ORDINAL DEFINABLE SUBSETS OF SINGULAR CARDINALS

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Abstract. A remarkable result by Shelah states that if κ is a singular strong limit cardinal of uncountable cofinality then there is a subset x of κ such that HOD_x contains the power set of κ. We develop a version of diagonal extender-based supercompact Prikry forcing, and use it to show that singular cardinals of countable cofinality do not in general have this property, and in fact it is consistent that for some singular strong limit cardinal κ of countable cofinality κ^+ is supercompact in HOD_x for all $x \subseteq \kappa$.

1. Introduction

It is a familiar phenomenon in the study of singular cardinal combinatorics that singular cardinals of countable cofinality can behave very differently from those of uncountable cofinality. For example in the area of cardinal arithmetic we may contrast Silver's theorem [8] with the many consistency results producing models where GCH first fails at a singular cardinal of countable cofinality [2]. The results in this paper show that there is a similar sharp dichotomy involving questions about the definability of subsets of a cardinal rather than the size of its powerset.

Shelah [9] proved that if κ is a singular strong limit cardinal of uncountable cofinality then there is a subset $x \subseteq \kappa$ such that $P(\kappa) \subseteq \mathrm{HOD}_x$. In this paper we show that the hypothesis is essential, by proving:

Theorem. *Suppose that $\kappa < \lambda$ where $\mathrm{cf}(\kappa) = \omega$, λ is inaccessible and κ is a limit of λ-supercompact cardinals. There is a forcing poset $\mathbb{Q}$ such that if G is $\mathbb{Q}$-generic then:*

- *The models V and $V[G]$ have the same bounded subsets of κ.*
- *Every infinite cardinal μ with $\mu \leq \kappa$ or $\mu \geq \lambda$ is preserved in $V[G]$.*
- $\lambda = (\kappa^+)^{V[G]}$.
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{\mathrm{HOD}_x} < \lambda$.*

From stronger assumptions we can use $\mathbb{Q}$ to obtain a model in which κ is a singular strong limit cardinal of cofinality ω, and κ^+ is supercompact in HOD_x for all $x \subseteq \kappa$.

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Notation. Our notation is mostly standard. We write $\ell(s)$ for the length of a sequence s. When $\mathbb{Q}$ is a forcing poset and $q \in \mathbb{Q}$ we write $\mathbb{Q} \downarrow q$ for $\{r \in \mathbb{Q} : r \leq q\}$, with the partial ordering inherited from $\mathbb{Q}$.

2. A proof of Shelah's theorem

For the reader's benefit we give a short and self-contained proof of the result by Shelah quoted in the Introduction. Let κ be a strong limit singular cardinal of uncountable cofinality μ. Fix a set $x \subseteq \kappa$ such that $L[x]$ contains an enumeration $(t_\eta)_{\eta<\kappa}$ of H_κ, together with a sequence of ordinals $(\alpha_i)_{i<\mu}$ which is increasing and cofinal in κ.

To each set $X \subseteq \kappa$ we associate a function $f_X : \mu \to \kappa$, where $f_X(i) = \eta$ for the unique η such that $X \cap \alpha_i = t_\eta$. It is easy to see that if $X \neq Y$ then $f_X(i) \neq f_Y(i)$ for all large i. We now impose a strict partial ordering on $P(\kappa)$ by defining $X \lhd Y$ if and only if $f_X(i) < f_Y(i)$ for all large i; since $\mu = \mathrm{cf}(\mu) > \omega$, the ordering $\lhd$ is well-founded.

Let R_α be the set of elements of $P(\kappa)$ with $\lhd$-rank α, and note that if X and Y are distinct elements of R_α then there are unboundedly many $i < \mu$ with $f_X(i) > f_Y(i)$, since otherwise $f_X(i) < f_Y(i)$ for all large i and it would follow that $X \lhd Y$. We claim that $|R_\alpha| \leq 2^\mu$, so that in particular $|R_\alpha| < \kappa$. Supposing for contradiction that $|R_\alpha| > 2^\mu$ we fix a sequence $(X_j)_{j<(2^\mu)^+}$ of distinct elements of R_α, and then define a colouring c of pairs from $(2^\mu)^+$ by setting $c(j, j') = i$ for the least $i < \mu$ with $f_{X_j}(i) > f_{X_{j'}}(i)$. By the Erdős-Rado theorem c has a homogeneous set of order type μ^+, which is impossible because it would yield an infinite decreasing sequence of ordinals.

We now define a well-ordering $\prec_\alpha$ of R_α. To do this we let $A_\alpha = \bigcup\{\mathrm{range}(f_X) : X \in R_\alpha\}$, note that $|A_\alpha| < \kappa$, and define g_α to be the order-isomorphism between A_α and $\mathrm{ot}(A_\alpha)$. If $X \in R_\alpha$ then $g_\alpha \circ f_X \in H_\kappa$, and we define $X \prec_\alpha Y$ if and only if the index of $g_\alpha \circ f_X$ is less than that of $g_\alpha \circ f_Y$ in the enumeration $(t_\eta)_{\eta<\kappa}$ of H_κ.

Forming the sum of the orderings $\prec_\alpha$ in the natural way, we obtain a well-ordering $\prec$ of $P(\kappa)$ which is clearly definable from x. Now every element of $P(\kappa)$ is ordinal definable from x as "the η^{th} element of $P(\kappa)$ in the ordering $\prec$", so that $P(\kappa) \subseteq \mathrm{HOD}_x$.

3. The main theorem

3.1. **Setup.** Let $\kappa < \lambda$ with λ strongly inaccessible and $\kappa = \sup_n \kappa_n$, where $(\kappa_n)_{n<\omega}$ is a strictly increasing sequence of λ-supercompact cardinals. Fix U_n a κ_n-complete fine normal ultrafilter on $P_{\kappa_n}\lambda$, and for $\kappa \leq \alpha < \lambda$ let $U_{n,\alpha}$ be the projection of U_n to $P_{\kappa_n}\alpha$ via the map $x \mapsto x \cap \alpha$.

We need a form of diagonal intersection lemma.

Lemma 1. *Let $k < l < \omega$, let $(\alpha_i)_{k\leq i<l}$ be a $\leq$-increasing sequence of elements of $[\kappa, \lambda)$, and let $S \subseteq \prod_{k\leq i<l} P_{\kappa_i}\alpha_i$ be a set of $\subseteq$-increasing sequences.*

Let $m \in [l, \omega)$, let $\alpha \in [\alpha_{l-1}, \lambda)$, and let $\vec{A} = (A_s)_{s \in S}$ be an S-indexed family with $A_s \in U_{m,\alpha}$ for each $s \in S$. Then the following set is in $U_{m,\alpha}$:

$$D = \{x \in P_{\kappa_m}\alpha : \forall (x_k, \dots, x_{l-1}) \in S\ [x_{l-1} \subseteq x \implies x \in A_{(x_k, \dots x_{l-1})}]\}.$$

Proof. Let $j : V \to M$ be the ultrapower map computed from U_m. To show that $D \in U_{m,\alpha}$ it suffices to show that $j``\alpha \in j(D)$.

Let $t = (y_k, \dots y_{l-1}) \in j(S)$ with $y_{l-1} \subseteq j``\alpha$. Write $\vec{B} = j(\vec{A})$. We need to show that $j``\alpha \in B_t$.

For each i with $k \leq i < l$, $y_i \subseteq j``\alpha_i$ and $|y_i| < \kappa_i < \kappa_m = \mathrm{crit}(j)$, so that $y_i = j``x_i = j(x_i)$ for some $x_i \in P_{\kappa_i}\alpha_i$. Write $s = (x_k, \dots x_{l-1})$. Then $t = j(s)$ and hence $s \in S$. In particular $A_s \in U_{m,\alpha}$, and so $j``\alpha \in j(A_s) = B_t$. □

In the setting of Lemma 1 we will refer to D as the *diagonal intersection* of $\vec{A}$ and we will write "$s \Subset x$" as shorthand for "$s = (x_k, \dots x_{l-1})$ with $x_{l-1} \subseteq x$".

With a view to the proof of Lemma 12 below we also need a technical lemma about measure one sets:

Lemma 2. *Let $i < \omega$ and let $\alpha_i \leq \beta_i \leq \beta_{i+1}$ be ordinals Let $B \in U_{i,\beta_i}$ and let $C \in U_{i+1,\beta_{i+1}}$. Let C' be the set of $y \in C$ such that for every $x \in B$ with $x \cap \alpha_i \subseteq y$ there is $x' \in B \cap P(y)$ with $x' \cap \alpha_i = x \cap \alpha_i$. Then $C' \in U_{i+1,\beta_{i+1}}$.*

Proof. Let $j : V \to M$ be the ultrapower map computed from U_{i+1}. It suffices to prove that $j``\beta_{i+1} \in j(C')$.

Since $C \in U_{i+1,\beta_{i+1}}$, we have $j``\beta_{i+1} \in j(C)$. Suppose that $X \in j(B)$ satisfies $X \cap j(\alpha_i) \subseteq j``\beta_{i+1}$; we shall find $X' \in j(B \cap j``\beta_{i+1})$ such that $X' \cap j(\alpha_i) = X \cap j(\alpha_i)$.

Since $B \in P_{\kappa_i}\beta_i$, $\kappa_i < \kappa_{i+1} = \mathrm{crit}(j)$ and $X \in j(B)$, it follows that $X \cap j(\alpha_i) = j(t) = j``t$ for some $t \in P_{\kappa_i}\alpha_i$. By elementarity there is $x' \in B$ such that $x' \cap \alpha_i = t$. Now $j(x') \in j(B)$, $j(x') = j``x' \subseteq j``\beta_{i+1}$, and $j(x') \cap j(\alpha_i) = j(t) = X \cap j(\alpha_i)$, so $X' = j(x')$ is as required. □

3.2. **The poset $\mathbb{Q}$.** The poset $\mathbb{Q}$ is a diagonal extender-based forcing of the general type introduced by Gitik and Magidor [4]. For the experts, we note the main innovations:

- The extender used here on level n is formed from the approximations $U_{n,\alpha}$ to the supercompactness measure U_n, rather than from measures approximating some short extender.
- The supports of the components of $\mathbb{Q}$ are quite large. This is necessary because the proofs below require long diagonalisations, which are only possible because the coordinates of the extender on level n are linearly ordered and all the measures $U_{n,\alpha}$ are normal.

A note on notation: The definitions of the poset $\mathbb{Q}$ and its ordering $\leq$ contain quite a number of clauses. We will write for example "b_q" as shorthand for "Clause b in the definition of $q \in \mathbb{Q}$" and "$6_{q,r}$" as shorthand for "Clause 6 in the definition of $q \leq r$".

Conditions in $\mathbb{Q}$ are sequences $(q_k)_{k<\omega}$ such that for some $m < \omega$ we have $q_k = f^q_k$ for $k < m$ and $q_k = (a^q_k, A^q_k, f^q_k)$ for $k \geq m$, where:

a) f^q_k is a function with $\mathrm{dom}(f^q_k) \subseteq [\kappa, \lambda)$, $|\mathrm{dom}(f^q_k)| < \lambda$, and $f^q_k(\eta) \in P_{\kappa_k}\eta$ for all $k < \omega$ and all $\eta \in \mathrm{dom}(f^q_k)$.
b) For $k \geq m$ we have $a^q_k \subseteq [\kappa, \lambda)$, $|a^q_k| < \lambda$, a^q_k and $\mathrm{dom}(f^q_k)$ are disjoint sets and a^q_k has a maximum element α^q_k.

c) For $k \geq m$ we have $A^q_k \in U_{k,\alpha^q_k}$.
d) $(a^q_k)_{m \leq k < \omega}$ is $\subseteq$-increasing with k.

We call the integer m the *length of* q and write $m = \ell(q)$.

Remark 1. We note that there is very little interaction among the "components" q_k of the condition q: it is only d_q that connects entries in q_k on different levels k.

Remark 2. Note that by b_q and d_q, if $\ell(q) \leq k < l$ then $\alpha^q_k \in a^q_l$ and $\alpha^q_k \leq \alpha^q_l$.

Given r and q in $\mathbb{Q}$, $r \leq q$ if and only if:

(1) $\ell(r) \geq \ell(q)$.
(2) For all k, $f^r_k \supseteq f^q_k$ (that is, $\mathrm{dom}(f^r_k) \supseteq \mathrm{dom}(f^r_k)$ and $f^r_k \restriction \mathrm{dom}(f^q_k) = f^q_k$).
(3) For k with $\ell(q) \leq k < \ell(r)$, $a^q_k \subseteq \mathrm{dom}(f^r_k)$, $f^r_k(\alpha^q_k) \in A^q_k$, and $f^r_k(\eta) = f^r_k(\alpha^q_k) \cap \eta$ for all $\eta \in a^q_k$.
(4) $(f^r_k(\alpha^q_k))_{\ell(q) \leq k < \ell(r)}$ is $\subseteq$-increasing.
(5) For $k \geq \ell(r)$, we have $a^q_k \subseteq a^r_k$, and $x \cap \alpha^q_k \in A^q_k$ for all $x \in A^r_k$.
(6) For $k \geq \ell(r)$, if $\ell(q) < \ell(r)$, then $f^r_{\ell(r)-1}(\alpha^q_{\ell(r)-1}) \subseteq x$ for all $x \in A^r_k$.

Remark 3. Note that if $r \leq q$ then $\alpha^r_k \geq \alpha^q_k$ for all $k \geq \ell(r)$.

Lemma 3. *The ordering $\leq$ on $\mathbb{Q}$ is transitive.*

Proof. Let $r \leq q \leq p$ be conditions in $\mathbb{Q}$. Note that $1_{r,p}$, $2_{r,p}$ and $5_{r,p}$ are immediate.

To verify $3_{r,p}$, let $\ell(p) \leq k < \ell(r)$ and distinguish two cases:

- $\ell(p) \leq k < \ell(q)$: In this case $f^r_k(\alpha^p_k) = f^q_k(\alpha^p_k)$ by $2_{r,q}$ and $f^q_k(\alpha^p_k) \in A^p_k$ by $3_{q,p}$, so $f^r_k(\alpha^p_k) \in A^p_k$. Moreover if $\eta \in a^p_k$ then $f^q_k(\eta) = f^q_k(\alpha^p_k) \cap \eta$ by $3_{q,p}$ for $q \leq p$. Also $f^r_k(\eta) = f^q_k(\eta)$ and $f^r_k(\alpha^p_k) = f^q_k(\alpha^p_k)$ by $2_{r,q}$, so $f^r_k(\eta) = f^r_k(\alpha^p_k) \cap \eta$.
- $\ell(q) \leq k < \ell(r)$: We have $\alpha^p_k \in a^q_k$ by $5_{q,p}$, so $f^r_k(\alpha^p_k) = f^r_k(\alpha^q_k) \cap \alpha^p_k$ by $3_{r,q}$. Now $f^r_k(\alpha^q_k) \in A^q_k$ by $3_{r,q}$, and so $f^r_k(\alpha^q_k) \cap \alpha^p_k \in A^p_k$ by $5_{q,p}$, hence $f^r_k(\alpha^p_k) \in A^p_k$. Since we will use this information again, we summarise it:

$$(*)\ f^r_k(\alpha^p_k) = f^r_k(\alpha^q_k) \cap \alpha^p_k \text{ whenever } \ell(q) \leq k < \ell(r).$$

  Moreover if $\eta \in a^p_k$ then $\eta, \alpha^p_k \in a^q_k$ by $5_{q,p}$, $f^r_k(\eta) = f^r_k(\alpha^q_k) \cap \eta$ and $f^r_k(\alpha^p_k) = f^r_k(\alpha^q_k) \cap \alpha^p_k$ by $3_{r,q}$, so $f^r_k(\eta) = f^r_k(\alpha^p_k) \cap \eta$.

To verify $4_{r,p}$, first note that $(f^q_k(\alpha^p_k))_{\ell(p) \leq k < \ell(q)}$ is $\subseteq$-increasing with k by $4_{q,p}$, and also that $(f^r_k(\alpha^q_k))_{\ell(q) \leq k < \ell(r)}$ is $\subseteq$-increasing by $4_{r,q}$. We now distinguish three cases:

- For $\ell(p) \leq k < \ell(q)$, by $2_{r,q}$, we have $f^q_k(\alpha^p_k) = f^r_k(\alpha^p_k)$ so that the sequence $(f^r_k(\alpha^p_k))_{\ell(p) \leq k < \ell(q)}$ is $\subseteq$-increasing.
- For $\ell(q) \leq k_0 < k_1 < \ell(r)$, we have that $f^r_{k_0}(\alpha^q_{k_0}) \subseteq f^r_{k_1}(\alpha^q_{k_1})$ and also (by Remark 2) that $\alpha^p_{k_0} \leq \alpha^p_{k_1}$, so using $(*)$ we get

$$f^q_{k_0}(\alpha^p_{k_0}) = f^r_{k_0}(\alpha^q_{k_0}) \cap \alpha^p_{k_0} \subseteq f^r_{k_1}(\alpha^q_{k_1}) \cap \alpha^p_{k_1} = f^q_{k_1}(\alpha^p_{k_1}).$$

  It follows that $(f^q_k(\alpha^p_k))_{\ell(q) \leq k < \ell(r)}$ is $\subseteq$-increasing.
- For $\ell(p) < \ell(q) < \ell(r)$, by $3_{r,q}$ we have $f^r_{\ell(q)}(\alpha^q_{\ell(q)}) \in A^q_{\ell(q)}$, so by $6_{q,p}$ we have $f^q_{\ell(q)-1}(\alpha^p_{\ell(q)-1}) \subseteq f^r_{\ell(q)}(\alpha^q_{\ell(q)}) \cap \alpha^p_{\ell(q)-1}$. By $(*)$ with $k = \ell(q)$ we have $f^r_{\ell(q)}(\alpha^p_{\ell(q)}) = f^r_{\ell(q)}(\alpha^q_{\ell(q)}) \cap \alpha^p_{\ell(q)}$, and we also have $f^q_{\ell(q)-1}(\alpha^p_{\ell(q)-1}) =$

$f^r_{\ell(q)-1}(\alpha^p_{\ell(q)-1})$ by $2_{r,q}$. Since $\alpha^p_{\ell(q)-1} \leq \alpha^p_{\ell(q)}$ by Remark 2, we see that

$$f^r_{\ell(q)-1}(\alpha^p_{\ell(q)-1}) = f^q_{\ell(q)-1}(\alpha^p_{\ell(q)-1}) \subseteq \\ \subseteq f^r_{\ell(q)}(\alpha^q_{\ell(q)}) \cap \alpha^p_{\ell(q)-1} \subseteq f^r_{\ell(q)}(\alpha^q_{\ell(q)}) \cap \alpha^p_{\ell(q)} = f^r_{\ell(q)}(\alpha^p_{\ell(q)}).$$

We thus conclude that $(f^r_k(\alpha^p_k))_{\ell(p)\leq k<\ell(r)}$ is $\subseteq$-increasing.

Finally, to verify $6_{r,p}$, suppose that $\ell(p) < \ell(r) \leq k$ and let $x \in A^r_k$. We distinguish two cases:

- $\ell(p) < \ell(q) = \ell(r)$. In this case, $x \cap \alpha^q_k \in A^q_k$ by $5_{r,q}$. By $6_{q,p}$ and the observation that $\alpha^p_{\ell(r)-1} \leq \alpha^p_{\ell(r)} \leq \alpha^q_{\ell(r)} \leq \alpha^q_k$, $x \cap \alpha^p_{\ell(r)-1} \supseteq f^q_{\ell(r)-1}(\alpha^p_{\ell(r)-1})$. Finally $f^q_{\ell(r)-1}(\alpha^p_{\ell(r)-1}) = f^r_{\ell(r)-1}(\alpha^p_{\ell(r)-1})$ by $2_{r,q}$, and so $f^r_{\ell(r)-1}(\alpha^p_{\ell(r)-1}) \subseteq x \cap \alpha^p_{\ell(r)-1}$.
- $\ell(p) \leq \ell(q) < \ell(r)$. In this case, by $6_{r,q}$, $f^r_{\ell(r)-1}(\alpha^q_{\ell(r)-1}) \subseteq x \cap \alpha^q_{\ell(r)-1}$. The ordinals $\alpha^p_{\ell(r)-1}$ and $\alpha^q_{\ell(r)-1}$ lie in $a^q_{\ell(r)-1}$ by $5_{q,p}$. Now $f^r_{\ell(r)-1}(\alpha^p_{\ell(r)-1}) = f^r_{\ell(r)-1}(\alpha^q_{lh(r)-1}) \cap \alpha^p_{\ell(r)-1}$ by $3_{r,q}$, and so $f^r_{\ell(r)-1}(\alpha^p_{\ell(r)-1}) \subseteq x \cap \alpha^q_{\ell(r)-1} \cap \alpha^p_{\ell(r)-1} = x \cap \alpha^p_{r-1}$, using the fact that $\alpha^q_{r-1} \geq \alpha^p_{r-1}$ by Remark 3.

This completes the proof. □

It is easy to see that for each $m < \omega$ and each α with $\kappa \leq \alpha < \lambda$, the set of conditions q with $\ell(q) > m$ is dense, as is the set of q with $\alpha \in \mathrm{dom}(f^q_m)$. It follows that we may view $\mathbb{Q}$ as adding a matrix of sets $(z_{m,\alpha})_{m<\omega,\kappa\leq\alpha<\lambda}$ such that $z_{m,\alpha} \in P_{\kappa_m}\alpha$ for all m and α.

Lemma 4. *The forcing poset $\mathbb{Q}$ collapses all cardinals μ with $\kappa < \mu < \lambda$ and preserves all cardinals μ with $\mu > \lambda$.*

Proof. It is easy to see that $|\mathbb{Q}| = \lambda$, so that $\mathbb{Q}$ trivially has λ^+-cc and preserves cardinals greater than λ.

Given μ with $\kappa < \mu < \lambda$, we will do a density argument to show that μ is collapsed. Let $p \in \mathbb{Q}$ arbitrary, and extend p to obtain q so that there is $\alpha \geq \mu$ with $\alpha \in \bigcup_{\ell(q)\leq k<\omega} a^q_k$. We claim that $q \Vdash \mu \subseteq \bigcup_m \dot{z}_{m,\alpha}$; to see this let $\beta < \mu$ be arbitrary, choose $y \in A^q_{\alpha^q_{\ell(q)}}$ with $\beta \in y$, and extend q to a condition r such that $\ell(r) = \ell(q) + 1$ and $f^r_{\ell(q)}(\alpha^q_{\ell(q)}) = y$, so that $\beta \in f^r_{\ell(q)}(\alpha) = y \cap \alpha$. □

The argument that the cardinal λ and cardinals μ with $\mu \leq \kappa$ are preserved will require a deeper analysis of the forcing poset $\mathbb{Q}$, which will be given in subsection 3.4. The intuition is that the ω-sequences $(z_{m,\alpha})_{m<\omega}$ for differing values of α are related, so that no unwanted information can be computed by comparing them.

3.3. **Basic properties of $\mathbb{Q}$.** Let us consider a secondary ordering on $\mathbb{Q}$. Let $r \leq^* q$ (r is a *direct extension* of q) if and only if $r \leq q$ and $\ell(r) = \ell(q)$.

Lemma 5. *Let $\vec{q} = (q_i)_{i<\mu}$ be a $\leq^*$-decreasing sequence of conditions where $\mu < \kappa_{\ell(q_0)}$. Then the sequence $\vec{q}$ has a $\leq^*$-lower bound.*

Proof. We define a lower bound r with $\ell(r) = \ell(q_0)$ as follows:

- For all $k < \omega$, let $f^r_k = \bigcup_{i<\mu} f^{q_i}_k$.
- Choose some ordinal $\beta < \lambda$ such that $\bigcup_{i<\mu}(a^{q_i}_k \cup \mathrm{dom}(f^{q_i}_k)) \subseteq \beta$ for all $k \geq \ell(r)$, and define $a^r_k = \bigcup_{i<\mu} a^{q_i}_k \cup \{\beta\}$ for all such k, so that $\alpha^r_k = \beta$.
- For $k \geq \ell(r)$, set $A^r_k = \{x \in P_{\kappa_k}\beta : \forall i < \mu\ x \cap \alpha^{q_i}_k \in A^{q_i}_k\}$.

It is routine to check that r is a condition with $r \leq^* q_i$ for all i. Note that Clauses 3, 4 and 6 from the definition of the ordering $\leq$ are irrelevant here. □

We will also need a stronger version of $\leq^*$. For $i < \omega$, write $r \leq^*_i q$ if and only if $r \leq^* q$ and in addition $(a^r_k, A^r_k) = (a^q_k, A^q_k)$ whenever $\ell(q) \leq k < \ell(q) + i$. Note that $\leq^* = \leq^*_0$ and $\leq^*_i$ is transitive for all i.

Lemma 6 (Fusion)**.** *Let $(q_i)_{i<\omega}$ be a sequence of conditions such that $q_{i+1} \leq^*_i q_i$ for all $i < \omega$. Then there exists a condition q_∞ such that $q_\infty \leq^*_i q_i$ for all i.*

Proof. Define q_∞ with $\ell(q_\infty) = \ell(q_0)$ as follows:

- $f^{q_\infty}_k = \bigcup_{i<\omega} f^{q_i}_k$ for all $k < \omega$;
- $(a^{q_\infty}_k, A^{q_\infty}_k) = (a^{q_{k+1}}_k, A^{q_{k+1}}_k)$ for $k \geq \ell(q_0)$.

It is routine to verify that q_∞ is a condition and $q_\infty \leq^*_i q_i$ for all i. □

Remark 4. Of course Lemma 5 already implies that the decreasing sequence from Lemma 6 has a $\leq^*$-lower bound, the point here is that $q_\infty \leq^*_i q_i$ rather than just $q_\infty \leq^* q_i$.

Definition 1. If r and q are conditions with $r \leq q$, then $\mathrm{stem}(r, q)$ is the finite sequence $(f^r_i(\alpha^q_i))_{\ell(q)\leq i<\ell(r)}$.

Note that by $3_{r,q}$ and $4_{r,q}$, $\mathrm{stem}(r, q) \in \prod_{\ell(q)\leq i<\ell(r)} A^q_i$, and is $\subseteq$-increasing.

Definition 2. Let q be a condition, and let $s \in \prod_{\ell(q)\leq i<l} A^q_i$ be a $\subseteq$-increasing sequence for some $l \in (\ell(q), \omega)$. We define $q + s$ as the ω-sequence $(r_k)_{k<\omega}$ such that:

- For $k < \ell(q)$, $r_k = f^q_k$.
- For $\ell(q) \leq k < l$, r_k is the function with domain $\mathrm{dom}(f^q_k) \cup a^q_k$ such that $r_k(\eta) = f^q_k(\eta)$ for $\eta \in \mathrm{dom}(f^q_k)$ and $r_k(\eta) = s_k \cap \eta$ for $\eta \in a^q_k$.
- For $k \geq l$, $r_k = (f^q_k, a^q_k, B_k)$ where $B_k = \{x \in A^q_k : s_{l-1} \subseteq x\}$.

By convention we also define $q + \langle\rangle = q$.

We shall soon establish that $q+s$ is a condition, but let us first point out that $q+s$ may be obtained by adding in one entry of s at a time.

Lemma 7. *Let q and $s = (s_i)_{\ell(q)\leq i<l}$ be as in Definition 2. Let $n = l - \ell(q)$. Define a decreasing sequence of conditions $(r_i)_{i\leq n}$ as follows: $r_0 = q$, and then $r_{i+1} = r_i + \langle s_i\rangle$ for $i < n$. Then $r_n = q + s$.*

Proof. The proof is straightforward by induction on the length of s. □

Lemma 8. *Let q and s be as in Definition 2. Then $q + s$ is a condition extending q. Moreover $r \leq^* q + \mathrm{stem}(r, q)$ for all $r \leq q$.*

Proof. By definition $\ell(q + s) = \ell(q) + \ell(s)$, so $1_{q+s,q}$ holds. Clearly $f^{q+s}_k \supseteq f^q_k$ for all k, so $2_{q+s,q}$ holds.

By construction $f^{q+s}_k(\alpha^q_k) = s_k$ for $\ell(q) \leq k < \ell(q) + \ell(s)$, so $f^{q+s}(\alpha^q_k) \in A^q_k$ by our assumptions on s. Also by definition $f^{q+s}_k(\eta) = s_k \cap \eta = f^{q+s}_k(\alpha^q_k) \cap \eta$ for all such k, so that $3_{q+s,q}$ holds. The sequence $(f^{q+s}_k(\alpha^q_k))_{\ell(q)\leq k<\ell(q+s)}$ is equal to s, which is $\subseteq$-increasing by our assumptions on s, so $4_{q+s,q}$ holds.

We have $a^{q+s}_k = a^q_k$ and $A^{q+s}_k \subseteq A^q_k$ for all $k \geq \ell(q+s)$, so $5_{q+s,q}$ holds. Finally we have by definition that $A^{q+s}_k = \{x \in A^q_k : s_{\ell(q+s)-1} \subseteq x\}$ for $k \geq \ell(q+s)$, and $f^{q+s}_{\ell(q+s)-1}(\alpha^q_{\ell(q+s)-1}) = s_{\ell(q+s)-1}$, so $6_{q+s,q}$ holds.

Now suppose that $r \leq q$ and $s = \mathrm{stem}(r,q)$. If $\ell(r) = \ell(q)$ then $s = \langle\rangle$ and $r \leq^* q = q+s$, so we may as well assume that $\ell(r) > \ell(q)$. We have $\ell(r) = \ell(q+s)$, so $1_{r,q+s}$ holds. By the definition of $q+s$ and $3_{r,q}$, $f^r_k \supseteq f^{q+s}_k$ whenever $\ell(q) \leq k < \ell(r)$. For $k \geq \ell(r)$, $f^{q+s}_k = f^q_k$ and $f^r_k \supseteq f^q_k$ by $2_{r,q}$, so $2_{r,q+s}$ holds.

Since $\ell(r) = \ell(q+s)$, $3_{r,q+s}$, $4_{r,q+s}$ and $6_{r,q+s}$ hold vacuously. For $k \leq \ell(q+s)$ we have $a^{q+s}_k = a^q_k$ by definition, and also $a^q_k \subseteq a^r_k$ by $5_{r,q}$. Finally for all $k \geq \ell(r) = \ell(q+s)$ and all $x \in A^r_k$, $x \cap \alpha^{q+s}_k = x \cap \alpha^q_k \in A^q_k$, and additionally $s_{\ell(q+s)-1} \subseteq x \cap \alpha^q_{\ell(q+s)-1}$; it follows from the definition that $x \cap \alpha^q_{\ell(q+s)} \in A^{q+s}_{\ell(q+s)}$. □

Lemma 9. *Let $q \leq^*_n p$ and let $r \leq q$ with $\ell(r) = \ell(q) + n$. Then* $\mathrm{stem}(r,q) = \mathrm{stem}(r,p)$*, and* $q + \mathrm{stem}(r,q) \leq^* p + \mathrm{stem}(r,p)$.

Proof. Let $s = \mathrm{stem}(r,q)$, that is $s_k = f^r_k(\alpha^q_k)$ for $\ell(q) \leq k < \ell(r)$. Since $q \leq^*_n p$ we have that $\alpha^q_k = \alpha^p_k$ for such k, and so $s_k = f^r_k(\alpha^p_k)$ and hence $\mathrm{stem}(r,p) = s = \mathrm{stem}(r,q)$. If $n = 0$ then $q = q + s \leq^* p = p + s$, so we may assume that $n > 0$.

Now $\ell(q+s) = \ell(r) = \ell(p+s)$, so Clauses 1, 3, 4 and 6 are easily seen to hold.

For $k < \ell(p)$ or $k \geq \ell(r)$, we have $f^{q+s}_k = f^q_k \leq f^p_k = f^{p+s}_k$ by the definitions and $2_{q,p}$. For $\ell(p) \leq k < \ell(r)$, we have that $f^{q+s}_k \leq f^{p+s}_k$ because $f^q_k \leq f^p_k$ by $2_{q,p}$, and $f^{q+s}_k(\eta) = s_k \cap \eta = f^{p+s}_k(\eta)$ for every $\eta \in a^q_k = a^p_k$.

For $k \geq \ell(r)$ we have $a^{p+s}_k = a^p_k \subseteq a^q_k = a^{q+s}_k$ and $x \in A^q_k \implies x \cap \alpha^p_k \in A^p_k$ by $5_{q,p}$. If $x \in A^{q+s}_k$ then $x \in A^q_k$ and $x \supseteq s_{\ell(r)-1}$, so $x \cap \alpha^p_k \in A^p_k$ and also $x \cap \alpha^p_k \supseteq s_{\ell(r)-1}$ using the fact that $\alpha^p_k \geq \alpha^p_{\ell(r)-1}$. □

3.4. **Main technical lemma.** If $\dot{\tau}$ is a name for an object in the ground model and r is a condition, then we say that r *decides* $\dot{\tau}$ if and only if there is $x \in V$ such that $r \Vdash \dot{\tau} = \check{x}$. We write $r \parallel \dot{\tau}$ for "r decides $\dot{\tau}$".

We now state and prove the key technical lemma about $\mathbb{Q}$.

Lemma 10. *Let $p \in \mathbb{Q}$ and let $\dot{\tau}$ be a name for an element of V. Then there is a direct extension $q \leq^* p$ such that for every $r \leq q$ such that $r \parallel \dot{\tau}$, we have* $q + \mathrm{stem}(r,q) \parallel \dot{\tau}$.

Proof. We will build an ω-sequence of conditions $(p_n)_{n<\omega}$ such that:

- $p_0 = p$.
- $p_{n+1} \leq^*_n p_n$ for all n.
- For every $r \leq p_{n+1}$ with $\ell(r) = \ell(p)+n$, if $r \parallel \dot{\tau}$, then $p_{n+1}+\mathrm{stem}(r,p_{n+1}) \parallel \dot{\tau}$.

Before giving the construction we verify that building $(p_n)_{n<\omega}$ is sufficient. Appealing to Lemma 6, we find p_∞ such that $p_\infty \leq^*_n p_n$ for all n. Let $r \leq p_\infty$ with $r \parallel \dot{\tau}$ and $\ell(r) = \ell(p) + n$. Then $p_\infty \leq^*_n p_{n+1}$, and so $p_\infty + \mathrm{stem}(r,p_\infty) \leq^* p_{n+1}+\mathrm{stem}(r,p_{n+1})$ by Lemma 9. By construction $p_{n+1}+\mathrm{stem}(r,p_{n+1}) \parallel \dot{\tau}$, and so $p_\infty + \mathrm{stem}(r,p_\infty) \parallel \dot{\tau}$. It follows that p_∞ will serve as a witness to the conclusion.

To begin the construction we set $p_0 = p$, and then ask whether there is a direct extension of p which decides $\dot{\tau}$, and set p_1 equal to such an extension if one exists, otherwise $p_1 = p$. If there is $r \leq^* p_1$ with $r \parallel \dot{\tau}$ then $r \leq^* p$, and $p_1 = p_1 + \mathrm{stem}(r,p_1) \parallel \dot{\tau}$.

Suppose now that we have constructed p_n for some $n > 0$. Let $(s^n_j)_{j<\mu}$ be an enumeration of all sequences s such that $\ell(s) = n$ and $p_n + s$ is well defined, that is to say all $\subseteq$-increasing sequences in $\prod_{\ell(p)\le k<\ell(p)+n} A^{p^n}_k$. Since λ is inaccessible, $\mu < \lambda$.

Note to the reader. The objects f^j_k, a^j_k, α^j_k, A^j_k built during the construction also depend on n, but we have suppressed the index n to lighten the notation.

We will construct by recursion for each $j < \mu$:

- Functions f^j_k for $k < \omega$.
- Sets a^j_k and A^j_k for $\ell(p) + n \le k < \omega$.

We will maintain the following hypotheses:

- $a^{p_n}_{\ell(p)+n-1} \subseteq a^0_{\ell(p)+n}$.
- For all k and all $j < j'$, $f^{j'}_k \supseteq f^j_k$ and $a^j_k \subseteq a^{j'}_k$.
- For all k and j, $a^j_k \subseteq [\kappa, \lambda)$ with $|a^j_k| < \lambda$.
- For all j, the sequence $(a^j_k)_{\ell(p)+n\le k}$ is $\subseteq$-increasing with k.
- a^j_k has a maximum element α^j_k and $A^j_k \in U_{k,\alpha^j_k}$.
- For all k with $\ell(p) \le k < \ell(p) + n$ and all j, $\mathrm{dom}(f^j_k)$ is disjoint from $a^{p_n}_k$.
- For all k with $\ell(p) + n \le k$ and all j, $\mathrm{dom}(f^j_k)$ is disjoint from a^j_k.

To begin the construction we set $f^0_k = f^{p_n}_k$ for all k, and $(a^0_k, A^0_k) = (a^{p_n}_k, A^{p_n}_k)$ for $k \ge \ell(p) + n$.

Suppose that we have constructed f^j_k, a^j_k and A^j_k. We define various auxiliary conditions in $\mathbb{Q}$:

- p^j is the condition such that $p^j_k = f^j_k$ for $k < \ell(p)$, $p^j_k = (a^{p_n}_k, A^{p_n}_k, f^j_k)$ for $\ell(p) \le k < \ell(p) + n$, and $p^j_k = (a^j_k, P_{\kappa_k}\alpha^j_k, f^j_k)$ for $\ell(p) + n \le k$.
- $q^j = p^j + s^n_j$.
- If there is $r \le^* q_j$ such that $r \parallel \dot\tau$, then r^j is some such condition r, otherwise $r^j = q^j$.

We now define:

- ▶ $f^{j+1}_k = f^{r^j}_k$ for $k < \ell(p)$ or $k \ge \ell(p) + n$.
- ▶ $f^{j+1}_k = f^{r^j}_k \restriction \mathrm{dom}(f^{r^j}_k) \setminus a^{p_n}_k$ for $\ell(p) \le k < \ell(p) + n$.
- ▶ $(a^{j+1}_k, A^{j+1}_k) = (a^{r^j}_k, A^{r^j}_k)$ for $k \ge \ell(p) + n$.

For j limit we proceed roughly as in the proof of Lemma 5, with the important caveat that we do not aim to make the sets A^j_k decrease with j:

- ▶ $f^j_k = \bigcup_{i<j} f^i_k$ for all k.
- ▶ For $k \ge \ell(p) + n$:
  - We choose a^j_k as in the final stage of the proof of Lemma 5, that is to say we choose some large enough β^j and set $a^j_k = \bigcup_{i<j} a^i_k \cup \{\beta^j\}$ for all k, so that $\alpha^j_k = \beta^j$.
  - We set $A^j_k = P_{\kappa_k}\beta^j$.

After defining f^j_k, a^j_k and A^j_k for $j < \mu$, we are ready to define p_{n+1}.

- ▶ $f^{p_{n+1}}_k = \bigcup_{i<\mu} f^i_k$ for all k.
- ▶ $(a^{p_{n+1}}_k, A^{p_{n+1}}_k) = (a^{p_n}_k, A^{p_n}_k)$ for $\ell(p) \le k < \ell(p) + n$.
- ▶ For $k \ge \ell(p) + n$:

- As at the limit stages below μ, we choose some large enough β^* and set $a_k^{p_{n+1}} = \bigcup_{i<\mu} a_k^i \cup \{\beta^*\}$, so that $\alpha_k^{p_{n+1}} = \beta^*$ for all k.
- We define $A_k^{p_{n+1}}$ by diagonal intersection, as the set of those $x \in P_{\kappa_k}\beta^*$ such that $x \cap \alpha_k^{p_n} \in A_k^{p_n}$ and, for all $i < \mu$ if $s_i^n \Subset x$ then $x \cap \alpha_k^{i+1} \in A_k^{i+1}$.

Note that $\{x \in P_{\kappa_k}\beta^* : x \cap \alpha_k^{i+1} \in A_k^{i+1}\} \in U_{k,\beta^*}$, because $A_k^{i+1} \in U_{k,\alpha_k^{i+1}}$ and this is the projection of U_{k,β^*} under the map $x \mapsto x \cap \alpha_k^{i+1}$. So $A_k^{p_{n+1}} \in U_{k,\beta^*}$ by Lemma 1.

It is routine to verify that p_{n+1} is a condition and that $p_{n+1} \leq_n^* p_n$. To finish the proof we must verify that for every $r \leq p_{n+1}$ such that $\ell(r) = \ell(p) + n$ and $r \parallel \dot\tau$, $p_{n+1} + \mathrm{stem}(r, p_{n+1}) \parallel \dot\tau$. Let $s = \mathrm{stem}(r, p_{n+1})$, and pick some $j < \mu$ such that $s = s_j^n$.

We claim that $p_{n+1} \leq_n^* p^j$. This is a routine verification: note that $A_k^{p_j} = P_{\kappa_k}\alpha_k^j$ for $\ell(p) + n \leq k$, so that there is no problem with Clause 5 for $p_{n+1} \leq p^j$.

By Lemma 8 we have $r \leq^* p_{n+1}+s$, and by Lemma 9 we have that $\mathrm{stem}(r, p^j) = s$ and $p_{n+1} + s \leq^* p^j + s = q^j$. So $r \leq^* q^j$ and $r \parallel \dot\tau$, hence we chose $r^j \leq^* q^j$ such that $r^j \parallel \dot\tau$ and used r^j in the definitions of f_k^{j+1}, a_k^{j+1}, and A_k^{j+1}.

We claim that $p_{n+1} + s \leq r^j$. The verification is fairly straightforward, the main point is to check the second part of Clause 5. So let $k \geq \ell(p) + n$, and recall that by definition $A_k^{p_{n+1}+s}$ is the set of $x \in A_k^{p_{n+1}}$ such that $s \subseteq x$. By the construction of $A_k^{p_{n+1}}$ as a diagonal intersection, for every $x \in A_k^{p_{n+1}}$ such that $s \Subset x$ we have that $x \cap \alpha_k^{j+1} \in A_k^{j+1}$. Since $A_k^{j+1} = A_k^{r_j}$, this is exactly what is needed for Clause 5. $\square$

3.5. **Prikry lemma.**

Lemma 11 (Prikry lemma). *For every condition p and every sentence ϕ in the forcing language there is $q \leq^* p$ such that $q \parallel \phi$.*

Proof. Let $\dot\tau$ name an ordinal which is 0 if ϕ is true and 1 if ϕ is false. Appealing to Lemma 10 we find $q_0 \leq^* p$ which is such that for all $r \leq q_0$, if r decides ϕ then $q_0 + \mathrm{stem}(r, q)$ decides ϕ.

Let S be the set of sequences s such that $q_0 + s$ is well-defined, and for each $s \in S$ define $F(s)$ as follows:

$$F(s) = \begin{cases} 0, & \text{if } q_0 + s \Vdash \phi \\ 1, & \text{if } q_0 + s \Vdash \neg\phi \\ 2, & \text{otherwise.} \end{cases}$$

For each $t \in S$ we may find a measure one set $B_t \subseteq A_{\ell(q_0)+\ell(t)}^{q_0}$ and $G(t)$ such that $F(t^\frown\langle x\rangle) = G(t)$ for all $x \in B_t$.

Now for each $k \geq \ell(q_0)$, let C_k be the diagonal intersection of B_t for all sequences t of length $k - \ell(q_0)$, that is the set of $x \in A_k^{q_0}$ such that $x \in B_t$ for all $t \Subset x$. Let q_1 be obtained from q_0 by replacing $A_k^{q_0}$ by C_k for all $k \geq \ell(q_0)$, and note that $\alpha_k^{q_1} = \alpha_k^{q_0}$ for all $k \geq \ell(q_0) = \ell(q_1)$. Let $q_2 \leq q_1$ be a condition deciding ϕ with $\ell(q_2)$ chosen minimal.

We claim that $\ell(q_2) = \ell(q_1)$, so that $q_2 \leq^* p$ and we are done. Suppose for a contradiction that $\ell(q_2) > \ell(q_1)$, and let $s = \mathrm{stem}(q_2, q_1)$. By construction $q_1+s \leq^*$

$q_0 + s$ and $q_0 + s$ decides ϕ, so $q_1 + s$ decides ϕ. We will assume that it forces ϕ; the argument for the case when $q_1 + s$ forces $\neg\phi$ is identical.

Let $x = s(\ell(s) - 1)$ and let $t = s \restriction \ell(s) - 1$, so that $s = t^\frown\langle x\rangle$. By construction $x \in B_t$, and so by definition $G(t) = F(s) = 1$. By Lemma 8 it is easy to see that every extension of $q_1 + t$ is compatible with $q_1 + t^\frown\langle y\rangle$ for some $y \in B_t$, and since $G(t) = 1$ we have that $q_1 + t^\frown\langle y\rangle \Vdash \phi$ for all $y \in B_t$, so that $q_1 + t$ forces ϕ. This contradicts the minimal choice of $\ell(q_2)$. □

Corollary 1. *The forcing poset $\mathbb{Q}$ adds no bounded subsets of κ.*

Proof. Let $p \Vdash \dot{x} \subseteq \mu$ with $\mu < \kappa$, and find $q \leq p$ such that $\ell(q) = k$ for some k large enough that $\kappa_k > \mu$. Appealing to Lemmas 11 and 5 we may build a $\leq^*$-decreasing sequence $(q_i)_{i<\mu}$ such that $q_0 = q$ and q_{i+1} decides whether $i \in \dot{x}$ for all i. By Lemma 5 again there is a condition r such that $r \leq q_i$ for all i, and $r \Vdash \dot{x} = \check{y}$ for some $y \in V$. □

3.6. **Projected forcing.** Let $k < \omega$ and let $\vec{\alpha} = (\alpha_i)_{k\leq i<\omega}$ be a $\leq$-increasing sequence from $[\kappa, \lambda)$. We define a forcing poset $\mathbb{Q}_{\vec{\alpha}}$. Conditions in $\mathbb{Q}_{\vec{\alpha}}$ are sequences $(q_i)_{k\leq i<\omega}$ such that some $l \geq k$: $q_i \in P_{\kappa_i}\alpha_i$ for $i < l$ and $q_i \in U_{i,\alpha_i}$ for $i \geq l$, and $(q_i)_{k\leq i<l}$ is $\subseteq$-increasing. We say that l is the *length* of q and write $l = \ell(q)$. If $r, q \in \mathbb{Q}_{\vec{\alpha}}$ then $r \leq q$ if and only if:

- $\ell(r) \geq \ell(q)$.
- $r_i = q_i$ for $k \leq i < \ell(q)$.
- $r_i \in A^q_i$ for $\ell(q) \leq i < \ell(r)$.
- $A^r_i \subseteq A^q_i$ for $\ell(r) \leq i < \omega$.

We note that $\mathbb{Q}_{\vec{\alpha}}$ has the λ-cc, because there are fewer than λ possible "stems" $(q_i)_{k\leq i<\ell(q)}$ and any two conditions with the same stem are compatible. It is also not hard to prove that $\mathbb{Q}_{\vec{\alpha}}$ obeys a version of the Prikry Lemma, but we will not need this (it actually follows from the next lemma). It is useful to note that the generic object added by $\mathbb{Q}_{\vec{\alpha}}$ is a $\subseteq$-increasing sequence $(z_i)_{k\leq i<\omega}$ such that $z_i \in P_{\kappa_i}\alpha_i$.

Lemma 12. *Let $q \in \mathbb{Q}$, and let $\vec{\alpha} = (\alpha_i)_{\ell(q)\leq i<\omega}$ be a $\leq$-increasing sequence such that $\alpha_i \in a^q_i$ for all i. Let $\pi : \mathbb{Q} \downarrow q \to \mathbb{Q}_{\vec{\alpha}}$ be the map defined by:*

- *$\pi(r)_i = f^r_i(\alpha_i)$ for $\ell(q) \leq i < \ell(r)$.*
- *$\pi(r)_i = \{x \cap \alpha_i : x \in A^r_i\}$ for $i \geq \ell(r)$.*

Then π is order-preserving and has the following property: for every $r \leq q$ there is $r^* \leq r$ such that for all $q_0 \leq \pi(r^*)$ there is $r' \leq r$ with $\pi(r') \leq q_0$.

Proof. It is routine to check that π is order-preserving. To verify the rest of the conclusion let $r \leq q$. We obtain a condition $r^* \leq^* r$ by shrinking measure one sets in r as follows:

- $\ell(r^*) = \ell(r)$.
- $f^{r^*}_i = f^r_i$ for all i.
- $a^{r^*}_i = a^r_i$ for all $i \geq \ell(r)$.
- We define $A^{r^*}_k$ for $k \geq \ell(r)$ by recursion on k, arranging that $A^{r^*}_k \subseteq A^r_k$ and $A^{r^*}_k \in U_{k,\alpha^r_k}$.

We start by setting $A^{r^*}_{\ell(r)} = A^r_{\ell(r)}$. For $i > 0$ we set $A^{r^*}_{\ell(r)+i}$ to be the set of $y \in A^r_{\ell(r)+i}$ such that for all $x \in A^{r^*}_{\ell(r)+i-1}$ with $x \cap \alpha_i \subseteq y \cap \alpha_{i+1}$ there is $x' \in A^{r^*}_{\ell(r)+i-1}$ with $x' \cap \alpha_i = x \cap \alpha_i$ and $x' \subseteq y$. By Lemma 2 we have $A^{r^*}_{\ell(r)+i} \in U_{\ell(r)+i,\beta_{i+1}}$.

Let $q_0 \leq \pi(r^*)$, and choose $y_k \in B^{r^*_k}$ such that $(q_0)_k = y_k \cap \alpha_k$ for $\ell(r) \leq k < \ell(q_0)$. Using the definition of $B^{r^*}_k$ we will choose $y'_{\ell(q_0)-i}$ for $0 < i \leq \ell(r)$ by induction on i as follows: $y'_{\ell(q_0)-1} = y_{\ell(q_0)-1}$, and $y'_{\ell(q_0)-(i+1)} \in B^{r^*}_{\ell(q_0)-(i+1)}$ with $y'_{\ell(q_0)-(i+1)} \subseteq\in y_{\ell(q_0)-i}$ and $y'_{\ell(q_0)-(i+1)} \cap \alpha_{\ell(q_0)-(i+1)} = y_{\ell(q_0)-(i+1)} \cap \alpha_{\ell(q_0)-(i+1)} = (q_0)_{\ell(q_0)-(i+1)}$.

Let $t = (y'_k)_{\ell(r) \leq k < \ell(q_0)}$, and form $r + t$. By shrinking measure one sets appropriately we obtain a condition $r' \leq r + t \leq r$ such that $\pi(r') \leq^* q_0$. □

The conclusion of Lemma 12 is a weakening of the standard property of being a projection between forcing posets, and was first isolated by Foreman and Woodin [1]. The following corollary of Lemma 12 is routine:

Corollary 2. *With the same hypotheses as in Lemma 12, if G is $\mathbb{Q}$-generic with $q \in G$ then $\pi[G]$ generates a $\mathbb{Q}_{\vec{\alpha}}$-generic filter.*

Remark 5. It is possible to give an alternative argument for Corollary 2 by first proving a characterisation of $\mathbb{Q}_{\vec{\alpha}}$-generic sequences in the style of Mathias' theorem on Prikry forcing [5]. This characterisation, which we will not prove, states that an increasing sequence $(z_i)_{k \leq i < \omega}$ is generic if and only if for every sequence $(A_i)_{k \leq i < \omega}$ in V with $A_i \in U_{i,\alpha_i}$ for all i, we have $z_i \in A_i$ for all large i. It is straightforward to verify that the induced filter $G_{\vec{\alpha}}$ corresponds to a sequence $(z_i)_{k \leq i < \omega}$ with this property.

Lemma 13. *Let G be $\mathbb{Q}$-generic and let $x \subseteq \kappa$ with $x \in V[G]$. Then there exists $\vec{\alpha}$ such that $x \in V[G_{\vec{\alpha}}]$.*

Proof. Note that by Corollary 1 we have $x \cap \kappa_n \in V$ for all $n < \omega$. Let $p \in \mathbb{Q}$ and let $\dot{\tau}_n$ be a name for $x \cap \kappa_n$. Appealing to Lemmas 10 and 5 we may find $q \leq^* p$ such that for all $r \leq q$, if r decides $\dot{\tau}_n$ then then $q + \operatorname{stem}(r, q)$ decides τ_n. Now let $\vec{\alpha} = (\alpha^q_k)_{\ell(q) \leq k < \omega}$. It is routine to verify that q forces that x can be computed from $G_{\vec{\alpha}}$. Explicitly, if $q \in G$ and $(z_i)_{k \leq i < \omega}$ is the generic sequence added by $G_{\vec{\alpha}}$ then $\beta \in x \iff \exists \bar{k} \geq k \; q + (z_i)_{k \leq i < \bar{k}} \Vdash \beta \in \dot{x}$. □

Lemma 14. *Let G be $\mathbb{Q}$-generic, then $\lambda = (\kappa^+)^{V[G]}$.*

Proof. Suppose for a contradiction that λ is not a cardinal in $V[G]$. By Lemma 4, cardinals μ with $\kappa < \mu < \lambda$ are all collapsed in $V[G]$, so that if λ is also collapsed then there is a set $x \subseteq \kappa$ in $V[G]$ coding a well-ordering of κ with order type λ. By Lemma 13 $x \in V[G_{\vec{\alpha}}]$ for some $\vec{\alpha}$, which gives an immediate contradiction since $\mathbb{Q}_{\vec{\alpha}}$ is λ-cc. □

3.7. **Definability.** Let G be $\mathbb{Q}$-generic. The last step in the proof of our main result is to analyze the class of sets HOD_x where $x \subseteq \kappa$ with $x \in V[G]$. This will require a rather technical result which appears below as Lemma 16. In order to motivate the statement of Lemma 16, we will state a lemma which contains the necessary information about HOD_x and prove it modulo one missing fact, which will then be provided by Lemma 16.

Lemma 15. *Let $x \subseteq \kappa$ with $x \in V[G]$, then there exists $\vec{\alpha}$ such that $(\mathrm{HOD}_x)^{V[G]} \subseteq V[G_{\vec{\alpha}}]$. In particular $(\kappa^+)^{(\mathrm{HOD}_x)^{V[G]}} < \kappa^+$.*

Proof of Lemma 15, Part One. By Lemma 13 find $q \in G$, $\vec{\alpha}$ with $\alpha_k \in a^q_k$ for all k and $\dot{x}$ a $\mathbb{Q}_{\vec{\alpha}}$-name such that $i_{G_{\vec{\alpha}}}(\dot{x}) = x$. Since $\mathrm{HOD}^{V[G]}_x$ is a model of ZFC, to show that $\mathrm{HOD}_x^{V[G]} \subseteq V[G_{\vec\alpha}]$ it will be sufficient to show that every set of ordinals $y \in \mathrm{HOD}_x^{V[G]}$ lies in $V[G_{\vec\alpha}]$.

Let y be such a set and fix a definition of y from x and some ordinal parameter δ, say

$$Y = \{\gamma : V[G] \models \psi(\gamma, \delta, x)\}.$$

In order to define y in $V[G_{\vec\alpha}]$, we define an auxiliary set $H \subseteq \mathbb{Q}$. Let $k = \ell(q)$ and let $(z_i)_{k \le i < \omega}$ be the generic sequence corresponding to the filter $G_{\vec\alpha}$. We define H to be the set of conditions $r \in \mathbb{Q}$ such that $r \le q$, and for every $\bar k \ge k$ there exists $r_1 \le r$ such that $f_i^{r_1}(\alpha_i) = z_i$ for $k \le i < \bar k$. Clearly $G \subseteq H$ and $H \in V[G_{\vec\alpha}]$.

We claim that $Y = \{\gamma : \exists r \in H \ r \Vdash \psi(\gamma, \delta, \dot x)\}$. If $\bar Y$ is the set defined on the right hand side of this equation then clearly $Y \subseteq \bar Y$ and $\bar Y \in V[G_\alpha]$, so we need only show that $\bar Y \subseteq Y$. Suppose for a contradiction that this is not the case, fix $\gamma \in \bar Y \setminus Y$, and then choose conditions $r \in H$ and $s \in G$ such that $r \Vdash \psi(\gamma, \delta, \dot x)$ and $s \Vdash \neg\psi(\gamma, \delta, \dot x)$.

We may extend s to find $s^* \le s, q$ such that $s^* \in G$ and $\ell(s^*) \ge \ell(r)$, and then use the fact that $r \in H$ to find $r^* \le r$ (not necessarily lying in H) such that $\ell(r^*) = \ell(s^*)$ and $f_i^{r^*}(\alpha_i) = z_i = f_i^{s^*}(\alpha_i)$ for all i with $\ell(q) = k \le i < \ell(r^*) = \ell(s^*)$. At this point we will be done if we can show that r^* and s^* can not force contradictory information about $\psi(\gamma, \delta, \dot x)$, and this is exactly the point of Lemma 16. Accordingly we interrupt the proof of Lemma 15 to state and prove Lemma 16.

Lemma 16. *Let $q \in \mathbb{Q}$, and let $\vec\alpha = (\alpha_i)_{\ell(q) \le i < \omega}$ be a $\le$-increasing sequence such that $\alpha_i \in a_i^q$ for all i. Let r^* and s^* be conditions such that $r^*, s^* \le q$, $\ell(r^*) = \ell(s^*)$, and $f_k^{r^*}(\alpha_k^q) = f_k^{s^*}(\alpha_k^q)$ for $\ell(q) \le k < \ell(r^*)$. Then there exist conditions $r' \le^* r^*$ and $s' \le^* s^*$ and $\rho : \mathbb{Q} \downarrow r' \to \mathbb{Q} \downarrow s'$ such that:*

- *ρ is an isomorphism.*
- *If G_0 is $\mathbb{Q}$-generic with $r' \in G_0$, and G_1 is the $\mathbb{Q}$-generic filter generated by $\rho[G_0 \cap \mathbb{Q} \downarrow r']$, then $G_{0,\vec\alpha} = G_{1,\vec\alpha}$.*

Proof. We choose direct extensions $r' \le^* r$ and $s' \le^* s$ such that:

- ▶ $\mathrm{dom}(f_k^{r'}) = \mathrm{dom}(f_k^{s'})$ for $k < \ell(r)$.
- ▶ $\mathrm{dom}(f_k^{r'}) \cup a_k^{r'} = \mathrm{dom}(f_k^{s'}) \cup a_k^{s'}$, $\alpha_k^{r'} = \alpha_k^{s'}$, $A_k^{r'} = A_k^{s'}$ for $k \ge \ell(r)$.

To help readability let β_k be the common value of $\alpha_k^{r'}$ and $\alpha_k^{s'}$, let B_k be the common value of $A_k^{r'}$ and $A_k^{s'}$, and let d_k be the common value of $\mathrm{dom}(f_k^{r'})$ and $\mathrm{dom}(f_k^{s'})$ for $k < \ell(r)$ and the common value of $\mathrm{dom}(f_k^{r'}) \cup a_k^{r'}$ and $\mathrm{dom}(f_k^{s'}) \cup a_k^{s'}$ for $\ell(r) \le k$.

We will now define maps $\rho : \mathbb{Q} \downarrow r' \to \mathbb{Q} \downarrow s'$ and $\sigma : \mathbb{Q} \downarrow s' \to \mathbb{Q} \downarrow r'$ with $\rho(r') = s'$, and argue that they are order-preserving and mutually inverse. The reader may find the following slogan useful: "ρ alters r''_k on d_k to make it look like the k-level of an extension s'' of s' with $\mathrm{stem}(s'', s') = \mathrm{stem}(r'', r')$".

Let $r'' \le r'$. We define $\rho(r'')$ to be the condition s'' such that:

- ▶ For $k < \ell(r')$, $\mathrm{dom}(f_k^{s''}) = \mathrm{dom}(f_k^{r''})$, $f_k^{s''} \restriction \mathrm{dom}(f_k^{s'}) = f_k^{s'}$, $f_k^{s''} \restriction \mathrm{dom}(f_k^{r''}) \setminus \mathrm{dom}(f_k^{s'}) = f_k^{r''} \restriction \mathrm{dom}(f_k^{r''}) \setminus \mathrm{dom}(f_k^{s'})$.
- ▶ For $\ell(r') \le k < \ell(r'')$, $\mathrm{dom}(f_k^{s''}) = \mathrm{dom}(f_k^{r''})$, $f_k^{s''} \restriction \mathrm{dom}(f_k^{s'}) = f_k^{s'}$, $f_k^{s''}(\eta) = f_k^{r''}(\beta_k) \cap \eta$ for $\eta \in a_k^{s'}$, $f_k^{s''} \restriction \mathrm{dom}(f_k^{r''}) \setminus d_k = f_k^{r''} \restriction \mathrm{dom}(f_k^{r''}) \setminus d_k$.
- ▶ For $\ell(r'') \le k$:

- $\mathrm{dom}(f_k^{s''}) = \mathrm{dom}(f_k^{s'}) \cup (\mathrm{dom}(f_k^{r''}) \setminus d_k)$, $f_k^{s''} \restriction \mathrm{dom}(f_k^{s'}) = f_k^{s'}$ and $f_k^{s''} \restriction (\mathrm{dom}(f_k^{r''}) \setminus d_k) = f^{r''} \restriction (\mathrm{dom}(f_k^{r''}) \setminus d_k)$.
- $a_k^{s''} = a_k^{s'} \cup (a_k^{r''} \setminus d_k)$.
- $A_k^{s''} = A_k^{r''}$.

$\sigma(s'')$ for $s'' \leq s'$ is defined in an exactly similar way with the roles of r', r'' and s', s'' reversed. Note in particular that:

- For $\ell(r') \leq k < \ell(r'')$, $\beta_k = \alpha_k^{r'} = \alpha_k^{s'} \in a_k^{r'} \cap a_k^{s'}$, and $f_k^{\alpha(r'')}(\beta_k) = f_k^{r''}(\beta_k)$.
- The definitions of ρ and σ are "level by level" in the sense that $\rho(r'')_k$ (resp $\sigma(s'')_k$) depends only on r''_k (resp s''_k).
- On each level k of r'' (resp s'') ρ (resp σ) only alters $a_k^{r''}$ and $f_k^{r''}$ (resp $a_k^{s''}$ and $f_k^{s''}$) inside the set d_k.

It is very easy to verify that $\rho(r') = s'$. We claim that ρ and σ are order-preserving, by symmetry we only need verify this for ρ. From Lemmas 8 and 7 we see that it is enough to verify that ρ is order-preserving for extensions of the form $r'' + \langle x \rangle \leq r''$ and $r''' \leq^* r''$.

Claim 16.1. *Let $r'' \leq r'$ and let x be such that $r'' + \langle x \rangle$ is defined. then $\rho(r'') + \langle x \rangle$ is well-defined and $\rho(r'' + \langle x \rangle) = \rho(r'') + \langle x \rangle$.*

Proof. We note that $A_k^{\rho(r'')} = A_k^{r''}$ for all $k \geq \ell(r'')$, so that $\rho(r'') + \langle x \rangle$ is well-defined. Now we do a case analysis:

- ► $k < \ell(r')$: $f_k^{\rho(r'')}$ is obtained by altering the values of $f_k^{r''}$ on $\mathrm{dom}(f_k^{s'})$ to agree with $f_k^{s'}$, and by definition $f_k^{\rho(r'')+\langle x \rangle} = f_k^{\rho(r'')}$. Also $f_k^{r''+\langle x \rangle} = f_k^{r''}$ and then $f_k^{\rho(r''+\langle x \rangle)}$ is obtained by altering its values on $\mathrm{dom}(f_k^{s'})$ to agree with $f_k^{s'}$, so easily $f_k^{\rho(r''+\langle x \rangle)} = f_k^{\rho(r'')+\langle x \rangle}$.
- ► $\ell(r') \leq k < \ell(r'')$: Note that $d_k \subseteq \mathrm{dom}(f_k^{r''})$. By definition, $f_k^{\rho(r'')}$ is obtained by altering the values of $f_k^{r''}$ on d_k, so as to agree with $f_k^{s'}$ on $\mathrm{dom}(f_k^{s'})$ and to agree with $\eta \mapsto f_k^{r''}(\beta_k) \cap \eta$ on $a_k^{s'}$. By definition $f_k^{\rho(r'')+\langle x \rangle} = f_k^{\rho(r'')}$. Also $f_k^{r''+\langle x \rangle} = f_k^{r''}$, and $f_k^{\rho(r''+\langle x \rangle)}$ is obtained from it by the scheme of alteration described above, so $f_k^{\rho(r''+\langle x \rangle)} = f_k^{\rho(r'')+\langle x \rangle}$.
- ► $k = \ell(r'')$: This is the most complicated case. Start by recalling that $d_k = \mathrm{dom}(f_k^{r'}) \cup a_k^{r'} = \mathrm{dom}(f_k^{s'}) \cup a_k^{s'} \subseteq \mathrm{dom}(f_k^{r''}) \cup a_k^{r''} = \mathrm{dom}(f_k^{r''+\langle x \rangle})$, and also that $r'' \leq r'$.

  By definition $f^{r''+\langle x \rangle}(\eta) = f_k^{r''}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{r''})$ and $f^{r''+\langle x \rangle}(\eta) = x \cap \eta$ for $\eta \in \mathrm{dom}(a_k^{r''})$. When we alter values to obtain $f_k^{\rho(r''+\langle x \rangle)}$, we obtain:
  - $f^{\rho(r''+\langle x \rangle)}(\eta) = f_k^{r''}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{r''}) \setminus d_k$.
  - $f^{\rho(r''+\langle x \rangle)}(\eta) = x \cap \eta$ for $\eta \in a_k^{r''} \setminus d_k$.
  - $f^{\rho(r''+\langle x \rangle)}(\eta) = f_k^{s'}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{s'})$.
  - $f^{\rho(r''+\langle x \rangle)}(\eta) = f_k^{r''+\langle x \rangle}(\beta_k) \cap \eta$ for $\eta \in a_k^{s'}$.

  In the last of these cases we note that $r'' \leq r'$ and $\ell(r'') = k$, so that $\beta_k = \alpha_k^{r'} \in a_k^{r''}$ and so $f_k^{r''+\langle x \rangle}(\beta_k) = x \cap \beta_k$. It follows that $f^{\rho(r''+\langle x \rangle)}(\eta) = x \cap \eta$ for $\eta \in a_k^{s'}$, so we conclude that:
  - $f^{\rho(r''+\langle x \rangle)}(\eta) = f_k^{r''}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{r''}) \setminus d_k$.
  - $f^{\rho(r''+\langle x \rangle)}(\eta) = x \cap \eta$ for $\eta \in a_k^{s'} \cup (a_k^{r''} \setminus d_k)$.

- $f^{\rho(r''+\langle x\rangle)}(\eta) = f_k^{s'}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{s'})$.

By definition again we have $a_k^{\rho(r'')} = a_k^{s'} \cup (a_k^{r''} \setminus d_k)$, $\mathrm{dom}(f_k^{\rho(r'')}) = \mathrm{dom}(f_k^{s'}) \cup (\mathrm{dom}(f_k^{r''}) \setminus d_k)$ and:

- $f_k^{\rho(r'')}(\eta) = f_k^{r''}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{r''}) \setminus d_k$.
- $f_k^{\rho(r'')}(\eta) = f_k^{s'}(\eta)$ for $\eta \in \mathrm{dom}(f_k^{s'}) \setminus d_k$.

Since $f_k^{\rho(r'')+\langle x\rangle} \supseteq f_k^{\rho(r'')}$ and $f_k^{\rho(r'')+\langle x\rangle}(\eta) = x \cap \eta$ for $\eta \in a_k^{\rho(r'')}$, we see that $f_k^{\rho(r''+\langle x\rangle)} = f_k^{\rho(r'')+\langle x\rangle}$.

▶ $k > \ell(r'')$: By definition $a_k^{r''+\langle x\rangle} = a_k^{r''}$ and $f_k^{r''+\langle x\rangle} = f_k^{r''}$, so that by the definition of the operation ρ we have $a_k^{\rho(r''+\langle x\rangle)} = a_k^{\rho(r'')}$ and $f_k^{\rho(r''+\langle x\rangle)} = f_k^{\rho(r'')}$. Also $a_k^{\rho(r'')+\langle x\rangle} = a_k^{\rho(r'')}$ and $f_k^{\rho(r'')+\langle x\rangle} = f_k^{\rho(r'')}$, from which it follows that $a_k^{\rho(r'')+\langle x\rangle} = a_k^{\rho(r''+\langle x\rangle)}$ and $f_k^{\rho(r'')+\langle x\rangle} = f_k^{\rho(r''+\langle x\rangle)}$. We also have

$$A_k^{\rho(r''+\langle x\rangle)} = A_k^{r''+\langle x\rangle} = \{y \in A_k^{r''} : x \subseteq y\} = \{y \in A_k^{\rho(r'')} : x \subseteq y\} = A_k^{\rho(r'')+\langle x\rangle}.$$

□

Claim 16.2. *Let $r''' \le^* r'' \le r'$, then $\rho(r''') \le^* \rho(r'')$.*

Proof. For $k < \ell(r'')$ we have $d_k \subseteq \mathrm{dom}(f_k^{r''}) \subseteq f_k^{r'''}$, while for $k \ge lh(r'')$ we have $d_k \subseteq \mathrm{dom}(f_k^{r''}) \cup a^{r''_k} \subseteq \mathrm{dom}(f_k^{r'''}) \cup a^{r'''_k}$. The definition of $\rho(r'')_k$ involves altering the values of $f_k^{r''}$ (for $k < \ell(r'')$) or $f_k^{r''}$ and $a_k^{r''}$ (for $k \ge \ell(r'')$) in a uniform manner inside d_k, using only the values of $f_k^{r'}$ and (for $lh(r') \le k < \ell(r'')$) the values of $f_k^{r''}(\alpha_k^{r'})$, and similarly for $\rho(r''')$. Since $f_k^{r''}(\alpha_k^{r'}) = f_k^{r'''}(\alpha_k^{r'})$, it is now routine to verify that $\rho(r''') \le \rho(r'')$. □

Claim 16.3. *ρ and σ are mutually inverse.*

Proof. Let $\rho(r'') = s''$. To verify that $\sigma(s'') = r''$, the key points are that $r'' \le r'$ and that for $lh(r') \le k < \ell(r'')$ we have $f_k^{s''}(\beta_k) = f_k^{r''}(\beta_k)$. It follows that for such values of k and for all $\eta \in \mathrm{dom}(f_k^{r'}) \cap a_k^{s'}$ we have $f_k^{\sigma(s'')}(\eta) = f_k^{s''}(\beta_k) \cap \eta = f_k^{r''}(\beta_k) \cap \eta = f_k^{r''}(\eta)$, which is the only difficult step in the proof that $\sigma(s'') = r''$. □

Claim 16.4. *If G_0 is $\mathbb{Q}$-generic with $r' \in G_0$, and G_1 is the $\mathbb{Q}$-generic filter generated by $\rho[G_0 \cap \mathbb{Q} \downarrow r']$, then $G_{0,\vec{\alpha}} = G_{1,\vec{\alpha}}$.*

Proof. Let $r'' \le r'$ and recall that $r', s' \le q$. we claim that $f_k^{\rho(r'')}(\alpha_k) = f_k^{r''}(\alpha_k)$ for all k with $\ell(q) \le k < \ell(r'')$.

▶ If $k < \ell(r')$ then $f_k^{\rho(r'')}(\alpha_k) = f_k^{s'}(\alpha_k) = f_k^{r'}(\alpha_k) = f_k^{r''}(\alpha_k)$.

▶ If $\ell(r') \le k < \ell(r'')$ then since $\alpha_k \in a_k^q \subseteq a_k^{s'}$ and $r'' \le r'$, we have $f_k^{\rho(r'')}(\alpha_k) = f_k^{r''}(\beta_k) \cap \alpha_k = f_k^{r'}(\alpha_k)$.

□

This concludes the proof of Lemma 16. □

We are now ready to complete the proof of Lemma 15.

Proof of Lemma 15, Part Two. Recall that we need to derive a contradiction from the assumption that there are r^* and s^* satisfying the hypotheses of Lemma 16 such that $r^* \Vdash \psi(\gamma, \delta, \dot{x})$ and $s^* \Vdash \neg\psi(\gamma, \delta, \dot{x})$, where (crucially) $\dot{x}$ is a $\mathbb{Q}_{\vec{\alpha}}$-name.

Let us force with $\mathbb{Q}$ below r^* to obtain a generic object G_0 containing r^*, and then use ρ to obtain a generic object G_1 such that G_1 contains s^*. Since $\rho \in V$, $V[G_0] = V[G_1]$ and since $G_{0,\vec{\alpha}} = G_{1,\vec{\alpha}}$, $i_{G_0}(\dot{x}) = i_{G_1}(\dot{x}) = y$ say. This is an immediate contradiction. □

Combining the results we have obtained, we have proved the Main Theorem.

Theorem. *Suppose that $\kappa < \lambda$ where $\mathrm{cf}(\kappa) = \omega$, λ is inaccessible and κ is a limit of λ-supercompact cardinals. There is a forcing poset $\mathbb{Q}$ such that if G is $\mathbb{Q}$-generic then:*

- *The models V and $V[G]$ have the same bounded subsets of κ.*
- *Every infinite cardinal μ with $\mu \leq \kappa$ or $\mu \geq \lambda$ is preserved in $V[G]$.*
- $\lambda = (\kappa^+)^{V[G]}$.
- *For every $x \subseteq \kappa$ with $x \in V[G]$, $(\kappa^+)^{\mathrm{HOD}_x} < \lambda$.*

As we mentioned in the Introduction, we can strengthen the conclusion of the Main Theorem. We start with $(\kappa_n)_{n<\omega}$ and λ as before, but we now assume that the cardinals κ_n and λ are supercompact. By doing a suitable preparatory class forcing we may assume in addition that for every set-generic extension W of V, $V \subseteq \mathrm{HOD}^W$. The idea of the preparation, which is essentially due to McAloon [6], is that we will make the κ_n and λ Laver-indestructible, and then do λ-directed closed forcing above λ to arrange that every sets of ordinals is coded unboundedly often into the values of the continuum function.

We claim that if G is $\mathbb{Q}$-generic then in $V[G]$ we have that λ is supercompact in HOD_x for all $x \subseteq \kappa$. To see this find $\vec{\alpha}$ as before such that $\mathrm{HOD}_x^{V[G]} \subseteq V[G_{\vec{\alpha}}]$, so that we have a chain of inclusions

$$V \subseteq \mathrm{HOD}^{V[G]} \subseteq \mathrm{HOD}_x^{V[G]} \subseteq V[G_{\vec{\alpha}}].$$

By the Intermediate Models Theorem the model $\mathrm{HOD}_x^{V[G]}$ is a set-generic extension of V via some complete Boolean subalgebra $\mathbb{B}$ of $\mathrm{r.o.}(\mathbb{Q}_{\vec{\alpha}})$. Since $|\mathbb{B}| < \lambda$, it follows by the Levy-Solovay theorem that λ is supercompact in $\mathrm{HOD}_x^{V[G]}$.

Remark 6. Gitik [3] has informed us that the same effect should be possible using Merimovich's supercompact extender-based forcing [7]. This would weaken the hypotheses of the Main Theorem to one supercompact cardinal with an inaccessible cardinal above it.

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