POTENTIAL ISOMORPHISM OF ELEMENTARY SUBSTRUCTURES OF A STRICTLY STABLE HOMOGENEOUS MODEL

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Abstract. This results herein are part of a larger project to characterize the classification properties of the class of submodels of a homogeneous stable diagram in terms of the solvability of the potential isomorphism problem for this class of submodels.

We assume we work with a large strictly stable homogeneous monster model $\mathfrak{M}$, and that $(\pi = \mathrm{cf}(\pi) > \lambda_r(\mathfrak{M}))^{\mathbb{L}}$. Let ${}^{ca\mathscr{P}(\lambda_r(\mathfrak{M}))}PIP^{\mathfrak{M}}_{\pi}$ be the collection of pairs $(\mathscr{A}, \mathscr{B}) \in \mathbb{L}$ of locally $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-saturated elementary substructures of $\mathfrak{M}$ with universe π such that there is a cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{M}))$-preserving extension of $\mathbb{L}$ in which $\mathscr{A} \cong \mathscr{B}$. We show that ${}^{ca\mathscr{P}(\lambda_r(\mathfrak{M}))}PIP^{\mathfrak{M}}_{\pi}$ is equiconstructible with $0^{\#}$.

The proof uses a novel method that does away with the need for a linear order on the skeleton.

Contents

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§1. Introduction. The results we give here are part of a larger project to prove strong non-structure results for non-elementary classes. The original impetus comes from work to generalize the results of [2] to the Homogeneous Model Theory context. The main theorem of that earlier work was:

THEOREM ([2]). *Assume $0^{\#}$ exists, and let T be a constructible first-order theory which is countable in Gödel's constructible universe $\mathbb{L}$. Then the following are equivalent:*

1. *The collection*

$$\begin{aligned}\{\langle \mathscr{A}, \mathscr{B}\rangle \in \mathbb{L} : \mathscr{A} \models T, \mathscr{B} \models T, \mathscr{A}, \mathscr{B} &\textit{ have universe } (\aleph_2)^{\mathbb{L}}, \\ \textit{and are isomorphic in an extension of } \mathbb{L} &\textit{ with the same} \\ &\textit{cardinals and reals as } \mathbb{L}\}\end{aligned} \tag{1}$$

   is constructible.
2. *The theory T is superstable with NOTOP and NDOP.*

This result was proved using strong non-structure theorems, following the cases found in the Main Gap Theorem [12].

We chose the Homogeneous Model Theory context to extend this result because of its well developed Main Gap Theorem [8]. Much of the difficulty lies in finding strong non-structure theorems in the Homogeneous Model Theory (HMT) context. While one can prove strong non-structure theorems in non-elementary contexts (e.g. Abstract Elementary Classes) with the order property in exactly the same way as was done for unstable first order theories, strong non-structure theorems have not been proved for almost any other non-elementary classes. This is because the only first-order strong non-structure that can be generalized is the one stemming from the order property.

In this paper, we prove a strong non-structure theorem for the strictly stable (stable but not superstable) case in HMT. In the first-order context, non-structure theorems for the strictly stable case are proved by first finding tree indiscernibles, and then using them as skeleta in an Ehrenfeucht-Mostowski model construction. In the HMT context, a major problem arises in simply generalizing the approach used in the first-order context: without large cardinals one cannot find tree-indiscernibles. In particular, even if one were willing to assume large cardinals, if one wants to make the constructions in $\mathbb{L}$, as we do in this paper, Ehrenfeucht-Mostowski model constructions cannot be used.

§2. Preliminaries.

2.1. Notation. Gödel's constructible universe will be denoted as $\mathbb{L}$. To differentiate, similarity types (languages) will be denoted with the calligraphic $\mathcal{L}$.

2.2. Set theory.

2.2.1. *Relative constructibility.* This paper is concerned with examining the solvability (in the sense of [1]) of certain problems in the classification of structures that are not-first order axiomatizable. Our intuition is that if the collection of constructible objects that satisfy a particular condition is constructible (i.e. in $\mathbb{L}$), then we say that the condition's problem is **solvable**. On the other hand, if the collection is not in $\mathbb{L}$, then we say that the condition's problem is **unsolvable**.

We will demonstrate the unsolvability of a problem by **reducing** to it sets that are known to be non-constructible, indeed sets that are equiconstructible with $0^{\#}$.

First, some notation:

DEFINITION 2.1. We have the following notion of **reduction**:
Suppose that $\langle X_0, X_1\rangle$, $\langle Y_0, Y_1\rangle$ are pairs of disjoint subsets of the constructible universe L. That is, that they are disjoint collections of constructible sets. Note that $\langle X_0, X_1\rangle$ and $\langle Y_0, Y_1\rangle$ need not be constructible themselves. We write that

$$\langle X_0, X_1\rangle \xrightarrow{\mathbb{L}} \langle Y_0, Y_1\rangle$$

if there exists a constructible function $g \in L$ such that

$$x \in X_0 \Rightarrow g(x) \in Y_0 \text{ and } x \in X_1 \Rightarrow g(x) \in Y_1.$$

We write X_0 instead of $\langle X_0, X_1\rangle$ in the case that X_0 is the complement of X_1 within some constructible set. We employ the analogous convention for the Ys.

The idea behind this notion of reduction is that if $\langle X_0, X_1\rangle$ is non-constructible, $X_0 \cup X_1$ is constructible, and $\langle X_0, X_1\rangle \xrightarrow{\mathbb{L}} \langle Y_0, Y_1\rangle$, then $\langle Y_0, Y_1\rangle$ is also non-constructible.

DEFINITION 2.2.

1. A **cardinal preserving extension** of $\mathbb{L}$ is a transitive model satisfying the Axiom of Choice containing all the ordinals, and which is contained in a set-generic extension of V and has the same cardinals as $\mathbb{L}$.
2. A **cardinal- and real-preserving extension** of $\mathbb{L}$ is a transitive model satisfying the Axiom of Choice containing all the ordinals, and which is contained in a set-generic extension of V and has the same cardinals and real numbers as $\mathbb{L}$.
3. For ν a cardinal, a **cardinal- and** $\mathscr{P}(\nu)$-preserving extension is defined analogously.

We also remind the reader of the following highly non-constructible object:

DEFINITION 2.3. If there exists a non-trivial elementary embedding of the constructible universe $\mathbb{L}$ into itself, then there is a closed unbounded proper class of ordinals that are indiscernible for the structure $(\mathbb{L}, \in)$. Then, we can define $0^{\#}$ (**"zero-sharp"**) to be the real number that codes in the canonical way the Gödel numbers of the formulas that are true about the indiscernibles in $\mathbb{L}$.

The existence of $0^{\#}$ is independent of the axioms of set theory, ZFC. If ZFC is consistent, then so is ZFC with the assumption that $0^{\#}$ does not exist. It is commonly assumed that ZFC is consistent with the assumption that $0^{\#}$ does exist.

We assume throughout that $0^{\#}$ exists.

The real number $0^{\#}$ is highly non-constructible object. Our intuition will be to show that a class of models is non-constructible by reducing $0^{\#}$ to it, in the sense above. In particular, we will use the following theorem.

THEOREM 2.4 ([1]). *Denote by $\mathcal{S}(\kappa)$ [resp.$\mathcal{S}_r(\kappa)$] the collection of sets $S \in \mathbb{L}$ such that $S \subseteq (S^{\nu}_{\omega})^{\mathbb{L}}$ is stationary in $\mathbb{L}$ and in a cardinal- [and real-] preserving extension, $\nu \setminus S$ contains a club.*

Then, if κ is an uncountable regular cardinal in $\mathbb{L}$ and $(\kappa^+ = \nu)^{\mathbb{L}}$, then

$$0^{\#} \xrightarrow{\mathbb{L}} \mathcal{S}(\kappa)$$

and

$$0^{\#} \xrightarrow{\mathbb{L}} \mathcal{S}_r(\kappa).$$

2.3. Homogeneous model theory.

2.3.1. *Introduction and motivation for homogeneous model theory.* **Homogeneous Model Theory (HMT)**, introduced in [10] as "finite diagrams stable in power", is an approach to the model theoretic classification of classes of non-elementary structures (i.e. structures not axiomatizable using a first-order theory). The motivation behind the development of this approach, as explained in [8, 3], was the aim to classify the class of models of an $\mathcal{L}_{\gamma^+\omega}$ sentence ψ, with $\preccurlyeq_{\mathcal{L}_{\gamma^+\omega}}$ as the substructure relation. We wish this class of models to be "well behaved" and so add the requirement that the class satisfies the amalgamation property. It was proved in [10] that it is equivalent to consider the class of elementary submodels of a homogeneous monster model $\mathfrak{M}$.

Thus, in practice the contrast to elementary (first-order) model theory where one assumes that all considerations take place within a large **saturated** monster model, is that we take away the assumption that the monster is saturated, and instead only insist that it be **homogeneous**. However, in the HMT context, a major difficulty arises because the compactness theorem fails. In return for this concession, we do gain a widening of the possible structures under consideration as opposed to elementary model theory. For example, the class of existentially closed models of an inductive theory can be studied within the framework of homogeneous model theory. In fact, for some γ big enough the class of submodels of a homogeneous model can be axiomatized in some theory $T^* \subset \mathcal{L}_{\gamma^+\omega}$. (Specifically, where $\gamma \geqslant |D(\mathrm{Th}(\mathfrak{M})) \setminus D|$, where D is the finite diagram. For more specifics, see [10, 3].)

2.3.2. *Types and homogeneous monsters.* We assume we work within very large homogeneous model which can serve as a monster model. We will then be interested in the class of elementary submodels of this monster.

We work with $\mathfrak{M}$-consistent types:

DEFINITION 2.5 ([8]). Let $A \subseteq \mathfrak{M}$, and let p be a (first-order) type over A. We say that p is **$\mathfrak{M}$-consistent** if it is realized in $\mathfrak{M}$.

We write $\mathrm{tp}_{\mathfrak{M}}(a/A)$ to indicate the $\mathfrak{M}$-consistent type of a over A. Similarly, we take $\mathrm{S}^m_{\mathfrak{M}}(A) = \{\mathrm{tp}_{\mathfrak{M}}(a/A) : a \in \mathfrak{M}, length(a) = m\}$, and $\mathrm{S}_{\mathfrak{M}}(A) = \bigcup_{m<\omega} \mathrm{S}^m_{\mathfrak{M}}(A)$.

DEFINITION 2.6. A homogeneous monster model $\mathfrak{M}$ is said to be **stable in λ** if for every $B \subset \mathrm{dom}(\mathfrak{M})$ of cardinality at most λ, and for every $n < \omega$, we have $|\mathrm{S}^n_{\mathfrak{M}}(B)| \leqslant \lambda$.

The monster model $\mathfrak{M}$ is **stable** if it is stable in some λ.

The monster model $\mathfrak{M}$ is **unstable** if it is not stable.

We denote by $\lambda(\mathfrak{M})$ the least λ in which $\mathfrak{M}$ is stable, if it exists [7]. Denote by $\lambda_r(\mathfrak{M})$ the first regular cardinal $\geqslant \lambda(\mathfrak{M})$.

2.3.3. *Indiscernibles and strong splitting independence.* A standard notion from model theory follows. We include this definition to make the terminology clear, as the set-theoretic usage is sometimes at odds to accepted usage among model-theorists.

Definition 2.7. An (indexed) set of tuples $\{\bar{a}_i : i < \alpha\}$ is called an n-**indiscernible sequence over** A , for $n < \omega$, if

$$\mathrm{tp}(\bar{a}_0, \dots, \bar{a}_{n-1}/A) = \mathrm{tp}(\bar{a}_{i_0}, \dots, \bar{a}_{i_{n-1}}/A),$$

for every $i_0 < \dots < i_{n-1} < \alpha$. The set of tuples $\{\bar{a}_i : i < \alpha\}$ is an **indiscernible sequence over** A if it is an n-indiscernible sequence over A for every $n < \omega$. It is said to be an **indiscernible set** if the ordering induced by the indices does not matter.

Definition 2.8 ([12] III, p. 85, Def. 1.2). A type $p \in \mathrm{S}^n(A)$ **splits strongly** over $B \subseteq A$ if there exists $\{\bar{a}_i : i < \omega\}$ an indiscernible sequence over B and a formula ϕ such that $\phi(\bar{x}, \bar{a}_0), \neg\phi(\bar{x}, \bar{a}_1) \in p$.

The following definitions are very similar to the definitions of independence and $\kappa(T)$ in the first-order context. However, here we use strong splitting instead of forking in the definitions. In the first order context, the definitions using forking and the definition as stated here are equivalent. In the HMT context, forking is ill-defined, so we take the strong splitting definition. Consequently, we lose some nice properties, such as transitivity of the independence relation.

Definition 2.9 ([7], p. 2). We define $\kappa(\mathfrak{M})$ to be the least infinite cardinal such that there are no a, b_i, and c_i, $i < \kappa(\mathfrak{M})$, such that

(i) for all $i < \kappa(\mathfrak{M})$, there is an infinite indiscernible set I_i over $\bigcup_{j<i}(b_j \cup c_j)$ such that $b_i, c_i \in I_i$,

(ii) for all $i < \kappa(\mathfrak{M})$, there is $\phi_i(x, y)$ such that $\models \phi_i(a, b_i) \wedge \neg\phi_i(a, c_i)$.

Note that $\kappa(\mathfrak{M}) \leqslant \lambda(\mathfrak{M})$ by Corollary 1.3 of [7].

Definition 2.10 ([7], p. 17, remarks before Lemma 5.1). We say that a monster model is **superstable** if $\kappa(\mathfrak{M}) = \aleph_0$. We will call a monster model **strictly stable** if it is stable, but not superstable.

Now we can define the notion of independence that we use in the HMT context.

Definition 2.11 ([7], Def. 3.1(i)). We write $a \downarrow_A B$ if there is $C \subseteq A$, $|C| < \kappa(\mathfrak{M})$, such that for all $D \supseteq A \cup B$ there is b which satisfies $\mathrm{tp}_{\mathfrak{M}}(b/A \cup B) = \mathrm{tp}_{\mathfrak{M}}(a/A \cup B)$ and $\mathrm{tp}_{\mathfrak{M}}(b/D)$ does not split strongly over C. We write $C \downarrow_A B$ if for all $a \in C$, $a \downarrow_A B$.

2.3.4. *Primary model constructions.* Most of the following definitions are given only in very general terms that allow one to apply the notions to a very wide range of contexts. We give here these definitions specifically in the way we need them in our context.

Definition 2.12. For the following, ν is a cardinal.

- We say that $\mathrm{tp}_{\mathfrak{M}}(a/A)$ is $\mathbf{F}^{\mathfrak{M}}_{\nu}$**-isolated** over B if there is $B \subseteq A$, $|B| < \nu$, such that for all b, $\mathrm{tp}_{\mathfrak{M}}(b/B) = \mathrm{tp}_{\mathfrak{M}}(a/B)$ implies $\mathrm{tp}_{\mathfrak{M}}(b/A) = \mathrm{tp}_{\mathfrak{M}}(a/A)$. ([7], Def. 5.2)

- We say that an (elementary sub-)model $\mathscr{A}$ (of $\mathfrak{M}$) is **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-saturated** if for all $A \subseteq \mathscr{A}$, $|A| < \nu$, and a, there is $b \in \mathscr{A}$ such that $\mathrm{tp}_{\mathfrak{M}}(b/A) = \mathrm{tp}_{\mathfrak{M}}(a/A)$. ([7], Def. 1.8(i))
- An **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-construction** is a triple
$$\mathscr{A} = \langle A, \{\bar{a}_i : i < \alpha\}, \langle B_i : i < \alpha\rangle\rangle,$$
such that $\mathrm{tp}_{\mathfrak{M}}(\bar{a}_i / \bigcup\{\bar{a}_j : j < i\} \cup A)$ is $\mathbf{F}^{\mathfrak{M}}_{\nu}$-isolated over B_i. ([12] IV, p. 155, Def. 1.2(1))
- We say that C_0 is **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-constructible over** A_0 if there is some $\mathbf{F}^{\mathfrak{M}}_{\nu}$-construction
$$\mathscr{A} = \langle A_0, \{\bar{a}_i : i < \alpha\}, \langle B_i : i < \alpha\rangle\rangle$$
such that
$$C_0 = \bigcup\{\bar{a}_i : i < \alpha\} \cup A_0.$$
([12] IV, p. 156, Def. 1.3)
- If C is $\mathbf{F}^{\mathfrak{M}}_{\nu}$-constructible over A and C is $\mathbf{F}^{\mathfrak{M}}_{\nu}$-saturated then we say that C is **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-primary over** A. ([12] IV, p. 156, Def. 1.4(1))
- We say that C is **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-primitive** over A if $A \subseteq C$, and for every $\mathbf{F}^{\mathfrak{M}}_{\nu}$-saturated C' such that $A \subseteq C'$, there is an elementary mapping f from C into C', where $f \restriction_A$ is the identity. ([12] IV, p. 156, Def. 1.4(2))
- We say that C is **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-prime** over A if it is $\mathbf{F}^{\mathfrak{M}}_{\nu}$-primitive over A and $\mathbf{F}^{\mathfrak{M}}_{\nu}$-saturated.
- We say A is **$\mathbf{F}^{\mathfrak{M}}_{\nu}$-atomic** over B if $B \subseteq A$ and for every $\bar{a} \in A$, $\mathrm{tp}_{\mathfrak{M}}(\bar{a}/B)$ is $\mathbf{F}^{\mathfrak{M}}_{\nu}$-isolated. ([12] IV, p. 157, Def. 1.5)

Remark 2.13. *On the surface, the isolation notion $\mathbf{F}^{\mathfrak{M}}_{\nu}$ above is quite similar to the isolation notion $\mathbf{F}^{p}_{\nu}$ of IV Definition 2.6 (p. 168) of [12], an isolation notion that does not satisfy certain axioms key in constructions.*

However, as was noted in the last paragraph of the introduction to [8], under the assumption that $\mathfrak{M}$ is stable, one can easily show that the isolation notion $\mathbf{F}^{\mathfrak{M}}_{\nu}$, for $\nu \geqslant \lambda_r(\mathfrak{M})$ reduces to a notion closely resembling the much better-behaved notion $\mathbf{F}^{s}_{\nu}$, a definition of which can be found in [12] IV Definitions 2.1.1.ii and 2.1.2.

Definition 2.14 ([8] Def. 0.1). A model $\mathscr{A}$ is said to be **locally $\mathbf{F}^{\mathfrak{M}}_{\nu}$-saturated** if for all finite sets $A \subset \mathscr{A}$ there is a $\mathbf{F}^{\mathfrak{M}}_{\nu}$-saturated model $\mathscr{B}$ such that $A \subset \mathscr{B} \subset \mathscr{A}$.

§3. The Strictly Stable Case.

Theorem 3.1. *Assume $0^{\#}$ exists.*

Suppose $\mathcal{L} \in \mathbb{L}$ is a signature such that $(|\mathcal{L}| \leqslant \omega)^{\mathbb{L}}$. Let $\mathfrak{M} \in \mathbb{L}$ be a strictly stable (stable, but not superstable) homogeneous monster model in similarity type $\mathcal{L}$ such that $(|\mathfrak{M}| = \mu)^{\mathbb{L}}$, for μ be a sufficiently large.

Let π be such that $\pi = \mathrm{cf}(\pi) > \lambda_r(\mathfrak{M})$. Let ${}^{carp}PIP^{\mathfrak{M}}_{\pi}$ be the collection of pairs $(\mathscr{A}, \mathscr{B}) \in \mathbb{L}$ of locally $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-saturated elementary substructures of $\mathfrak{M}$ with universe π such that there is a cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{M}))$-preserving extension of $\mathbb{L}$ in which $\mathscr{A} \cong \mathscr{B}$.

Then, ${}^{carp}PIP^{\mathfrak{M}}_{\pi}$ is equiconstructible with $0^{\#}$.

We will show that for each stationary set $S \subseteq S^\pi_\omega$, one can find two models $\mathscr{A}, \mathscr{B} \in \mathbb{L}$ of size π such that in any CAP-extension of $\mathbb{L}$, $\mathscr{A} \cong \mathscr{B}$ iff $\pi \setminus S$ contains a club set. We do this by constructing two trees of small height J_0, J_1, differing from one another only in that one incorporates S while the other does not. We will then perform a primary model constructions along these trees. We show then that these models are not isomorphic in the ground model, but become isomorphic in an suitable extension only if S is no longer stationary in that extension.

3.1. Defining the trees and other orderings. We define two trees I_0 and I_1, which will be used to define two trees J_0 and J_1. The trees I_i, J_i, $i = 0, 1$ all belong to a certain general family of trees K^ω_{tr}, defined below. Note that the trees we define here are precisely the trees that were used for the Ehrenfeucht-Mostowski constructions in the first order strictly stable case as analyzed in [2] and papers cited there.

As opposed to the first-order case, non-structure results for strictly stable theories have only been shown for weakly $\mathbf{F}^{\mathfrak{m}}_{\lambda_r(\mathfrak{m})}$-saturated models and Ehrenfeucht-Mostowski constructions yield models that are not sufficiently saturated, we will instead use the technique of primary model constructions. We cannot use Ehrehnfeucht-Mostowski constructions in this case because we would need to find tree indiscernibles in the model, and to do so we would need large cardinals that are not available to us in $\mathbb{L}$. Because we need this different technique, we need to further define $K_i = \mathscr{P}^{<\omega}(J_i)$, the set of all finite subsets of J_i, $i = 0, 1$, as well as an ordering on the K_i. We will then carry out primary model constructions using sets indexed by the K_i.

We define first a general family of trees:

Definition 3.2. Let θ be a linear order, and let ${}^{\leqslant\omega}\theta$ be the set of all suborders of θ of length at most ω. We let $K^\omega_{tr}(\theta)$ be the class of models that are isomorphic to a model of the form

$$\mathcal{I} = (I, \lessdot, \mathrm{DOM}_\alpha, <_{lex}, \mathrm{MaxInSg})_{\alpha \leqslant \omega},$$

where

1. $I \subseteq {}^{\leqslant\omega}\theta$ and is closed under initial segments;
2. $\lessdot$ is the initial segment relation;
3. $\mathrm{DOM}_\alpha = \{\eta \in I : \mathrm{dom}\,\eta = \alpha\}$;
4. $<_{lex}$ denotes the lexicographic ordering on I;
5. $\mathrm{MaxInSg}(\zeta, \eta)$ is the maximal common initial segment of ζ and η.

Trees in the class $K^\omega_{tr}(\theta)$ are called **ordered trees** in the literature. We define

$$K^\omega_{tr} = \bigcup \{K^\omega_{tr}(\theta) : \theta \text{ is a linear order}\}.$$

3.1.1. *The first generation of trees.* We fix some notation.

- Let $(\lambda = \lambda_r(\mathfrak{m}))^{\mathbb{L}}$. Because we have assumed that $\mathfrak{m}$ is strictly stable, $\lambda \geqslant \aleph_1$.
- Let $\pi \geqslant \lambda^+ \geqslant \aleph_2$ be an uncountable regular cardinal such that $\pi^\omega = \pi$.
- Let $S \subseteq (S^\pi_\omega)^L$ be a stationary set in $\mathbb{L}$;

- Let $\bar{S} = \langle \eta_\alpha : \alpha \in S \rangle$, where each η_α is an increasing cofinal sequence in α of order type ω (i.e., a π-club guessing sequence). We are guaranteed the existence of this club guessing sequence because $\pi \geqslant \aleph_2$.

We define our first pair of trees.

Definition 3.3.

- Let
$$I_0 = I(\pi, \bar{S})$$
be an ordered tree in $K^\omega_{tr}(\pi)$, with cardinality $|I_0| = \pi$, having universe
$$^{<\omega}\pi \cup \{\eta_\alpha : \eta_\alpha \in \bar{S}\} \subset {}^{\leqslant\omega}\pi,$$
where the relations are as always on ordered trees.
- Let
$$I_1 = I(\pi, \langle\rangle) = {}^{<\omega}\pi.$$
The tree I_1 is also in $K^\omega_{tr}(\pi)$, and $|I_1| = \pi$.

3.1.2. *The second generation of trees.* Now, we define the domains of our next generation of trees. This construction is due to Shelah [11].

Let

- $\mathrm{LEX}(^{<\omega}\pi)$ be a linear order with universe $^{<\omega}\pi$, ordered lexicographically.
- $\mathrm{OT}_\pi(^{<\omega}\pi)$ be a linear (well) order with universe $^{<\omega}\pi$, ordered with order type π.
- $\theta = \mathrm{OT}_\pi(^{<\omega}\pi) \cdot \mathrm{LEX}(^{<\omega}\pi)$ be the product of the linear orders $\mathrm{OT}_\pi(^{<\omega}\pi)$ and $\mathrm{LEX}(^{<\omega}\pi)$ whose universe is $\mathrm{OT}_\pi(^{<\omega}\pi) \times \mathrm{LEX}(^{<\omega}\pi)$.

Let
$$\bar{I}_0 = \langle I_0 \cap {}^{\leqslant\omega}\alpha : \alpha < \pi \rangle,$$
$$\bar{I}_1 = \langle I_1 \cap {}^{\leqslant\omega}\alpha : \alpha < \pi \rangle = \langle {}^{<\omega}\pi \cap {}^{\leqslant\omega}\alpha : \alpha < \pi \rangle = \langle {}^{<\omega}\alpha : \alpha < \pi \rangle$$
be π-representations of I_0 and I_1, respectively.

Lemma 3.4 ([4] Lemma 7.24 or [9] Lemma 8.17). *Suppose π is a cardinal and $\mathrm{LEX}(^{<\omega}\pi)$ is as above. Then there is $E \subseteq \mathrm{LEX}(^{<\omega}\pi)$ of cardinality π such that for any $a, b \in E$ there is an automorphism $g_{a,b}$ of $\mathrm{LEX}(^{<\omega}\pi)$ which maps a to b.*

Let $E \subseteq \mathrm{LEX}(^{<\omega}\pi)$ be as given by Lemma 3.4. Fix $c \in E$. Let g be a bijection $g : \{R : R \in \bar{I}_0 \cup \bar{I}_1\} \longrightarrow E \setminus \{c\}$.

Definition 3.5. Let $J_0 = J(c, g, \bar{I}_0, \bar{I}_1)$ have a universe consisting of functions $\eta \in {}^{\leqslant\omega}\theta$, such that one of the following holds

1. $\eta \in {}^{<\omega}\theta$ (in other terms, $\eta \in \mathrm{DOM}_n$ for some $n \in \omega$; i.e. η is of finite length);
2. There is $s \in I_0$ such that $\mathrm{dom}(s) = \omega$, and for all $n < \omega$,
$$\eta(n) = \langle s\restriction_{(n+1)}, c \rangle;$$
3. there are $m < \omega$, $R \in \bar{I}_0 \cup \bar{I}_1$, and $s \in R$ with $\mathrm{dom}(s) = \omega$ such that for all finite $n \geqslant m$, $\eta(n) = \langle s\restriction_{(n+1)}, g(R) \rangle$.

Let $J_1 = J(c, g, \bar{I}_1, \bar{I}_0)$ be defined analogously. Note that it is almost the same as the above definition, but with the second possible condition omitted.

The trees J_0 and J_1 are isomorphic to ordered trees in $K^{\omega}_{tr}(\theta)$, thus we assume that $J_0, J_1 \in K^{\omega}_{tr}(\theta)$.

Lemma 8.20 of [9] establishes that J_0 and J_1 are $\mathcal{L}_{\infty\pi}$-equivalent.

3.1.3. *The third generation: a quasi-order.* At this point in the construction, we can lose the $<_{lex}$ ordering on J_i, since we do not need it for the primary model construction that follows. Indeed, we could have used a different construction in the second generation that did not feature $<_{lex}$. However, we chose to take advantage of the existing construction from [11] to save some effort.

Let $K_i = \mathscr{P}^{<\omega}(J_i)$ be the set of all finite subsets of J_i, $i = 0, 1$, respectively.

We define relations as in [6]. Let $u, v \in K_i$. We define the "minimum" set of initials $\mathrm{MinSetIn}(u, v)$ to be the largest set X such that:

1. $X \subseteq \{\mathrm{MaxInSg}(\zeta, \eta) : \zeta \in u, \eta \in v\}$;
2. if $\eta_i, \eta_j \in X$ and η_i is an initial segment of η_j, then $\eta_i = \eta_j$.

Note that

$$\mathrm{MinSetIn}(u, u) = \{\zeta \in u : \neg\exists\eta \in u\,(\zeta \text{ is a proper initial segment of } \eta)\}.$$

The elements of K_i are ordered by $<^K$: $u <^K v$ iff for every $\zeta \in u$ there is $\eta \in v$ such that ζ is an initial segment of η. In other terms,

$$u \leq^K v \text{ iff } \mathrm{MinSetIn}(u, v) = \mathrm{MinSetIn}(u, u).$$

Note that $(K_i, <^K)$ cannot have infinite descending chains.

Definition 3.6. We call $s \in K_i$ **semi-good** if s is an antichain with regard to the $\lessdot$ relation in J_i.

Denote by $\bar{s}$ the downwards closure of s. We say that $r \in K_i$ is **good** if it is downwards closed and $r \subset \bar{s}$, where s is semi-good. We denote by $G(K_i)$ the collection of good elements of K_i.

3.2. Building the models: putting fat on the trees. We will base a primary model construction based on the trees J_i, using the quasi-order K_i.

3.2.1. *Cardinal assumptions.* Recall that we assume in this section that we work within $\mathfrak{M}$, a strictly stable homogeneous monster model of cardinality $||\mathfrak{M}|| = \mu$. We let $\lambda(\mathfrak{M})$ be the first cardinal in which $\mathfrak{M}$ is stable, and we let $\lambda = \lambda_r(\mathfrak{M})$ be the first regular cardinal $\geqslant \lambda(\mathfrak{M})$. By our assumption that $\mathfrak{M}$ is strictly stable, $\kappa(\mathfrak{M}) \neq \omega$ (see 3.7 below). Thus, $\aleph_1 \leqslant \kappa(\mathfrak{M}) \leqslant \lambda(\mathfrak{M}) \leqslant \lambda$. Further, let π be a regular cardinal such that $\pi^{\omega} = \pi$ and $\lambda < \pi < \mu$. Thus $\pi \geqslant \aleph_2$. This π is the size of the models that we will be building, and is the cardinal upon which our trees have been built.

We proceed with the construction similarly to [6].

3.2.2. *An initial ω-sequence of models.* We restate the following lemma, which provides the seed for our construction:

Lemma 3.7 (Lemma 5.1 [7]). *The following are equivalent:*

1. $\mathfrak{M}$ *is not superstable.*
2. $\kappa(\mathfrak{M}) \neq \omega$.
3. *There is an increasing sequence $\mathscr{A}_n$, $n < \omega$ of $\mathbf{F}^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated models and an element a such that for all $n < \omega$, $a \not\downarrow_{\mathscr{A}_n} \mathscr{A}_{n+1}$.*

Remark 3.8. *The sequence $\mathscr{A}_n$, $n < \omega$ in Lemma 3.7 can be chosen to consist of models of size λ.*

Proof. Let $\mathscr{A}_n$, $n < \omega$ be the sequence of models given by Lemma 3.7. It is easy to find such models that are quite large.

Each $\mathscr{A}_n$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda}$-saturated, and hence strongly $\mathbf{F}^{\mathfrak{M}}_{\kappa(\mathfrak{M})}$-saturated by Lemma 1.9(iv) of [7]. Thus, by the monotonicity given by Lemmas 1.2(vi) and 1.13, and the proof of Lemma 3.2(iii) of that same paper, there exist $B_n \subset \mathscr{A}_n$ an increasing sequence of sets of size $< \kappa(\mathfrak{M})$ such that

$$a \underset{B_n}{\downarrow} \mathscr{A}_n.$$

We also have that $a \not\downarrow_{B_i} \mathscr{A}_{i+1}$. By the finite character of independence in our setting (Corollary 3.5(i) of [7]), there exist finite $b_{n+1} \in \mathscr{A}_{n+1}$ that witness $a \not\downarrow_{B_n} \mathscr{A}_{n+1}$ such that

$$a \underset{B_n}{\not\downarrow} b_{n+1}.$$

Choose $\mathbf{F}^{\mathfrak{M}}_{\lambda}$-saturated models $\mathscr{C}_n$ of size λ so that $B_n \subset \mathscr{C}_n \subset \mathscr{A}_n$ and $b_{n+1} \in \mathscr{C}_{n+1}$. We can do this by Theorem 3.14 of [7].

We claim that $(\mathscr{C}_n)_{n<\omega}$ satisfy the requirements of Lemma 3.7: Assume the contrary, that $a \downarrow_{\mathscr{C}_n} \mathscr{C}_{n+1}$. Since $a \downarrow_{B_n} \mathscr{A}_n$, $a \downarrow_{B_n} \mathscr{C}_n$ by monotonicity. By transitivity and monotonicity, $a \downarrow_{B_n} \mathscr{C}_{n+1}$. Finally, monotonicity gives us

$$a \underset{B_n}{\downarrow} b_{n+1},$$

and hence a contradiction. $\square_{3.8}$

Construction Element. Thus, fix $(\mathscr{A}_j)_{j \leqslant \omega}$, a sequence of $\mathbf{F}^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated models of size λ, and an element a with the properties as in Lemma 3.7.

Construction Element. Let $\mathscr{A}_\omega$ be a $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-primary model over

$$a \cup \bigcup_{i<\omega} \mathscr{A}_i,$$

the existence of which is guaranteed by Theorem 5.3 of [7] (proof is in [10]).

3.2.3. *The construction.*

Construction Element. For all $\eta \in \pi^{\leqslant \omega}$, using analogous reasoning to that found in Section 1 of [5] (discussion of which begins after Theorem 1.15 and continues through the proof of Lemma 1.17 of that paper), we define models $\mathscr{A}_\eta$ such that

- for all $\eta \in {}^{\leqslant\omega}\pi$, there is an automorphism f_η of $\mathfrak{M}$ such that
$$f_\eta(\mathscr{A}_{\mathrm{length}(\eta)}) = \mathscr{A}_\eta;$$
- if η is an initial segment of ζ, then
$$f_\zeta \restriction_{\mathscr{A}_{\mathrm{length}(\eta)}} = f_\eta \restriction_{\mathscr{A}_{\mathrm{length}(\eta)}};$$

- if $\eta \in {}^{<\omega}\pi$, $\alpha \in \pi$, and X is the set of those $\eta \in {}^{\leqslant\omega}\pi$ such that $\eta \smile (\alpha)$ is an initial segment of ζ, then

$$\bigcup_{\zeta \in X} \mathscr{A}_\zeta \underset{\mathscr{A}_\eta}{\perp} \bigcup_{\zeta \in ({}^{\leqslant\omega}\pi \setminus X)} \mathscr{A}_\zeta;$$

- for all $\eta \in {}^{\omega}\pi$, we let $a_\eta = f_\eta(a)$.

We recall a definition which will allow us to carry out the construction in an orderly and controlled manner.

Definition 3.9 (Definition 3 of [6]). Assume $J \subseteq {}^{\leqslant\omega}\pi$ is closed under initial segments and $K = \mathscr{P}^{<\omega}(J)$. Let $\Sigma = \{A_u : u \in K\}$ be an indexed family of subsets of $\mathfrak{M}$ of cardinality $< \mu$. We say that Σ is **strongly independent** if

1. for all $u, v \in K$, $u \leq^K v \to A_u \subseteq A_v$;
2. if $u, u_i \in I$, $i < n \in \omega$, and $B \subseteq \bigcup_{i<n} A_{u_i}$ has cardinality $< \pi$, then there is an automorphism $f = f^{\Sigma,B}_{(u,u_0,\dots,un-1)}$ of $\mathfrak{M}$ such that $f \restriction_{(B\cap A_u)} = id_{B\cap A_u}$, and $f(B \cap A_{u_i}) \subseteq A_{\mathrm{MinSetIn}(u,u_i)}$.

Construction Element. Define

$$A^i_u = \bigcup_{\eta \in u} \mathscr{A}_\eta,$$

for $u \in K_i$.

We can now apply Lemma 6 of [6] to find that $\{A^i_u : u \in K_i\}$ is strongly independent.

Construction Element. We apply Lemma 4 of [6] to $\{A^i_u : u \in K_i\}$, and so find models $\mathscr{A}^i_u \preccurlyeq \mathfrak{M}$, $u \in K_i$ which satisfy the following properties:

1. For all $u, v \in K_i$, $u \leq^K v$ implies $\mathscr{A}^i_u \subseteq \mathscr{A}^i_v$;
2. for all $u \in K_i$, $\mathscr{A}^i_u$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-primary over A^i_u. This implies that $\bigcup_{u\in K_i} \mathscr{A}^i_u$ is a model.
3. if $v \leq u$, then $\mathscr{A}^i_u$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-atomic over $\bigcup_{u\in K_i} A^i_u$, and $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-primary over $\mathscr{A}^i_v \cup A^i_u$.

These models $\mathscr{A}^i_u$ arise via a $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-construction, with points a_γ, and sets B_γ, $\gamma < \alpha$ chosen appropriately. See proof of Lemma 4, [6] for full details.

In addition, note that by the proof of [6] Lemma 4 (Claim), the families of models $\{\mathscr{A}^i_u : u \in K_i\}$, where $i = 0$ or $i = 1$ are strongly independent.

Construction Element. Denote by

$$\mathscr{A}^{J_i} = \bigcup_{u \in K_i} \mathscr{A}^i_u$$

the resulting constructed models given by Lemma 4 of [6].

3.3. Non-isomorphism when symmetric difference of S-invariants is stationary. Lemma 4 of [6] guarantees that $\mathscr{A}^{J_0}$ and $\mathscr{A}^{J_1}$ have certain properties that we will need to show non-isomorphism.

LEMMA 3.10 (Lemma 4 of [6]). *Assume $\Sigma = \{A_u : u \in K\}$ is strongly independent with notation as in the definition above. Then there are sets $\mathscr{A} \subset \mathfrak{M}$ such that*

1. *for all $u, v \in I$, $u \leq^K v \to \mathscr{A}_u \subseteq \mathscr{A}_v$;*
2. *for all $u \in I$, $\mathscr{A}_u$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-primary over A_u (and so $\bigcup_{u \in I} \mathscr{A}_u$ is a model),*
3. *if $v \leq^K u$, then $\mathscr{A}_u$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda}$-atomic over $\bigcup_{u \in I} A_u$ and $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-primary over $\mathscr{A}_v \cup A_u$,*
4. *if $J' \subseteq J$ is closed under initial segments, and $u \in P^{<\omega}(J')$, then the union $\bigcup_{v \in P^{<\omega}(J')} \mathscr{A}_v$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-constructible over $\mathscr{A}_u \cup \bigcup_{v \in P^{<\omega}(J')} A_v$.*

Furthermore, we have much information about the structure of the trees J_0, J_1.

DEFINITION 3.11. Denote by I_{NS^π} be the ideal of non-stationary sets on π.
For $J \subseteq \pi^{\leqslant\omega}$, let $J^\alpha = J \cap \alpha^{\leqslant\omega}$.
For $K = \mathscr{P}^{\leqslant\omega}(J)$, let $K^\alpha = \mathscr{P}^{\leqslant\omega}(J^\alpha)$.
Define the S-**invariant** of J to be:

$$S(\bar{J}) = \{\delta : \exists \eta \in J^\delta \, (\eta \notin \bigcup_{\alpha<\delta} J^\alpha)\} \text{ modulo } I_{NS^\pi}.$$

LEMMA 3.12. *Let $\mathscr{A}^J$ and $\mathscr{A}^{J'}$ be models constructed as above for trees $J, J' \subseteq \pi^{\leqslant\omega}$. Assume that $S(J) \triangle S(J') = (S(J) \setminus S(J')) \cup (S(J') \setminus S(J))$ is stationary. Then $\mathscr{A}^J \ncong \mathscr{A}^{J'}$.*

PROOF. We follow Lemma 8 of [6].

Assume for a contradiction that $f : \mathscr{A}^J \longrightarrow \mathscr{A}^{J'}$ is an isomorphism.

Let $\bar{J} = (J^\alpha)_{\alpha<\pi}$, $\bar{J}' = ({J'}^\alpha)_{\alpha<\pi}$. Let $K = \mathscr{P}^{\leqslant\omega}(J)$, $K' = \mathscr{P}^{\leqslant\omega}(J')$, and let $K^\alpha = \mathscr{P}^{\leqslant\omega}(J^\alpha)$, ${K'}^\alpha = \mathscr{P}^{\leqslant\omega}({J'}^\alpha)$.

Let $\mathscr{A}^\alpha_J = \bigcup_{s \in G(K^\alpha)} \mathscr{A}_s$, where $G(K^\alpha)$ is the collection of good elements of K^α, as defined in Definition 3.6.

We can find α and α_i, $i < \omega$ such that

- $\eta = (\alpha_i)_{i<\omega}$ is strictly increasing for all $i < \omega$,
- $\alpha = \bigcup_{i<\omega} \alpha_i \in S(J) \triangle S(J')$,
- $$f \restriction_{\mathscr{A}^\alpha_J} : \mathscr{A}^\alpha_J \xrightarrow{\cong} \mathscr{A}^\alpha_{J'}$$

  and

  $$f \restriction_{\mathscr{A}^{\alpha_i}_J} : \mathscr{A}^{\alpha_i}_J \xrightarrow{\cong} \mathscr{A}^{\alpha_i}_{J'}, \forall i < \omega$$

  are isomorphisms.

Without loss of generality, we can assume that $\alpha \in S(J) \setminus S(J')$ and thus that $\eta \in J \setminus J'$.

Claim 3.12.1.

$$a_\eta \underset{\mathscr{A}^{\alpha_i}_J}{\downarrow} \mathscr{A}^{\alpha_{i+1}}_J$$

Recall from the construction that

$$a_\eta \underset{\mathscr{A}_{\eta\restriction_i}}{\downarrow} \mathscr{A}_{\eta\restriction_{i+1}}.$$

Since $\mathscr{A}_{\eta\restriction_i} \subset \mathscr{A}_J^{\alpha_i}$ and $\mathscr{A}_{\eta\restriction_{i+1}} \subset \mathscr{A}_J^{\alpha_{i+1}}$, and $\mathscr{A}_{\eta\restriction_{i+1}} \not\subset \mathscr{A}_J^{\alpha_i}$, by monotonicity (Lemma 3.2 (i), [7]), we have

$$a_\eta \underset{\mathscr{A}_{\eta\restriction_i}}{\not\downarrow} \mathscr{A}_{\eta\restriction_{i+1}} \Rightarrow a_\eta \underset{\mathscr{A}_{\eta\restriction_i}}{\not\downarrow} \mathscr{A}_J^{\alpha_{i+1}}.$$

Claim (3.12.1*). Thus, to prove Claim 3.12.1, it is enough to show that

$$a_\eta \underset{\mathscr{A}_J^{\alpha_i}}{\not\downarrow} \mathscr{A}_{\eta\restriction_{i+1}}.$$

Assume for a contradiction that $a_\eta \downarrow_{\mathscr{A}_J^{\alpha_i}} \mathscr{A}_{\eta\restriction_{i+1}}$.

Claim 3.12.2.

$$a_\eta \underset{\mathscr{A}_J^{\alpha_i}}{\downarrow} \mathscr{A}_{\eta\restriction_{i+1}} \Rightarrow \mathscr{A}_J^{\alpha_i} \underset{\mathscr{A}_{\eta\restriction_i}}{\not\downarrow} \mathscr{A}_{\eta\restriction_{i+1}}.$$

By assumption, $a_\eta \downarrow_{\mathscr{A}_J^{\alpha_i}} \mathscr{A}_{\eta\restriction_{i+1}}$. This implies that $\mathscr{A}_{\eta\restriction_{i+1}} \downarrow_{\mathscr{A}_J^{\alpha_i}} a_\eta$. We get this symmetry by using monotonicity to find that $a_\eta \downarrow_{\mathscr{A}_J^{\alpha_i}} \bar{b}$ for any finite $\bar{b} \in \mathscr{A}_{\eta\restriction_{i+1}}$. Then, since $\mathscr{A}_J^{\alpha_i}$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated by construction ([6]), and hence strongly $\mathbf{F}^{\mathfrak{M}}_{\kappa(\mathfrak{M})}$-saturated, by Lemma 3.6 of [7], $\bar{b} \downarrow_{\mathscr{A}_J^{\alpha_i}} a_\eta$. Since this is true for all $\bar{b} \in \mathscr{A}_{\eta\restriction_{i+1}}$, we get

$$\mathscr{A}_{\eta\restriction_{i+1}} \underset{\mathscr{A}_J^{\alpha_i}}{\downarrow} a_\eta.$$

Now, assume for a contradiction that $\mathscr{A}_J^{\alpha_i} \downarrow_{\mathscr{A}_{\eta\restriction_i}} \mathscr{A}_{\eta\restriction_{i+1}}$. By a similar symmetry argument, $\mathscr{A}_{\eta\restriction_{i+1}} \downarrow_{\mathscr{A}_{\eta\restriction_i}} \mathscr{A}_J^{\alpha_i}$.

Thus, we have

$$\mathscr{A}_{\eta\restriction_{i+1}} \underset{\mathscr{A}_J^{\alpha_i}}{\downarrow} a_\eta \text{ and } \mathscr{A}_{\eta\restriction_{i+1}} \underset{\mathscr{A}_{\eta\restriction_i}}{\downarrow} \mathscr{A}_J^{\alpha_i}.$$

In addition, by [6] Lemma 3.2 (iii), we have $a_\eta \downarrow_{\mathscr{A}_J^{\alpha_i}} \mathscr{A}_J^{\alpha_i}$. We can thus apply [6] Lemma 3.8 (iii) to find that

$$\mathscr{A}_{\eta\restriction_{i+1}} \underset{\mathscr{A}_{\eta\restriction_i}}{\downarrow} a_\eta \cup \mathscr{A}_J^{\alpha_i}.$$

By monotonicity and symmetry, we get $a_\eta \downarrow_{\mathscr{A}_{\eta\restriction_i}} \mathscr{A}_{\eta\restriction_{i+1}}$, a contradiction.

$\boxtimes_{\text{Claim 3.12.2}}$

Thus, with our assumptions so far, we have $\mathscr{A}_{\eta\restriction_{i+1}} \not\downarrow_{\mathscr{A}_{\eta\restriction_i}} \mathscr{A}_J^{\alpha_i}$. We now show that this dependence causes a contradiction.

Since $\mathscr{A}_{\eta\restriction_i}$ is sufficiently saturated, by [6] Corollary 3.5 (i), there is $c \in \mathscr{A}_J^{\alpha_i}$ such that

$$\mathscr{A}_{\eta\restriction_{i+1}} \underset{\mathscr{A}_{\eta\restriction_i}}{\not\downarrow} c.$$

Since $\mathscr{A}_J^{\alpha_i} = \bigcup_{s \in G(K^{\alpha_i})} \mathscr{A}_s$, there is a good $s \in K^{\alpha_i}$ such that $c \in \mathscr{A}_s$.

Now, let $r = \{\eta \restriction_j : j \leqslant i+1\}$. Then, r is good and $r \cap J^{\alpha_i} = \{\eta \restriction_j : j \leqslant i\}$. Without loss of generality, we can assume that $\eta \restriction_i \in s$, since $\mathscr{A}_s$ cannot get smaller with this assumption.

However, by strong independence (see [6]), $\mathscr{A}_r \downarrow_{\mathscr{A}_{r\cap s}} \mathscr{A}_s$, which by definition, written otherwise

$$\mathscr{A}_{\eta\restriction_{i+1}} \underset{\mathscr{A}_{\eta\restriction_i}}{\downarrow} \mathscr{A}_s.$$

This gives a contradiction since $c \in \mathscr{A}_s$. $\boxtimes_{\text{Claim 3.12.1}}$

Thus, there is $u \in K'$ such that for all $i < \omega$, $\mathscr{A}_u \not\downarrow_{\mathscr{A}^{\alpha_i}_{J'}} \mathscr{A}^{\alpha_{i+1}}_{J'}$. However, since $\alpha \notin S(J')$, this contradicts Lemma 7 of [6]. $\square_{3.12}$

Corollary 3.13. *Let $\mathscr{A}^J$ and $\mathscr{A}^{J'}$ be models constructed as above for trees $J, J' \subseteq \pi^{\leqslant\omega}$. Assume that $S(J) = S \subset S^\pi_\omega$ and $S(J') = \emptyset$, thus $S(J) \triangle S(J') = S$ is stationary. Then $\mathscr{A}^J \not\cong \mathscr{A}^{J'}$ in any cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{m}))$-preserving extension of the universe where the symmetric difference $S(J) \triangle S(J')$ remains stationary.*

Notice that the proof of Lemma 3.12 is in ZFC. In particular, the notion of independence is absolute for models where no small (of size $< \lambda_r(\mathfrak{m})$) subsets are added. Thus, two models $\mathscr{A}_J$ and $\mathscr{A}_{J'}$ which are non-isomorphic in the ground model remain non-isomorphic in any cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{m}))$-preserving extension of the universe where the symmetric difference $S(J) \triangle S(J')$ remains stationary.

It is easy to see that $S(J_0) = S$ and $S(J_1) = \emptyset$. Thus, we can apply the previous lemma to find that in $\mathbb{L}$, $\mathscr{A}_{J_0} \not\cong \mathscr{A}_{J_1}$.

3.4. Isomorphism of the models when S is killed.

Theorem 3.14. *Assume that in some extension of the set theoretic universe which preserves cardinals and $\mathscr{P}(\lambda_r(\mathfrak{m}))$, $J_0 \cong J_1$. Then in that extension $\mathscr{A}_{J_0} \cong \mathscr{A}_{J_1}$.*

Proof. Assume that $F : J_0 \longrightarrow J_1$ is an isomorphism. We aim to find an isomorphism between $\mathscr{A}_{J_0}$ and $\mathscr{A}_{J_1}$.

We proceed by induction on good elements of K_0 along the ordering $\leq^K$ by building elementary maps G_u, $u \in K_0$. We ensure in this induction that if $u_i \leq^K u_j$ and $u_j \not\leq^K u_i$ then G_{u_i} is constructed before G_{u_j}.

Base case – isomorphism for the first level of the tree G_0: For all $u \in K_0 = \mathscr{P}^{<\omega}(J_0)$, let $F(u) = \{F(\eta) : \eta \in u\}$. For $\eta \in J_0$, let $G_0 \restriction_{\mathscr{A}_\eta} = f_F^{-1}(\eta) \circ f_\eta$, where the f_η are as defined on page 10. Thus,

$$G_0 : \bigcup_{\eta\in J_0} \mathscr{A}_\eta \longrightarrow \bigcup_{\eta\in J_1} \mathscr{A}_\eta.$$

Claim 3.14.1. The function G_0, which maps one strongly independent family to the other, is elementary.

We prove the claim by induction on good elements $s \in K_0$ along the ordering $\leq$. Denote by $G_0^\eta = f^{-1}_{F(\eta)} \circ f_\eta$, and by $G_0^s = \bigcup_{\xi\in s} G_0^\xi$, for $s \in G(J_0)$.

By construction, G_0^η, $\eta \in J_0$ is elementary.

Now, assume that G_0^s has been shown to be elementary. We wish to show that $G_0^{s'}$ for $s' \geq^K s$ is also elementary. Our ordering of $G(J_0)$ implies that it is enough to consider $s' = s \cup \{\eta\}$ for some $\eta \in J_0$. We thus have two cases: $\eta \in \pi^{<\omega}$ or $\eta \in \pi^\omega$. The arguments for both are similar.

If $\eta \in \pi^{<\omega}$, denote by $\eta^- = \eta \restriction_{(\text{length}(\eta)-1)}$. If $\eta \in \pi^\omega$ is an infinite branch, then we can then find $i < \omega$ such that $\forall \xi \in s$, $\xi \not> \eta \restriction_i$. Denote by $\eta^- = \eta \restriction_{i-1)}$.

Since we are working in a homogeneous monster model $\mathfrak{M}$, we can assume without loss of generality that $G_0^s \restriction_{A_s} = id_{A_s}$.

In addition, we know from the construction that

$$\mathrm{tp}_{\mathfrak{M}}(\mathscr{A}_\eta/\mathscr{A}_{\eta^-}) = \mathrm{tp}_{\mathfrak{M}}(G_0^\eta(\mathscr{A}_\eta)/\mathscr{A}_{\eta^-}),$$

because G_0^η is elementary and $G_0^\eta \restriction_{\mathscr{A}_\eta} = id$. We thus want to show that

$$\mathrm{tp}_{\mathfrak{M}}(\mathscr{A}_\eta/A_s) = \mathrm{tp}_{\mathfrak{M}}(G_0^\eta(\mathscr{A}_\eta)/A_s)$$

. Since $\mathscr{A}_\eta$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda(\mathfrak{M})}$-saturated, these types are stationary. Therefore, by definition of stationarity (Def. 3.3. [7]) it is enough to show that

$$\mathscr{A}_\eta \underset{\mathscr{A}_{\eta^-}}{\downarrow} A_s \text{ and } G_0^\eta(\mathscr{A}_\eta) \underset{\mathscr{A}_{\eta^-}}{\downarrow} A_s.$$

However, note that $\eta^- \in s = f(s)$, thus we have the independence by construction, and so the embedding is elementary.

$\square_{\text{Claim 3.14.1}}$

Before we continue with the next step of the induction, we give some notation and reminders. Denote $\mathscr{A}_{\{\eta\}} = \mathscr{A}_\eta$. Recall that by the construction, $A_u = \bigcup_{\eta \in u} \mathscr{A}_\eta$ for $u \in K_i$, and $\mathscr{A}_u$ is $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-prime over A_u.

Ultimately, we aim to build an isomorphism $G : \mathscr{A}^{J_0} \longrightarrow \mathscr{A}^{J_1}$ such that $G \restriction_{\mathscr{A}_\eta} = id_{\mathscr{A}_\eta}$ for all $\eta \in J_i$. Since $\mathscr{A}^{J_i} = \bigcup_{u \in K_i} \mathscr{A}_u^i$, it is enough to construct $G_u : \mathscr{A}_u^0 \longrightarrow \mathscr{A}_u^1$ such that if $t \leq^K u$, then $G_t \subseteq G_u$. If we can show that $\bigcup_{t \leq^K u} G_t$ is elementary, then using homogeneity of $\mathfrak{M}$, we can find the desired isomorphism G_u. The full isomorphism will then be $G = \bigcup_{u \in G(K_i)} G_u$.

Inductive step: Assume we have shown that for all $t \lneq u$, G_t are isomorphisms. We build an isomorphism $G_u : \mathscr{A}_u^0 \longrightarrow \mathscr{A}_u^1$.

Claim 3.14.2. The function $\bigcup_{t <^K u} G_t$ is elementary.

We assume for a contradiction that the inductive step fails at some point. Let u be the $\leq^K$-smallest such that $\bigcup_{t \leq^K u} G_t = G^*$ is not elementary.

This failure of elementariness is witnessed by some set of finitely many points $a_0, \dots, a_m \in \bigcup_{t \leq^K u} \mathscr{A}_t$. Then, in particular, $G^* \restriction_{\{a_0, \dots, a_m\}}$ is not elementary.

Subclaim 3.14.2.1. The points $a_0, \dots, a_m$ can be replaced with tuples $\bar{a}_i$, $i = 0, \dots n$ which appear all at once at a given step in the construction, that is, $\bar{a}_i \in \mathscr{A}_{t_i}$ and $\bar{a}_i \cap \bigcup_{t <^K t_i} \mathscr{A}_i = \emptyset$.

Consider $a_0, \dots, a_m$. For all $i \leqslant m$, there is $t^i <^K u$ such that $a_i \in \mathscr{A}_{t^i} \setminus \bigcup_{t <^K t^i} \mathscr{A}_t$. Let $\{t_0, \dots, t_n\}$ be an enumeration of the t^i so that $t_i \neq t_j$ if $i \neq j$ (i.e., we get rid of repetitions). In addition, we can assume without loss of generality that t_n is maximal in $\{t_0, \dots, t_n\}$ with respect to the ordering $\leq^K$.

Define $\bar{a}_i = \{a_j : t^j = t_i\}$. Then $\bar{a}_i$ is the desired tuple. such that $\bar{a}_i \in \mathscr{A}_{t_i}$ and $\bar{a}_i \cap \bigcup_{t < t_i} \mathscr{A}_i = \emptyset$. $\square_{\text{Subclaim 3.14.2.1}}$

To save ink, we will denote the tuples $\bar{a}_i$ as a_i, and now consider the finite set of tuples $\{a_0, \dots, a_n\}$.

We wish to refine this choice of witnesses $\{a_0, \dots, a_n\}$ to minimize the t_n and the number n. To this end, we devise an ordering on $\mathscr{P}^{\leqslant\omega}(K_i)$:

DEFINITION 3.15. For $t_i, u_i \in K_i$, we say that $\{t_i : i \leqslant n\} \Subset \{u_i : i \leqslant m\}$ iff for all $i \leqslant n$ there is $j \in m$ such that $t_i \leq u_j$ and there is u_j such that $u_j \not\leq t_i$ for every $i \leqslant n$.

We can minimize the choice of witnesses $\{a_0, \dots, a_n\}$ easily if there are only finitely many candidates which may be smaller than our initial choice. We will assume otherwise, and, using Ramsey's Theorem, come to a contradiction. Thus, assume for a contradiction, that there are infinitely many choices of witnesses $\{a_0, \dots, a_n\} = \{a_0^0, \dots, a_{n_0}^0\}$, $\{a_0^1, \dots, a_{n_1}^1\}$, ... , $\{a_0^j, \dots, a_{n_j}^j\}$, ... from $\mathscr{P}^{<\omega}(K)$ for which the associated $\{t_0, \dots, t_n\} = \{t_0^0, \dots, t_{n_0}^0\}$, $\{t_0^1, \dots, t_{n_1}^1\}$, ... , $\{t_0^j, \dots, t_{n_j}^j\}$, ... , are $\Subset$ than our original choice. These are quasi-ordered by $\Subset$.

Subclaim 3.15.0.2. The collection

$$\{t_0^0, \dots, t_{n_0}^0\}, \{t_0^1, \dots, t_{n_1}^1\}, \dots, \{t_0^j, \dots, t_{n_j}^j\}, \dots$$

is a quasi-ordering with no $\Subset$-infinite descending sequences.

For notational simplicity, we will write $X_j = \{t_i^j : i < n_j\}$, and consider them with the ordering $\Subset$.

Assume for a contradiction that there is an infinite descending chain. We assume, without loss of generality, that this chain is enumerated so that $X_{j+1} \Subset X_j$.

Let $u_j \in X_j$ be such that $u_j \not\leq^K t_k^{j+1}$ for every $k < n_{j+1}$ (by definition of $\Subset$, there is at least one such $u_j \in X_j$ for every j).

Thus, for all $j < i < \omega$, $u_j \not\leq^K u_i$. This is because if $i = j+1$, then this is simply the definition of u_j, and otherwise, we can find $k < n_{j+1}$ such that $u_i \leq^K t_k^{j+1}$. So, if $u_j \leq^K u_i$, then $u_j \leq^K t_k^{j+1}$, a contradiction with the definition of u_i.

Since the u_j are finite antichains in J_i, it is easy to see that $\bigcup\{u_j : j < \omega\}$ does not contain infinite decreasing $\leq^J$-chains. By the same argument, there are also no infinite increasing $\leq^J$-sequences.

By Ramsey's Theorem, there must thus be an infinite $\leq^J$-antichain. Thus, we can find t_i^0, $i < n_0$, and an infinite set $X \subseteq \omega$ such that $\{u_j : j \in X\}$ is an $\leq^K$-antichain, and $u_j \leq^K t_i^0$ for all $j \in X$.

Let T be the tree composed of $\eta \in J$, such that $\eta < \xi$ for some $\xi \in t_i^0 \subset J$. We show that since such a tree has no maximal branches, the existence of an infinite $\leq^K$-antichain is not possible.

Note that for all $j < i$ and k, there is n such that $t_k^i \leq^K t_n^j$.

Without loss of generality, we can assume that $u_j = \{u_i^j : i < m\}$. To ensure this, we may need to make X smaller so that $|u_j| \leqslant n_0$, for all $i \in X$.

By applying the Ramsey Theorem m times, we can assume the one of the following for all $i < m$:

1. for all $j < k$, $u_i^k <^K u_i^k$;
2. for all $j < k$, $u_i^j \perp^K u_i^k$;

3. for all $j < k$, $u_i^j \geq^K u_i^k$.

Clearly case 1 is not possible. Furthermore, it is not possible for case 3 holds for all $i < m$. Thus, let $i < m$ be such that 2 holds. Then $\{u_i^j : j \in X\}$ is an infinite $\leq^J$-antichain in T, a contradiction.

$\boxtimes_{\text{Subclaim 3.15.0.2}}$

Assume now that our choice of $\{a_0, \ldots, a_n\}$ and $\{t_0, \ldots, t_n\}$ is minimal in $<^K$.

There is $C \subset \bigcup_{t<t_n} \mathscr{A}_t$, $|C| = \lambda_r(\mathfrak{M})$ such that

$$\mathrm{tp}(a_n/C) \models \mathrm{tp}(a_n/\bigcup_{t<t_n} \mathscr{A}_t).$$

Let $B = C \cup \{a_0, \ldots, a_{n-1}\}$.

On the one hand, let $H = f^B_{(t_n, t_0, \ldots, t_{n-1})}$ be as in Definition 3.9. That is, H is an automorphism of $\mathfrak{M}$ such that $H \restriction_{(B\cap\mathscr{A}_{t_n})} = id_{B\cap\mathscr{A}_{t_n}}$ and for $i < n$,

$$H(B \cap \mathscr{A}_{t_i}) \subseteq \mathscr{A}_{\mathrm{MinSetIn}(t_n,t_i)}.$$

Then, $H(a_i) \in \mathscr{A}_{\mathrm{MinSetIn}(t_n,t_i)}$. Since $\mathrm{MinSetIn}(t_n, t_i) < t_n$, $H(a_i) \in \bigcup_{t'<t_n} \mathscr{A}_{t'}$. Since $H \restriction_C = id$, we have

$$\mathrm{tp}(a_0, \ldots, a_{n-1}/C) = \mathrm{tp}(H(a_0), \ldots, H(a_{n-1})/C)$$

and

$$\mathrm{tp}(a_n/C) \models \mathrm{tp}(a_n/C \cup \{H(a_0), \ldots, H(a_{n-1})\}),$$

so

$$\mathrm{tp}(a_n/C) \models \mathrm{tp}(a_n/C \cup \{a_0, \ldots, a_{n-1}\}).$$

On the other hand, consider G^*. Since $\{a_0, \ldots, a_n\}$ is a minimal witness that G^* is not elementary, the function $G^* \restriction_{C\cup\{a_0,\ldots,a_{n-1}\}}$ must be elementary.

Let G^+ be an automorphism of $\mathfrak{M}$ such that $G^+ \circ G^* \restriction_{C\cup\{a_0,\ldots,a_{n-1}\}} = id$.

Since $G^* \restriction_{\mathscr{A}_{t_n}} = G_{t_n}$ and $C \subseteq \mathscr{A}_{t_n}$, $G^* \restriction_{C\cup a_n}$ is elementary. Thus

$$\mathrm{tp}(G^+(G^*(a_n))/C) = \mathrm{tp}(a_n/C).$$

However,

$$\mathrm{tp}(G^+(G^*(a_n)), a_0, \ldots, a_{n-1}/C) = \mathrm{tp}(G^*(a_n), G^*(a_0), \ldots, G^*(a_{n-1})/G^*(C)),$$

thus

$$\mathrm{tp}(G^*(a_n), G^*(a_0), \ldots, G^*(a_{n-1})/\emptyset) \models \mathrm{tp}(a_n, a_0, \ldots, a_{n-1}/\emptyset).$$

This means that

$$\mathrm{tp}(a_n/C) \not\models \mathrm{tp}(a_n/C \cup a_0, \ldots, a_{n-1}),$$

a contradiction.

$\boxtimes_{\text{Claim 3.14.2}}$

$\square_{3.14}$

Corollary 3.16. *Let $\mathscr{A}^{J_0}$ and $\mathscr{A}^{J_1}$ be models constructed as above for trees J_0 and J_1. Assume that in a cardinal-preserving extension of the universe, $S(J_0)$ is not stationary. Then $\mathscr{A}^{J_0} \cong \mathscr{A}^{J_1}$.*

Proof. Lemmas 7.15 and 7.31 of [4] demonstrate that in the extension, $J_0 \cong J_1$. We can then apply the previous theorem 3.14. $\square_{3.16}$

3.5. Constructibility with respect to $0^{\#}$. We now have all the necessary ingredients to prove Theorem 3.1.

Proof. The result is a direct result of Theorem 2.4 and Corollaries 3.13 and 3.16. $\square_{3.1}$

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