, there is a condition in $\mathbb{P}^{[\kappa,\alpha)}$ extending every condition in $\pi``G_{[\kappa,\alpha)}$, including in particular p, since G was assumed to contain $\bar{p}$. This condition is the desired master condition extending p.

For singular $\alpha \in I \smallsetminus K$, the argument is not too different. Let κ be the least cardinal that is α^+-subcompact. In this case σ will be a $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{T}}_\alpha$-name, so $\bar{\sigma}$ will be $\mathbb{P}_{\bar{\alpha}} * \dot{\mathbb{T}}_{\bar{\alpha}}$-name. As in the proof of Theorem 13, we may take π witnessing α^+-subcompactness of κ such that no κ' is $\bar{\alpha}^+$-subcompact. Thus, the forcing will be $\mathbb{T}_{\bar{\alpha}} * \dot{\mathbb{S}}_{\bar{\alpha}}$ at stage $\bar{\alpha}$, and from G we get a $\mathbb{P}_{\bar{\alpha}} * \dot{\mathbb{T}}_{\bar{\alpha}}$-generic, which gives rise to master condition in $\mathbb{P}^{[\kappa,\alpha+1)}$. As usual this argument can be run below any condition $p \in \mathbb{P}_{\alpha^+}$, so such master conditions are dense, and α^+-subcompactness of $\bar{\kappa}$ is preserved. $\dashv$

In this case our forcing will be non-trivial on certain singular cardinals between κ and λ (as in Definition 9) for κ an ω-super strong cardinal. However, it seems likely that a careful homogeneity argument, using a homogeneity iteration result like those in [9], will show that ω-superstrong cardinals are again preserved under this forcing; we leave the details to the interested reader.

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