An Introduction to β-Recursion Theory

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Section 1. Background for the Theory.

α-Recursion Theory, or Recursion Theory on (Σ_1)-admissible ordinals, is certainly a very successful generalization of Recursion Theory on ω. It has demonstrated that constructions exploiting the Σ_2 or Σ_3 admissibility of ω can be replaced by finer constructions which rely only on Σ_1 admissibility.

The minimal degree construction however has resisted generalization to arbitrary admissible ordinals, though important progress was made by Richard Shore ([14]) who treated the Σ_2-admissible case. It is instructive to examine his argument, as the attempt to adapt it to all admissible ordinals led to the development of β- Recursion Theory.

Shore's construction can be outlined as follows: If α is Σ_2 admissible, then the structure $< L_\alpha, \epsilon, C >$ is admissible where C is a complete α-recursively enumerable set. A minimal α-degree can then be constructed by applying the α-finite injury method to this structure, much in the way Sacks and Simpson ([13]) first used it in their solution to Post's Problem.

What if α is only Σ_1 admissible? Then the structure $< L_\alpha, \epsilon, C >$ is no longer admissible. Nonetheless, is it still possible to apply the α-finite injury method to this structure, yielding a minimal α-degree?

This leads to the study of Recursion Theory on possibly inadmissible structures $< L_\beta[A], \epsilon, A >$, or β-Recursion Theory. There are two goals for this theory:

1) To produce new constructions of recursively enumerable sets which are not

dependent on any admissibility assumption.

2) To clarify the concepts and techniques used in Admissible Recursion Theory.

In both cases we hope to provide new data in the search for the axioms needed to

do Recursion Theory. In this paper we report on the progress that has been

made in these directions.

Section 2. Basic Notions.

          The correct general setting for β-Recursion Theory is Jensen's

S-Hierarchy for L. For limit ordinals β, S_β has all of the important pro-

perties shared by limit levels of Gödel's L-Hierarchy. We proceed to define the

S-Hierarchy and list these properties, referring the reader to [1] for a more

thorough treatment. For ordinals β such that ω^ω divides β, we have

$S_\beta = L_\beta$, so in this case one may work with the usual L-Hierarchy.

          A function $f: V^n \to V$ is rudimentary if it can be generated from the

following schemata:

$\quad$ i) $f(x_1, \ldots, x_n) = x_i$

$\quad$ ii) $f(\overline{x}) = \{x_i, x_j\}$ $\qquad \}$ $1 \le i, j \le n$

$\quad$ iii) $f(\overline{x}) = x_i - x_j$

$\quad$ iv) $f(\overline{x}) = h(g_1(\overline{x}), \ldots, g_k(\overline{x}))$

$\quad$ v) $f(y, \overline{x}) = \bigcup_{z \in y} g(z, \overline{x})$

$R \subseteq V^n$ is rudimentary if for some rudimentary function f, $\overline{x} \in R \longleftrightarrow f(\overline{x}) = \emptyset$.

If X is transitive, the rudimentary closure of X = the closure of X under the

rudimentary functions. Also define $\text{rud}(X)$ = rudimentary closure of $X \cup \{X\}$.

          Lemma. There is a rudimentary function $\underline{S}$ such that for transitive X,

$\underline{S}(X)$ is transitive, $X \cup \{X\} \subseteq \underline{S}(X)$ and $\text{rud}(X) = \bigcup_{n \in \omega} \underline{S}^n(X)$.

The S-Hierarchy is now defined by:

$$S_0 = \emptyset \, ,$$

$$S_{\alpha+1} = \underline{S}(S_\alpha) \, ,$$

$$S_\lambda = \bigcup_{\alpha < \lambda} S_\alpha \, .$$

Properties of the S-Hierarchy

1) β limit $\longrightarrow S_\beta$ is closed under rudimentary functions, $On(S_\beta) = \beta$.

2) $S_{\beta+\omega} \cap \mathcal{P}(S_\beta) = \{X \subseteq S_\beta \,|\, X \text{ is first-order definable over } <S_\beta, \epsilon >\}$.

3) $L_\beta = S_\beta \longleftrightarrow \omega^\omega$ divides β, $\bigcup_\beta S_\beta = L$.

4) Suppose β is a limit ordinal and $Tr_n = \{<i, x_1, ..., x_n>| \text{ the } i^{th} \Sigma_n \text{ formula } \varphi \text{ is } n\text{-ary and } <S_\beta, \epsilon > \models \varphi(x_1, ..., x_n)\}$. Then Tr_n is Σ_n over $<S_\beta, \epsilon >$, uniformly in β.

5) There is a well-ordering $<$ of L such that for limit β, $< |S_\beta \times S_\beta$ is Σ_1 over $<S_\beta, \epsilon >$, uniformly in β.

The above properties are sufficient to safely adapt the definitions of α-Recursion Theory to an arbitrary $<S_\beta, \epsilon >$, β limit.

For $A \subseteq S_\beta$, we define:

A is β-recursively enumerable (β-r.e.) $\longleftrightarrow A$ is Σ_1-Definable over $<S_\beta, \epsilon >$.

A is β-recursive (β-rec) $\longleftrightarrow A, \overline{A} = S_\beta - A$ are both β-r.e.

A is β-finite $\longleftrightarrow A \in S_\beta$.

By virtue of property 4) above, there is a universal β-r.e. set $W(e, x)$ such that any β-r.e. set is of the form $W_e = \{x | W(e, x)\}$ for some e.

The reducibilities of β-Recursion Theory are also derived from the admissible case. For any $A \subseteq S_\beta$, let $A^* = \{<x, y>| \ x \subseteq A, \ y \subseteq \overline{A} , \ x, y \ \beta\text{-finite}\}$ and $A_f^* = \{<x, y>| \ x \subseteq A, \ y \subseteq \overline{A}, \ x, y \ \text{finite}\}$. Then:

A is $\underline{\text{finitely } \beta\text{-reducible}}$ to B $\longleftrightarrow$ (for some β-r.e. R,

$$(A \leq_{f\beta} B) \qquad x \in A_f^* \longleftrightarrow \exists \ y \in B_f^* [<x, y> \in R])$$

A is <u>weakly β-reducible</u> to B $\leftrightarrow$ $A_f^* \leq_{f\beta} B^*$

$\qquad$ $(A \leq_{w\beta} B)$

A is <u>β-reducible</u> to B $\leftrightarrow$ $A^* \leq_{f\beta} B^*$

$\qquad$ $(A \leq_\beta B)$

Both finitely β-reducible and β-reducible imply weakly β-reducible (but not conversely in general) and these two notions are in general incomparable.

β-reducibility is the fundamental reducibility for β-Recursion Theory.

β-reducibility is reflexive and transitive and thus we may define the β-degree of $A = \{B \mid A \leq_\beta B, B \leq_\beta A\}$; the β-degree are partially-ordered by $\leq_\beta$. There is a smallest β-degree 0 and a largest β-degree of a β-r.e. set, denoted 0' .

Tameness and Regularity

$\qquad$ β-r.e. sets are constructed in stages. As computations from a set A are determined by pairs $< x, y >$, where x and y are β-finite and satisfy $x \subseteq A$, $y \subseteq \overline{A}$, it is convenient to know that such pairs are satisfied by some stage of the construction; i.e., if A^σ denotes the amount of A enumerated by stage σ, we would like to know that:

(*)x $\quad$ β-finite, $x \subseteq A \longrightarrow$ For some σ, $x \subseteq A^\sigma$.

In this case, the enumeration $\{A^\sigma\}_{\sigma < \beta}$ is said to be <u>tame</u>. More precisely, if $\exists y \varphi(x, y)$ (φ a Δ_0 formula) defines A over $<S_\beta, \epsilon >$, then it gives rise to the enumeration $\{A^\sigma\}$ where $x \epsilon A^\sigma \leftrightarrow \exists y \epsilon S_\sigma \varphi(x, y) \wedge x \epsilon S_\sigma$. A is <u>tamely-r.e.</u> if A has such an enumeration with the property (*) above. This is equivalent to the assertion that $\{\sigma\text{-finite } x \mid x \subseteq A\}$ is β-r.e. From this it follows that deg $A = 0 \leftrightarrow A, \overline{A}$ are both tamely-r.e.

$\qquad$ <u>Theorem 1</u> ([3]). Assume that β is inadmissible. Let W be a universal β-r.e. set. Then there is a β-recursive set A such that $0 <_\beta A <_\beta W$ and every β-recursive or tamely-r.e. set is β-reducible to A.

It follows that in the inadmissible case there are β-recursive sets of non-zero β-degree. In particular, t.r.e. $\neq$ r.e. and $\leq_{w\beta} \neq \leq_\beta$. Theorem 1 provides a weak solution to Post's Problem in β-Recursion Theory (the question of the existence of β-r.e. degrees between 0 and 0'). The solution is weak because it does not provide incomparable β-r.e. degrees; moreover, in this case we have $W \leq_{w\beta} A \leq_{w\beta} 0$. Denoting deg(A) by $0^{1/2}$, we are led to the following picture of the β-r.e. degrees:

In general, however, nonzero tamely-r.e. degrees will not exist (though $0^{1/2}$ provides an example of a nonzero β-recursive degree). Simple questions regarding these degrees remain unsettled; in particular, it is not known if the t.r.e. degrees or the β-recursive degrees form an initial segment of the β-r.e. degrees. It follows from [10], though, that the t.r.e. degrees are always contained in the β-recursive degrees when β is inadmissible.

$A \subseteq S_\beta$ is <u>regular</u> if $A \cap x$ is β-finite whenever x is β-finite. It is a theorem of Sacks ([12]) in the case that β is admissible that every β-r.e. degree has a regular β-r.e. representative. Regular β-r.e. sets are more rare for inadmissible β; in fact, for some β's, every regular β-r.e. set (even every regular set) has degree 0 (see [3]). However, a slight extension of t.r.e.-ness is enough to guarantee the existence of regular, β-r.e. representatives:

Theorem 2 ([3], [10]).   If $\{x \mid x \text{ is } \beta\text{-finite, } x \cap A \neq \emptyset\}$ is t.r.e., then

A has the same β-degree as a regular β-r.e. set.

However, for some β's there are t.r.e. sets which do not lie in the same

β-degree as some regular set. Thus, there appears to be no simple characteri-

zation of the regular β-r.e. degrees.

The reader is referred to [10] for proofs of the above facts as well as

further information concerning the β-r.e., t.r.e., and regular β-r.e. degrees.

In the further development of the theory, the limit ordinals fall into two

classes determined by their degree of admissibility. This split into cases was

first revealed in Jensen's proof of Σ_2-Uniformization for S_β, and is determined

by the values of certain key parameters which we now proceed to define.

A relation on S_β is Σ_n if it can be defined over $<S_\beta, \epsilon>$ by a formula

consisting of n alternating unbounded quantifiers (beginning with an existential)

followed by a limited formula.   A function is Σ_n if its graph is.

The first type of parameter that we define measures the extent to which β

is not a cardinal. The $\underline{\Sigma_n\text{-projectum}}$ of β, ρ_n^β, is the least ordinal γ such

that there is a Σ_n injection of β into γ. Jensen shows ([9]) that this is the same

as the least γ such that some Σ_n subset of γ is not β-finite. As there is always

a Σ_1 bijection between β and S_β (see [1]), we can in fact inject S_β into ρ_n^β via a

Σ_n function.

Our second set of parameters describes the extent to which β is singular.

The $\underline{\Sigma_n\text{-cofinality}}$ of β, $\Sigma_n \text{cf} \beta$, is the least γ such that some Σ_n function with

domain γ has range unbounded in β. If β is Σ_{n-1}-admissible, then this is the

same as the least γ such that some Σ_n function with domain γ is not β-finite

(though this equivalence is not true for all β).

In case $n = 1$, ρ_1^β and $\Sigma_1 cf \beta$ are alternatively written β^* and $R cf \beta$,

respectively. (R cf abbreviates "Recursive Cofinality".) We shall be mostly

concerned with β^*, $R cf \beta$, ρ_2^β and $\Sigma_2 cf \beta$. Note that if β is admissible and A

is a regular β-r.e. set of degree $0'$, then ρ_2^β and $\Sigma_2 cf \beta$ are just the Σ_1-

projectum and Σ_1 cofinality of the relativized structure $< L_\beta, \epsilon, A >$.

In case $R cf \beta \geq \beta^*$ we say that β is weakly admissible. In this case,

many of the arguments from admissibility theory apply. The reason for this is

that many priority arguments use β^* to index a listing of requirements and the

above assumption allows one to perform Σ_1 inductions of length β^*.

Σ_2-Uniformization is also easy in this case. If β is admissible and

$\Sigma_2 cf \beta \geq \rho_2^\beta$, then we say that β is weakly Σ_2-admissible. In this case, one can

carry out the construction of minimal β-degrees, minimal pairs of β-r.e.

degrees and major subsets of β-r.e. sets.

If $R cf \beta < \beta^*$ we say that β is strongly inadmissible. In this case, the

arguments of admissibility theory do not apply and new techniques are needed.

This is the difficult case of Σ_2-Uniformization. If β is admissible and

$\Sigma_2 cf \beta < \rho_2^\beta$, then we say that β is strongly Σ_2-inadmissible. The constructions

of minimal β-degrees, minimal pairs of β-r.e. degrees and major subsets of

β-r.e. sets are all very difficult for such β and have only been accomplished in

very special cases. However, the techniques of β-Recursion Theory are now

beginning to apply themselves to this case (see Section 5).

Section 3. Weak Admissibility

As mentioned before, the methods of α-Recursion Theory apply in this

case. In particular, the method of Shore blocking (see [17]) was used in [3] to

prove:

Theorem 3.   If β is weakly admissible, then there are regular t.r.e.
sets A, B such that $A \not\leq_{w\beta} B$, $B \not\leq_{w\beta} A$.

W. Maass (in [10]) has found a technique for transferring many results
from α-Recursion Theory to arbitrary weakly admissible ordinals. He asso-
ciates to each weakly admissible β an admissible structure $\mathcal{U}$ such that
$\mathcal{U}$-r.e. degrees embed into the β-r.e. degrees.   In this way, known results
about the admissible structure $\mathcal{U}$ have consequences about the β-r.e. degrees.
We now describe his construction in more detail.

Let $\kappa = \Sigma_1 \mathrm{cf}\,\beta$. As $\kappa \geq \beta^*$, there certainly is a Σ_1 injection of β into κ.
In fact, more is true: there is a Σ_1 bijection of β onto κ (see [3], p. 15).
Let $f: \beta \to \kappa$ be such a bijection. Let

$$< e, x, \sigma > \in \widetilde{T} \longleftrightarrow x \in W_e^\sigma$$

and $T = f[\widetilde{T}]$. Then $T \subseteq \kappa$, T is β-recursive and $\mathcal{U} = <L_\kappa, \epsilon, T>$ is ad-
missible. Moreover, if $A \subseteq \kappa$, then A is β-r.e. if and only if A is $\mathcal{U}$-r.e.
(Σ_1 over $\mathcal{U}$). Define $\leq_{\mathcal{U}}$ analogously to $\leq_\beta$. Then these two reducibilities
do not necessarily agree on subsets of κ. $A \subseteq \kappa$ is β-immune if

  x β-finite, $x \subseteq A \longrightarrow x \in L_\kappa$,

  x β-finite, $x \subseteq \kappa - A \longrightarrow x \in L_\kappa$.

Then $\leq_{\mathcal{U}}$ and $\leq_\beta$ do agree on β-immune sets.   Maass shows that every
$\mathcal{U}$-r.e. degree has a β-immune $\mathcal{U}$-r.e. representative. This gives an em-
bedding E of the $\mathcal{U}$-r.e. degrees 1-1 into the β-r.e. degrees.

Theorem 4 (Maass) (β weakly admissible).   The range of E = the t.r.e.
degrees = the recursive degrees. E (complete $\mathcal{U}$-r.e. set) $= 0^{1/2}$.

An application of admissibility theory to $\mathcal{U}$ ([15] and [16]) yields:

Corollary.   Any nonzero t. r. e degree is the join of two lesser t. r. e.
degrees.   If one t. r. e. degree is below another, then there is a t. r. e. degree
in between.

Section 4.   Strong Inadmissibility

This is the most challenging case for β-Recursion Theory, for the lack of
admissibility is now so strong that many of the ideas from the admissible case
become useless.   The alternative is to employ deeper techniques from the Fine
Structure of L as developed initially by Gödel [8] and more extensively by
Jensen [9].   All of these techniques emanate from two basic lemmas due to
Godel:

Lemma.   For each limit ordinal β, there is a partial function
$h: \omega \times S_\beta \to S_\beta$ which is Σ_1 over S_β such that for any Σ_1 formula $\varphi(x, p)$,

$$<S_\beta, \epsilon> \models \exists\, x\varphi(x, p) \longrightarrow \exists\, i \in \omega\varphi(h(i, p), p) \,.$$

Proof.   Recall the canonical Σ_1 well-ordering $<$ of S_β.   Then if the
i^{th} Σ_1 formula is $\exists\, y \psi(x, p, y)$,   define $h'(i, p) \simeq$ least (in the sense of $<$)
pair $<x, y>$ such that $\psi(x, p, y)$.   Then $h(i, p)$ = first component of $h'(i, p)$.   ⊣

The h above is called the canonical Σ_1 skolem function for S_β .

Transitive Collapse Lemma.   If $X \prec_1 S_\beta$ (i.e., $X \subseteq S_\beta$ and any Σ_1
formula with parameters from X and a solution in S_β has a solution in X) then
$< X, \epsilon >$ is isomorphic to a unique $<S_\gamma, \epsilon >$.

Using these two lemmas, we can now illustrate in a simple example how
Fine Structure technique can be used to generalize to arbitrary β a result whose
"recursion-theoretic" proof only succeeds for admissible β.

Proposition (Jensen).   Suppose $A \subseteq \gamma < \beta^*$ and A is β-r.e.   Then

A is β-finite.

Proof Number 1, β admissible.   If A is not β-finite, then it has a 1-1

β-recursive listing $f: \beta \to A$.   But then $\beta^* \leq \sup A \leq \gamma < \beta^*$,   contradiction.

Proof Number 2, β arbitrary.   Let $p \in S_\beta$ be a parameter defining A as a

set Σ_1 over S_β. Form X = Range h on $\omega \times (\gamma \cup \{p\})$, where h is from the

Lemma.   Then $X \prec_1 S_\beta$, so apply the Transitive Collapse Lemma to get

$j: X \simeq S_\delta$.   Let $g = j \circ h$.

Now g is Σ_1 over S_δ (simply transfer the Σ_1 definition for h over X

to S_δ). Then so is g^{-1}. But if f uniformizes g^{-1}, f Σ_1 over S_δ, we see that

f injects S_δ into $\omega \times (\gamma \cup \{p\})$; hence into γ. Since $\gamma < \beta^*$, we have proved

that $\delta < \beta$.

But A is Σ_1 definable over S_δ, so $A \in S_\beta$.   ⊣

Further ideas of Jensen, in particular an effectivized version of his ◇

principle, were used in [4] to establish:

Theorem 5.   If β^* is regular with respect to β-recursive functions,

then there are β-r.e. sets $A, B \subseteq \beta^*$ such that $A \not\leq_{w\beta} B$, $B \not\leq_{w\beta} A$.

This is the best solution to Post's Problem so far known in the strongly

inadmissible case. This covers the case where $S_\beta \models$ "β^* is a successor

cardinal".

Open Problem.   Does the conclusion of Theorem 5 hold for arbitrary strongly

inadmissible β ?

Forcing can be used to achieve a stronger and more model-theoretic in-

comparability than that in Theorem 5. The following result will appear in [5]:

Theorem 6.   Assume β^* is regular with respect to β-recursive functions

and $S_\beta \models$ "β^* is the largest cardinal."   Then there are β-r.e. sets $A, B \subseteq \beta^*$

such that

     A is not Δ_1 over $<S_\beta[B], \epsilon>$,

     B is not Δ_1 over $<S_\beta[A], \epsilon>$.

($S_\beta[A]$ is the βth level of the $S[A]$-hierarchy.   This hierarchy is defined exactly as the S-hierarchy except the function $f(x) = A \cap x$ is added to the schemes for the rudimentary functions.)

We conclude this section by sketching the proof of a theorem which illustrates the use of Skolem Hulls and $\diamondsuit$ in β-Recursion Theory.

    <u>Theorem 7.</u>   There are β-r.e. sets A, B such that $A \not\leq_{f\beta} B$, $B \not\leq_{f\beta} A$.

The proof of this theorem is not uniform in the sense that it divides into cases depending on the nature of β. Thus, the sets A, B will be defined relative to the choice of a parameter $p \in S_\beta$.

<u>Open Problem.</u>   Can Theorem 7 be made uniform in that the sets A, B have parameter-free Σ_1 definitions independent of β?

We believe that the answer is "yes." In fact, we

<u>Conjecture.</u>   There are integers m, n such that for all limit ordinals β,

$$W_m^\beta \text{ is not } \Delta_1 \text{ over } <S_\beta[W_n^\beta], \epsilon> ,$$

$$W_n^\beta \text{ is not } \Delta_1 \text{ over } <S_\beta[W_m^\beta], \epsilon> ,$$

where W_n^β = the nth parameter-free β-r.e. set.

Before giving our proof sketch of Theorem 7, we make some preliminary definitions and remarks. In view of Theorem 3, it suffices to prove Theorem 7 in the strongly inadmissible case. Choose β-recursive functions $f_0 : S_\beta \xrightarrow{1-1} \beta^*$ and $g_0 : \mathrm{Rcf}\,\beta \twoheadrightarrow \beta$ such that g_0 is order-preserving, Range g_0 unbounded in β. Let $p_0' \in S_\beta$ be such that both f_0 and g_0 are Σ_1 over S_β in the parameter p_0'. Let $p_0 = <p_0', \beta^*>$.

Let $h(i, p)$ be the canonical Σ_1 skolen function for S_β and for $\gamma < \beta^*$

define $H(\gamma) = \{h(i, <\gamma', p_0>) \mid i \in \omega, \gamma' < \gamma\}$. Thus $H(\gamma)$ is the "Σ_1 Skolem Hull

of $\gamma \cup \{p_0\}$". Note that $\bigcup_{\gamma < \beta^*} H(\gamma) = S_\beta$. In our construction, $H(\gamma)$ consists

of those reduction procedures e of priority γ. (Thus the construction is re-

dundant in that each reduction procedure is assigned a final segment of different

priorities.) Of special importance are those $\gamma < \beta^*$ such that $\gamma \notin H(\gamma)$.

<u>Claim.</u>   Let $\kappa < \beta^*$ be a β-cardinal (i.e., $S_\beta \models$ "κ is a cardinal"). Let $\kappa^+ =$

the next β-cardinal. Then $\{\gamma < \kappa^+ \mid \gamma \notin H(\gamma)\}$ is closed, unbounded in κ^+.

<u>Proof.</u>   See [4], Page 24. $\dashv$

Let $\mathcal{J} = \{\gamma < \beta^* \mid \gamma \notin H(\gamma)$ and γ is not a β-cardinal$\}$. The form of

$\Diamond$ that we need (which more resembles $\Diamond'$ in fact) reads as follows:

$\Diamond_{\beta^*}$:   There is a sequence $<D_\gamma \mid \gamma < \beta^*>$, Σ_1-definable without parameter

over L_{β^*}, such that

  1) $D_\gamma \subseteq$ Power Set of γ

  2) $D_\gamma \in L_{\gamma^+}$ (where $\gamma^+ =$ least β-cardinal $> \gamma$)

  3) If $A \subseteq \beta^*$ is β-r.e. with parameter p_0, then

$$\gamma \in \mathcal{J} \longrightarrow A \cap \gamma \in D_\gamma.$$

<u>Proof.</u>   See [4], Page 25. $D_\gamma = \{x \subseteq \gamma \mid x \in L_{\hat{\gamma}}\}$ where $\hat{\gamma} =$ least δ such that

γ is not a δ-cardinal. $\dashv$

We are now ready to outline the construction. We wish to satisfy the

requirements:

R_e^A : $\overline{B} \neq \{x \mid \exists$ finite $y \subseteq \overline{A}$ $(<x, y> \in W_e)\}$

R_e^B : $\overline{A} \neq \{x \mid \exists$ finite $y \subseteq \overline{B}$ $(<x, y> \in W_e)\}$.

It is easy to see that satisfying these requirements for each $e \in S_\beta$ guarantees

$A \not\leq_{f\beta} B$, $B \not\leq_{f\beta} A$. Here, $W_e = e^{th}$ β-r.e. set. Our method for attempting to

satisfy R_e^A is to put some x into B at stage σ if some finite $y \subseteq \overline{A^\sigma}$ satisfies

$<x, y> \in W_e^\sigma$. ($A^\sigma =$ part of A enumerated by stage σ, similarly for W_e^σ.)

If we can succeed in guaranteeing $y \subsetneq \overline{A^{\sigma'}}$ for all $\sigma' > \sigma$, R_e^A will be satisfied.

These attempts at the above requirements conflict with each other. The solution is to order the requirements in a list, the requirements lower on the list having higher priority. In this construction, each requirement R_e^A (or R_e^B) is assigned all of the priorities $\gamma < \beta^*$ such that $e \in H(\gamma)$.

The construction proceeds in $R \text{ cf } \beta$ steps. Recall the function $g_0 : R \text{ cf } \beta \rightarrow \beta$. At each stage σ we will use an approximation to the set $\mathcal{A}$ and the function H. Let $H^\sigma(\gamma) = \{ h^\sigma(i, <\gamma', p_0>) | \, i \in \omega \, , \gamma' < \gamma \}$ where h^σ is the canonical Σ_1 skolem function for $S_{g_0(\sigma)}$. (If β is of the form $\beta' + \omega$, let $H^\sigma(\gamma) = H(\gamma) \cap g_0(\sigma)$.) Then define $\mathcal{A}_\sigma = \{ \gamma < \beta^* | \, \gamma \in H^\sigma(\gamma) \}$.

Stage σ.   Do the following for each $\gamma \in \mathcal{A}_\sigma$: Form all pairs $<R_e^A, z>$, $<R_e^B, z>$ where $z \in D_\gamma$, $e \in H^\sigma(\gamma)$. Order such pairs in a list and choose the least β-finite bijection j between $H^\sigma(\gamma)$ and γ. Given that earlier pairs have been considered, attack $<R_e^A, z>$ as follows: See if there is an $x \notin H^\sigma(\gamma)$, $x > \gamma$, x not being restrained from entering B by γ, and a finite $y \subseteq \overline{A^\sigma}$ such that $<x, y> \in W_e^\sigma$ and $y \cap j^{-1}[z] = \emptyset$.   Then for the least such pair $<x, y>$, put x into B and have γ restrain the members of y from entering A. The pairs $<R_e^B, z>$ are handled similarly.   This ends the construction.

The idea, then, is that the members of D_γ (via the bijection $j : H^\sigma(\gamma) \rightarrow \gamma$) provide "guesses" at $A \cap H^\sigma(\gamma)$, $B \cap H^\sigma(\gamma)$. Of course, since the parameter p_0 defines the entire construction, $\diamondsuit_{\beta^*}$ implies that $j[A \cap H^\sigma(\gamma)]$, $j[B \cap H^\sigma(\gamma)] \in D_\gamma$ if $\gamma \in \mathcal{A}$. So for $\gamma \in \mathcal{A}$, one of the "guesses" is correct. Then these guesses are each used to search for an x and y which attempt to satisfy R_e^A (or R_e^B).

Now each pair $<R^A_e, z>$, $e \in H(\gamma)$, $z \in D_\gamma$ is attacked <u>at most once</u> at each stage of the construction; thus, any x put into A or B by γ and any y restrained from intersecting A or B by γ must belong to $H(\gamma+1)$.

Lemma. Suppose $y \in H^\sigma(\gamma+1) - H^\sigma(\gamma)$ and $\gamma \in \mathscr{L}$. Then $y \notin H(\gamma)$.

Proof. Otherwise, let $y \in H^{\sigma'}(\gamma')$, $\gamma' < \gamma$, σ' least. Assume that γ' and γ have the same β-cardinality κ. Let δ_0 be the least $\delta < \kappa^+$ so that for some $\tau < \sigma'$, $y \in H^\tau(\delta) - H^\tau(\gamma')$. Then $\delta \in H^{\sigma'}(\gamma'+1)$ and $\delta \geq \gamma$. But as $\kappa \leq \gamma'$, $\delta \subseteq H^{\sigma'}(\gamma'+1)$ and so $\gamma \in H^{\sigma'}(\gamma'+1)$. This contradicts $\gamma \in \mathscr{L}$. $\dashv$

Now any y restrained by γ at stage σ must belong to $H^\sigma(\gamma+1) - H^\sigma(\gamma)$. Thus, by the Lemma, if $\gamma \in \mathscr{L}$, we have $y \notin H(\gamma)$. But then as each $\gamma' < \gamma$ only puts members of $H(\gamma)$ into A or B, no member of y can ever be put into A or B. If in addition the attempt associated with y used a correct guess z for $A \cap H^\sigma(\gamma)$ (or $B \cap H^\sigma(\gamma)$), then this attempt will succeed and the corresponding requirement R^A_e (or R^B_e) will be satisfied.

Lastly, note that no $\gamma \in \mathscr{L}$ can ever be put into A or B, by construction. Thus we may argue for $B \not\leq_{f\beta} A$ as follows ($A \leq_{f\beta} B$ is similar): If $\overline{B} = \{x \mid \exists \text{ finite } y \subseteq \overline{A}, <x, y> \in W_e\}$, then choose $\gamma \in \mathscr{L}$ and σ so that if $\gamma' = $ least member of $\mathscr{L}$ greater than γ,

1) $\exists \text{ finite } y \subseteq \overline{A^\sigma}, <\gamma', y> \in W^\sigma_e$,

2) $y \cap (A \cap H^\sigma(\gamma)) = \emptyset$,

3) $e \in H^\sigma(\gamma)$.

Such a γ and σ exist since $\mathscr{L} \subseteq \overline{B}$. Then there must be an attempt made at this stage or an earlier stage for the pair $<R^A_e, j[A \cap H^\sigma(\gamma)]>$. By earlier Remarks, this attempt will succeed. End of proof sketch.

Section 5.   Minimal α-Degrees Revisited

          We return now to the original problem which motivated our study.

Have we learned anything new concerning minimal α-degrees through the study

of β-Recursion Theory?   The following result gives an affirmative answer:

          Theorem 8.   If $\alpha^* = \alpha$ and ρ_2^α is a successor α-cardinal, then there

is a minimal α-degree which is α-r.e. in $0'$.

          The proof, which applies the techniques of β-Recursion Theory to the

structure $<L_\alpha, \epsilon, C>$ (C a complete α-r.e. set), will appear in [6].

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