# BPFA and Projective Well-orderings of the Reals


Andrés Eduardo Caicedo
Boise State University
Department of Mathematics—1910 University Drive
Boise, ID 83725
Homepage: `http://math.boisestate.edu/~caicedo/`
Email: `caicedo@math.boisestate.edu`

Sy-David Friedman
Kurt Gödel Research Center for Mathematical Logic
University of Vienna
Vienna, Austria
Homepage: `http://www.logic.univie.ac.at/~sdf/`
Email: `sdf@logic.univie.ac.at`


November 25, 2009


**Abstract**

If the bounded proper forcing axiom BPFA holds and $\omega_1 = \omega_1^L$, then there is a lightface $\Sigma^1_3$ well-ordering of the reals. The argument combines a coding due to Caicedo-Veličković with "David's trick." We also present a general coding scheme that, in particular, establishes the following weaker result: BPFA is equiconsistent with the additional requirement that there is a lightface $\Sigma^1_4$ well-ordering of the reals. This is accomplished through a use of David's trick and a coding through the $\Sigma_2$ stable ordinals of $L$, and has the advantage of not requiring the theory of the mapping reflection principle, unlike the optimal result.


## 1 Introduction

BPFA denotes the Bounded Proper Forcing Axiom introduced in Goldstern-Shelah [6]. In this paper we show that BPFA implies the existence of well-orderings of descriptive set theoretic optimal complexity under the anti-large cardinal assumption that $\omega_1 = \omega_1^L$.

Recall that $\vec{C} = (C_\alpha : \alpha < \omega_1)$ is a ***C-sequence*** (or a *ladder system*) iff $C_\alpha \subseteq \alpha$ is cofinal in $\alpha$ and of least possible order type, for all $\alpha < \omega_1$.

In Caicedo-Veličković [3] it is shown that BPFA implies that for any C-sequence $\vec{C}$ there is a $\Sigma_1$ well-ordering of $\mathbb{R}$ in $\vec{C}$ as a parameter.

Here, we combine this result with a coding method of David (see Friedman [4] or [5, §6.2]) to prove:

**Theorem 1.** *If* BPFA *holds and* $\omega_1 = \omega_1^L$*, then there is a lightface* $\Sigma^1_3$ *well-ordering of the reals.*

This is best possible in the sense that already MA implies that there are no $\utilde{\mathbf{\Sigma}}^1_2$ well-orderings. Notice that we obtain an implication rather than merely a consistency result. The coding used in Caicedo-Veličković [3] requires an understanding of the theory of the Mapping Reflection Principle MRP, see Moore [8]. To provide a further illustration of the use of "David's trick" for those readers not familiar with MRP, we include the following weaker result:

Recall that a cardinal $\kappa$ is ***reflecting*** iff $\kappa$ is regular and $V_\kappa$ is $\Sigma_2$-elementary in the universe $V$. In Goldstern-Shelah [6] it is shown that BPFA is equiconsistent over ZFC with the existence of a reflecting cardinal.

**Theorem 2.** *The following are equiconsistent:*

1. *There is a reflecting cardinal.*
2. BPFA *holds, and there is a (lightface)* $\Sigma^1_4$ *well-ordering of the reals.*

**Remark 3.** *Actually, the argument of Theorem 2 allows us (consistently) to code in a* $\Sigma^1_4$ *fashion many relations on* $\mathbb{R}$ *that can be "locally certified" in a certain sense, see Remark 9.*

It was shown in Caicedo [2] that BPFA is consistent with the existence of projective well-orderings of the reals, and it was already noted in Caicedo-Veličković [3] that if $\omega_1^L = \omega_1$ and BPFA holds, then there is a *lightface* projective well-ordering. However, the coding arguments used in these papers do not seem to suffice to obtain a well-ordering of smaller complexity than $\Sigma^1_6$.

As explained in Section 3, one can obtain a well-ordering of smaller complexity by enhancing the standard (Goldstern-Shelah) iteration that forces BPFA, by including stages at which certain trees are *specialised*, following a method of Baumgartner [1], and at which "$\Pi^1_2$ witnesses" to these specialisations are added, following the method of David. To prevent the witnessing of BPFA from damaging the codings, we are forced to concentrate the iteration on stages $\alpha$ that are $\Sigma^L_2$ ***stable***, i.e., such that $L_\alpha$ is $\Sigma_2$-elementary in $L$. Unfortunately, this forces us to also introduce $\Pi^1_2$ witnesses to failures of $\Sigma^L_2$ stability. These last witnesses lead us to a $\Sigma^1_4$, rather than $\Sigma^1_3$, definition of the well-ordering.

In Section 2 we review the notion of $\mathcal{S}$-properness which will be needed in the argument, and prove a combinatorial lemma that will be used to carry out the coding.

It is shown in Friedman [5, Theorem 8.51] that $\mathsf{MA} + \omega_1 = \omega_1^L$ is consistent with a $\Sigma^1_3$ well-ordering. The argument uses an iteration of Jensen-like codings. A natural attempt by the second author at generalizing this approach failed because we do not have the kind of reflection needed to ensure BPFA at the end of the iteration—while the kind of reflection required by MA poses no difficulties.

The well-ordering of optimal complexity is exhibited in Section 4. We show that, in the presence of $\mathsf{BPFA} + \omega_1 = \omega_1^L$, David's trick allows one to convert a well-ordering of $\mathbb{R}$ that is $\Sigma_1$ over $H(\omega_2)$ in $\omega_1$ as a parameter, into a $\Sigma^1_3$ well-ordering.

The proof of Theorem 2, in particular the fact that we seem forced to use $\Sigma_2$-stable stages, suggested initially that $\mathsf{BPFA}$ would rule out the existence of $\Sigma^1_3$ well-ordering of the reals.

There were other obstacles: Assuming that every real has a sharp, the existence of a $\underset{\sim}{\mathbf{\Sigma}}^1_3$ well-ordering of the reals implies $\mathsf{CH}$. In addition, in the presence of sharps, $\mathsf{MA}_{\omega_1}$ (Martin's axiom for partial orders of size $\omega_1$) implies that every $\underset{\sim}{\mathbf{\Sigma}}^1_3$ set of reals is Lebesgue measurable. These two statements are proved in Hjorth [7].

This suggested that if there were at all a model of $\mathsf{BPFA}$ with a $\Sigma^1_3$ well-ordering of the reals, then there was likely one satisfying $\omega_1 = \omega_1^L$. This led us to reexamine the coding in Caicedo-Veličković [3] and eventually to Theorem 1.

### 1.1 Acknowledgements

The first author thanks the National Science Foundation for partial support through grant DMS-0801189. The second author wishes to thank the Austrian Science Fund (FWF) for its generous support through Project Number P 19375-N18.

## 2 Preliminaries

To prove Theorem 2, we define in Section 3 a countable support iteration that forces $\mathsf{BPFA}$. Unlike the usual argument, and for reasons having to do with the forcings that add localising witnesses (as explained in Section 3), the factors in the iteration will not be proper but only $\mathcal{S}$-*proper*, in the sense described below.

**Definition 4.** *Say that a class $\mathcal{S}$ is* **closed under truncation** *iff for all regular uncountable cardinals $\theta$ and all $x \in S$, we have that $x \cap H(\theta) \in S$.*

*A class $\mathcal{S}$ is* **everywhere stationary** *iff $S$ is closed under truncation, and its intersection with $[H(\theta)]^\omega$ is stationary for all uncountable regular cardinals $\theta$.*

*Suppose that $\mathcal{S}$ is everywhere stationary. A partial order $\mathbb{P}$ is* **$\mathcal{S}$-proper** *iff for all regular cardinals $\theta > \omega_1$ such that $\mathbb{P} \in H(\theta)$, there is a club of countable elementary substructures $x$ of $H(\theta)$ with the property that if $x \in \mathcal{S}$ and $p \in \mathbb{P} \cap x$, then there is $q \leq p$ in $\mathbb{P}$ which forces the generic to intersect $D \cap x$ for any $D \in x$ that is dense in $\mathbb{P}$.*

$\mathcal{S}$-properness is a $\Sigma_2$ notion (in the predicate $\mathcal{S}$), as "all regular cardinals $\theta$" can be replaced by "the least regular cardinal $\theta$" in the above definition. This is because if $\theta > \omega_1$ is the least regular cardinal such that $\mathbb{P} \in H(\theta)$, $C$ witnesses the desired property for $\theta$, and $\tau > \theta$ is regular, then (using closure under truncation) we have that

$$C^* = \{x \colon x \cap H(\theta) \in C\}$$

witnesses the desired property for $\tau$.

Just as with the usual notion of properness, $\mathcal{S}$-proper forcing notions preserve $\omega_1$, and $\mathcal{S}$-properness is preserved under countable support iterations (see Shelah [9]).

Our method for obtaining a definable well-ordering is based on the following lemma. For $\beta$ a regular uncountable cardinal, let $T(\beta)$ be the tree $(\beta^+)^{<\beta}$ of sequences through $\beta^+$ of length less than $\beta$.

**Lemma 5.** *Assume $V = L$ and that $\beta > \omega_1$ is regular. Let $\mathcal{S}$ be an everywhere stationary class. Suppose that $\mathbb{Q}$ is an $\mathcal{S}$-proper forcing, that $|\mathbb{Q}| < \beta$, and that $G$ is $\mathbb{Q}$-generic over $L$. Then:*

1. *$T(\beta)$, viewed as a forcing, is $\mathcal{S}$-proper in $L[G]$.*
2. *There is a proper forcing $\mathbb{R}$ in $L[G]$ of size $\beta^{++}$ that destroys the $\mathcal{S}$-properness of $T(\beta)$; in fact, if $H$ is $\mathbb{R}$-generic over $L[G]$, then in any $\omega_1$-preserving outer model of $L[G][H]$ there is no branch through $T(\beta)$ which is $T(\beta)$-generic over $L$.*

*Proof.* (1) It suffices to show that $\mathbb{Q}$ is $\mathcal{S}$-proper in $T(\beta)$-generic extensions of $L$. But the forcing $T(\beta)$ is $\beta$-closed and therefore does not add subsets of $\max\{|\mathbb{Q}|, \omega_1\}$; it follows that any witness to the $\mathcal{S}$-properness of $\mathbb{Q}$ in $L$ is still a witness to its $\mathcal{S}$-properness in any $T(\beta)$-generic extension of $L$.

(2) First add $\beta^{++}$ Cohen reals with a finite support product over $L[G]$, producing $L[G][H_0]$. Then Lévy collapse $\beta^{++}$ to $\omega_1$ with countable conditions, producing $L[G][H_0][H_1]$. As ccc and $\omega$-closed forcings are proper, this is a proper forcing extension of $L[G]$.

Note that (as originally shown by Silver) in $L[G][H_0][H_1]$, any $\beta$-branch through $T(\beta)$ in fact belongs to $L[G][H_0]$: Otherwise we choose an $L[G][H_0]$-name $\dot{b}$ for the new branch and build a binary $\omega$-tree $U$ of conditions in the Lévy collapse, each branch of which has a lower bound, such that distinct branches force different interpretations of the name $\dot{b}$. It follows that in $L[G][H_0]$, $T(\beta)$ has $2^{\aleph_0} = \beta^{++}$ nodes on a fixed level, which is impossible because $\mathsf{GCH}$ holds in $L$.

Thus the tree $T(\beta)$ has at most $\omega_1$-many branches in $L[G][H_0][H_1]$, none of which contains ordinals cofinal in $\beta^+$ and therefore none of which is $T(\beta)$-generic over $L$. Also, every node of $T(\beta)$ belongs to a $\beta$-branch.

Now we use Baumgartner's general method of "specialising a tree off a small set of branches".

**Fact 6.** *If $T$ is a tree of height $\omega_1$ with at most $\aleph_1$ cofinal branches (and every node of $T$ belongs to a cofinal branch of $T$) then there is a ccc forcing $\mathbb{P}$ such that if $G$ is $\mathbb{P}$-generic over $V$ then in any $\omega_1$-preserving outer model of $V[G]$, all cofinal branches through $T$ belong to $V$.*

*Proof.* We outline the argument and refer the reader to Baumgartner [1] for details.

List the branches as $(b_i \mid i < \omega_1)$ and write $T$ as the disjoint union of $b_i(x_i)$, where the $x_i$ are distinct nodes of the tree chosen so that each $x_i$ is a node on $b_i$ and $b_i(x_i)$ denotes the tail of $b_i$ starting at $x_i$. Now force to add a function $f$ with finite conditions from $\{x_i \mid i < \omega_1\}$ into $\omega$ such that if $x_i$ is below $x_j$ in $T$ then $f(x_i)$ is different from $f(x_j)$. Baumgartner [1] shows that this forcing is ccc. Now if $b$ is a cofinal branch through $T$ distinct from the $b_i$'s in an $\omega_1$-preserving outer model of $V[f]$, then $b$ must intersect uncountably many of the $b_i(x_i)$'s and therefore contains uncountably many $x_i$'s. But then the $f(x_i)$'s are distinct for these uncountably many $x_i$'s, contradicting the fact that $f$ maps into $\omega$.

This completes the proof of Fact 6. □

Now use Fact 6 to create a ccc extension $L[G][H_0][H_1][H_2]$ of $L[G][H_0][H_1]$ to ensure that $T(\beta)$ (viewed as a tree of height $\omega_1$ using a cofinal $\omega_1$-sequence through $(\beta^+)^L$) will have no new branches in any $\omega_1$-preserving outer model. As no $\beta$-branch through $T(\beta)$ in $L[G][H_0]$ is $T(\beta)$-generic over $L$ and all cofinal branches through $T(\beta)$ in any $\omega_1$-preserving outer model of $L[G][H_0][H_1][H_2] = L[G][H]$ belong to $L[G][H_0]$, we are done.

This completes the proof of Lemma 5. □

# 3 BPFA and a $\Sigma^1_4$ well-ordering

We now begin the proof of the direction Con(1) → Con(2) of Theorem 2; the other direction follows from Goldstern-Shelah [6].

Assume $V = L$ and let $\kappa$ be reflecting. Fix an appropriate bookkeeping function $f : \kappa \to H(\kappa)$ (so that $f$ "guesses" every object in $H(\kappa)$ stationarily often). We will use $f$ throughout the argument to select certain objects. We use a countable support iteration of length $\kappa$. As mentioned in the previous section, the factors in our iteration will be $\mathcal{S}$-proper for a suitable everywhere stationary class $\mathcal{S}$, that we now proceed to describe. As usual, by $\mathsf{ZF}^-$ we denote $\mathsf{ZF}$ without the power set axiom.

Suppose that $\theta$ is regular and uncountable, and that $x$ is a countable elementary substructure of $L_\theta$. Let $(x, \in)$ be isomorphic to $L_\alpha$. We say that $x$ ***collapses nicely*** iff for all $\beta \geq \alpha$, if $L_\beta$ is a model of $\mathsf{ZF}^-$ and $x \cap \omega_1$ is a cardinal in $L_\beta$, then every cardinal of $L_\alpha$ is also a cardinal of $L_\beta$.

Let $\mathcal{S}$ be the class of all $x$ in $L$ which collapse nicely.

**Lemma 7.** *$\mathcal{S}$ is everywhere stationary.*

*Proof.* Let $\theta$ be regular and uncountable, and let $C \subseteq [L_\theta]^\omega$ be club, so $C \in L_{\theta^+}$. Let $x$ be the least elementary substructure of $L_{\theta^+}$ that contains $C$ as an element. Then $x \cap L_\theta \in C$. Let $L_\alpha$ be the transitive collapse of $(x, \in)$. Then there is an $L_{\alpha+1}$-definable injection from $L_\alpha$ into $\omega$ and, therefore, there is *no* $\beta > \alpha$ such that $L_\beta \models \mathsf{ZF}^-$ and $x \cap \omega_1$ is a cardinal of $L_\beta$. It follows that $x \in S$ and therefore $x \in S \cap C$. Since $S$ is clearly closed under truncation, we are done. □

Let $C$ enumerate the closed unbounded subset of $\kappa$ consisting of those $\alpha$ such that $L_\alpha$ is $\Sigma_2$-elementary in $L_\kappa$. (As $\kappa$ is regular, $C$ is indeed unbounded in $\kappa$.) We perform an $\mathcal{S}$-proper iteration of length $\kappa$ with countable support which is nontrivial at stages $\alpha$ in $C$. The iteration $\mathbb{P}_\alpha * \mathbb{Q}(\alpha)$ up to and including stage $\alpha$ will belong to $L_\beta$ where $\beta$ is the least element of $C$ greater than $\alpha$. In particular, $|\mathbb{P}_\alpha| < \kappa$ for each $\alpha < \kappa$, and therefore $\kappa$ remains reflecting throughout the iteration.

Suppose that $\alpha$ belongs to $C$. We proceed to describe the forcing $\mathbb{Q}(\alpha)$ as a six-step iteration $\mathbb{Q}^0(\alpha) * \mathbb{Q}^1(\alpha) * \mathbb{Q}^2(\alpha) * \mathbb{Q}^3(\alpha) * \mathbb{Q}^4(\alpha) * \mathbb{Q}^5(\alpha)$.

## 3.1 $\mathbb{Q}^0(\alpha)$

Inductively, $\mathbb{P}_\alpha$ has size at most $(\alpha^+)^L$. By Lemma 5, we know that the forcing $T(\beta)$, consisting of $(< \beta)$-sequences through $\beta^+$, is $\mathcal{S}$-proper in $L[G_\alpha]$ when $\beta$ is regular and at least $(\alpha^{++++})^L$. In addition, there is a forcing $\mathbb{R}(\beta)$ of size $\beta^{++}$ in $L[G_\alpha]$ which guarantees that there is no $T(\beta)$-generic over $L$.

Now let $\alpha_n$ be $(\alpha^{+4(n+1)})^L$ for each finite $n$, and let $T(n), \mathbb{R}(n)$ denote $T(\alpha_n), \mathbb{R}(\alpha_n)$. Then both $T(n)$ and $\mathbb{R}(n)$ are $\mathcal{S}$-proper in any extension of $L[G_\alpha]$ obtained by forcing with $U(0) * U(1) * \cdots * U(n-1)$ where each $U(i)$ is either $T(i)$ or $\mathbb{R}(i)$.

Let $<_\alpha$ denote the natural well-ordering of $L[G_\alpha]$ and let $x_\alpha <_\alpha y_\alpha$ be the pair of reals in $L[G_\alpha]$ provided by the bookkeeping function (which guarantees that any pair $(x, y)$ of reals which appears in the iteration is of the form $(x_\alpha, y_\alpha)$ for some $\alpha$, provided it satisfies $x <_\beta y$ where $\beta$ is least such that $x, y$ both belong to $L[G_\beta]$).

Now take $\mathbb{Q}^0(\alpha)$ to be the (fully supported) $\omega$-iteration $U(0) * U(1) * \ldots$ where $U(n)$ equals $T(n)$ if $n$ belongs to $x_\alpha * y_\alpha$ (the join of $x_\alpha$ and $y_\alpha$) and equals $\mathbb{R}(n)$ otherwise. This is an $\mathcal{S}$-proper forcing and $\mathbb{P}_\alpha * \mathbb{Q}^0(\alpha)$ belongs to $L_\beta$, where $\beta$ is the least element of $C$ greater than $\alpha$.

## 3.2 $\mathbb{Q}^1(\alpha)$

Now we consider the $\Sigma_1$ sentence with parameter from $L[G_\alpha] \cap \mathcal{P}(\omega_1)$, provided by the bookkeeping function (which ensures that all $\Sigma_1$ sentences with parameter from the final $\mathcal{P}(\omega_1)$ will be considered at some stage $\alpha < \kappa$ in $C$).

Ask of this sentence whether it holds in an $\mathcal{S}$-proper forcing extension of $L[G_\alpha][H^0]$, where $H^0$ is our $\mathbb{Q}^0(\alpha)$-generic. If so, then as $\kappa$ is reflecting in $L[G_\alpha][H^0]$, there is such an $\mathcal{S}$-proper forcing in $L_\kappa[G_\alpha][H^0]$, and also the witness to the $\Sigma_1$ sentence can be assumed to have a name in $L_\kappa[G_\alpha][H^0]$. Let $\beta$ be the least element of $C$ greater than $\alpha$; then as $L_\beta$ is $\Sigma_2$-elementary in $L_\kappa$, it follows that $L_\beta[G_\alpha][H^0]$ is $\Sigma_2$-elementary in $L_\kappa[G_\alpha][H^0]$. Thus we can choose our $\mathcal{S}$-proper forcing $\mathbb{Q}^1(\alpha)$ witnessing the $\Sigma_1$ sentence to be an element of $L_\beta[G_\alpha][H^0]$, necessary to satisfy the requirement that $\mathbb{P}_\alpha * \mathbb{Q}(\alpha)$ belong to $L_\beta$. Let $H^1$ denote the generic for $\mathbb{Q}^1(\alpha)$.

### 3.3 $\mathbb{Q}^2(\alpha)$

The forcing $\mathbb{Q}^2(\alpha)$ is the Lévy collapse with countable conditions of a sufficiently large ordinal less than $\kappa$ to $\omega_1$, to ensure that the resulting extension $L[G_\alpha][H^0][H^1][H^2]$ is of the form $L[X_\alpha]$ where $X_\alpha$ is a subset of $\omega_1$ which codes the ordinal $\alpha$ as well as the generic $H^0 * H^1 * H^2$. Then we have:

> $(*)$ If $M = L_\delta[X_\alpha]$ is a model of $\mathsf{ZF}^-$, then $(\alpha^{+\omega})^L$ is an ordinal of $M$, and in $M$ there is a branch through $T((\alpha^{+4(n+1)})^L)$ whose ordinals are cofinal in $(\alpha^{+4(n+1)})^L$ iff $n$ belongs to $x_\alpha * y_\alpha$.

### 3.4 $\mathbb{Q}^3(\alpha)$

The purpose of the forcing $\mathbb{Q}^3(\alpha)$ is to add $Y_\alpha \subseteq \omega_1$ that "localises" property $(*)$ in the following sense. Let $\mathrm{Even}(Y_\alpha)$ denote $\{\delta \mid 2\delta \in Y_\alpha\}$. Then:

> $(**)$ For any $\gamma < \omega_1$ and countable $\mathsf{ZF}^-$ model $M$ containing $Y_\alpha \cap \gamma$ as an element: If $\gamma = \omega_1^M = (\omega_1^L)^M$ then $\mathrm{Even}(Y_\alpha \cap \gamma)$ codes an $L$-cardinal $\bar{\alpha}$ of $M$ such that there is a branch through the $T((\bar{\alpha}^{+4(n+1)})^L)$ of $M$ whose ordinals are cofinal in the $(\bar{\alpha}^{+4(n+1)})^L$ of $M$ iff $n$ belongs to $x_\alpha * y_\alpha$.

We now describe the forcing $\mathbb{Q}^3(\alpha)$ for adding the witness $Y_\alpha$ to $(**)$. A condition in $\mathbb{Q}^3(\alpha)$ is an $\omega_1$-Cohen condition $r : |r| \to 2$ in $L[X_\alpha]$ with the following properties:

1. The domain $|r|$ of $r$ is a countable limit ordinal.
2. $X_\alpha \cap |r|$ is the even part of $r$, i.e., for $\gamma < |r|$, $\gamma$ belongs to $X_\alpha$ iff $r(2\gamma) = 1$.
3. $(**)$ holds for all limit $\gamma \leq |r|$ with $Y_\alpha \cap \gamma$ replaced by $r \restriction \gamma$, i.e.:

    > $(**)_r$ For any limit $\gamma \leq |r|$ and countable $\mathsf{ZF}^-$ model $M$ containing $r \restriction \gamma$ as an element: If $\gamma = \omega_1^M = (\omega_1^L)^M$ then $\mathrm{Even}(r \restriction \gamma)$ codes some $\bar{\alpha}$, an $L$-cardinal of $M$, such that there is a branch through the $T((\bar{\alpha}^{+4(n+1)})^L)$ of $M$ whose ordinals are cofinal in the $(\bar{\alpha}^{+4(n+1)})^L$ of $M$ iff $n$ belongs to $x_\alpha * y_\alpha$.

**Lemma 8.** *$\mathbb{Q}^3(\alpha)$ is $\mathcal{S}$-proper.*

*Proof.* First note that we have the following ***extendibility property***: Given $r$ and a countable limit $\gamma$ greater than $|r|$, we can extend $r$ to $r^*$ of length $\gamma$.

This is because we can take the odd part of $r^*$ on the interval $[|r|, |r| + \omega)$ to code $\gamma$ and to consist only of 0's on $[|r| + \omega, \gamma)$; then there are no new instances of requirement (3) for being a condition to check because no $\mathsf{ZF}^-$ model containing $r^* \restriction |r| + \omega$ can have its $\omega_1$ in the interval $(|r|, \gamma]$.

Now in $L[X_\alpha]$ let $\theta$ be large and regular, let $M$ be countable and elementary in $H(\theta)$ with $M \cap L$ in $\mathcal{S}$ and let $r$ belong to $\mathbb{Q}^3(\alpha) \cap M$. Successively extend $r$ to $r = r_0 \geq r_1 \geq \cdots$ in $M$ so that if $D$ in $M$ is dense on $\mathbb{Q}^3(\alpha)$ then for some $k$, $r_k$ meets $D$. (In particular, $r_k$ forces the $\mathbb{Q}^3(\alpha)$-generic to meet $D$ in a condition belonging to $M$.) By extendibility, $\sup r_k$ converges to $\delta := M \cap \omega_1$.

We want to show that the $r_k$'s admit the lower bound $r_\omega = \bigcup_k r_k$. For this, it suffices to verify property $(**)_{r_\omega}$ when $\gamma = \delta$, i.e.:

> $(***)$ For any countable $\mathsf{ZF}^-$ model $N$ containing $r_\omega$ as an element: If $\delta = \omega_1^N = (\omega_1^L)^N$ then $\text{Even}(r_\omega)$ codes some $\bar{\alpha}$, an $L$-cardinal of $N$, such that there is a branch through the $T((\alpha^{+4(n+1)})^L)$ of $N$ whose ordinals are cofinal in the $(\alpha^{+4(n+1)})^L$ of $N$ iff $n$ belongs to $x_\alpha * y_\alpha$.

$M$ is elementary in $H(\theta) = L_\theta[X_\alpha]$. Let $\bar{M} = L_{\bar{\theta}}[X_\alpha \cap \delta]$ be the transitive collapse of $M$, where $\alpha$ is sent to $\bar{\alpha}$ under the transitive collapse map. As $X_\alpha$ codes the generic $G_\alpha * H^0 * H^1 * H^2$, it ensures that in $L_\theta[X_\alpha]$ there is a branch through $T((\alpha^{+4(n+1)})^L)$ whose ordinals are cofinal in $(\alpha^{+4(n+1)})^L$ iff $n$ belongs to $x_\alpha * y_\alpha$. By elementarity, in $\bar{M}$ there is a branch through the $T((\alpha^{+4(n+1)})^L)$ of $\bar{M}$ whose ordinals are cofinal in the $(\alpha^{+4(n+1)})^L$ of $\bar{M}$ iff $n$ belongs to $x_\alpha * y_\alpha$.

Now if $\bar{N}$ is any countable $\mathsf{ZF}^-$ model containing $r_\omega$ as an element such that $\omega_1^{\bar{N}} = \delta$, $\bar{N}$ also contains $X_\omega \cap \delta$ as an element (as $X_\omega \cap \delta$ is the even part of $r_\omega$) and as $M \cap L = L^M$ collapses nicely, the $(\alpha^{+4(n+1)})^L$, $T((\alpha^{+4(n+1)})^L)$ of $\bar{M}$ are equal to those of $\bar{N}$. It follows that also in $\bar{N}$, there is a branch through the $T((\alpha^{+4(n+1)})^L)$ of $\bar{N}$ whose ordinals are cofinal in the $(\alpha^{+4(n+1)})^L$ of $\bar{N}$ iff $n$ belongs to $x_\alpha * y_\alpha$, establishing $(***)$. □

### 3.5 $\mathbb{Q}^4(\alpha)$

We next code the $\mathbb{Q}^3(\alpha)$-generic $Y_\alpha$ by a real using $\mathbb{Q}^4(\alpha)$, a ccc almost disjoint coding with finite conditions.

To each countable ordinal $\beta$ associate the set $b_\beta$ of numbers that code a finite initial segment of the $\beta$-th real in the natural well-ordering of the reals in $L$. Then distinct $b_\beta$'s have a finite intersection. A condition in $\mathbb{Q}^4(\alpha)$ is a pair $(s, A)$ where $s$ is a finite subset of $\omega$ and $A$ is a finite subset of $\{b_\beta \mid \beta \in Y_\alpha\}$. Extension is defined by: $(s, A) \leq (t, B)$ iff $s$ end-extends $t$, $A$ contains $B$ as a subset and $s \setminus t$ is disjoint from each element of $B$.

This forcing is ccc because any two conditions with the same first component are compatible and there are only countably many first components. The generic produces a subset $R_\alpha$ of $\omega$ that is almost disjoint from $b_\beta$ exactly if $\beta$ belongs to $Y_\alpha$.

As the sequence of $b_\beta$'s belongs to $L$, it follows that $Y_\alpha$ belongs to $L[R_\alpha]$. Most importantly, as for any countable $\mathsf{ZF}^-$ model $M$ containing $R_\alpha$ as an element, $Y_\alpha \cap (\omega_1^L)^M$ can be decoded from $R_\alpha$ in $M$, we have:

> $(*)_{R_\alpha}$ For any countable $\mathsf{ZF}^-$ model $M$ containing $R_\alpha$ as an element and such that $\omega_1^M = (\omega_1^L)^M$, $R_\alpha$ codes in $M$ some $\bar{\alpha}$, an $L$-cardinal of $M$, such that $T((\bar{\alpha}^{+4(n+1)})^L)$ has a branch whose ordinals are cofinal in $(\bar{\alpha}^{+4(n+1)})^L$ iff $n$ belongs to $x_\alpha * y_\alpha$.

### 3.6 $\mathbb{Q}^5(\alpha)$

To complete stage $\alpha$ of the iteration we apply a forcing $\mathbb{Q}^5(\alpha)$ introducing $\Pi^1_2$ witnesses to failures of $\Sigma^L_2$ stability.

Let $z_\alpha$ be the real in $L[G_\alpha]$ provided by the bookkeeping function (so that each real that appears anywhere in the iteration is equal to $z_\alpha$ for some $\alpha \in C$).

We say that $z_\alpha$ is a ***coding witness for*** $x < y$ (where $x, y$ are reals in $L[G_\alpha]$) iff $(*)_{z_\alpha,x,y}$ holds (where $(*)_{z_\alpha,x,y}$ is $(*)_{R_\alpha}$ with $R_\alpha, x_\alpha, y_\alpha$ replaced by $z_\alpha, x, y$).

Note that by reflection, $(*)_{z_\alpha,x,y}$ holds without the restriction that $M$ be countable. Let $\delta$ be the $L$-cardinal witnessing $(*)_{z_\alpha,x,y}$ for the model

$$M = L_{\kappa^+}[G_\alpha][H^0][H^1][H^2][H^3][H^4],$$

where $H^i$ is the generic for $\mathbb{Q}^i(\alpha)$. Then if $\delta$ is *not* $\Sigma^L_2$ stable, the forcing $\mathbb{Q}^5(\alpha)$ introduces a real $w_\alpha$ such that:

> $(****)_{z_\alpha,w_\alpha}$ For all countable $\mathsf{ZF}^-$ models $M$ containing $z_\alpha, w_\alpha$ as elements and such that $\omega_1^M = (\omega_1^L)^M$, $w_\alpha$ codes in $M$ some $\bar{\beta}$, an $L$-cardinal of $M$, such that $L_{\bar{\alpha}}$, where $\bar{\alpha}$ is the $L$-cardinal of $M$ coded by $z_\alpha$, is not $\Sigma_2$-elementary in $L_{\bar{\beta}}$.

The forcing $\mathbb{Q}^5(\alpha)$ is defined analogously to the two-step iteration $\mathbb{Q}^3(\alpha) * \mathbb{Q}^4(\alpha)$, and like that forcing, it is $\mathcal{S}$-proper.

This completes stage $\alpha$ of the iteration.

### 3.7 The well-ordering

The iteration so defined is $\mathcal{S}$-proper, forces $\kappa$ to be at most $\omega_2$, and is $\kappa$-cc. It follows that $\kappa = \omega_2$ in the generic extension $L[G]$, and the standard argument shows that $\mathsf{BPFA}$ (indeed, the bounded forcing axiom for $\mathcal{S}$-proper forcings) holds there.

To describe the desired $\Sigma^1_4$ well-ordering of the reals, say that a real $z$ is a ***good coding witness for*** $x < y$ iff it is a coding witness for $x < y$, and there is *no* $w$ witnessing the failure of the $\Sigma^L_2$ stability of the $L$-cardinal coded by $z$, i.e., there is no real $w$ such that $(****)_{z,w}$ holds.

The set of good witnesses is $\Pi^1_3$. Thus the desired well-ordering of the reals in $L[G]$ is given by:

> $x < y$ iff for some $\alpha$ in $C$, $(x, y) = (x^G_\alpha, y^G_\alpha)$ iff there exists a good coding witness for $x < y$.

This completes the proof of Theorem 2.

**Remark 9.** *Although we organized the presentation of the argument above around the proof of Theorem 2, it should be clear that it is really more general. In fact, it allows us to code in a $\Sigma^1_4$ way many relations $R$ on $\mathbb{R}$ that can be added in a local fashion throughout an iteration as the one we described.*

*More specifically, work in L, and suppose that $\kappa$ is reflecting. Suppose that there is a countable support iteration $\mathbb{P}$ of size $\kappa$ with intermediate stages $\mathbb{P}_\alpha$ of size below $\kappa$ that are $\mathcal{S}$-proper, where $\mathcal{S}$ is as above. Suppose there is a definable relation $R'$ such that, uniformly in the ground model, whenever $G$ is $\mathbb{P}$-generic, for each tuple $\vec{r}$ of reals of $V[G]$, we can identify an intermediate stage $\alpha$ such that $\vec{r}$ already belongs to the $\alpha$-th intermediate model $V[G_\alpha]$, and $V[G_\alpha] \models R'(\vec{r})$. Suppose that the forcing $\mathbb{Q}_\alpha$ at stage $\alpha$ uses David trick as in Subsections 3.4–3.6 above, to code $R'(\vec{r})$ and, if necessary, the failure of $\Sigma^L_2$-stability of $\alpha$, by reals. Let $R$ in $V[G]$ be the relation given by $R(\vec{r})$ iff $V[G_\alpha] \models R'(\vec{r})$ for $\alpha$ as above. Then $R$ is $\Sigma^1_4$ in $V[G]$.*

*Note that Theorem 2 is a particular instance of this scheme.*

# 4 BPFA and a well-ordering of optimal complexity

Here we prove Theorem 1.

We begin by noticing that the argument from Caicedo-Veličković [3] shows that whenever BPFA holds and $\omega_1 = \omega_1^L$, then there is a $\Sigma_1$ well-ordering of $H(\omega_2)$ in $\omega_1$ as a parameter, since any transitive model $M$ of an appropriate fragment of $\mathsf{ZFC} + \mathsf{BPFA}$ that computes $\omega_1$ correctly would be able to compute correctly the $L$-least C-sequence $\vec{C}$, which is also a C-sequence in $V$ and $M$.

But now David's trick allows us to turn this into a $\Sigma^1_3$ well-ordering: Say that $x < y$ iff $\phi(x, y, \omega_1)$, where $\phi$ is $\Sigma_1$. Then, $x < y$ iff

> $(*)$ For some real $z$, $M \models \phi(x, y, \omega_1^M)$ for each countable transitive model $M$ of $\mathsf{ZF}^-$ containing $x$, $y$, and $z$ such that $\omega_1^M = (\omega_1^L)^M$.

The point is that with countably closed forcings, we can first collapse $\kappa = \beth_{\omega_1}$ to size $\omega_1$ and then, using the fact that $\kappa^+ = (\kappa^+)^L$ (which holds by covering), code the resulting $H(\omega_2)$ into a subset of $\omega_1$, i.e., arrange that

$$H(\omega_2) = L_{\omega_2}[A]$$

for some $A \subseteq \omega_1$. Then, over this model, the forcing that produces the real $z$ (given a witness to $\phi(x, y, \omega_1)$, as in Section 3) is proper and of size $\omega_1$; the appropriate version of $(**)$ from Subsection 3.4 is:

> For any $\gamma < \omega_1$ and countable $M \models \mathsf{ZF}^-$ containing $Y \cap \gamma$ as an element, we have that if $\gamma = \omega_1^M = (\omega_1^L)^M$, then $\mathrm{Even}(Y \cap \gamma)$ codes a witness to $\phi(x, y, \gamma)$ in $M$.

As the $L$-cardinals are not being used in the coding, the notion of "collapsing nicely" is no longer needed and the forcing to add such a $Y$ is fully proper.

Then, as in Subsection 3.5, we apply a ccc forcing to obtain the desired witnessing real $z$ satisfying $(*)$ above. Finally, as the forcing to produce $z$ is proper and BPFA holds, such a $z$ must exist in $V$.

## 5 Open questions

We close the paper with some natural problems suggested by the results above:

1. In Theorem 1, can the hypothesis $\omega_1 = \omega_1^L$ be weakened to $0^\sharp$ does not exist?

2. Is $\mathsf{MA} + \omega_1 = \omega_1^L$ consistent with the *nonexistence* of a projective well-ordering of the reals?

## References


[1] Baumgartner, James. *Applications of the proper forcing axiom.* In **Handbook of set-theoretic topology**, Kunen, Kenneth; Vaughan Jerry, eds., North-Holland, Amsterdam, 1984, 913–959, .

[2] Caicedo, Andrés E. *Projective well-orderings and bounded forcing axioms.* J. Symbolic Logic **70 (2)** (2005), 557–572.

[3] Caicedo, Andrés E.; Veličković, Boban. *The bounded proper forcing axiom and well-orderings of the reals.* Mathematical Research Letters **13 (2-3)** (2006), 393–408.

[4] Friedman, Sy-D. *David's trick.* In **Sets and proofs (Leeds, 1997)**, Cooper, Barry; Truss John, eds., London Math. Soc. Lecture Note Ser., **258**, Cambridge Univ. Press, Cambridge, 1999, 67–71.

[5] Friedman, Sy-D. **Fine structure and class forcing**, de Gruyter Series in Logic and its Applications, **3**, Walter de Gruyter, Berlin, 2000.

[6] Goldstern, Martin; Shelah, Saharon. *The bounded proper forcing axiom.* J. Symbolic Logic **60 (1)** (1995), 58–73.

[7] Hjorth, Greg. *The size of the ordinal* $u_2$. J. London Math. Soc. (2) **52 (3)** (1995), 417–433.

[8] Moore, Justin. *Set mapping reflection.* Journal of Mathematical Logic, **5 (1)** (2005), 87–98.

[9] Shelah, Saharon. **Proper and improper forcing**, Second edition. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.