Slow Consistency

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Abstract. The fact that "natural" theories, i.e. theories which have something like an "idea" to them, are almost always linearly ordered with regard to logical strength has been called one of the great mysteries of the foundation of mathematics. However, one easily establishes the existence of theories with incomparable logical strengths using self-reference (Rosser-style). As a result, **PA** + Con(**PA**) is not the least theory whose strength is greater than that of **PA**. But still we can ask: is there a sense in which **PA** + Con(**PA**) is the least "natural" theory whose strength is greater than that of **PA**? In this paper we exhibit natural theories in strength strictly between **PA** and **PA** + Con(**PA**) by introducing a notion of slow consistency.

Key words: Peano arithmetic, consistency strength, fast growing function, slow consistency
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1 Preliminaries

PA is Peano Arithmetic. $\mathbf{PA}\restriction_k$ denotes the subtheory of **PA** usually denoted by $\mathbf{I}\Sigma_k$. It consists of a finite base theory $\mathbf{P}^-$ (which are the axioms for a commutative discretely ordered semiring) together with a single Π_{k+2} axiom which asserts that induction holds for Σ_k formulae. For functions $F:\mathbb{N}\to\mathbb{N}$ we use exponential notation $F^0(x)=x$ and $F^{k+1}(x)=F(F^k(x))$ to denote repeated compositions of F.

In what follows we require an ordinal representation system for ε_0. Moreover, we assume that these ordinals come equipped with specific fundamental sequences $\lambda[n]$ for each limit ordinal $\lambda\le\varepsilon_0$. Their definition springs forth from their representation in Cantor normal form (to base ω). For an ordinal α such that $\alpha>0$, α has a unique representation :

$$\alpha=\omega^{\alpha_1}\cdot n_1+\cdots+\omega^{\alpha_k}\cdot n_k,$$

where $0<k,n_1,\dots,n_k<\omega$, and $\alpha_1,\dots,\alpha_k$ are ordinals such that $\alpha_1>\cdots>\alpha_k$.

Definition 1.1 For α an ordinal and n a natural number, let ω_n^α be defined inductively by $\omega_0^\alpha:=\alpha$, and $\omega_{n+1}^\alpha:=\omega^{\omega_n^\alpha}$.

We also write ω_n for ω_n^1. In particular, $\omega_0=1$ and $\omega_1=\omega$.

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Definition 1.2 For each limit ordinal $\lambda \leq \varepsilon_0$, define a strictly monotone sequence, $\lambda[n]$, of ordinals converging to λ from below. The definition is by induction on λ.

Case 1. $\lambda = \omega^{\alpha+1} \cdot (\beta + 1)$.
Put $\lambda[n] = \omega^{\alpha+1} \cdot \beta + \omega^{\alpha} \cdot n$. (Remark: In particular, $\omega[n] = n$.)

Case 2. $\lambda = \omega^{\gamma} \cdot (\beta + 1)$, and $\gamma < \lambda$ is a limit ordinal.
Put $\lambda[n] = \omega^{\gamma} \cdot \beta + \omega^{\gamma[n]}$.

Case 3. $\lambda = \varepsilon_0$.
Put $\varepsilon_0[0] = \omega$ and $\varepsilon_0[n+1] = \omega^{\varepsilon_0[n]}$. (Remark: Thus $\varepsilon_0[n] = \omega_{n+1}$.)

It will be convenient to have $\alpha[n]$ defined for non-limit α. we set $(\beta+1)[n] = \beta$ and $0[n] = 0$.

Definition 1.3 By "a fast growing " hierarchy we simply mean a transfinitely extended version of the Grzegorczyk hierarchy i.e. a transfinite sequence sequence of number-theoretic functions $F_\alpha : \mathbb{N} \rightarrow \mathbb{N}$ defined recursively by iteration at successor levels and diagonalization over fundamental sequences at limit levels. We use the following hierarchy:

$$\begin{aligned} F_0(n) &= n+1 \\ F_{\alpha+1}(n) &= F_\alpha^{n+1}(n) \\ F_\alpha(n) &= F_{\alpha[n]}(n) \quad \text{if } \alpha \text{ is a limit.} \end{aligned}$$

It is closely related to the Hardy hierarchy:

$$\begin{aligned} H_0(n) &= n \\ H_{\alpha+1}(n) &= H_\alpha(n+1) \\ H_\alpha(n) &= H_{\alpha[n]}(n) \quad \text{if } \alpha \text{ is a limit.} \end{aligned}$$

Their relationship is as follows:

$$H_{\omega^\alpha} = F_\alpha \tag{1}$$

for every $\alpha < \varepsilon_0$. If $\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k$ is in Cantor normal form and $\beta < \omega^{\alpha_k+1}$, then

$$H_{\alpha+\beta} = H_\alpha \circ H_\beta. \tag{2}$$

Ketonen and Solovay [**6**] found an interesting combinatorial characterization of the H_α's. Call an interval $[k, n]$ 0-large if $k \leq n$, $\alpha + 1$-large if there are $m, m' \in [k, n]$ such that $m \neq m'$ and $[m, n]$ and $[m', n]$ are both α-large; and λ-large (where λ is a limit) if $[k, n]$ is $\lambda[k]$-large.

Theorem 1.4 (Ketonen, Solovay [**6**]) *Let* $\alpha < \varepsilon_0$.

$$\begin{aligned} H_\alpha(n) &= \textit{least } m \textit{ such that } [n,m] \textit{ is } \alpha\textit{-large} \\ F_\alpha(n) &= \textit{least } m \textit{ such that } [n,m] \textit{ is } \omega^\alpha\textit{-large.} \end{aligned}$$

The order of growth of F_{ε_0} is essentially the same as that of the Paris-Harrington function F_{PH}.

Definition 1.5 Let X be a finite set of natural numbers and $|X|$ be the number of elements in X. X, is **large** if X if X is non-empty, and, letting s be the least element of X, X has at least s elements. If $d \in \mathbb{N}$ then $[X]^d$ denotes the set of all subsets of X of cardinality d. If $g : [X]^d \to Y$, a subset Z of Y is **homogeneous** for g if g is constant on $[Z]^d$. Identify $n \in \mathbb{N}$ with the set $\{0, \dots, n-1\}$.

Let $a, b, c \in \mathbb{N}$. Then $a \to (\text{large})^b_c$ if for every map $g : [a]^b \to c$, there is a large homogeneous set for g of cardinality greater than b.

Let $\sigma(b, c)$ be the least integer a such that $a \to (\text{large})^b_c$ and $f_{PH}(n) = \sigma(n, n)$.

Theorem 1.6 (i) (Harrington, Paris [**10**]) *The function* f_{PH} *dominates all* **PA**-*provably recursive functions.*

(ii) (Ketonen, Solovay [**6**]) *For* $n \geq 20$*:*

$$F_{\varepsilon_0}(n-3) \leq \sigma(n, 8) \leq F_{\varepsilon_0}(n-2)$$
$$f_{PH}(n) \leq F_{\varepsilon_0}(n-1).$$

The computation of $F_\alpha(x)$ is closely connected with the step-down relations of [**6**] and [**13**]. For $\alpha < \beta \leq \varepsilon_0$ we write $\beta \xrightarrow[n]{} \alpha$ if for some sequence of ordinals $\gamma_0, \dots, \gamma_r$ we have $\gamma_0 = \beta$, $\gamma_{i+1} = \gamma_i[n]$, for $0 \leq i < r$, and $\gamma_r = \alpha$.

Lemma 1.7 *There is a* Δ_0*-formula expressing* $F_\alpha(x) = y$ *(as a predicate of* α, x, y*).*

Proof: This is shown in [**16**, 5.2]. □

Lemma 1.8 *The following are provable in* $\mathrm{I}\Sigma_1$*:*

(i) *If* $\beta \xrightarrow[x]{} \alpha$ *and* $F_\beta(x) \downarrow$*, then* $F_\alpha(x) \downarrow$ *and* $F_\beta(x) \geq F_\alpha(x)$.

(ii) *If* $F_\beta(x) \downarrow$ *and* $x > y$*, then* $F_\beta(y) \downarrow$ *and* $F_\beta(x) > F_\beta(y)$.

(iii) *(i) and (ii) hold with* H_β *and* H_α *in place of* F_β *and* F_α*, respectively.*

Proof: (i) follows by induction on the length r of the sequence $\gamma_0, \dots, \gamma_r$ with $\gamma_0 = \beta$, $\gamma_{i+1} = \gamma_i[n]$, for $0 \leq i < r$, and $\gamma_r = \alpha$. In the proof one uses the fact that '$F_\delta(x) = y$' is Δ_0 as a relation with arguments δ, x, y, and also uses [**16**, Theorem 5.3] (or rather Claim 1 in Appendix A of [**15**]).

(ii) follows from [**16**, Proposition 5.4(v)]. □

Lemma 1.9 *For all* $x < \omega$, $\omega_{x+1} \xrightarrow[2]{} \omega_x + \omega_x$.

Proof: We use induction on x. As $\omega_1 = \omega$, $\omega_0 = 1$ and $\omega[2] = 2$ this holds for $x = 0$. Now suppose $x > 0$. Note that $\omega_{x+1} = \omega^{\omega_x}$, thus we have $\omega_{x+1}[2] = \omega^{\omega_x[2]}$. Inductively we also have $\omega_x[2] \xrightarrow[2]{} \omega_{x-1} + \omega_{x-1}$. By [**6**] Lemma 5 (p. 282) we conclude that

$$\omega_{x+1}[2] = \omega^{\omega_x[2]} \xrightarrow[2]{} \omega^{\omega_{x-1}+\omega_{x-1}}. \tag{3}$$

We also have $\omega_{x-1} + \omega_{x-1} \xrightarrow[2]{} \omega_{x-1} + 1$ by [**6**] Lemma 1 (p. 281), and hence

$$\omega^{\omega_{x-1}+\omega_{x-1}} \xrightarrow[2]{} \omega^{\omega_{x-1}+1}, \tag{4}$$

using [**6**] Lemma 5 (p.282) again. As $\omega^{\omega_{x-1}+1} \xrightarrow[2]{} \omega^{\omega_{x-1}} \cdot 2 = \omega^{\omega_{x-1}} + \omega^{\omega_{x-1}} = \omega_x + \omega_x$, it follows from (3) and (4), owing to the transitivity of $\xrightarrow[2]{}$, that

$$\omega_{x+1} \xrightarrow[2]{} \omega_x + \omega_x.$$

□

Corollary 1.10 *For all integers* x *and* $y \geq 2$ *we have:*

(i) $\varepsilon_0[x+1] \xrightarrow[2]{} \varepsilon_0[x] + \varepsilon_0[x]$.

(ii) $F_{\varepsilon_0[x+1]}(y+1) > F_{\varepsilon_0[x+1]}(y) \geq F_{\varepsilon_0[x]+1}(y) \geq F_{\varepsilon_0[x]}(F_{\varepsilon_0[x]}(y))$.

Proof: As $\varepsilon_0[u] = \omega_{u+1}$, (i) is a consequence of Lemma 1.9.

We have

$$F_{\varepsilon_0[x]}(F_{\varepsilon_0[x]}(y)) = H_{\omega^{\varepsilon_0[x]}}(H_{\omega^{\varepsilon_0[x]}}(y)) = H_{\omega^{\varepsilon_0[x]}+\omega^{\varepsilon_0[x]}}(y) \overset{(*)}{\leq} H_{\omega^{\varepsilon_0[x+1]}}(y)$$
$$\overset{(**)}{=} F_{\varepsilon_0[x+1]}(y) \overset{(***)}{<} F_{\varepsilon_0[x+1]}(y+1).$$

Here the first and second equality hold by (1) and (2), respectively. $(*)$ follows from (i) with the help of Lemma 1.8(iii) since

$$\omega^{\varepsilon_0[x+1]} = \varepsilon_0[x+2] \underset{2}{\to} \varepsilon_0[x+1] + \varepsilon_0[x+1] = \omega^{\varepsilon_0[x]} + \omega^{\varepsilon_0[x]}.$$

$(**)$ is again a consequence of (1) whilst $(***)$ follows from Lemma 1.8(ii). □

2 Slow consistency

To motivate our notion of slow consistency we recall the concept of interpretability of one theory in another theory. Let S and S' be arbitrary theories. S' is **interpretable in** S or S **interprets** S' (in symbols $S' \lhd S$) "if roughly speaking, the primitive concepts and the range of the variables of S' are defined in such a way as to turn every theorem of S' into a theorem of S" (quoted from [**8**] p. 96; for details see [**8**, section 6]).

To simplify matters, we restrict attention to theories T formulated in the language of **PA** which contain the axioms of **PA** and have a primitive recursive axiomatization, i.e. the axioms are enumerated by such a function. For an integer $k \geq 0$, we denote by $T\restriction_k$ the theory consisting of the first k axioms of T. Let $\mathrm{Con}(T)$ be the arithmetized statement that T is consistent.

A theory T is **reflexive** if it proves the consistency of all its finite subtheories, i.e. $T \vdash \mathrm{Con}(T\restriction_k)$ for all $k \in \mathbb{N}$. Note that theories satisfying the conditions spelled out above will always be reflexive.

Another interesting relationship between theories we shall consider is $T_1 \subseteq_{\Pi^0_1} T_2$, i.e. every Π^0_1 theorem of T_1 is also a theorem of T_2.

Theorem 2.1 *Let S, T be theories that satisfy the conditions spelled out above. Then:*

(5) $S \lhd T$ *if and only if* $T \vdash \mathrm{Con}(S\restriction_n)$ *holds for all* $n \in \mathbb{N}$

(6) *if and only if* $S \subseteq_{\Pi^0_1} T$.

Proof: (5) seems to be due to Orey [**9**]. Another easily accessible proof of (5) can be found in [**8**, Section 6, Theorem 5]. (6) was first stated in [**5**] and [**7**]. A proof can also be found in [**8**, Section 6, Theorem 6]. □

We know that

$$\mathrm{Con}(\mathbf{PA}) \leftrightarrow \forall x\, \mathrm{Con}(\mathbf{PA}\restriction_x).$$

Given a function $f : \mathbb{N} \to \mathbb{N}$ (say provably total in **PA**) we are thus led to the following consistency statement:

$$\mathrm{Con}_f(\mathbf{PA}) \quad := \quad \forall x\, \mathrm{Con}(\mathbf{PA}\restriction_{f(x)}). \tag{7}$$

It is perhaps worth pointing out that the exact meaning of $\mathrm{Con}_f(\mathbf{PA})$ depends on the representation that we choose for f.

Statements of the form (7) are interesting only if the function f grows extremely slowly, though still has an infinite range but **PA** cannot prove that fact.

Definition 2.2 Define

$$F_{\varepsilon_0}^{-1}(n) \quad = \quad \max(\{k \leq n \mid \exists y \leq n\, F_{\varepsilon_0}(k) = y\} \cup \{0\}).$$

Note that, by Lemma 1.7, the graph of $F_{\varepsilon_0}^{-1}$ has a Δ_0 definition. Thus it follows that $F_{\varepsilon_0}^{-1}$ is a provably recursive function of **PA**.

Let $\mathrm{Con}^*(\mathbf{PA})$ be the statement $\forall x\, \mathrm{Con}(\mathbf{PA}\restriction_{F_{\varepsilon_0}^{-1}(x)})$. Of course, in the definition of $\mathrm{Con}^*(\mathbf{PA})$ we have in mind some standard representation of F_{ε_0} referred to in Lemma 1.7. Note that $\mathrm{Con}^*(\mathbf{PA})$ is equivalent to the statement

$$\forall x\, [F_{\varepsilon_0}(x) \downarrow \rightarrow \mathrm{Con}(\mathbf{PA}\restriction_x)].$$

Proposition 2.3 $\mathbf{PA} \nvdash \mathrm{Con}^*(\mathbf{PA})$.

Proof: Aiming at a contradiction, suppose $\mathbf{PA} \vdash \mathrm{Con}^*(\mathbf{PA})$. Then $\mathbf{PA}\restriction_k \vdash \mathrm{Con}^*(\mathbf{PA})$ for all sufficiently large k. As $\mathbf{PA}\restriction_k \vdash F_{\varepsilon_0}(k)\downarrow$ on account of $F_{\varepsilon_0}(k)\downarrow$ being a true Σ_1 statement, we arrive at $\mathbf{PA}\restriction_k \vdash \mathrm{Con}(\mathbf{PA}\restriction_k)$, contradicting Gödel's second incompleteness theorem. □

Proposition 2.3 holds in more generality.

Corollary 2.4 *If T is a recursive consistent extension of* **PA** *and f is a total recursive function with unbounded range, then*

$$T \nvdash \forall x\, \mathrm{Con}(T\restriction_{f(x)})$$

where $f(x)\downarrow$ is understood to be formalized via some Σ_1 representation of f.

Proof: Basically the same proof as for Proposition 2.3. □

It is quite natural to consider another version of slow consistency where the function $f : \mathbb{N} \rightarrow \mathbb{N}$, rather than acting as a bound on the fragments of **PA**, restricts the lengths of proofs. Let $\perp$ be a Gödel number of the canonical inconsistency and let $\mathrm{Proof}_{\mathbf{PA}}(y, z)$ be the primitive recursive predicate expressing the concept that *"y is the Gödel number of a proof in* **PA** *of a formula with Gödel number z"*.

$$\mathrm{Con}_f^{\ell}(\mathbf{PA}) \quad := \quad \forall x\, \forall y < f(x)\, \neg \mathrm{Proof}_{\mathbf{PA}}(y, \perp) \tag{8}$$

Let $\mathrm{Con}^{\#}(\mathbf{PA})$ be the statement $\mathrm{Con}^{\ell}_{F_{\varepsilon_0}^{-1}}(\mathbf{PA})$.

Note that $\mathrm{Con}^{\#}(\mathbf{PA})$ is equivalent to the following formula:

$$\forall u\, [F_{\varepsilon_0}(u)\downarrow \rightarrow \ \forall y < u\, \neg \mathrm{Proof}_{\mathbf{PA}}(y, \perp)].$$

As it turns out, by contrast with $\mathrm{Con}^*(\mathbf{PA})$, $\mathrm{Con}^{\#}(\mathbf{PA})$ is not very interesting.

Lemma 2.5 $\mathbf{PA} \vdash \mathrm{Con}^{\#}(\mathbf{PA})$.

Proof: First recall that Gentzen showed how to effectively transform an alleged **PA**-proof of an inconsistency (the empty sequent) in his sequent calculus into another proof of the empty sequent such that the latter gets assigned a smaller ordinal than the former. More precisely, there is a reduction procedure $\mathcal{R}$ on proofs P of the empty sequent together with an assignment *ord* of representations for ordinals $< \varepsilon_0$ to proofs such that $ord(\mathcal{R}(P)) \prec ord(P)$. Here $\prec$ is the ordering on ordinal representations induced by the ordering $<$ of the pertaining ordinals. The functions $\mathcal{R}$ and *ord* and the relation $\succ$ are primitive recursive (when viewed as acting on codes for the syntactic objects). With $g(n) = ord(\mathcal{R}^n(P))$, the n-fold iteration of $\mathcal{R}$ applied to P, one has $g(0) \succ g(1) \succ g(2) \succ \ldots \succ g(n)$ for all n, which is absurd as the ordinals are well-founded.

We will now argue in **PA**. Suppose that $F_{\varepsilon_0}(u)\!\downarrow$. Aiming at a contradiction assume that there is a $p < u$ such that $\mathrm{Proof}_{\mathbf{PA}}(p, \bot)$. We have not said anything about the particular proof predicate $\mathrm{Proof}_{\mathbf{PA}}$ we use, however, whatever proof system is assumed, p will be larger than the Gödel numbers of all formulae occurring in the proof. The proof that p codes, can be primitive recursively transformed into a sequent calculus proof P of the empty sequent in such a way that $ord(P) < \omega_p$ since p is larger than the number of logical symbols occurring in any cut or induction formulae featuring in P (for details see [**17**, Ch.2]). Inspection of Gentzen's proof, as e.g. presented in [**17**, 2.12.8], shows there is a primitive recursive function ℓ such that the number of steps it takes to get from $ord(P)$ to 0 by applying the reduction procedure $\mathcal{R}$ is majorized by $\ell(F_{\varepsilon_0}(u))$. As a result we have a contradiction since there is no proof P_0 of the empty sequent with ordinal $ord(P_0) = 0$.

The authors realize that the foregoing proof is merely a sketch. An alternative proof can be obtained by harking back to [**1**]. The reader will be assumed to have access to [**1**]. That paper uses an infinitary proof system with the ω-rule (of course). But this system is also quite peculiar in that the ordinal assignment adhered to is very rigid and, crucially, it has a so-called accumulation rule. To deal with infinite proofs in **PA**, though, one has to use primitive recursive proof trees instead of arbitrary ones (for details see [**3**]). The role of the repetition rule (or trivial rule) (cf. [**3**]) is of central importance to capturing the usual operations on proofs, such as inversion and cut elimination, by primitive recursive functions acting on their codes. In the proof system of [**1**] the accumulation rule takes over this role. Now assume that everything in [**1**] has been recast in terms of primitive recursive proof trees. Then the cut elimination for infinitary proofs with finite cut rank (as presented in [**3**, Theorem 2.19]) can be formalized in **PA**. Working in **PA**, suppose that $F_{\varepsilon_0}(u)\!\downarrow$. Aiming at a contradiction assume that there is a $p < u$ such that $\mathrm{Proof}_{\mathbf{PA}}(p, \bot)$. As above, the proof that p codes, can be primitive recursively transformed into a proof P of $\bot$ in the sequent calculus of [**1**] with ordinal ω_p and cut-degree 0 (in the sense of [**1**, Definition 5]). The plan is to reach a contradiction by constructing an infinite descending sequence of ordinals $(\alpha_i)_{i\in\mathbb{N}}$ such that $\alpha_0 = \omega_p$ and $\alpha_{i+1} <_{l_{i+1}} \alpha_i$ for some $l_{i+1} < F_{\omega_p}(2)$. This is absurd since it imlpies that $F_{\alpha_i}(k^*) > F_{\alpha_{i+1}}(k^*)$ where $k^* = F_{\omega_p}(2)$. The definedness of $F_{\alpha_i}(k^*)$ follows from the following facts: $F_{\varepsilon_0}(u)\!\downarrow$ implies $F_{\varepsilon_0}(p-1)\downarrow$ and hence $F_{\omega_p}(p-1)\downarrow$, thus $F_{\omega_p}(2)\downarrow$ by [**16**, 5.4(v)]. By induction on i, using [**16**, 5.3)] as well as [**16**, 5.4(v)], one concludes that $F_{\alpha_i}(l_{i+1})\downarrow$ for all i.

It remains to determine $(\alpha_i)_{i\in\mathbb{N}}$. To this end we construct a branch of the proof-tree P with $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ being the i-th node of the branch (bottom-up). The sequent Γ_i contains only closed elementary prime formulas and formulas of the form $n \in N$ whereas Δ_i is of the form $\{n_1 \notin N, \ldots, n_r \notin N\}$ or $\emptyset$. We set $k_{\Delta_i} := \max(\{2\} \cup \{3\cdot n_1, \ldots, 3\cdot n_r\})$ in the former and $k_{\Delta_i} := 2$ in the latter case. We say that Γ_i is true in m if Γ_i is true when N is interpreted as the finite set $\{n \mid 3 \cdot n < m\}$. Let $\Gamma_0 = \{0 = 1\}$ and $\Delta_0 = \emptyset$. Clearly, Γ_0 is false in $F_{\alpha_0}(2)$. Now assume $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ has been constructed in such a way that Γ_i is false in $F_{\alpha_i}(k_{\Delta_i})$ and $F_{\alpha_i}(k_{\Delta_i}) \leq F_{\alpha_0}(2)$. Since Γ_i is false in $F_{\alpha_i}(k_{\Delta_i})$

and $F_{\alpha_i}(k_{\Delta_i}) > k_{\Delta_i}$, it follows that Δ_i, Γ_i is not an axiom. Thus $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ is not an end-node in P and therefore it is the result of an application of an inference rule. As the cut-rank of P is 0, the only possible rules are a cut rank 0, an N-rule, and Accumulation.

If it is an N-rule, Γ_i contains "$Sn \in N$" for some n and $\vdash^{\beta} \Delta_i, \Gamma'_i, n \in N$ will be a node in P immediately above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ with $\Gamma'_i \subseteq \Gamma_i$ and $\beta + 1 = \alpha_i$. We let $\alpha_{i+1} = \beta$, $l_{i+1} = 1$, $\Delta_{i+1} = \Delta_i$ and $\Gamma_{i+1} = \Gamma_i, n \in N$. Since Γ_i is false in $F_{\alpha_i}(k_{\Delta_i})$ and $F_{\alpha_{i+1}}(k_{\Delta_i}) + 3 \leq F_{\alpha_i}(k_{\Delta_i})$ it follows that Γ_{i+1} is false in $F_{\alpha_i}(k_{\Delta_{i+1}})$.

If the last rule is Accumulation, $\vdash^{\beta} \Delta_i, \Gamma_i$ will be a node in P immediately above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ for some $\beta <_{k_{\Delta_i}} \alpha_i$. Then let $\Delta_{i+1} = \Delta_i$, $\Gamma_{i+1} = \Gamma_i$, $\alpha_{i+1} = \beta$, and $l_{i+1} = k_{\Delta_i}$. Since $F_\beta(k_{\Delta_i}) \leq F_{\alpha_i}(k_{\Delta_i})$, Γ_{i+1} is false in $F_{\alpha_{i+1}}(k_{\Delta_{i+1}})$, too. Inductively we also have $F_{\alpha_i}(k_{\Delta_i}) \leq F_{\alpha_0}(2)$, and hence $l_{i+1} < F_{\alpha_0}(2)$.

If the last rule is a cut with a closed elementary prime formula A, the immediate nodes above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ in P are of the form $\vdash^{\beta} \Delta_i, \Gamma_i, A$ and $\vdash^{\beta} \Delta_i, \Gamma_i, \neg A$, respectively, where $\beta + 1 = \alpha_i$. Let $\Delta_{i+1} = \Delta_i$, $\alpha_{i+1} = \beta$, and $l_{i+1} = 1$. If A is false let $\Gamma_{i+1} = \Gamma_i, A$. If A is true, let $\Gamma_{i+1} = \Gamma_i, \neg A$. Clearly, Γ_{i+1} will be false in $F_{\alpha_{i+1}}(k_{\Delta_{i+1}})$ since this value is smaller than $F_{\alpha_i}(k_{\Delta_i})$.

Finally suppose the last rule is a cut with cut formula "$n \in N$". Then the immediate nodes above $\vdash^{\alpha_i} \Delta_i, \Gamma_i$ in P are of the form $\vdash^{\beta} \Delta_i, n \in N, \Gamma_i$ and $\vdash^{\beta} \Delta_i, n \notin N, \Gamma_i$, respectively, where $\beta + 1 = \alpha_i$. Set $\alpha_{i+1} = \beta$ and and $l_{i+1} = 1$. If $F_\beta(k_{\Delta_i}) \leq 3 \cdot n$, then "$n \in N$" will be false in $F_\beta(k_{\Delta_i})$, and hence, as $F_\beta(k_{\Delta_i}) < F_{\alpha_i}(k_{\Delta_i})$, it follows that $n \in N, \Gamma_i$ will be false in $F_\beta(k_{\Delta_i})$ as well. So in this case let $\Delta_{i+1} = \Delta_i$ and $\Gamma_{i+1} = n \in N, \Gamma_i$.

If on the other hand $3 \cdot n < F_\beta(k_{\Delta_i})$, we compute that

$$F_\beta(k_{\Delta_i, n \notin N}) < F_\beta(F_\beta(k_{\Delta_i})) \leq F_{\alpha_i}((k_{\Delta_i}).$$

Hence Γ_i will be false in $F_\beta(k_{\Delta_i, n \notin N})$, and we put $\Delta_{i+1} = \Delta_i, n \notin N$ and $\Gamma_{i+1} = \Gamma_i$.

□

The next goal will be to show that Con(**PA**) is not derivable in **PA** + Con*(**PA**). We need some preparatory definitions.

Definition 2.6 Let E denote the "stack of two's" function, i.e. $E(0) = 0$ and $E(n+1) = 2^{E(n)}$.

Given two elements a and b of a non-standard model $\mathfrak{M}$ of **PA**, we say that **'** b **is much larger than** a**'** if for every standard integer k we have $E^k(a) < b$.

If $\mathfrak{M}$ is a model of **PA** and $\mathfrak{I}$ is a substructure of $\mathfrak{M}$ we say that $\mathfrak{I}$ is an **initial segment** of $\mathfrak{M}$, if for all $a \in |\mathfrak{I}|$ and $x \in |\mathfrak{M}|$, $\mathfrak{M} \models x < a$ implies $x \in |\mathfrak{I}|$. We will write $\mathfrak{I} < b$ to mean $b \in |\mathfrak{M}| \setminus |\mathfrak{I}|$. Sometimes we write $a < \mathfrak{I}$ to indicate $a \in |\mathfrak{I}|$.

Theorem 2.7 *Let $\mathfrak{N}$ be a non-standard model of* **PA** *(or $\Delta_0(\exp)$), n be a standard integer, and $e, d \in |\mathfrak{N}|$ be non-standard such that $\mathfrak{N} \models F_{\omega_n^e}(e) = d$. Then there is an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < d$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction.*

Proof: This follows e.g. from [**16**, Theorem 5.25], letting $\alpha = 0$, $c = e$, $a = e$ and $b = d$. The technique used to prove Theorem 5.25 in [**16**] is a variation of techniques used by Paris in [**12**]. □

Corollary 2.8 *Let $\mathfrak{N}$ be a non-standard model of* **PA***, $e, r \in |\mathfrak{N}|$ be non-standard such that $\mathfrak{N} \models F_{\varepsilon_0}(e) = r$. Then for every standard n there is an initial segment $\mathfrak{I}$ of $\mathfrak{M}$ such $e < \mathfrak{I} < r$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction.*

Proof: In view of the previous Theorem we just have to ensure that $F^{\mathfrak{N}}_{\omega^e_n}(e)\downarrow$, i.e., $\mathfrak{N} \models F_{\omega^e_n}(e) = d$ for some d with $d \le r$. To show this we utilize the fact that the computation of $F_\alpha(x)$ is closely connected with the step-down relation $\beta \underset{n}{\longrightarrow} \alpha$.

In what follows we argue in $\mathfrak{N}$. By induction on x one readily verifies that $\varepsilon_0[x] = \omega^\omega_x$. By [**6**, Theorem 2.4] we have $\varepsilon_0[x] \underset{1}{\longrightarrow} \varepsilon_0[y]$ whenever $x > y$. As $\varepsilon_0[0] = \omega$ and $\omega \underset{e}{\longrightarrow} e$ we arrive at $\varepsilon_0[x] \underset{e}{\longrightarrow} e$ for all x by [**6**] (Proposition (a), p. 281). Thus $\varepsilon_0[e-n] \underset{e}{\longrightarrow} e$, so that by iterated applications of [**6**] Lemma 5 (p. 282), we get

$$\varepsilon_0[e] = \omega_n^{\varepsilon_0[e-n]} \underset{a}{\longrightarrow} \omega^e_n.$$

By [**6**] (Proposition (d), p. 283), the latter yields $F_{\omega^e_n}(e)\downarrow$ and $F_{\varepsilon_0}(e) \ge F_{\omega^e_n}(e)$. □

Definition 2.9 Below we shall need the notion of two models $\mathfrak{M}$ and $\mathfrak{N}$ of **PA** 'agreeing up to e'. For this to hold, the following conditions must be met:

(1) e belongs to both models.
(2) e has the same predecessors in both $\mathfrak{M}$ and $\mathfrak{N}$.
(3) If d_0, d_1, and c are $\le e$ (in one of the models $\mathfrak{M}$ and $\mathfrak{N}$), then $\mathfrak{M} \models d_0 + d_1 = c$ iff $\mathfrak{N} \models d_0 + d_1 = c$.
(4) If d_0, d_1, and c are $\le e$ (in one of the models $\mathfrak{M}$ and $\mathfrak{N}$), then $\mathfrak{M} \models d_0 \cdot d_1 = c$ iff $\mathfrak{N} \models d_0 \cdot d_1 = c$.

If $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e, $d \le e$ and $\theta(x)$ is a Δ_0 formula, it follows that $\mathfrak{M} \models \theta(d)$ iff $\mathfrak{N} \models \theta(d)$ (cf. [**2**, Proposition 1]).

Theorem 2.10 $\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) \nvdash \mathrm{Con}(\mathbf{PA})$.

Proof: Let $\mathfrak{M}$ be a countable non-standard model of $\mathbf{PA} + F_{\varepsilon_0}$ *is total.* Let M be the domain of $\mathfrak{M}$ and $a \in M$ be non-standard. Moreover, let $e = F^{\mathfrak{M}}_{\varepsilon_0}(a)$. As a result of the standing assumption, $\mathfrak{M} \models \mathrm{Con}(\mathbf{PA}\restriction_a)$. Owing to a result of Solovay's [**14**, Theorem 1.1], there exists a countable model $\mathfrak{N}$ of **PA** such that:

(i) $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e (in the sense of Definition 2.9).
(ii) $\mathfrak{N}$ thinks that $\mathbf{PA}\restriction_a$ is consistent.
(iii) $\mathfrak{N}$ thinks that $\mathbf{PA}\restriction_{a+1}$ is inconsistent. In fact there is a proof of $0 = 1$ from $\mathbf{PA}\restriction_{a+1}$ whose Gödel number is less than 2^{2^e} (as computed in $\mathfrak{N}$).

In actuality, to be able to apply [**14**, Theorem 1.1] we have to ensure that e is much larger than a, i.e., $E^k(a) < e$ for every standard number k. It follows from [**6**, p. 269] that $E(s) \le F_3(s)$ holds for all non-standard elements s of $\mathfrak{M}$ and hence

$$E^k(s) \le F^k_3(s) \le F^s_3(s) \le F_4(s) < F_{\varepsilon_0}(s),$$

so that $E^k(a) < e$ holds for all standard k, leading to e being much larger than a.

We will now distinguish two cases.

Case 1: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\uparrow$. Then also $\mathfrak{N} \models F_{\varepsilon_0}(d)\uparrow$ for all $d > a$ by Lemma 1.8(ii). Hence, in light of (ii), $\mathfrak{N} \models \mathrm{Con}^*(\mathbf{PA})$. As (iii) yields $\mathfrak{N} \models \neg\mathrm{Con}(\mathbf{PA})$, we have

$$\mathfrak{N} \models \mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) + \neg\mathrm{Con}(\mathbf{PA}). \tag{9}$$

Case 2: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\downarrow$. We then also have $e = F^{\mathfrak{N}}_{\varepsilon_0}(a)$, for $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e and the formula '$F_{\varepsilon_0}(x) = y$' is Δ_0 by Lemma 1.7. Let $c := F^{\mathfrak{N}}_{\varepsilon_0}(a+1)$.

In view of Theorem 2.7 we just have to ensure that for each standard n, $F^{\mathfrak{N}}_{\omega^e_n}(e)\downarrow$ with value not bigger than c, i.e., $\mathfrak{N} \models F_{\omega^e_n}(e) = d$ for some d with $d \le c$. To show this we utilize Corollary 1.10. In what follows we argue in $\mathfrak{N}$. By [**6**, Theorem 2.4] we have $\varepsilon_0[x] \xrightarrow[1]{} \varepsilon_0[y]$ whenever $x > y$. As $\varepsilon_0[0] = \omega$ and $\omega \xrightarrow[e]{} e$ we arrive at $\varepsilon_0[x] \xrightarrow[e]{} e$ for all x by [**6**] (Proposition (a), p. 281). Thus $\varepsilon_0[a-n] \xrightarrow[e]{} e$, so that by repeated applications of [**6**] Lemma 5 (p. 282), we have

$$\varepsilon_0[a] = \omega_n^{\varepsilon_0[a-n]} \xrightarrow[e]{} \omega_n^e. \tag{10}$$

By Corollary 1.10 we have

$$c = F_{\varepsilon_0[a+1]}(a+1) \geq F_{\varepsilon_0[a]+1}(a) \geq F_{\varepsilon_0[a]}(F_{\varepsilon_0[a]}(a)) = F_{\varepsilon_0[a]}(e). \tag{11}$$

In particular, $F_{\varepsilon_0[a]}(e) \downarrow$. By [**6**] (Proposition (d), p. 283), the latter together with (10) and (11) entails that $F_{\omega_n^e}(e) \downarrow$ and $c \geq F_{\omega_n^e}(e)$.

As a consequence of Theorem 2.7 there is thus an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows from the properties of $\mathfrak{N}$ and the fact that $2^{2^e} < \mathfrak{I}$, that

1. $\mathfrak{I}$ thinks that $\mathbf{PA}\restriction_a$ is consistent.
2. $\mathfrak{I}$ thinks that $\mathbf{PA}\restriction_{a+1}$ is inconsistent.
3. $\mathfrak{I}$ thinks that $F_{\varepsilon_0}(a+1)$ is not defined.

Consequently, $\mathfrak{I} \models \mathrm{Con}^*(\mathbf{PA}) + \neg\mathrm{Con}(\mathbf{PA}) + \Pi_{n+1}$-induction. Since n was arbitrary, this shows that $\mathbf{PA} + \mathrm{Con}^*(\mathbf{PA}) + \neg\mathrm{Con}(\mathbf{PA})$ is a consistent theory. □

Proposition 2.3 and Theorem 2.10 can be extended to theories $\mathbf{T} = \mathbf{PA} + \psi$ where ψ is a true Π^0_1 statement.

Theorem 2.11 *Let* $\mathbf{T} = \mathbf{PA} + \psi$ *where* ψ *is a* Π_1 *statement such that* $T+$'F_{ε_0} is total' *is a consistent theory. Let* $\mathbf{T}\restriction_k$ *to be the theory* $\mathbf{PA}\restriction_k + \psi$ *and* $\mathrm{Con}^*(\mathbf{T}) := \forall x \mathrm{Con}(\mathbf{T}\restriction_{F^{-1}_{\varepsilon_0}(x)})$. *Then the strength of* $\mathbf{T} + \mathrm{Con}^*(T)$ *is strictly between* $\mathbf{T}$ *and* $\mathbf{T} + \mathrm{Con}(\mathbf{T})$, *i.e.*

- (i) $\mathbf{T} \nvdash \mathrm{Con}^*(\mathbf{T})$.
- (ii) $\mathbf{T} + \mathrm{Con}^*(\mathbf{T}) \nvdash \mathrm{Con}(\mathbf{T})$.
- (iii) $\mathbf{T} + \mathrm{Con}(\mathbf{T}) \vdash \mathrm{Con}^*(\mathbf{T})$.

Proof: For (i) the same proof as in Proposition 2.3 works with **PA** replaced by **T**. (iii) is obvious. For (ii) note that Solovay's Theorem also works for **T** so that the proof of case 1 of Theorem 2.10 can be copied. To deal with case 2, observe that $\mathfrak{I} \models \psi$ since ψ is Π_1, $\mathfrak{N} \models \psi$ and $\mathfrak{I}$ is an initial segment of $\mathfrak{N}$. □

The methods of Theorem 2.10 can also be used to produce two 'natural' slow growing functions f and g such that the theories $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA})$ and $\mathbf{PA} + \mathrm{Con}_g(\mathbf{PA})$ are mutually non-interpretable in each other.

Definition 2.12 The even and odd parts of F_{ε_0} are defined as follows:

$$F^{even}_{\varepsilon_0}(2n) = F_{\varepsilon_0}(2n), \qquad F^{even}_{\varepsilon_0}(2n+1) = F_{\varepsilon_0}(2n) + 1\,,$$
$$F^{odd}_{\varepsilon_0}(2n+1) = F_{\varepsilon_0}(2n+1), \qquad F^{odd}_{\varepsilon_0}(2n) = F_{\varepsilon_0}(2n+1) + 1 \text{ if } n > 0.$$

$$\begin{aligned} f(n) &= \max(\{k \leq n \mid \exists y \leq n\, F^{even}_{\varepsilon_0}(k) = y\} \cup \{0\}) \\ g(n) &= \max(\{k \leq n \mid \exists y \leq n\, F^{odd}_{\varepsilon_0}(k) = y\} \cup \{0\}). \end{aligned}$$

By Lemma 1.7, the graphs of f and g are Δ_0 and both functions are provably recursive functions of **PA**.

Remark 2.13 In a much more elaborate form, the method of defining variants of a given computable functions (such as F_{ε_0}) in a piecewise manner has been employed in [**11**] to obtain results about degree structures of computable functions and in [**4**] to obtain forcing-like results about provably recursive functions.

Theorem 2.14 (i) $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA}) \nvdash \mathrm{Con}_g(\mathbf{PA})$.
(ii) $\mathbf{PA} + \mathrm{Con}_g(\mathbf{PA}) \nvdash \mathrm{Con}_f(\mathbf{PA})$.

Proof: (i) The proof is a variant of that of Theorem 2.10. Let $\mathfrak{M}$ be a countable non-standard model of $\mathbf{PA} + F_{\varepsilon_0}$ *is total*. Let M be the domain of $\mathfrak{M}$ and $a \in M$ be non-standard such that $\mathfrak{M}$ thinks that a is **odd**. Let $e = F^{\mathfrak{M}}_{\varepsilon_0}(a)$. As before, there exists a countable model $\mathfrak{N}$ of $\mathbf{PA}$ such that:

(i) $\mathfrak{M}$ and $\mathfrak{N}$ agree up to e.
(ii) $\mathfrak{N}$ thinks that $\mathbf{PA}\upharpoonright_a$ is consistent.
(iii) $\mathfrak{N}$ thinks that $\mathbf{PA}\upharpoonright_{a+1}$ is inconsistent. In fact there is a proof of $0 = 1$ from $\mathbf{PA}\upharpoonright_{a+1}$ whose Gödel number is less than 2^{2^e} (as computed in $\mathfrak{N}$).

Again we distinguish two cases.

Case 1: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\uparrow$. Then also $\mathfrak{N} \models F_{\varepsilon_0}(d)\uparrow$ for all $d > a$ by Lemma 1.8(ii). Since $\mathfrak{M}$ thinks that $a+1$ is even, so does $\mathfrak{N}$, as both models agree up to e. Thus $\mathfrak{N} \models F^{even}_{\varepsilon_0}(d)\uparrow$ for all $d > a$. As a result, $\mathfrak{N} \models \forall x\, f(x) \leq a$, and hence, $\mathfrak{N} \models \mathrm{Con}_f(\mathbf{PA})$. On the other hand, since $\mathfrak{N} \models F^{odd}_{\varepsilon_0}(a+1) = e+1$ and $\mathfrak{N}$ thinks that $\mathbf{PA}\upharpoonright_{a+1}$ is inconsistent, it follows that $\mathfrak{N} \not\models \mathrm{Con}_g(\mathbf{PA})$.

Case 2: $\mathfrak{N} \models F_{\varepsilon_0}(a+1)\downarrow$. As in the proof of Theorem 2.10, letting $c := F^{\mathfrak{N}}_{\varepsilon_0}(a+1)$, for each n we find an initial segment $\mathfrak{I}$ of $\mathfrak{N}$ such $e < \mathfrak{I} < c$ and $\mathfrak{I}$ is a model of Π_{n+1}-induction. Moreover, it follows from the properties of $\mathfrak{N}$ and the fact that $2^{2^e} < \mathfrak{I}$, that

(1) $\mathfrak{I}$ thinks that $\mathbf{PA}\upharpoonright_a$ is consistent.
(2) $\mathfrak{I}$ thinks that $\mathbf{PA}\upharpoonright_{a+1}$ is inconsistent.
(3) $\mathfrak{I}$ thinks that $F_{\varepsilon_0}(a+1)$ is not defined.

Consequently as $\mathfrak{I}$ thinks that $a+1$ is even, $\mathfrak{I} \models \forall x\, f(x) \leq a$, whence $\mathfrak{I} \models \mathrm{Con}_f(\mathbf{PA})$. On the other hand, since $\mathfrak{I} \models F^{odd}_{\varepsilon_0}(a+1) = e+1$, we also have that $\mathfrak{N} \not\models \mathrm{Con}_g(\mathbf{PA})$. Since n was arbitrary, this shows that $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA}) + \neg\mathrm{Con}_g(\mathbf{PA})$ is a consistent theory.

(ii). The argument is completely analogous, the only difference being that we start with a non-standard $a \in M$ such that $\mathfrak{M}$ thinks that a is even. □

Corollary 2.15 *Neither is* $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA})$ *interpretable in* $\mathbf{PA} + \mathrm{Con}_g(\mathbf{PA})$ *nor* $\mathbf{PA} + \mathrm{Con}_g(\mathbf{PA})$ *interpretable in* $\mathbf{PA} + \mathrm{Con}_f(\mathbf{PA})$.

Proof: This follows from Theorem 2.14 and Theorem 2.1. □

2.1 A natural Orey sentence

A sentence φ of $\mathbf{PA}$ is called an **Orey sentence** if both $\mathbf{PA} + \varphi \lhd \mathbf{PA}$ and $\mathbf{PA} + \neg\varphi \lhd \mathbf{PA}$ hold.

Corollary 2.16 *The sentence* $\exists x\, (F_{\varepsilon_0}(x)\uparrow \wedge \forall y < x\, F_{\varepsilon_0}(y)\downarrow \wedge\, x \texttt{ is even})$ *is an Orey sentence.*

Proof: Let ψ be the foregoing sentence. In view of Theorem 2.1, it suffices to show that $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\upharpoonright_k + \psi)$ and $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\upharpoonright_k + \neg\psi)$ hold for all k. Fix $k > 0$.

First we show that $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\upharpoonright_k + \psi)$. Note that $\mathbf{PA}$ proves the consistency of $\mathbf{PA}\upharpoonright_k + \forall x\, F_{\omega_{k+1}}(x) \downarrow + \exists x F_{\varepsilon_0}(x) \uparrow$. Arguing in $\mathbf{PA}$ we thus find a non-standard model $\mathfrak{N}$ such that

$$\mathfrak{N} \models \mathbf{PA}\upharpoonright_k + \forall x\, F_{\omega_{k+1}}(x) \downarrow + \exists x F_{\varepsilon_0}(x) \uparrow .$$

In particular there exists a least $a \in |\mathfrak{N}|$ in the sense of $\mathfrak{N}$ such that $\mathfrak{N} \models F_{\varepsilon_0}(a) \uparrow$. If $\mathfrak{N}$ thinks that a is even, then $\mathfrak{N} \models \psi$, which entails that $\mathrm{Con}(\mathbf{PA}\upharpoonright_k + \psi)$. If $\mathfrak{N}$ thinks that a is odd, we define a cut $\mathfrak{I}$ such that $\mathfrak{I} \models \mathbf{PA}\upharpoonright_k$ and $F^{\mathfrak{N}}_{\varepsilon_0}(a-2) < \mathfrak{I} < F^{\mathfrak{N}}_{\varepsilon_0}(a-1)$, applying Theorem 2.7. Then $\mathfrak{I} \models \psi$ which also entails $\mathrm{Con}(\mathbf{PA}\upharpoonright_k + \psi)$.
Next we show that $\mathbf{PA} \vdash \mathrm{Con}(\mathbf{PA}\upharpoonright_k + \neg\psi)$. As $\mathbf{PA}$ proves $\mathrm{Con}(\mathbf{PA}\upharpoonright_k + \forall x\, F_{\omega_{k+1}}(x) \downarrow)$, we can argue in $\mathbf{PA}$ and assume that we have a model $\mathfrak{M} \models \mathbf{PA}\upharpoonright_k + \forall x\, F_{\omega_{k+1}}(x) \downarrow$. If $\mathfrak{M} \models \forall x F_{\varepsilon_0}(x)\downarrow$ then $\mathfrak{M} \models \neg\psi$, and $\mathrm{Con}(\mathbf{PA}\upharpoonright_k + \neg\psi)$ follows. Otherwise there is a least a in the sense of $\mathfrak{M}$ such that $F^{\mathfrak{M}}_{\varepsilon_0}(a) \uparrow$. If $\mathfrak{M}$ thinks that a is odd we have $\mathfrak{M} \models \neg\psi$, too. If $\mathfrak{M}$ thinks that a is even we introduce a cut $F^{\mathfrak{M}}_{\varepsilon_0}(a-2) < \mathfrak{I}' < F^{\mathfrak{M}}_{\varepsilon_0}(a-1)$ such that $\mathfrak{I}' \models \mathbf{PA}\upharpoonright_k$. Since $\mathfrak{I}' \models F_{\varepsilon_0}(a-1) \uparrow$ we have $\mathfrak{I}' \models \neg\psi$, whence $\mathrm{Con}(\mathbf{PA}\upharpoonright_k + \neg\psi)$. $\square$

References

[1] W. Buchholz, S. Wainer: *Provably computable functions and the fast growing hierarchy.* In: S. Simpson (ed.): *Logic and combinatoris.* Contemporary Mathematics 65 (AMS, Providence, 1987) 179–198.

[2] C. Dimitracopoulos, J.B. Paris: *Truth definitions for Δ_0 formulae*, in: Logic and Algorithmic, L'Enseignement Mathematique 30 (Univ. Genève, Geneva, 1982) 317–329.

[3] H. Friedman and S. Sheard: *Elementary descent recursion and proof theory*, Annals of Pure and Applied Logic 71 (1995) 1–45.

[4] S.-D. Friedman, M. Rathjen, A. Weiermann: *Some results on PA-provably recursive functions*, preprint 2011.

[5] D. Guaspari: *Partially conservative extensions of arithmetic*, Trans.Amer.Math.Soc. 254 (1979) 47–68.

[6] J. Ketonen, R.M. Solovay: *Rapidly growing Ramsey functions*, Annals of Mathematics 113 (1981) 267–314.

[7] P. Lindström: *Some results on interpretability*, in: Proceedings of the 5th Scandinavian Logic Symposium 1979 (Aalborg University Press, Aoalborg, 1979) 329–361.

[8] P. Lindström: *Aspects of incompleteness*, Lecture notes in logic 10, second edition (Association for Symbolic Logic, 2003).

[9] S. Orey: *Relative interpretations*, Zeitschrift für mathematische Logik 7 (1961) 146–153.

[10] J. Paris, L. Harrington: *A mathematical incompleteness in Peano arithmetic.* In: J. Barwise (ed.): *Handbook of Mathematical Logic* (North Holland, Amsterdam, 1977) 1133-1142.

[11] L. Kristiansen: *Subrecursive degrees and fragments of Peano arithmetic*, Archive for Mathematical Logic 40 (2001) 365–397.

[12] J.B. Paris: *A hierarchy of cuts in models of arithmetic*, in: Lecture notes in Mathematics, Vol. 834 (Springer, Berlin, 1980) 312–337.

[13] D. Schmidt: *Built-up systems of fundamental sequences and hierarchies of number-theoretic functions*, Arch. Math. Logik 18 (1976) 47–53.

[14] R.M. Solovay; *Injecting inconsistencies into models of PA*, Annals of Pure and Applied Logic 44 (19890 101–132.

[15] R. Sommer: *Transfinite induction and hierarchies generated by transfinite recursion within Peano arithmetic.* Ph.D., U.C. Berkeley, 1990.

[16] R. Sommer: *Transfinite induction within Peano arithmetic.* Annals of Pure and applied Logic 76 (1995) 231–289.

[17] G. Takeuti: *Proof Theory* 2nd edition (North-Holland, Amsterdam, 1987).