

AN INTRODUCTION TO THE ADMISSIBILITY SPECTRUM

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The admissibility spectrum provides a useful invariant for studying definability properties of reals. An ordinal α is *R-admissible* if $L_\alpha(R)$ obeys Σ_1 replacement. If R is a subset of ω , let $\Lambda(R)$ denote the class of all *R-admissible* ordinals greater than ω . Then $\Lambda(R)$ is a proper class containing all $L(R)$ -cardinals. The least element of $\Lambda(R)$ is precisely ω_1^R , the least non-*R*-recursive ordinal.

The ordinal ω_1^R has received a great deal of attention in the literature. It can be characterized in many equivalent ways: the least *R*-admissible greater than ω , the least non-*R*-recursive ordinal, the closure ordinal for *R*-arithmetical positive inductive definitions, the least α such that the logic \mathcal{L}_α , $A = L_\alpha(R)$, is Σ_1 compact. A beautiful relationship between ω_1^R and the hyperdegree of R was discovered by Spector.

SPECTOR CRITERION. $\omega_1^R > \omega_1^{ck}$ iff $\mathcal{O} \leq_h R$ (where \mathcal{O} is Kleene's complete Π_1^1 set of integers and \leq_h is hyperarithmetical reducibility).

It is reasonable to expect that other elements of the admissibility spectrum $\Lambda(R)$ would provide further information concerning definability properties of R . This is illustrated below; in particular there is a natural generalization of Spector's Criterion which relates $\Lambda(R)$ to the *L*-degree of R .

1. Early results

Work of SACKS [1976] and JENSEN [1972] characterizes the countable sets which can occur as an initial segment of $\Lambda(R)$ for some real R . We present

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proofs of these results in this section which are somewhat simpler than the original ones. (SACKS [1976] actually proves a result stronger than what we consider here. See the discussion at the end of this section.)

THEOREM 1 (Sacks). *If $\alpha > \omega$ is admissible and countable then there is a real R such that $\omega_1^R = \alpha$.*

PROOF (Almost Disjoint Forcing). We can assume that α is a limit of admissibles as otherwise if $\beta = \sup(\alpha \cap \text{Adm})$, we can force over L_α with finite conditions from ω into β ; this produces a generic real R so that α is the least R -admissible greater than ω . (Admissibility is always preserved when forcing with a set of conditions which is an element of the ground model.)

Now the desired real R is obtained in two steps.

Step 1. Find $A \subseteq \alpha$ so that $\beta \in \text{Adm} \cap \alpha \rightarrow L_\beta[A]$ is inadmissible.

Step 2. “Code” A by a real R so that $\beta \in \text{Adm} \cap \alpha \rightarrow A \cap \beta$ is $\Delta_1(L_\beta(R))$.

In both steps we of course want to preserve the admissibility of α .

To accomplish Step 1 first force $A_0 \subseteq \alpha$ so that $L_\alpha[A_0]$ is *locally countable*; i.e., $L_\alpha[A_0] \models$ “Every set is countable”. This can be done by forcing with finite conditions p from $\alpha \times \omega$ into α with the property that $p(\beta, n) < \beta$. Note that if \mathcal{P}_0 denotes this forcing and $\beta \in \text{Adm} \cap \alpha$ then any maximal antichain M for $\mathcal{P}_0^\beta = \mathcal{P}_0 \cap L_\beta$ is also a maximal antichain for \mathcal{P}_0 . It follows that the \mathcal{P}_0 -forcing relation is Σ_1 when restricted to ranked sentences and that given p such that $p \Vdash \exists \beta \phi$, $\phi \Delta_0$, one can effectively produce a maximal antichain M below p so that $M \in L_\alpha$ and $q \in M \rightarrow q \Vdash \phi(\beta_q)$ for some β_q . These facts imply that if A_0 is \mathcal{P}_0 -generic over L_α then $L_\alpha[A_0]$ is admissible.

Second, we add $A_1 \subseteq \alpha$ so that $L_\alpha[A_0, A_1]$ is admissible but $\beta \in \text{Adm} \cap \alpha \rightarrow L_\beta[A_0, A_1]$ is inadmissible. This is done by forcing with \mathcal{P}_1 consisting of all conditions $p: \beta_p \rightarrow 2$ in $L_\alpha[A_0]$ so that $\beta \in \text{Adm} \cap (\beta_p + 1) \rightarrow L_\beta[A_0, p]$ is inadmissible. Using the fact that $L_\alpha[A_0]$ is locally countable it is easy to see that $p \in \mathcal{P}_1$, $\beta < \alpha \rightarrow \exists q \leq p$, $\beta_q \geq \beta$. It is easy to see that the forcing relation is Σ_1 when restricted to pairs (p, ϕ) , ϕ a ranked sentence of rank $< \beta_p$, as in this case $p \Vdash \phi$ iff $L_{\beta_p}[p] \models \phi$. Lastly if $\langle D_i \mid i < \omega \rangle$ is a uniformly $\Sigma_1(L_\alpha[A_0])$ sequence of dense open sets, $p \in \mathcal{P}_1$ then we can effectively define $p = p_0 \geq p_1 \geq \dots$ so that $p_{i+1} \in D_i$ and $\langle p_i \mid i < \omega \rangle$ is $\Sigma_1(L_\beta[A_0])$, $\beta = \bigcup \{\beta_{p_i} \mid i < \omega\}$. Thus $L_\beta[A_0]$ is inadmissible and $p = \bigcup \{p_i \mid i < \omega\}$ is a condition. This form of distributivity suffices to show that if A_1 is \mathcal{P}_1 -generic over $L_\alpha[A_0]$ then $L_\alpha[A_0, A_1]$ is admissible. To complete Step 1 define $A = A_0 \vee A_1$.

Step 2 is accomplished using almost disjoint forcing. We assign a real R_β to each $\beta < \alpha$ so that R_β is definable over $L_\beta[A]$ uniformly in β . Note that for any $\beta < \alpha$ there must be an $L_\beta[A]$ -definable bijection of ω and $L_\beta[A]$ as the least counterexample $L_\beta[A]$ to this assertion would have to be admissible, contrary to hypothesis. Thus we can in fact choose R_β to be Cohen generic over $L_\beta[A]$ as well, say for all $\Sigma_2(L_\beta[A])$ dense sets.

A condition in the forcing \mathcal{P} for coding A is a pair (r, \bar{r}) where r is a finite subset of ω and \bar{r} is a finite subset of $\{R_\beta^* \mid \beta \in A\} \cup \{r^* \mid r \text{ a finite subset of } \omega\}$. Here we make use of the canonical operation $R \mapsto R^* = \{\text{Code}(R \upharpoonright n) \mid n < \omega\} \subseteq \omega$ for converting distinct subsets of ω into almost disjoint ones. Write $(r', \bar{r}') \leq (r, \bar{r})$ if $r \subseteq r'$, $\bar{r} \subseteq \bar{r}'$ and $b \in \bar{r} \rightarrow b \cap r' \subseteq b \cap r$. Thus generically we produce a real R so that $\beta \in A$ iff R, R_β^* are almost disjoint. Also note that as each R_β is uniformly definable over $L_\beta[A]$ we obtain that $A \cap \beta$ is uniformly $\Delta_1(L_\beta(R))$, by induction on β . (To define $A \cap (\beta + 1)$ we need to know $A \cap \beta$ and R_β ; but the latter is definable over $L_\beta[A \cap \beta] = L_\beta[A]$.)

We need only show that \mathcal{P} preserves the admissibility of $L_\alpha[A]$. As in the first part of Step 1 it suffices to argue that if $M \subseteq \mathcal{P}^\beta = \mathcal{P} \cap L_\beta[A]$ is a maximal \mathcal{P}^β -antichain and Σ_1 -definable over $L_\beta[A]$ then M is a maximal antichain in \mathcal{P} . It is for the proof of this assertion that we chose R_β to be Cohen generic over $L_\beta[A]$. Indeed suppose $(r, \bar{r}_0 \cup \bar{r}_1)$ were incompatible with each element of M , where $\bar{r}_0 \subseteq L_\beta[A]$, $\bar{r}_1 \cap L_\beta[A] = \emptyset$. Note that the reals $\bar{r}_1 \subseteq \{R_{\beta_i} \mid R_{\beta_i}^* \in \bar{r}_1\}$ are *mutually* Cohen generic over $L_\beta[A]$ as if $\beta_1 < \beta_2 < \dots < \beta_k$ then R_{β_1} is Cohen generic over $L_{\beta_1}[A]$, R_{β_2} is Cohen generic over $L_{\beta_2}[A] \supseteq L_{\beta_1}[A][R_{\beta_1}]$, ... and we use the product lemma. So in fact the preceding assertion about $(r, \bar{r}_0 \cup \bar{r}_1)$ is forced by a Cohen condition c on \bar{r}_1 . But then $(r, \bar{r}_0 \cup \{s^* \mid s \in c\}) \in L_\beta[A]$ would be incompatible with each element of M , contradicting the maximality of M . This completes the proof of Theorem 1. \square

To be sure, there are many published proofs of the preceding result. We have included the above proof here, however, to serve as a model for the following proof of Jensen's result, as yet unpublished. To save notation we introduce:

CONVENTION. When writing $L_\alpha[X_1, \dots, X_n]$ we refer to the structure $\langle L_\alpha[X_1, \dots, X_n], X_1, \dots, X_n \rangle$.

THEOREM 2 (Jensen). *Suppose X is a countable set of countable admissibles greater than ω and $\alpha \in X \rightarrow L_\alpha[X]$ is admissible. Then for some real R , X is an initial segment of $\Lambda(R)$.*

PROOF. We can assume that X has a greatest element α . As in the proof of Theorem 1 we proceed in two steps.

Step 1. Find $A \subseteq \alpha$ so that $\beta < \alpha \rightarrow L_\beta[A]$ is admissible iff $\beta \in X$, $L_\beta[A]$ is not recursively Mahlo.

Step 2. Code A by a real R so that $\beta < \alpha \rightarrow A \cap \beta$ is $\Delta_1(L_\beta(R))$.

In both steps we want to preserve the admissibility of the elements of X .

To accomplish Step 1 first add $A_0 \subseteq \alpha$ so that $L_\alpha[A_0]$ is locally countable, as in the proof of Theorem 1 except over the ground model $L_\alpha[X]$. Then $L_\beta[X]$ admissible $\rightarrow L_\beta[X, A_0]$ admissible for all $\beta \leq \alpha$. (To see this note that if A_0 is \mathcal{P}_0 -generic then $A_0 \cap \beta$ is \mathcal{P}_0^β -generic.) Also if $\hat{\beta} =$ least p.r. closed ordinal greater than β then $\beta < \alpha \rightarrow \beta$ is countable in $L_{\hat{\beta}}[X, A_0]$. Second, add $A_1 \subseteq \alpha$ so that $L_\beta[X, A_0, A_1]$ is not recursively Mahlo for all $\beta \leq \alpha$. The collection of conditions Q_0 for doing this consists of all $p: \beta_p \rightarrow 2$ so that

- (i) $\beta \leq \beta_p \rightarrow p \upharpoonright \beta \in L_{\hat{\beta}}[X, A_0]$,
- (ii) $\beta \leq \beta_p$, $L_\beta[X, A_0]$ admissible $\rightarrow L_\beta[X, A_0, p]$ admissible,
- (iii) $\beta \leq \beta_p \rightarrow L_\beta[X, A_0]$ is not recursively Mahlo.

We must show that $p \in Q_0$, $\alpha > \beta > \beta_p \rightarrow$ there is a $q \leq p$, $\beta_q \geq \beta$. Then the argument of the second part to Step 1 in the proof of Theorem 1 shows that Q_0 is sufficiently distributive so as to preserve the admissibility of $L_\alpha[X, A_0]$.

The extendibility assertion is proved by induction on β . If β is a successor ordinal then the result is clear. If β is a limit ordinal but $L_\beta[X, A_0]$ is inadmissible then the construction of q is easy by induction, using the fact that β is countable in $L_{\hat{\beta}}[X, A_0]$. If $L_\beta[X, A_0]$ is admissible then first we force with $Q_0^\beta = Q_0 \cap L_\beta[X, A_0]$ to obtain $q': \beta \rightarrow 2$ so that $q' \in L_{\hat{\beta}}[X, A_0]$ and $q' \supseteq p$. (Note that $p \in Q_0^\beta$.) Then $L_\beta[X, A_0, q']$ is admissible as Q_0^β preserves admissibility just as does Q_0 . We must arrange that $L_\beta[X, A_0, q]$ is not recursively Mahlo. This requires one further forcing. Let Q_1^β consist of all closed $p \subseteq \beta$, $|p| = \max(p) \in p$ so that $p \in L_\beta[X, A_0, q']$ and

- (i) $\beta' \leq |p| \rightarrow p \cap \beta' \in L_{\hat{\beta}'}[X, A_0, q']$,
- (ii) $\beta' \leq |p|$, $L_{\beta'}[X, A_0, q']$ admissible $\rightarrow L_{\beta'}[X, A_0, p, q']$ admissible,
- (iii) $\beta' \in p \rightarrow L_{\beta'}[X, A_0, q']$ inadmissible.

(Note that (ii) is actually redundant due to (i), (iii) and the fact that p is closed.) Now force q'' to be Q_1^β -generic, $q'' \in L_{\hat{\beta}}[X, A_0]$. Then Q_1^β can be shown to preserve admissibility much as could Q_0^β . Clearly $L_\beta[X, A_0, q', q'']$ is not recursively Mahlo as q'' provides a closed unbounded set of $\beta' < \beta$ such that $L_{\beta'}[X, A_0, q']$ is inadmissible. Finally we define $q \leq p$ so as to code q' , q'' . Then $q \in Q_0$, $\beta_q = \beta$.

We now have that $L_\beta[X]$ admissible $\rightarrow L_\beta[X, A_0, A_1]$ admissible, $\beta < \alpha \rightarrow L_\beta[X, A_0, A_1]$ is not recursively Mahlo. In particular $\beta < \alpha \rightarrow$ there is an $L_\beta[X, A_0, A_1]$ -definable bijection of ω and $L_\beta[X, A_0, A_1]$. At last we now complete Step 1. We add $A_2 \subseteq \alpha$ so that $\beta < \alpha \rightarrow L_\beta[X, A_0, A_1, A_2]$ is admissible iff $\beta \in X$. The collection of conditions \mathcal{P}_1 for doing this consists of all $p: \beta_p \rightarrow 2$ in $L_\beta[X, A_0, A_1]$ so that

- (i) $\beta \leq \beta_p \rightarrow L_\beta[X, A_0, A_1, p]$ is admissible iff $\beta \in X$,
- (ii) $\beta \leq \beta_p \rightarrow p \upharpoonright \beta \in L_{\beta_p}[X, A_0, A_1]$.

We must show that for all $p \in \mathcal{P}_1$, $\beta < \alpha$ there exists $q \leq p$, $\beta_q \geq \beta$. Once this is accomplished we have completed Step 1 as the argument that \mathcal{P}_1 preserves admissibility is much like that for Q_0 .

The extendibility assertion is proved by induction on β . As before the nontrivial case is where $\beta \in X$. Then the desired $q \leq p$ is obtained by forcing with $\mathcal{P}_1^\beta = \mathcal{P}_1 \cap L_\beta[X, A_0, A_1]$. Such a q can be found in $L_{\beta_p}[X, A_0, A_1]$. And, \mathcal{P}_1^β preserves admissibility just as did Q_0^β . This completes Step 1: let $A = X \vee A_0 \vee A_1 \vee A_2$ where A_2 is \mathcal{P}_1 -generic.

Step 2 is precisely as in the proof of Theorem 1. Note that we can choose R_β to be definable over $L_\beta[A]$, as in that proof, since $\beta < \alpha \rightarrow$ there is an $L_\beta[A]$ -definable bijection of ω and $L_\beta[A]$. Lastly note that $\beta \in X$, R \mathcal{P} -generic over $L_\alpha[A] \rightarrow R$ \mathcal{P}^β -generic over $L_\beta[A]$ (for Σ_2 definable dense sets) so it follows that $L_\beta[R]$ is admissible. \square

As we mentioned earlier, SACKS [1976] establishes a result somewhat stronger than Theorem 1: If $\alpha > \omega$ is a countable admissible ordinal then $\alpha = \omega_1^R$ for some real R such that $S \lessdot_h R \rightarrow \omega_1^S < \omega_1^R$, where \lessdot_h refers to hyperarithmetical reducibility. Sacks uses pointed perfected forcing and in addition, when L_α is not locally countable, perfect trees of Lévy collapsing maps.

Recently, R. LUBARSKY [1984] has established a version of the preceding result in the context of Jensen's theorem. He shows that, assuming X as in Jensen's theorem has a greatest element α and in addition that $X \cap \beta$ is uniformly definable over L_β for $\beta \in X$, that there is a real R so that X is an initial segment of $\Lambda(R)$ and in addition, $S \in L_\alpha(R) \rightarrow R \in L_\alpha(S)$ or X is not an initial segment of $\Lambda(S)$. Lubarsky's proof is a significant extension of Sacks'; the key difference is that $\alpha = \omega_1^S \rightarrow L_\alpha(S)$ is locally countable, however $\beta \in \Lambda(S) \not\rightarrow L_\beta(S)$ is locally countable. Thus when establishing minimality for R , Lubarsky must consider that for $S \in L_\alpha(R)$ one need not have the local countability of $L_\beta(S)$ for $\beta \in X$ (though $L_\beta(R)$ is locally countable for $\beta \in X$). A new argument is required to rule out the possibility that such an S may obey " X is an initial segment of $\Lambda(S)$ ".

2. The full spectrum-limitations

The theme of this section is that $\Lambda(R)$ is a useful invariant for detecting the set-theoretic complexity of R . Let Λ denote $\Lambda(0) =$ all admissibles greater than ω .

THEOREM 3. *Suppose $R \in L$. Then $\Lambda(R)$ contains $\Lambda - \beta$ for some $\beta < \aleph_1^L$.*

PROOF. Choose β so that $R \in L_\beta$, $\beta < \aleph_1^L$. \square

Thus it follows from Theorem 2 that if $V = L$ then the possible admissibility spectra $\Lambda(R)$ can be completely characterized: they are of the form $X \cup (\Lambda - \alpha)$ where X is as in Jensen's theorem, $X \in L_\alpha$.

Note that if R is a Sacks real (R is generic for perfect set forcing over L) then a density argument shows that the conclusion of Theorem 3 fails. However we have the following.

THEOREM 4. *Suppose R is set-generic over L (R belongs to $L(G)$ where G is \mathcal{P} -generic over L , $\mathcal{P} \in L$). Then:*

(a) $\Lambda(R) \supseteq \Lambda - \beta$ for some β .

(b) For any $\alpha < \aleph_1$, there exist $\beta, \gamma < \aleph_1$ such that $\Lambda \cap (\beta, \gamma)$ has ordertype $\geq \alpha$ and is contained in $\Lambda(R)$.

PROOF. (a) Choose β so that $\mathcal{P} \in L_\beta$ where $R \in L_\beta(G)$, G is \mathcal{P} -generic over L . If $\alpha > \beta$ is admissible then $L_\alpha(G)$ is admissible as forcing with a set of conditions preserves admissibility. Thus $L_\alpha(R)$ is admissible since $R \in L_\beta(G) \subseteq L_\alpha(G)$.

(b) By the result of (a) we know that there exist $\beta, \gamma \in \text{ORD}$ such that $\Lambda \cap (\beta, \gamma)$ has ordertype $\geq \alpha$ and is contained in $\Lambda(R)$. But $\text{HC} =$ (the hereditarily countable sets) is a Σ_1 elementary substructure of V . So there must exist such β, γ which are countable. \square

The preceding result imposes severe restrictions on which admissibility spectra can be obtained via set-forcing over L . It implies that even when restricting to *countable* admissible ordinals, simple spectra such as $\{\alpha_{2i} \mid i \in \text{ORD}\} =$ (even admissibles) cannot be realized by $\Lambda(R)$ for set-generic R (where $\alpha_0 < \alpha_1 < \dots$ is the increasing enumeration of Λ).

The next result implies that certain spectra cannot be realized without the use of large cardinals.

THEOREM 5 (Silver). *Suppose $\Lambda(R) - \beta$ is contained in the class of all L -cardinals for some β . Then $0^* \leq_L R$.*

PROOF. Let κ be a singular cardinal greater than β . Then $(\kappa^+)^{L(R)} > (\kappa^+)^L$ since there are R -admissible ordinals between κ and $(\kappa^+)^{L(R)}$. By Jensen's Covering Theorem (see DEVLIN-JENSEN [1974]), $0^* \in L(R)$. \square

This result can in fact be strengthened to provide a natural generalization of Spector's Criterion, in the context of L -degrees.

DEFINITION. $X \subseteq \text{ORD}$ is Σ_1 -complete if whenever $Y \subseteq \text{ORD}$ is $\Sigma_1(L)$, Y is $\Delta_1(L[X], X)$.

THEOREM 6. $\Lambda(R)$ is Σ_1 -complete iff $0^* \leq_L R$.

PROOF. X is Σ_1 -complete whenever X is unbounded and $X \subseteq L\text{-Card} = \{\alpha \mid \alpha \text{ is an } L\text{-cardinal}\}$, as if Y is $\Sigma_1(L)$ with defining formula $\phi(y)$ then $y \notin Y$ iff $\exists \alpha \in X (L_\alpha \models \sim \phi(y) \text{ and } y, p \in L_\alpha)$ where p is the parameter in ϕ . (We are using the fact that α an L -cardinal $\rightarrow L_\alpha <_{\Sigma_1} L$; i.e., α is stable.) Thus $\Lambda(R)$ is Σ_1 -complete whenever $0^* \leq_L R$ as $\Lambda(0^*) \subseteq L\text{-Card}$. Conversely if $\Lambda(R)$ is Σ_1 -complete then $L\text{-Card}$ is $\Sigma_1(L(R))$ and as in the proof of Theorem 5, $(\kappa^+)^{L(R)} > (\kappa^+)^L$ for sufficiently large singular κ . (We are using the R -stability of $(\kappa^+)^{L(R)}$.) By the Covering Theorem, $0^* \leq_L R$. \square

Theorem 6 has the consequence that certain spectra X are ruled out entirely, even though the Jensen criterion ($\alpha \in X \rightarrow \langle L_\alpha[X], X \rangle$ is admissible) is satisfied.

COROLLARY. *There is no real R obeying any of the following:*

- (a) $\Lambda(R) = \Sigma_2$ -admissible L -cardinals,
- (b) $\Lambda(R) = \Sigma_2$ -admissible stables,
- (c) R is generic over L via an amenable class forcing, $\Lambda(R) \subseteq \text{stables}$.

PROOF. (a), (b) are clear, using Theorem 6. (c) follows from the fact that the condition on R contradicts $0^* \leq_L R$ (see Beller-Jensen-Welch [1982], p. 157). \square

We have left open the possibility of solutions to spectrum equations $\Lambda(R) = X$, where X is not Σ_1 -complete. We discuss this in the next section.

3. The full spectrum-positive results

The results of Section 2 imply that a real R satisfying $\Lambda(R) =$ (even admissibles) cannot be set-generic and cannot construct 0^* (i.e., $0^* \leq_L R$). Thus such reals are entirely ruled out by the following conjecture of Solovay.

SOLOVAY'S CONJECTURE. $0^* \leq_L R \rightarrow R$ is set-generic (over L).

Fortunately for our purposes, Solovay's conjecture is false. This was shown by Jensen (see BELLER-JENSEN-WELCH [1982]).

THEOREM 7 (Jensen). *If $A \subseteq \text{ORD}$ then there is an $\langle L[A], A \rangle$ -definable forcing for extending $L[A]$ to $L(R)$, $R \subseteq \omega$ so that $L(R) \models \text{ZFC}$ and A is definable over $L(R)$.*

COROLLARY. *The negation of Solovay's conjecture is consistent.*

PROOF. Choose $A \subseteq \text{ORD}$ to be amenable but not L -definable. By Jensen's theorem we can get $R \subseteq \omega$ so that A is definable over $L(R)$. Then R cannot be set-generic over L as otherwise there is a condition $p \in \mathcal{P}$ and a formula ϕ (where $R \in L(G)$, G \mathcal{P} -generic over L) such that for unboundedly many $\alpha \in \text{ORD}$, $p \Vdash \overline{A \cap \alpha}$ is an initial segment of $\{\beta \mid \phi(\beta)\}$. Thus $\beta \in A$ iff $\exists x \in L$ ($p \Vdash \beta \in x$, x an initial segment of $\{\beta \mid \phi(\beta)\}$). \square

As it turns out the technique used to prove Theorem 7, Jensen's coding method, suffices to get the first example of a nontrivial spectrum.

THEOREM 8 (David, Friedman). *There is an L -definable forcing for producing a real R so that $L(R) \models \text{ZFC}$ and $\Lambda(R) \subseteq$ (even admissibles).*

IDEA OF PROOF. The desired forcing is made up of certain "building blocks", which are not difficult to describe. Jensen coding is used to put these building blocks together.

We wish to arrange that α R -admissible $\rightarrow \alpha$ is an even admissible. Suppose that we have $D \subseteq \aleph_1$ so that: $L_\alpha[D]$ admissible $\rightarrow \alpha$ is even. Then we could hope to choose R so as to code D and satisfy the desired property.

The problem is that if we code D by R in the usual way (with almost

disjoint forcing) we only obtain the following: For all α , $D \cap (\aleph_1)^{L_\alpha}$ is $\Delta_1(L_\alpha(R))$. The reason is that to decode D from R we need to know the almost disjoint coding reals R_β and it is only for $\beta < (\aleph_1)^{L_\alpha}$ that we have $R_\beta \in L_\alpha$. Thus the recovery of D from R is not "fast enough". On the other hand we are in great shape if D has the following stronger properties:

$$L_\alpha(D \cap \xi) \text{ admissible, } L_\alpha(D \cap \xi) \models \xi = \aleph_1 \rightarrow \alpha \text{ is even.} \quad (*)$$

$$L_\alpha[D] \text{ admissible and locally countable} \rightarrow \alpha \text{ is even.} \quad (**)$$

For then we need only recover $D \cap (\aleph_1)^{L_\alpha}$ inside $L_\alpha(R)$ to guarantee that α is even (or inadmissible relative to R), a recovery that can be made.

The question is how to obtain $D \subseteq \aleph_1$ obeying (*), (**). The natural thing to do is force with conditions d which are initial segments of \aleph_1 obeying (*), (**) for $\xi \leq \text{sup}(d)$. We now come to the heart of the argument, which is contained in the following two observations:

(1) Extendibility for this forcing is trivial because given d and $\xi > \text{sup}(d)$ we are free to extend d to length ξ by *killing all admissibles* between $\text{sup}(d)$ and ξ . It is crucial for this argument that we are only concerned with killing admissibility, not in preserving it.

(2) Distributivity for this forcing is easily established assuming the following (!): There exists $D' \subseteq \aleph_2$ such that:

$$L_\alpha(D' \cap \xi) \text{ admissible, } L_\alpha(D' \cap \xi) \models \xi = \aleph_2 \rightarrow \alpha \text{ is even} \quad (*')$$

$$L_\alpha[D'] \text{ admissible, } L_\alpha[D'] = \forall x (\text{card}(x) \leq \aleph_1) \rightarrow \alpha \text{ is even.} \quad (**')$$

Thus we are faced with the original problem, but one cardinal higher!

Proof by induction does not look promising. However note that we need not already "have" all of D' before we can "start building" D ; thus the idea of the proof (as in all Jensen coding constructions) is to build R, D, D', D'', \dots simultaneously and check distributivity for any final segment of the forcing. \square

A proof of the preceding result will appear in DAVID [1984]. In that paper the above ideas are combined with some ideas from "strong coding" (mentioned below) to improve the conclusion of Theorem 8 to: $\Lambda(R) \subseteq \{\alpha \mid L \models \phi(\alpha)\}$, where ϕ is Σ_1 and $L \models \phi(\kappa)$ for all cardinals κ .

The next step in the study of admissibility spectra is to introduce the requirement of admissibility preservation into the above. Thus for example we wish to obtain solutions to the equation $\Lambda(R) = (\text{even admissibles})$. This requires the method of strong coding.

THEOREM 9. *There is a $\Delta_1(L)$ -definable forcing \mathcal{P} for producing a real R so that $L(R) \models \text{ZFC}$ and $\Lambda(R) = (\text{even admissibles})$.*

IDEA OF PROOF. We approach the problem as in Theorem 8. Of course now the extendibility property is much more difficult (distributivity is the same). Indeed the desired extension of d to d' of length $\geq \xi$ must be made generically, so as to preserve even admissibles. Thus we see that our conditions must be constructed out of generic sets for “local” versions of the very same forcing. Thus in fact we construct a strong coding $\mathcal{P}^\beta \subseteq L_\beta$ at each admissible β and then inductively build \mathcal{P}^β out of generic sets for various $\mathcal{P}^{\beta'}$, $\beta' < \beta$.

The main difficulty is in showing that the desired generic sets actually exist; note that we want a \mathcal{P}^β -generic over L_β where β may indeed be uncountable. The proof of generic existence is by a simultaneous induction with the proofs of extendibility, distributivity and requires use of the critical projecta of FRIEDMAN [1982]. (These projecta are closely related to Jensen’s notion of dependency in the theory of higher-gap morasses.)

The other difficulty in the extendibility argument is the conflict between the genericity requirement and the need to “avoid” the almost disjoint codes R_β^* : Recall that in almost disjoint forcing, $(r', \bar{r}') \leq (r, \bar{r})$ iff $r' \supseteq r$, $\bar{r}' \supseteq \bar{r}$ and $b \in \bar{r} \rightarrow b \cap r' \subseteq r$. This last requirement causes difficulty with the need for making r' generic. Solving this requires the construction of special “supergeneric” codes R_β . These codes will not be Cohen generic but instead generic for a suitable forcing, defined inductively. \square

4. Recent work

A complete characterization of those $A \subseteq \text{ORD}$ which can be realized as admissibility spectra $\Lambda(R)$ is not known. However some hints as to the nature of such a characterization are hinted at by the following examples.

(a) Suppose $A = L\text{-Card}$, the class of L -cardinals. Then A cannot be of the form $\Lambda(R)$ as A fails to satisfy: $\alpha \in A \rightarrow L_\alpha[A]$ is admissible.

(b) Suppose $A = (\text{all } \alpha \text{ such that } L_\alpha \models \text{Power set})$. Then A cannot be of the form $\Lambda(R)$ as then the $L(R)$ -cardinal successor to \aleph_ω would be greater than the L -cardinal successor to \aleph_ω , hence $0^* \in L(R)$; but then $\Lambda(R) - \beta \subseteq L\text{-Card}$ for some β .

(c) Suppose $A = \{\alpha \mid \alpha \text{ a successor admissible, } L\text{-Card}(\alpha) \text{ a successor } L\text{-Cardinal}\} \cup \{\alpha \mid \alpha \text{ recursively inaccessible, } L\text{-Card}(\alpha) \text{ a limit } L\text{-cardinal}\}$. Then A cannot be of the form $\Lambda(R)$ else $L\text{-Card}$ is $\Delta_1(L(R))$ and thus $0^* \in L(R)$; this is a contradiction as in (b).

(d) Suppose $A = \text{nonprojectibles} = \{\alpha \mid \Sigma_1 \text{ projectum}(\alpha) = \alpha\}$. Then A cannot be of the form $\Lambda(R)$ for then the least R -admissible α greater than \aleph_1 would have cofinality ω , but this is false since $\text{cof}(\alpha) = \text{cof}(\Sigma_1 \text{ projectum } \alpha \text{ relative to } R) = \aleph_1$.

Also note the following: If $A = \Lambda(R)$ then A is $\Delta_1(L(R))$ and hence A “collapses to itself” when transitively collapsing Σ_1^R Skolem hulls. More precisely, for any $x \in \mathcal{P}_{\omega_1}(\text{ORD}) = \{x \subseteq \text{ORD} \mid x \text{ is countable}\}$ let π_x be the unique order-preserving function from x onto $\text{ordertype}(x)$. Then in $L(R)$, $A^* = \{x \in \mathcal{P}_{\omega_1}(\text{ORD}) \mid \pi_x[A]$ is an initial segment of $A\}$ contains a closed unbounded class (namely $\{x \in \mathcal{P}_{\omega_1}(\text{ORD}) \mid x <_{\Sigma_1} L(R)\}$). Thus $\langle L[A], A \rangle \models A^*$ is stationary in $\mathcal{P}_{\omega_1}(\text{ORD})$, assuming $(\aleph_1)^{L[A]} = (\aleph_1)^{L(R)}$.

The above considerations lead us to conjecture what the situation is in a very special case of the general problem. Namely suppose $\alpha =$ (least α such that $L_\alpha \models \text{KP}$ and \aleph_2 exists). We conjecture the following.

(*) Suppose $A \subseteq \alpha$ is amenable and $\langle L_\alpha, A \rangle$ is admissible. Then there is a real R such that $A = \Lambda(R) \cap \alpha$, $L_\alpha(R) \models \text{KP} + \aleph_2$ exists iff:

- (i) $\aleph_1^{L_\alpha}, \aleph_2^{L_\alpha} \in A$,
- (ii) $\beta \in A \rightarrow \langle L_\beta[A], A \cap \beta \rangle$ is admissible,
- (iii) $\aleph_1^{L_\alpha} < \beta$, β a successor element of $A \rightarrow L_\alpha \models \text{cof}(\beta) = \aleph_1$,
- (iv) $\langle L_\alpha, A \rangle \models A^*$ is stationary on $\mathcal{P}_{\omega_1}(\text{ORD})$.

The key step in establishing this conjecture should be to obtain an $(\omega_1, 1)$ -morass of A -preserving maps, using property (iv) to show that the natural forcing for doing this is ω -distributive.

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