The tree property at $\aleph_{\omega+2}$

Sy-David Friedman, Ajdin Halilović*
Kurt Gödel Research Center
University of Vienna

June 19, 2010

Abstract

Assuming the existence of a weakly compact hypermeasurable cardinal we prove that in some forcing extension $\aleph_\omega$ is a strong limit cardinal and $\aleph_{\omega+2}$ has the tree property. This improves a result of Matthew Foreman (see [4]).

1 Introduction

For an infinite cardinal κ, a *κ-tree* is a tree T of height κ such that every level of T has size less than κ. A tree T is a *κ-Aronszajn tree* if T is a κ-tree which has no cofinal branches. We say that *the tree property holds at κ*, or TP(κ) *holds*, if every κ-tree has a cofinal branch, i.e. a branch of length κ through it. Thus, TP(κ) holds iff there is no κ-Aronszajn tree. TP($\aleph_0$) holds in ZFC, and it is actually exactly the statement of the well-known König's lemma. Aronszajn showed also in ZFC that there is an $\aleph_1$-Aronszajn tree. Hence, TP($\aleph_1$) fails in ZFC.

Large cardinals are needed once we consider trees of height greater than $\aleph_1$. Silver proved that for $\kappa > \aleph_1$ TP(κ) implies κ is weakly compact in L. Mitchell proved that given a weakly compact cardinal λ above a regular cardinal κ, one can make λ into κ^+ so that in the extension, κ^+ has the tree property. Thus, TP($\aleph_2$) is equiconsistent with the existence of a weakly compact cardinal.

For more of the relevant literature on the tree property we refer the reader to the following: Abraham [5], Cummings and Foreman [4], and Foreman, Magidor and Schindler [6] have done work on the tree property at two or more successive cardinals; Magidor and Shelah [7] have worked on the tree property at successors of singular cardinals.

Natasha Dobrinen and Sy-D. Friedman [1] used a generalization of Sacks forcing to reduce the large cardinal strength required to obtain the tree property at the double successor of a measurable cardinal from a supercompact to a weakly compact hypermeasurable cardinal (see Definition 3).

In this paper we extend the method of [1] to obtain improved upper bounds on the consistency strength of the tree property at the double successor of singular cardinals.

*The authors wish to thank the Austrian Science Fund (FWF) for its generous support via project P19898-N18.

2 The tree property at κ^{++}

Definition 1. Let ρ be a strongly inaccessible cardinal. Then Sacks(ρ) denotes the following forcing notion. A condition p is a subset of $2^{<\rho}$ such that:

1. $s \in p$, $t \subseteq s \to t \in p$.
2. Each $s \in p$ has a proper extension in p.
3. For any $\alpha < \rho$, if $\langle s_\beta : \beta < \alpha \rangle$ is a sequence of elements of p such that $\beta < \beta' < \alpha \to s_\beta \subseteq s_{\beta'}$, then $\bigcup \{s_\beta : \beta < \alpha\} \in p$.
4. Let Split(p) denote the set of $s \in p$ such that both $s^\frown 0$ and $s^\frown 1$ are in p. Then for some club denoted $C(p) \subseteq \rho$, $\text{Split}(p) = \{s \in p : \text{length}(s) \in C(p)\}$.

The conditions are ordered as follows: $q \leq p$ iff $q \subseteq p$, where $q \leq p$ means that q is stronger than p.

Given $p \in \text{Sacks}(\rho)$, let $\langle \gamma_\alpha : \alpha < \rho \rangle$ be the increasing enumeration of $C(p)$. For $\alpha < \rho$, the α-th splitting level of p, $\text{Split}_\alpha(p)$, is the set of $s \in p$ of length γ_α. For $\alpha < \rho$ we write $q \leq_\alpha p$ iff $q \leq p$ and $\text{Split}_\beta(q) = \text{Split}_\beta(p)$ for all $\beta < \alpha$.

Sacks(ρ) satisfies the following *ρ-fusion* property: Every decraesing sequence $\langle p_\alpha : \alpha < \rho \rangle$ of elements in Sacks(ρ) such that for each $\alpha < \rho$, $p_{\alpha+1} \leq_\alpha p_\alpha$, has a lower bound, namely $\bigcap_{\alpha<\rho} p_\alpha \in \text{Sacks}(\rho)$.

The forcing notion Sacks(ρ) is also $< \rho$ -closed, satisfies the ρ^{++}-c.c., and preserves ρ^+. For a proof see [3] or [1].

Definition 2. Let ρ be a strongly inaccessible cardinal and let $\lambda > \rho$ be a regular cardinal. Sacks(ρ, λ) denotes the λ-length iteration of Sacks(ρ) with supports of size $\leq \rho$.

Sacks(ρ, λ) satisfies the *generalized ρ-fusion* property which we describe next: For $\alpha < \rho$, $X \subseteq \rho$ of size less than ρ, and $p, q \in \text{Sacks}(\rho, \lambda)$, we write $q \leq_{\alpha,X} p$ iff $q \leq p$ (i.e. $q \restriction i \Vdash q(i) \leq p(i)$ for each $i < \lambda$) and in addition, for each $i \in X$, $q \restriction i \Vdash q(i) \leq_\alpha p(i)$. Every decraesing sequence $\langle p_\alpha : \alpha < \rho \rangle$ of elements in Sacks(ρ, λ) such that for each $\alpha < \rho$, $p_{\alpha+1} \leq_{\alpha,X_\alpha} p_\alpha$, where the X_α's form an increasing sequence of subsets of λ each of size less than ρ whose union is the union of the supports of the p_α's, has a lower bound. [The lower bound is q where $q(0) = \bigcap_{\alpha<\rho} p_\alpha(0)$, $q(1)$ is a name s.t. $q(0) \Vdash q(1) = \bigcap_{\alpha<\rho} p_\alpha(1)$, etc.]

Assuming $2^\rho = \rho^+$, Sacks(ρ, λ) is $< \rho$ -closed, satisfies the λ-c.c., preserves ρ^+, collapses λ to ρ^{++} and blows up 2^ρ to ρ^{++}. For a proof see [3] or [1].

Definition 3. We say that κ is *weakly compact hypermeasurable* if there is weakly compact cardinal $\lambda > \kappa$ and an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ such that $H(\lambda)^V = H(\lambda)^M$.

Let κ be a weakly compact hypermeasurable cardinal. Define a forcing notion P as follows. Let ρ_0 be the first inaccessible cardinal and let λ_0 be the least weakly compact cardinal above ρ_0. For $k < \kappa$, given λ_k, let ρ_{k+1} be the least inaccessible cardinal above λ_k and let λ_{k+1} be the least weakly compact cardinal above ρ_{k+1}. For limit ordinals $k < \kappa$, let ρ_k be the least inaccessible cardinal greater than or equal to $\sup_{l<k} \lambda_l$ and let λ_k be the least weakly compact

cardinal above ρ_k. Note that $\rho_\kappa = \kappa$ and λ_κ is the least weakly compact cardinal above κ.

Let $P_0 = \{1_0\}$. For $i < \kappa$, if $i = \rho_k$ for some $k < \kappa$, let $\dot{Q}_i$ be a P_i-name for the direct sum $\bigoplus_{\eta \leq \lambda_k} \text{Sacks}(\rho_k, \eta) := \{\langle \text{Sacks}(\rho_k, \eta), p \rangle \colon \eta$ is an inaccessible $\leq \lambda_k$ and $p \in \text{Sacks}(\rho_k, \eta)\}$, where $\langle \text{Sacks}(\rho_k, \eta), p \rangle \leq \langle \text{Sacks}(\rho_k, \eta'), p' \rangle$ iff $\eta = \eta'$ and $p \leq_{\text{Sacks}(\rho_k, \eta)} p'$. Otherwise let $\dot{Q}_i$ be a P_i-name for the trivial forcing. Let $P_{i+1} = P_i * \dot{Q}_i$. Let P_κ be the iteration $\langle \langle P_i, \dot{Q}_i \rangle : i < \kappa \rangle$ with reverse Easton support.

Theorem 1 (N. Dobrinen, S. Friedman)**.** *Assume that V is a model of ZFC in which GCH holds and κ is a weakly compact hypermeasurable cardinal in V. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with crit(j) $= \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$, witnessing the weakly compact hypermeasurability of κ. Let $G * g$ be a generic subset of $P = P_\kappa * \dot{Sacks}(\kappa, \lambda)$ over V. Then in $V[G][g]$, $2^\kappa = \kappa^{++}$, κ^{++} has the tree property, and κ is still measurable, i.e. the embedding $j : V \to M$ can be lifted to an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over M.*

For a proof see [1].

3 The tree property at the double successor of a singular cardinal

Theorem 2. *Assume that V is a model of ZFC and κ is a weakly compact hypermeasurable cardinal in V. Then there exists a forcing extension of V in which $cof(\kappa) = \omega$ and κ^{++} has the tree property.*

Proof. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with crit$(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$. We may assume that M is of the form $M = \{j(f)(\alpha) : \alpha < \lambda, f : \kappa \to V, f \in V\}$. First force as in Theorem 1 with $P = P_\kappa * \dot{\text{Sacks}}(\kappa, \lambda)$ over V to get a model $V[G][g]$ in which $2^\kappa = \kappa^{++}$, κ^{++} has the tree property, and κ is still measurable, i.e. there is an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over M.

Now force with the usual Prikry forcing which we will denote by $R := \{(s, A) : s \in [\kappa]^{<\omega}, A \in U\}$, where U is the normal measure on κ derived from j. We say that s is *the lower part* of (s, A). A condition (t, B) is stronger than a condition (s, A) iff s is an initial segment of t, $B \subseteq A$, and $t - s \subset A$. The Prikry forcing preserves cardinals and introduces an ω-sequence of ordinals which is cofinal in κ. It remains to show that it also preserves the tree property on $\kappa^{++} = \lambda$.

In order to get a contradiction suppose that there is a κ^{++}-Aronszajn tree in some R-extension of $V[G][g]$. Then in $V[G]$ there is a $\text{Sacks}(\kappa, \lambda) * \dot{R}$ - name $\dot{T}$ of size λ (because $\text{Sacks}(\kappa, \lambda) * \dot{R}$ satisfies λ-c.c.) and a condition $(p, \dot{r}) \in \text{Sacks}(\kappa, \lambda) * \dot{R}$ which forces $\dot{T}$ to be a κ^{++}-Aronszajn tree. Recall that λ is a weakly compact cardinal in $V[G]$. Therefore, there exist in $V[G]$ transitive ZF^--models N_0, N_1 of size λ and an elementary embedding $k : N_0 \to N_1$ with critical point λ, such that $N_0 \supseteq H(\lambda)^{V[G]}$ and $G, \dot{T} \in N_0$.

Since g is also Sacks(κ, λ)-generic over N_0 and the critical point of k is λ, k can be lifted to $k^* : N_0[g] \to N_1[g][K]$, where K is any $N_1[g]$-generic subset of Sacks$(\kappa, [\lambda, k(\lambda)))$ in some larger universe (and where Sacks$(\kappa, [\lambda, k(\lambda)))$ is the quotient Sacks$(\kappa, k(\lambda))/$Sacks(κ, λ), i.e. the iteration of Sacks(κ) indexed by ordinals between λ and $k(\lambda)$). Consider the forcing $R^* := k^*(\dot{R}^g)$ in $N_1[g][K]$ and choose any generic C^* for it such that $k^*(r) \in C^*$, where $r = \dot{r}^g$. Let $C := (k^*)^{-1}[C^*]$ be the pullback of C^* under k^*. Then C is an $N_0[g]$-generic subset of R, because if $\Delta \in N_0[g]$ is a maximal antichain of R then $k^*(\Delta) = k^*[\Delta]$ (since crit(k)=λ and R has the κ^+-c.c.) and by elementarity $k^*(\Delta)$ is maximal in $k^*(R) = R^*$, so $k^*[\Delta]$ meets C^* and hence Δ meets C. It follows that there is an elementary embedding $k^{**} : N_0[g][C] \to N_1[g][K][C^*]$ extending k^*.

We have $r \in C$. So it follows that the evaluation T of $\dot{T}$ in $N_0[g][C]$ is a λ-Aronszajn tree. By elementarity $k^{**}(T)$ is a $k^{**}(\lambda)$-Aronszajn tree in $N_1[g][K][C^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g][K][C^*]$. We will show that b has to belong to $N_1[g][C]$ (i.e. the quotient Q of the natural projection π : Sacks$(\kappa, k(\lambda)) * \dot{R}^* \to RO($Sacks$(\kappa, \lambda) * \dot{R})$ can not add a new branch), and thereby reach the desired contradiction!

Let us first analyse the quotient Q of the projection above. In $N_1[g][C]$ we have $Q = \{(p^*, (s^*, \dot{A}^*)) \in$ Sacks$(\kappa, k(\lambda)) * \dot{R}^* \mid$ for all $(p, (s, \dot{A})) \in g * C$, $(p, (s, \dot{A}))$ does not force that $(p^*, (s^*, \dot{A}^*))$ is not a condition in the quotient$\}$. Observe that $(p, (s, \dot{A}))$ forces that $(p^*, (s^*, \dot{A}^*))$ is not a condition in Q iff the two conditions are incompatible, which is the case iff one of the following holds:

1. $p^* \restriction \lambda$ is incompatible with p.
2. $s^* \subsetneq s$ and $s \subsetneq s^*$.
3. $p^* \restriction \lambda$ is compatible with p, $s^* \subseteq s$, and $p^* \cup p$ forces that $s - s^* \subsetneq \dot{A}^*$.
4. $p^* \restriction \lambda$ is compatible with p, $s \subseteq s^*$, and $p^* \restriction \lambda \cup p$ forces that $s^* - s \subsetneq \dot{A}$.

It follows that $Q = \{(p^*, (s^*, \dot{A}^*)) \in$ Sacks$(\kappa, k(\lambda)) * \dot{R}^* \mid (p^*, (s^*, \dot{A}^*))$ is compatible with all $(p, (s, \dot{A})) \in g * C\}$, i.e. Q is the set of all $(p^*, (s^*, \dot{A}^*)) \in$ Sacks$(\kappa, k(\lambda)) * \dot{R}^*$ such that for all $(p, (s, \dot{A})) \in g * C$ either

1. $p^* \restriction \lambda$ is compatible with p, $s^* \subseteq s$, and $p^* \cup p$ does not force that $s - s^* \subsetneq \dot{A}^*$, or
2. $p^* \restriction \lambda$ is compatible with p, $s \subseteq s^*$, and $p^* \restriction \lambda \cup p$ does not force that $s^* - s \subsetneq \dot{A}\}$.

Equivalently, Q is the set of all $(p^*, (s^*, \dot{A}^*)) \in$ Sacks$(\kappa, [\lambda, k(\lambda))) * \dot{R}^*$ such that

1. $p^* \in$ Sacks$(\kappa, [\lambda, k(\lambda)))$,
2. s^* is an initial segment of $S(C)$ (the Prikry ω-sequence arising from C)
3. p^* forces that $\dot{A}^*$ is in $\dot{U}^*$, and
4. for any finite subset x of $S(C)$, some extension q of p^* forces x to be a subset of $s^* \cup \dot{A}^*$.

We now again argue indirectly. Assume that b is not in $N_1[g][C]$, and let $\dot{b}$ in $N_1[g]$ be an $R * \dot{Q}$ - name for b. Identify $k(\dot{T})$ with the $R * \dot{Q}$ - name defined by interpreting the Sacks$(\kappa, k(\lambda)) * \dot{R}^*$ - name $k(\dot{T})$ in N_1 as an $R * \dot{Q}$ - name in $N_1[g]$. Let $((s_0, A_0), (p_0, (t_0, \dot{A}_0)))$ be an $R * \dot{Q}$ - condition forcing that the Prikry-name $\dot{T}$ is a λ-tree and that $\dot{b}$ is a branch through $\dot{T}$ not belonging to $N_1[g][\dot{C}]$.

Let us take a closer look at the condition $((s_0, A_0), (p_0, (t_0, \dot{A}_0)))$. Note that the forcing Q lives in $N_1[g][C]$, but its elements are in $N_1[g]$, so we can assume that $(p_0, (t_0, \dot{A}_0))$ is a real object and not just a Prikry-name. The Prikry condition (s_0, A_0) forces that p_0 is an element of Sacks$(\kappa, [\lambda, k(\lambda)))$, that t_0 is an initial segment of $S(\dot{C})$, and that for all finite subsets x of $S(\dot{C})$, some extension of p_0 forces x to be a subset of $t_0 \cup \dot{A}_0$. This simply means that t_0 is an initial segment of s_0 and for every finite subset x of $s_0 \cup A_0$, some extension of p_0 forces x to be a subset of $t_0 \cup \dot{A}_0$.

Moreover, we can assume that s_0 equals t_0. Namely, from the next claim follows that the set of conditions of the form $((s, A), (p, (s, \dot{A})))$ is dense in $R * \dot{Q}$.

Claim. Suppose that p is an element of Sacks$(\kappa, [\lambda, k(\lambda)))$ which forces that $\dot{A}$ is in $\dot{U}^*$. Then there is $A(p) \in U$ such that whenever x is a finite subset of $A(p)$, there is $q \leq p$ forcing x to be contained in $\dot{A}$.

Proof of the claim. Define the function $f : [\kappa]^{<\omega} \to 2$ by

$$f(x) = \begin{cases} 1 & \text{if } \exists q \leq p \; q \Vdash x \subseteq \dot{A} \\ 0 & \text{otherwise.} \end{cases}$$

By normality f has a homogeneous set $A(p) \in U$. It follows that for each $n \in \omega$, $f \restriction [A(p)]^n$ has the constant value 1: Assume on the contrary that there is some $n \in \omega$ such that $f \restriction [A(p)]^n$ has the constant value 0. Then $p \Vdash x \not\subseteq \dot{A}$ for every $x \in [A(p)]^n$, but this is in contradiction with the facts that the measure U^* extends U, $p \Vdash \dot{A} \in U^*$, and $A(p) \in U$.

It now follows easily that the set of conditions of the form $((s, A), (p, (s, \dot{A})))$ is dense in $R * \dot{Q}$. Assume that $((s, A), (p, (t, \dot{A})))$ is an arbitrary condition in $R * \dot{Q}$. We have $t \subseteq s$. There is some $q \leq p$ which forces that $x := s - t$ is contained in $\dot{A}$. Now by shrinking A to $A(q)$ we get that $((s, A(q)), (q, (s, \dot{A})))$ is a condition which is below $((s, A), (p, (t, \dot{A})))$. We will from now on work with this dense subset of $R * \dot{Q}$.

Now in $N_1[g]$ build a κ-tree E of conditions in Sacks$(\kappa, [\lambda, k(\lambda)))$, whose branches will be fusion sequences, together with a sequence of ordinals $\langle \lambda_\beta : \beta < \kappa \rangle$, each $\lambda_\beta < \lambda$, as follows:

Consider an enumeration $\langle s_\beta : \beta < \kappa \rangle$ of all possible lower parts of conditions in R, i.e. all finite increasing sequences of ordinals less than κ, in which every lower part appears cofinally often. Start building the tree E below the condition p_0 (p_0 was chosen such that $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$ forces $\dot{b}$ to be a bad branch). Assume that the tree E is built up to level β. Then, at stage β of the construction of the tree, at each node v (a condition in Sacks$(\kappa, [\lambda, k(\lambda)))$), is associated an $X_v \subset [\lambda, k(\lambda)), |X_v| < \kappa$; we will find stronger (incompatible) conditions v_0 and v_1 which on all indices in X_v equal v below level β (for purposes of fusion), i.e. $v_0, v_1 \leq_{\beta, X_v} v$. (The sets X_v can be chosen in different ways, the only condition they have to satisfy is that at the end of the construction of the tree E for every branch through the tree the union of the supports of the conditions (nodes) on the branch is equal to the union of the corresponding X's.) Before we start the construction of the level $\beta + 1$ of the tree E we need to set some notation.

Given $i \in [\lambda, k(\lambda))$, let S_i denote Sacks$(\kappa, [\lambda, i))$. For a node v on level β, let $\delta_v = o.t.(X_v)$ and $d_v = |{}^{\delta_v}({}^{\beta+1}2)|$. Let $\langle i^v_\epsilon : \epsilon < \delta_v \rangle$ be the strictly increasing enumeration of X_v and let $i_{\delta_v} = \sup\{i^v_\epsilon : \epsilon < \delta_v\}$. For each $\epsilon < \delta_v$ there are $S_{i^v_\epsilon}$-names $\dot{s}^v_{\epsilon,\zeta}$ $(\zeta \in {}^{\beta+1}2)$ such that $S_{i^v_\epsilon} \Vdash (\dot{s}^v_{\epsilon,\zeta}$ is the ζ-th node of $\text{Split}_{\beta+1}(v(i^v_\epsilon)))$, where the nodes of $\text{Split}_{\beta+1}(v(i^v_\epsilon))$ are ordered canonically lexicographically (by choosing an $S_{i^v_\epsilon}$- name for an isomorphism between $v(i^v_\epsilon)$ and ${}^{<\kappa}2$). Let $\langle u^v_l : l < d_v \rangle$ enumerate ${}^{\delta_v}({}^{\beta+1}2)$ (the δ_v-length sequences whose entries are elements of ${}^{\beta+1}2$) so that $u^v_l = \langle u^v_l(\epsilon) : \epsilon < \delta_v \rangle$, where each $u^v_l(\epsilon) \in {}^{\beta+1}2$. We now need the following two facts:

Fact 1. Suppose that v is a node and $l < d_v$. We can construct a condition $r \leq v$ called *v thinned through u_l*, denoted by $(v)^{u_l}$, in the following manner: $r \upharpoonright i^v_0 = v \upharpoonright i^v_0$, for each $\epsilon < \delta_v$, $r(i^v_\epsilon) = v(i^v_\epsilon) \upharpoonright \dot{s}^v_{\epsilon,u^v_l(\epsilon)}$, $r \upharpoonright (i^v_\epsilon, i^v_{\epsilon+1}) = v \upharpoonright (i^v_\epsilon, i^v_{\epsilon+1})$ and $r \upharpoonright (i_{\delta_v}, k(\lambda)) = v \upharpoonright (i_{\delta_v}, k(\lambda))$, where $v(i^v_\epsilon) \upharpoonright \dot{s}^v_{\epsilon,u^v_l(\epsilon)}$ is the subtree of $v(i^v_\epsilon)$ whose branches go through $\dot{s}^v_{\epsilon,u^v_l(\epsilon)}$.

Fact 2. Suppose that v and r are conditions in Sacks$(\kappa, [\lambda, k(\lambda)))$ with $r \leq (v)^{u_l}$. Then there is a condition v' such that $v' \leq_{\beta, X_v} v$ and $(v')^{u_l} \sim r$ (i.e. $(v')^{u_l} \leq r$ and $r \leq (v')^{u_l}$). We say that v' is *v refined through u_l* to r.

Let $\langle v_j : j < 2^{\beta+1} \rangle$ be an enumeration of level β of the tree E and let $\langle u_m \rangle_{m < \sum_{j<2^{\beta+1}} d_{v_j}}$ be an enumeration of $Y := \bigcup_{j<2^{\beta+1}} \{u^{v_j}_l : l < d_{v_j}\}$. In order to construct the next level of the tree we will first thin out all the nodes on level β (by considering all the pairs in Y) and then split each of them into two incompatible nodes. The thinning out is done as follows: Consider u_0 and u_1. If they belong to the same node, i.e. if there is $j < 2^{\beta+1}$ and $l_0, l_1 < d_{v_j}$ s.t. $u_0 = u^{v_j}_{l_0}$ and $u_1 = u^{v_j}_{l_1}$, then no thinning takes place. So assume that u_0 and u_1 belong to different nodes, say v_{j_0} and v_{j_1}, respectively. Use Fact 1 to construct conditions $r_{01} = (v_{j_0})^{u_0}$ and $r_{10} = (v_{j_1})^{u_1}$, i.e. thin v_{j_0} and v_{j_1} through u_0 and u_1 to r_{01} and r_{10}, respectively. Now ask whether there exist extensions r'_{01} and r'_{10} of r_{01} and r_{10}, respectively, such that for some $\gamma_{01} < \lambda$ and some A_{01}, A_{10}, $\dot{A}_{01}$, $\dot{A}_{10}$, $((s_\beta, A_{01}), (r'_{01}, (s_\beta, \dot{A}_{01})))$ and $((s_\beta, A_{10}), (r'_{10}, (s_\beta, \dot{A}_{10})))$ force different nodes on level γ_{01} of $\dot{T}$ to lie on $\dot{b}$. If the answer is 'yes', use Fact 2 to refine v_{j_0} and v_{j_1} through r'_{01} and r'_{10}, respectively, and continue with the next pair: u_0, u_2. And if the answer is 'no', go to the pair u_0, u_2 without refining v_{j_0} and v_{j_1}. The next pairs are u_1, u_2; u_0, u_3 and so on, i.e. all pairs of the form u_δ, u_η, for $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$. At the limit stages take lower bounds, they exist since the forcing is κ-closed. Let λ_β be the supremum of (the increasing sequence of) $\gamma_{\delta\eta}$'s. Now extend each node v on level β (after thinning out the whole level) to two incompatible conditions v_o and v_1, such that $v_0, v_1 \leq_{\beta, X_v} v$.

Let α be the supremum of λ_β's. Note that $\alpha < \lambda$, because $\lambda = (\kappa^{++})^{N_1[g]}$. Let p be the result of a fusion along a branch through E. By the claim we can choose $A_0(p) \subseteq A_0$ in U such that $((s_0, A_0(p)), (p, (s_0, \dot{A}_0)))$ is a condition. Extend this condition to some $((s_1(p), A_1(p)), (p^*, (s_1(p), \dot{A}_1(p))))$ which decides $\dot{b}(\alpha)$, say it forces $\dot{b}(\alpha) = x_p$.

As level α of $\dot{T}$ has size $< \lambda$, there exist limits p, q of κ-fusion sequences arising from distinct κ-branches through E for which x_p equals x_q and $s_1(p)$ equals $s_1(q)$. Moreover, we can intersect $A_1(p)$ and $A_1(q)$ to get a common A_1. Say, $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ and $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$ force $\dot{b}(\alpha) = x$.

Now choose a Prikry generic C containing (s_1, A_1) (and therefore containing (s_0, A_0)). As $\dot{b}$ is forced by $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$ to not belong to $N_1[g][\dot{C}]$ and $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ extends $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$, we can extend $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ to incompat. conditions $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$, $((s_{2_1}, A_{2_1}), (p_1^{**}, (s_{2_1}, \dot{A}_{2_1})))$, with $(s_{2_0}, A_{2_0}), (s_{2_1}, A_{2_1}) \in C$ and $p_0^{**}, p_1^{**} \leq p^*$, which force a disagreement about $\dot{b}$ at some level γ above α.

Now extend $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$ to some $((s_3, A_3), (q^{**}, (s_3, \dot{A}_3)))$ deciding $\dot{b}(\gamma)$ with (s_3, A_3) in C. Suppose w.l.o.g. that $((s_3, A_3), (q^{**}, (s_3, \dot{A}_3)))$ and $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$ disagree about $\dot{b}(\gamma)$. Also w.l.o.g. we can assume that $s_3 \supseteq s_{2_0}$.

Using the claim extend $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$ to some $((s_3, A_3'), (p^{***}, (s_3, \dot{A}_{2_0})))$ with $A_3' \in U$ and $p^{***} \leq p_0^{**}$.

Now, for some $\beta < \kappa$ we have $s_3 = s_\beta$ where s_β is the βth element of the enumeration of the lower parts (s_3 is not the third element!). Since s_β appears cofinally often in the construction of the tree E, we can assume that the branches which fuse to p and q split in E at some node below level β and go through some nodes v_{j_0} and v_{j_1} at level β. It follows that for some $l < d_{v_{j_0}}$ and $k < d_{v_{j_1}}$,

$$r_1 := ((s_3, A_3'((p^{***})^{u_l^{v_{j_0}}})), ((p^{***})^{u_l^{v_{j_0}}}, (s_3, \dot{A}_{2_0})))$$

and

$$r_2 := ((s_3, A_3((q^{**})^{u_k^{v_{j_1}}})), ((q^{**})^{u_k^{v_{j_1}}}, (s_3, \dot{A}_3)))$$

force different nodes to lie on $\dot{b}$ at level $\gamma > \alpha$. By construction, this means that for some $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$,

$$r_3 := ((s_\beta, A_{\delta\eta}), (r'_{\delta\eta}, (s_\beta, \dot{A}_{\delta\eta})))$$

and

$$r_4 := ((s_\beta, A_{\eta\delta}), (r'_{\eta\delta}, (s_\beta, \dot{A}_{\eta\delta})))$$

force different nodes on level $\gamma_{\delta\eta} (< \alpha)$ of $\dot{T}$ to lie on $\dot{b}$. Say, $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\gamma_{\delta\eta}) = y_1$, respectively.

On the other side, r_1 and r_2 extend $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ and $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$, respectively. Therefore we have that r_1 and r_2 also force $\dot{b}(\alpha) = x$.

Note that $(p^{***})^{u_l^{v_{j_0}}} \leq r'_{\delta\eta}$ and $(q^{**})^{u_k^{v_{j_1}}} \leq r'_{\eta\delta}$. Since any two $R * \dot{Q}$ conditions with the same lower part and compatible Sacks conditions are compatible, we have that $r_1 \parallel r_3$ and $r_2 \parallel r_4$. Let $((s_3, B'), (\bar{p}, (s_3, \dot{B}')))$ be a common lower bound of r_1 and r_3, and let $((s_3, B''), (\bar{q}, (s_3, \dot{B}'')))$ be a common lower bound of r_2 and r_4. The first condition forces $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\alpha) = x$, and the second condition forces $\dot{b}(\gamma_{\delta\eta}) = y_1$ and $\dot{b}(\alpha) = x$.

Finally, let $\bar{B} := B' \cap B''$. Then $(s_3, \bar{B})$ forces that $y_0, y_1 <_{\dot{T}} x$ in the ordering of the tree $\dot{T}$, because $\dot{T}$ is a Prikry-name, i.e. all the relations between the nodes of $\dot{T}$ are determined by the Prikry parts of the conditions above. Contradiction. $\square$

4 The tree property at $\aleph_{\omega+2}$

Using a forcing notion which makes κ into $\aleph_\omega$ instead of Prikry forcing in the proof of Theorem 2 one can get from the same assumptions the tree property at $\aleph_{\omega+2}$, $\aleph_\omega$ strong limit.

Theorem 3. *Assume that V is a model of ZFC and κ is a weakly compact hypermeasurable cardinal in V. Then there exists a forcing extension of V in which $\aleph_{\omega+2}$ has the tree property.*

Proof. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$. We may assume that M is of the form $M = \{j(f)(\alpha) : \alpha < \lambda, f : \kappa \to V, f \in V\}$. First force as in Theorem 1 with $P = P_\kappa * \mathrm{Sacks}(\kappa, \lambda)$ over V to get a model $V[G][g]$ in which $2^\kappa = \kappa^{++}$, κ^{++} has the tree property, and κ is still measurable, i.e. there is an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G*g*H*h$ is a generic subset of $j(P)$ over M. Let $M^* := M[G][g][H][h]$. Note that M^* is the ultrapower of $V[G][g]$ (by the normal measure U induced by j), i.e. every element in M^* is of the form $j(f)(\kappa)$ for some $f : \kappa \to V[G][g], f \in V[G][g]$. This is because every element in M^* is of the form $j(f)(\alpha)$ for some $\alpha < \lambda, f : \kappa \to V[G][g], f \in V[G][g]$, and every $\alpha < \lambda$ is of the form $j(g)(\kappa)$ for some $g : \kappa \to V[G][g], g \in V[G][g]$.

Claim. Define $Q' := \mathrm{Coll}((\kappa^{+++})^{M^*}, j(\kappa))^{M^*}$, the forcing that collapses each ordinal less than $j(\kappa)$ to $(\kappa^{+++})^{M^*}$ using conditions of size $\leq (\kappa^{++})^{M^*}$. There exists G' in $V[G][g]$, a generic subset of Q' over M^*.

Proof of the claim. Every maximal antichain $\Delta \subset Q'$ in M^* is actually in $M[G][g][H]$, and thus of the form σ^{G*g*H} for some $j(P_\kappa)$-name σ in M. It follows that Δ is of the form $j(f)(\alpha)^{G*g*H}$ for some $\alpha < \lambda = (\kappa^{++})^{M^*}$, and some $f : \kappa \to V, f \in V$. Since we can assume that $\sigma = j(f)(\alpha)$ is in $V_{j(\kappa)}$ (because $|j(P_\kappa)| = j(\kappa)$ and $j(P_\kappa)$ has $j(\kappa)$-c.c.), it follows that we can assume that $f : \kappa \to V_\kappa$.

For a fixed $f : \kappa \to V_\kappa$ we have that $F_f := \{\Delta \subset Q' \mid \Delta$ maximal antichain, $\Delta \in M[G][g][H]$, and $j(f)(\alpha)^{G*g*H} = \Delta$ for some $\alpha < (\kappa^{++})^{M^*}\}$ is an element of $M[G][g][H]$. Therefore, since Q' is $(\kappa^{+++})^{M^*}$-distributive in $M[G][g][H]$, there exists a single condition $p_f \in Q'$ which lies below every antichain in F_f.

Now, there are $2^\kappa = \kappa^+$ functions $f : \kappa \to V_\kappa$ in V. Enumerate them as f_1, f_2, f_3... We can find conditions $q_\gamma \in Q'$ for $\gamma < \kappa^+$ such that q_γ is a lower bound of $(p_{f_\beta})_{\beta<\gamma}$, because $M[G][g][H]^\kappa \cap V[G][g] \subseteq M[G][g][H]$ and Q' is $(\kappa^+)^V$-closed in $M[G][g][H]$. The sequence $\{q_\gamma \mid \gamma < \kappa^+\}$ generates a filter G' for Q' in $V[G][g]$, which is generic over $M[G][g][H]$. Here ends the proof of the claim.

We now define in $V[G][g]$ a κ^+-c.c. forcing notion $R(G', U)$, or just R, called *Collapse Prikry*, which makes κ into $\aleph_\omega$ and preserves the tree property on κ^{++} as follows: An element p of R is of the form $(\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ where

1. $\aleph_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessibles
2. $f_i \in \mathrm{Coll}(\alpha_i^{+++}, \alpha_{i+1})$ for $i < n-1$ and $f_{n-1} \in \mathrm{Coll}(\alpha_{n-1}^{+++}, \kappa)$

3. $A \in U$, $\min A > \alpha_{n-1}$

4. F is a function on A such that $F(\alpha) \in \mathrm{Coll}(\alpha^{+++}, \kappa)$

5. $[F]_U$, which is an element of $\mathrm{Coll}((\kappa^{+++})^{M^*}, j(\kappa))^{M^*}$, belongs to G'.

The conditions in R are ordered as follows:
$(\aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}, B, H) \leq (\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ iff

1. $m \geq n$

2. $\forall i < n$ $\beta_i = \alpha_i$, $g_i \supseteq f_i$

3. $B \subseteq A$

4. $\forall i \geq n$ $\beta_i \in A$, $g_i \supseteq F(\beta_i)$

5. $\forall \alpha \in B$ $H(\alpha) \supseteq F(\alpha)$.

We often abbreviate the lower part of a condition by a single letter and write (s, A, F) instead of $(\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ where $|s| = n$ denotes the length of the lower part. Let S denote the 'generic sequence', i.e. the Prikry sequence together with the generic collapsing functions.

Claim. R satisfies κ^+-c.c.

Proof of the claim. There are only κ lower parts and any two conditions with the same lower part are compatible, so no antichain has size bigger than κ.

Claim. Let $(s, A, F) \in R$ and let σ be a statement of the forcing language. There exists a stronger condition (s', A^*, F^*) with $|s| = |s'|$ which decides σ.

For a proof see [2].

Claim. Let C be a $V[G][g]$-generic subset of R and let $\langle \aleph_0, \alpha_1, ..., \alpha_n, ...\rangle$ be the Prikry sequence in κ introduced by R. For $j \in \omega$, define $R \restriction j := \mathrm{Coll}(\aleph_0^{+++}, \alpha_1) \times \mathrm{Coll}(\alpha_1^{+++}, \alpha_2) \times ... \times \mathrm{Coll}(\alpha_{j-1}^{+++}, \alpha_j)$. Then $V[G][g][C]$ and $V[G][g][C \restriction j]$ have the same cardinal structure below $\alpha_j + 1$, namely $\aleph_1, \aleph_2, \aleph_3, \alpha_1, \alpha_1^+, \alpha_1^{++}, \alpha_1^{+++}, ..., \alpha_{j-1}, \alpha_{j-1}^+, \alpha_{j-1}^{++}, \alpha_{j-1}^{+++}, \alpha_j$, where $C \restriction j$ is the restriction of C to $R \restriction j$.

Proof of the claim. Write R as $R \restriction j * R/(\dot{R} \restriction j)$, where the quotient $R/(\dot{R} \restriction j)$ is defined in the same way as R (using only inaccessibles between α_j and κ). We need to show that $R/(R \restriction j)$ does not add bounded subsets of α_j, but this follows immediately from the last claim.

So we proved that R makes κ into $\aleph_\omega$. It remains to show that it also preserves the tree property on $\kappa^{++} = \lambda$.

In order to get a contradiction suppose that there is a κ^{++}-Aronszajn tree in some R-extension of $V[G][g]$. Then in $V[G]$ there is a $\mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ -name $\dot{T}$ of size λ (because $\mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ satisfies λ-c.c.) and a condition $(p, \dot{r}) \in \mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ which forces $\dot{T}$ to be a κ^{++}-Aronszajn tree. Let $\dot{G}'$ be a $\mathrm{Sacks}(\kappa, \lambda)$-name in $V[G]$ for G' of size λ (there is such a name because $\mathrm{Sacks}(\kappa, \lambda)$ has the λ-c.c. and $|Q'| = \lambda$). We can assume w.l.o.g. that p forces $\dot{G}'$ to be generic over Q'. Recall that λ is a weakly compact cardinal in $V[G]$. Therefore, there exist in $V[G]$ transitive ZF^--models N_0, N_1 of size

λ and an elementary embedding $k : N_0 \to N_1$ with critical point λ, such that $N_0 \supseteq H(\lambda)^{V[G]}$ and $G, \dot{T}, \dot{G}' \in N_0$.

Since g is also Sacks(κ,λ)-generic over N_0 and the critical point of k is λ, k can be lifted to $k^* : N_0[g] \to N_1[g][K]$, where K is any $N_1[g]$-generic subset of Sacks$(\kappa, [\lambda, k(\lambda)))$ in some larger universe (and where Sacks$(\kappa, [\lambda, k(\lambda)))$ is the quotient Sacks$(\kappa, k(\lambda))$/Sacks(κ,λ), i.e. the iteration of Sacks(κ) indexed by ordinals between λ and $k(\lambda)$). Consider the forcing $R^* := k^*(R) = R(k(G'), k(U))$ in $N_1[g][K]$ and choose any generic C^* for it such that $k^*(r) \in C^*$, where $r = \dot{r}^g, R = \dot{R}^g, G' = \dot{G}'^g$. Let $C := (k^*)^{-1}[C^*]$ be the pullback of C^* under k^*. Then C is an $N_0[g]$-generic subset of R because crit(k)=λ and R has the κ^+-c.c. It follows that there is an elementary embedding $k^{**} : N_0[g][C] \to N_1[g][K][C^*]$ extending k^*.

We have $r \in C$. So it follows that the evaluation T of $\dot{T}$ in $N_0[g][C]$ is a λ-Aronszajn tree. By elementarity $k^{**}(T)$ is a $k^{**}(\lambda)$-Aronszajn tree in $N_1[g][K][C^*]$ which coincides with T up to level λ. Hence T has a cofinal branch b in $N_1[g][K][C^*]$. We will show that b has to belong to $N_1[g][C]$ and thereby reach the desired contradiction!

Let us first analyse the quotient Q arising from the natural projection π : Sacks$(\kappa, k(\lambda)) * \dot{R}^* \to RO($Sacks$(\kappa,\lambda) * \dot{R})$. As in the previous section, Q is the set of all $(p^*, (\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, \dot{A}^*, \dot{F}^*)) \in$ Sacks$(\kappa, k(\lambda)) * \dot{R}^*$ which are compatible with each $(p, (\aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}, \dot{A}, \dot{F})) \in g * C$, that is, either

1. $p^* \restriction \lambda$ is compatible with p,
2. $n < m$,
3. for all $i < n$ $\alpha_i = \beta_i \wedge f_i \parallel g_i$,
4. there is $q \leq p \cup p^*$ such that $q \Vdash$ "$\beta_n, ..., \beta_{m-1} \subset \dot{A}^*$ and $\dot{F}^*(\beta_i) \parallel g_i$ for $n \leq i < m$",

or

1. $p^* \restriction \lambda$ is compatible with p,
2. $n \geq m$,
3. for all $i < m$ $\alpha_i = \beta_i \wedge f_i \parallel g_i$,
4. there is $q \leq p \cup p^*$ such that $q \Vdash$ "$\alpha_m, ..., \alpha_{n-1} \subset \dot{A}$ and $\dot{F}(\alpha_i) \parallel f_i$ for $m \leq i < n$".

[Note that in both cases the condition q also forces $\dot{F}$ and $\dot{F}^*$ to be compatible on a measure one set. This is because the weaker condition p (by definition) forces $j(\dot{F})(\kappa)$ to be in $\dot{G}'$, and therefore, by elementarity, also forces $k(j)(k(\dot{F}))(\kappa)$ to be in $k(\dot{G}')$, but $k(j)(k(\dot{F}))(\kappa)$ is the same as $k(j)(\dot{F})(\kappa) = [\dot{F}]_{U^*}$, since the trivial condition forces $k(\dot{F}) = \dot{F}$.]

Equivalently, Q is the set of conditions $(p^*, (\aleph_0, f_0, ..., \alpha_{n-1}, f_{n-1}, \dot{A}^*, \dot{F}^*))$ in Sacks$(\kappa, [\lambda, k(\lambda))) * \dot{R}^*$ such that

1. $p^* \in$ Sacks$(\kappa, [\lambda, k(\lambda)))$,
2. $\langle \aleph_0, \alpha_1 ..., \alpha_{n-1} \rangle$ is an initial segment of $S(C)$ (the Prikry sequence arising from C),

3. the collapsing function $\bar{g}_i : \alpha_i^{+++} \to \alpha_{i+1}$ arising from C extends f_i, $i < n$,

4. p^* forces that $\dot{A}^*$ is in $\dot{U}^*$, and that $\dot{F}^*$ is a function on $\dot{A}^*$ such that $\dot{F}^*(\alpha) \in \mathrm{Coll}(\alpha^{+++}, \kappa)$ for each $\alpha \in \dot{A}^*$,

5. for every finite subset $x = \langle \beta_n, ..., \beta_{m-1} \rangle$ of $S(C)$ and every sequence of functions $\langle g_n, ..., g_{m-1} \rangle$ with $g_i \subseteq \bar{g}_i$, $n \leq i < m$, there is some extension q of p^* which forces that x is a subset of $\{\aleph_0, \alpha_1, ..., \alpha_{n-1}\} \cup \dot{A}^*$ and that $\dot{F}^*(\beta_i) \parallel g_i$ for $n \leq i < m$.

We now again argue indirectly. Assume that b is not in $N_1[g][C]$, and let $\dot{b}$ in $N_1[g]$ be an $R * \dot{Q}$ - name for b. Identify $k(\dot{T})$ with the $R * \dot{Q}$ - name defined by interpreting the $\mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ - name $k(\dot{T})$ in N_1 as an $R * \dot{Q}$ - name in $N_1[g]$. Let $((s_0, A_0, F_0), (p_0, (t_0, \dot{A}_0, \dot{F}_0)))$ be an $R * \dot{Q}$ - condition forcing that the Prikry-name $\dot{T}$ is a λ-tree and that $\dot{b}$ is a branch through $\dot{T}$ not belonging to $N_1[g][\dot{C}]$.

Let us take a closer look at the condition $((s_0, A_0, F_0), (p_0, (t_0, \dot{A}_0, \dot{F}_0)))$. Say, $s_0 = \langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1} \rangle$ and $t_0 = \langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1} \rangle$. Note that the forcing Q lives in $N_1[g][C]$, but its elements are in $N_1[g]$, so we can assume that $(p_0, (t_0, \dot{A}_0, \dot{F}_0))$ is a real object and not just an R-name. The condition (s_0, A_0, F_0) forces $(p_0, (t_0, \dot{A}_0, \dot{F}_0))$ to be an element of $\dot{Q}$. But this simply means that:

1. p_0 is an element of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$,

2. $\langle \aleph_0, \beta_1, ..., \beta_{m-1} \rangle$ is an initial segment of $\langle \aleph_0, \alpha_1, ..., \alpha_{n-1} \rangle$,

3. $g_i \subseteq f_i$ for $i < m$, and

4. for every finite subset $x = \langle \delta_1, ..., \delta_l \rangle$ of $\{\aleph_0, \alpha_1, ..., \alpha_{n-1}\} \cup A_0$ and every sequence of functions $\langle g_{\delta_1}, ..., g_{\delta_l} \rangle$ with $g_{\delta_i} \supseteq F_0(\delta_i)$ if $\delta_i > \alpha_{n-1}$, and $g_{\delta_i} \supseteq f_i$ if $\delta_i = \alpha_i$ (for some $i < n$), some extension of p_0 forces that x is a subset of $\{\aleph_0, \beta_1, ..., \beta_{m-1}\} \cup \dot{A}_0$ and that $\dot{F}_0(\delta_i) \parallel g_{\delta_i}$ for $i < l$.

Moreover, we can assume that $s_0 = t_0$. Namely, the following claim gives us a nice dense subset of $R * \dot{Q}$ on which we will work from now on.

Claim. Let $((s, A, F), (p, (t, \dot{A}, \dot{F})))$ be an arbitrary condition in $R * \dot{Q}$. There is a stronger condition $((s', A', F'), (p', (s', \dot{A}, \dot{F})))$ with the property that for each $\alpha \in A'$ $p' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$.

Proof of the claim. Say, s is of the form $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1} \rangle$ and t is of the form $\langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1} \rangle$. Let q be an extension of p which forces that $\{\alpha_m, ..., \alpha_{n-1}\}$ is a subset of $\dot{A}$ and that $f_i \parallel \dot{F}(\alpha_i)$ for $m \leq i < n$. Extend q further to q' to decide $\dot{F}(\alpha_i)$ and let $f'_i := f_i \cup \dot{F}(\alpha_i)$. Define s' to be $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{m-1}, f_{m-1}, \alpha_m, f'_m, ..., \alpha_{n-1}, f'_{n-1} \rangle$.

Using the fusion property of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ we can find a condition $q'' \leq q'$ and a ground model function F^* on A with $|F^*(\alpha)| \leq \alpha^{++}$ for each α such that $q'' \Vdash \dot{F}(\alpha) \in \mathrm{Coll}(\alpha^{+++}, \kappa) \cap F^*(\alpha)$. It follows that q'' forces that in $\mathrm{Ult}(N_1[g], U)$, the ultrapower of $N_1[g]$ by U, $j_U(\dot{F})(\kappa) \in \mathrm{Coll}(\kappa^{+++}, j_U(\kappa)) \cap j_U(F^*)(\kappa)$, where $|j_U(F^*)(\kappa)| \leq \kappa^{++}$, that is, q'' forces that there are fewer than κ^{+++} possibilities for $j_U(\dot{F})(\kappa)$. Note that $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ of $\mathrm{Ult}(N_1[g], U)$ is the same as $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ of $\mathrm{Ult}(N_0[g], U)$, because these two ultrapowers agree below $j_U(\kappa)$.

Since $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ is κ^{+++}-closed we can densely often find conditions in $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ which are either stronger than or incompatible with all elements in $j_U(F^*)(\kappa)$. Therefore we can choose some $j_U(F')(\kappa) \leq j_U(F)(\kappa)$ in G' with this property, i.e. $q'' \Vdash j_U(F')(\kappa) \leq j_U(\dot{F})(\kappa) \vee j_U(F')(\kappa) \perp j_U(\dot{F})(\kappa)$. But actually we have $q'' \Vdash j_U(F')(\kappa) \leq j_U(\dot{F})(\kappa)$, because for any generic K below q'', $j_U(F')(\kappa)$ and $j_U(\dot{F}^K)(\kappa)$ can not be incompatible as $k(j_U(F')(\kappa))$ and $k(j_U(\dot{F}^K)(\kappa)) = j_{k(U)}(\dot{F}^K)(\kappa)$ both belong to the guiding generic $k(G')$.

It follows that q'' forces that for some $B \in U, B \subseteq A$, for each $\alpha \in B$, $q'' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$. Extend q'' to some p' deciding B.

Finally, using the claim from the previous section, shrink B to some A' such that every finite subset of A' is forced by some extension of p' to belong to $\dot{A}$. Then we have $((s', A', F'), (p', (s', \dot{A}, \dot{F}))) \leq ((s, A, F), (p, (t, \dot{A}, \dot{F})))$ such that for each $\alpha \in A'$ $p' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$. This proves the claim.

Now in $N_1[g]$ build a κ-tree E of conditions in $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$, whose branches will be fusion sequences, together with a sequence of ordinals $\langle \lambda_\beta : \beta < \kappa \rangle$, each $\lambda_\beta < \lambda$, in the same way as in the last section (using the same notation, Fact 1 and Fact 2):

Let $\langle v_j : j < 2^{\beta+1} \rangle$ be an enumeration of level β of the tree E and let $\langle u_m \rangle_{m < \sum_{j<2^{\beta+1}} d_{v_j}}$ be an enumeration of $Y := \bigcup_{j<2^{\beta+1}} \{u_l^{v_j} : l < d_{v_j}\}$. In order to construct the next level of the tree we will first thin out all the nodes on level β (by considering all the pairs in Y) and then split each of them into two incompatible nodes. The thinning out is done as follows: Consider u_0 and u_1. If they belong to the same node, i.e. if there is $j < 2^{\beta+1}$ and $l_0, l_1 < d_{v_j}$ s.t. $u_0 = u_{l_0}^{v_j}$ and $u_1 = u_{l_1}^{v_j}$, then no thinning takes place. So assume that u_0 and u_1 belong to different nodes, say v_{j_0} and v_{j_1}, respectively. Use Fact 1 to construct conditions $r_{01} = (v_{j_0})^{u_0}$ and $r_{10} = (v_{j_1})^{u_1}$, i.e. thin v_{j_0} and v_{j_1} through u_0 and u_1 to r_{01} and r_{10}, respectively. Now ask whether there exist extensions r'_{01} and r'_{10} of r_{01} and r_{10}, respectively, such that for some $\gamma_{01} < \lambda$ and some A_{01}, A_{10}, F_{01}, F_{10}, $\dot{A}_{01}$, $\dot{A}_{10}$, $\dot{F}_{01}$, $\dot{F}_{10}$, $((s_\beta, A_{01}, F_{01}), (r'_{01}, (s_\beta, \dot{A}_{01}, \dot{F}_{01})))$ and $((s_\beta, A_{10}, F_{10}), (r'_{10}, (s_\beta, \dot{A}_{10}, \dot{F}_{10})))$ force different nodes on level γ_{01} of $\dot{T}$ to lie on $\dot{b}$. If the answer is 'yes', use Fact 2 to refine v_{j_0} and v_{j_1} through r'_{01} and r'_{10}, respectively, and continue with the next pair: u_0, u_2. And if the answer is 'no', go to the pair u_0, u_2 without refining v_{j_0} and v_{j_1}. The next pairs are u_1, u_2; u_0, u_3 and so on, i.e. all pairs of the form u_δ, u_η, for $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$. At the limit stages take lower bounds, they exist since the forcing is κ-closed. Let λ_β be the supremum of (the increasing sequence of) $\gamma_{\delta\eta}$'s. Now extend each node v on level β (after thinning out the whole level) to two incompatible conditions v_o and v_1, such that $v_0, v_1 \leq_{\beta, X_v} v$.

Let α be the supremum of λ_β's. Note that $\alpha < \lambda$, because $\lambda = (\kappa^{++})^{N_1[g]}$. Let p be the result of a fusion along a branch through E. As before we can choose $A_0(p) \subseteq A_0$ in U such that $((s_0, A_0(p), F_0), (p, (s_0, \dot{A}_0, \dot{F}_0)))$ is a condition. Extend this condition to some $((s_1(p), A_1(p), F_1(p)), (p^*, (s_1(p), \dot{A}_1(p), \dot{F}_1(p))))$ which decides $\dot{b}(\alpha)$, say it forces $\dot{b}(\alpha) = x_p$.

As level α of $\dot{T}$ has size $< \lambda$, there exist limits p, q of κ-fusion sequences arising from distinct κ-branches through E for which x_p equals x_q and $s_1(p)$ equals $s_1(q)$. Moreover, we can extend $(s_1(p), A_1(p), F_1(p))$ and $(s_1(q), A_1(q), F_1(q))$ to get a common (s_1, A_1, F_1). Say, $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ and $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$ force $\dot{b}(\alpha) = x$.

Now choose a Collapse Prikry generic C containing (s_1, A_1, F_1) (and hence containing (s_0, A_0, F_0)). As $((s_0, A_0, F_0), (p_0, (s_0, \dot{A}_0, \dot{F}_0))) \Vdash \dot{b} \notin N_1[g][\dot{C}]$ and $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ extends $((s_0, A_0, F_0), (p_0, (s_0, \dot{A}_0, \dot{F}_0)))$, we can extend $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ to two incompatible conditions, $((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$ and $((s_{2_1}, A_{2_1}, F_{2_1}), (p_1^{**}, (s_{2_1}, \dot{A}_{2_1}, \dot{F}_{2_1})))$, with $(s_{2_0}, A_{2_0}, F_{2_0})$, $(s_{2_1}, A_{2_1}, F_{2_1}) \in C$ and $p_0^{**}, p_1^{**} \leq p^*$, which force a disagreement about $\dot{b}$ at some level γ above α.

Now extend $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$ to some stronger condition $((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ which decides $\dot{b}(\gamma)$ with (s_3, A_3, F_3) in C. Say, $((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ and $((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$ do not agree about $\dot{b}(\gamma)$, and say, s_3 is of the form $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}\rangle$, and s_{2_0} is of the form $\langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}\rangle$.

We can assume w.l.o.g. that $m < n$. As both (s_3, A_3, F_3) and $(s_{2_0}, A_{2_0}, F_{2_0})$ are in C, we have that $\langle \aleph_0, \beta_1, ..., \beta_{m-1}\rangle$ is an initial segment of $\langle \aleph_0, \alpha_1, ..., \alpha_{n-1}\rangle$, $g_i \parallel f_i$ for $i < m$, $\{\alpha_m, ..., \alpha_{n-1}\} \subset A_{2_0}$, and $F_{2_0}(\alpha_i) \parallel f_i$ for $m \leq i < n$. Let $f_i' := f_i \cup g_i$ for $i < m$, and $f_i' := f_i \cup F_{2_0}(\alpha_i)$ for $m \leq i < n$. Define s_3' to be $\langle \aleph_0, f_0', \alpha_1, f_1', ..., \alpha_{n-1}, f_{n-1}'\rangle$.

Note that $((s_3', A_3, F_3), (q^{**}, (s_3', \dot{A}_3, \dot{F}_3))) \leq ((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ is also a condition.

Since $\{\alpha_m, ..., \alpha_{n-1}\} \subset A_{2_0}$, there exists some $p^{***} \leq p_0^{**}$ which forces that $\{\alpha_m, ..., \alpha_{n-1}\} \subset \dot{A}_{2_0}$. It follows that there is also some $A_3' \in U$ such that $((s_3', A_3', F_{2_0}), (p^{***}, (s_3', \dot{A}_{2_0}, \dot{F}_{2_0}))) \leq ((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$.

Now, for some $\beta < \kappa$ we have $s_3' = s_\beta$ where s_β is the βth element of the enumeration of the lower parts. Since s_β appears cofinally often in the construction of the tree E, we can assume that the branches which fuse to p and q split in E at some node below level β and go through some nodes v_{j_0} and v_{j_1} at level β. It follows that for some $l < d_{v_{j_0}}$ and $k < d_{v_{j_1}}$,

$$r_1 := ((s_3', A_3'((p^{***})^{u_l^{v_{j_0}}}), F_{2_0}), ((p^{***})^{u_l^{v_{j_0}}}, (s_3', \dot{A}_{2_0}, \dot{F}_{2_0})))$$

and

$$r_2 := ((s_3', A_3((q^{**})^{u_k^{v_{j_1}}}), F_3), ((q^{**})^{u_k^{v_{j_1}}}, (s_3', \dot{A}_3, \dot{F}_3)))$$

force different nodes to lie on $\dot{b}$ at level $\gamma > \alpha$. By construction, this means that for some $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$,

$$r_3 := ((s_\beta, A_{\delta\eta}, F_{\delta\eta}), (r_{\delta\eta}', (s_\beta, \dot{A}_{\delta\eta}, \dot{F}_{\delta\eta})))$$

and

$$r_4 := ((s_\beta, A_{\eta\delta}, F_{\eta\delta}), (r_{\eta\delta}', (s_\beta, \dot{A}_{\eta\delta}, \dot{F}_{\eta\delta})))$$

force different nodes on level $\gamma_{\delta\eta}(< \alpha)$ of $\dot{T}$ to lie on $\dot{b}$. Say, $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\gamma_{\delta\eta}) = y_1$, respectively.

On the other side, r_1 and r_2 extend $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ and $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$, respectively. Therefore we have that r_1 and r_2 also force $\dot{b}(\alpha) = x$.

Note that $(p^{***})^{u_l^{v_{j_0}}} \leq r_{\delta\eta}'$ and $(q^{**})^{u_k^{v_{j_1}}} \leq r_{\eta\delta}'$. Since any two $R * \dot{Q}$ conditions with the same lower part and compatible Sacks conditions are compatible (this follows by the same arguments used in the proof of the last claim), we have that $r_1 \parallel r_3$ and $r_2 \parallel r_4$. Let $((s_3', B', H'), (\bar{p}, (s_3', \dot{B}', \dot{H}')))$ be a common lower bound of r_1 and r_3, and let $((s_3', B'', H''), (\bar{q}, (s_3', \dot{B}'', \dot{H}'')))$ be a common lower bound of r_2 and r_4. The first condition forces $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\alpha) = x$, and the second condition forces $\dot{b}(\gamma_{\delta\eta}) = y_1$ and $\dot{b}(\alpha) = x$.

Finally, let $\bar{B} := B' \cap B''$ and $\bar{H} := H' \cap H''$. Then $(s_3', \bar{B}, \bar{H})$ forces that $y_0, y_1 <_{\dot{T}} x$ in the ordering of the tree $\dot{T}$, because $\dot{T}$ is a Collapse Prikry-name, i.e. all the relations between the nodes of $\dot{T}$ are determined by the Collapse Prikry parts of the conditions above. Contradiction. □

Open questions

1. What is the consistency strength of $\aleph_\omega$ strong limit with the tree property at $\aleph_{\omega+2}$? [The best known lower bound is a weakly compact λ such that for each $n < \omega$ there exists $\kappa < \lambda$ with $o(\kappa) = \kappa^{+n}$.]

2. What is the consistency strength of the tree property at every even successor cardinal?

3. Is it consistent with ZFC to have the tree property at each $\aleph_n$, $1 < n < \omega$, and $\aleph_{\omega+2}$?

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