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INFINITARY LOGIC AND $0^\#$

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In this paper we use infinitary model theory together with some ideas from Jensen coding (see Beller-Jensen-Welch [82]) to establish a perfect set theorem for Π^1_2 sets of reals. Assume throughout that $0^\#$ exists. If $R \subseteq \omega$ then we let I^R denote the canonical Silver indiscernibles for $L(R)$. It follows from either Paris' work (Paris [74]) or the Covering Theorem (Devlin-Jensen [74]) that if $0^\# \notin L(R)$ then $R^\#$ exists and in fact $R^\# \in L(R,0^\#)$.

DEFINITION Suppose $\kappa > \gamma$ are uncountable cardinals. Then $C \subseteq \kappa$ is CUB in (κ,γ) if C is closed unbounded (CUB) in κ and $C \cap \gamma$ is CUB in γ.

DEFINITION Suppose $\kappa > \gamma$ are uncountable cardinals and R is a real. Then $X \subseteq \kappa$ is k-indiscernible^R if for any formula $\phi(R,v_1,\ldots,v_k)$ and k-tuples $i_1 <\ldots< i_k$, $j_1<\ldots<j_k$ from X, $L(R) \models \phi(R,i_1,\ldots,i_k) \longleftrightarrow \phi(R,j_1,\ldots,j_k)$. X is almost k-indiscernible^R at (κ,γ) if $X \cap C$ is k-indiscernible^R for some constructible C which is CUB in (κ,γ).

We wish to consider reals which preserve k-indiscernibility in a certain sense.

DEFINITION Suppose $\kappa > \gamma$ are uncountable cardinals and R is a real. Then R preserves k-indiscernibles at κ if X k-indiscernible, $X \subseteq \kappa$, X constructible $\longrightarrow$ X k-indiscernible^R. R weakly preserves k-indiscernibles at (κ,γ) if X k-indiscernible, $X \subseteq \kappa$, X constructible $\longrightarrow$ X almost k-indiscernible^R at (κ,γ).

REMARK It is clear that if R weakly preserves k-indiscernibles at (κ,γ) then $0^\# \notin L(R)$; thus $R^\#$ exists and all uncountable cardinals belong to I^R. It follows that R weakly preserves k-indiscernibles at (κ',γ') for all pairs of uncountable cardinals $\kappa' > \gamma'$.

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DEFINITION R weakly preserves k-indiscernibles if R weakly preserves k-indiscernibles at (κ,γ) for some (and hence any) pair of uncountable cardinals $\kappa > \gamma$.

The following is our main result.

THEOREM Suppose A is a Π^1_2 set of reals containing a nonconstructible real which weakly preserves k-indiscernibles for infinitely many k. Then A contains a perfect closed subset.

COROLLARY Suppose R is a Π^1_2-singleton which weakly preserves k-indiscernibles for infinitely many k. Then R is constructible.

Solovay has conjectured that there is a nonconstructible Π^1_2-singleton R such that $0^\# \notin L(R)$.

Our first lemmas concern the indiscernible preservation hypothesis in the theorem. A class $X \subseteq \mathrm{ORD}$ is amenable if $X \cap \alpha \in L$ for all $\alpha \in \mathrm{ORD}$.

LEMMA 1 For any k there exists an amenable k-indiscernible $I_k \subseteq \mathrm{ORD}$ such that $I \subseteq I_k$.

PROOF For any $\gamma \in \mathrm{ORD}$ let $i^\gamma_1 < \ldots < i^\gamma_k$ be the first k elements of I greater than γ. Let $X = \{\gamma \mid$ For all $i < \gamma$ and all $\phi(x,x_1,\ldots,x_k)$, $L \models \phi(i,\gamma,i^\gamma_2,\ldots,i^\gamma_k) \longleftrightarrow \phi(i,i^\gamma_1,\ldots,i^\gamma_k)\}$. Note that $I \subseteq X$ and that X is amenable. Now suppose that $\gamma_1 < \ldots < \gamma_k$ is an increasing sequence from X and choose $i_1 < \ldots < i_k$ from I so that $i_1 > \gamma_k$. Then for any ϕ, $L \models \phi(\gamma_1,\ldots,\gamma_k) \longleftrightarrow \phi(\gamma_1,\ldots,\gamma_{k-1},i_k) \longleftrightarrow \phi(\gamma_1,\ldots,\gamma_{k-2},i_{k-1},i_k) \longleftrightarrow \ldots \longleftrightarrow \phi(\gamma_1,i_2,\ldots,i_k) \longleftrightarrow \phi(i_1,\ldots,i_k)$, by indiscernibility and the definition of X. So any two increasing k-tuples from X realize the same type and we are done. ——|

In the definition of "R weakly preserves k-indiscernibles at (κ,γ)" we now drop the assumption that $\kappa > \gamma$ are true cardinals; the same definition applies if $\kappa > \gamma$ are arbitrary L(R)-uncountable L(R)-cardinals.

LEMMA 2 Suppose that R weakly preserves k-indiscernibles. Then there exists an amenable $X \subseteq \mathrm{ORD}$ such that $I \subseteq X \cup \alpha$ for some $\alpha < \aleph_1$ and $\kappa > \gamma$ in $X \longrightarrow R$ weakly preserves k-indiscernibles at (κ,γ). Moreover X can be chosen independently of k.

PROOF Let I_2 be as in Lemma 1 where k = 2. Now for each $\kappa \in I^R$ we can choose a constructible CUB $C_\kappa \subseteq \kappa$ so that $I_2 \cap C_\kappa$ is 2-indiscernible^R. Assume that C_κ is chosen to be L-least with this property. Then by indiscernibility $\kappa < \kappa'$ in $I^R \longrightarrow C_\kappa = C_{\kappa'} \cap \kappa$ and hence $X = I_2 \cap C$ is 2-indiscernible^R where $C = \cup \{C_\kappa | \kappa \in I^R\}$. Also $I^R \subseteq C$ and hence $\kappa > \gamma$ in $X \longrightarrow R$ weakly preserves k-indiscernibles at (κ,γ).

To complete the proof we need only show that $I \subseteq I^R \cup \alpha$ for some $\alpha < \aleph_1$. But note that if we replace 2 by k in the above argument we obtain $X(k) \supseteq I^R$ so that $X(k)$ is amenable. Now for each k choose a countable α_k so that $X(k) \supseteq I\text{-}\alpha_k$. This is possible as indiscernibility implies that $X(k) \supseteq I\text{-}\alpha_k$ for some α_k and then we can get α_k to be countable in $L(R^{\#})$ by reflection. Finally let $\alpha = \cup\{\alpha_k | k \in \omega\}$. Then $I \subseteq (\bigcap_k X(k)) \cup \alpha$ yet $\bigcap_k X(k)$ must equal I^R since it is a class of R-indiscernibles. ——|

We will actually need some finer information about the classes I_k in Lemma 1 and X in Lemma 2. This is captured by the following definition.

<u>DEFINITION</u> $Y \subseteq ORD$ is <u>special amenable</u> if for some term t and countable $i_1<\ldots<i_n$ in I we have that $i_n < i \in I \longrightarrow Y \cap i = t(i_1,\ldots,i_n,i,j_1,\ldots,j_m)$ whenever $i < j_1<\ldots< j_m$ belong to I. We refer to $i_1,\ldots,i_n$ as the <u>parameters</u> for Y.

Any amenable $Y \subseteq ORD$ is of the above form if the restriction that the parameters $i_1<\ldots<i_n$ be countable is dropped. The proofs of Lemmas 1,2 show:

<u>LEMMA 3</u> I_k as definied in Lemma 1 is special amenable and X in Lemma 2 can be chosen to be special amenable.

We are now prepared to turn to the main ideas of the proof of the Theorem.

<u>INFINITARY LOGIC</u>

Let L' denote the language of set theory augmented by a new symbol $\underline{a}$ for each $a \in L$ and a unary predicate $\underline{R}$ (for denoting a real). We shall work with the following base theory T_0, formulated in the logic $L'_{\infty\omega}$: T_0 consists of ZFC + V = L($\underline{R}$) + $\forall x(\underline{R}(x) \longrightarrow x \in \underline{\omega})$ + Diagram(L) = $\{\forall x(x \in \underline{a} \longrightarrow \bigvee_{b\in a} x = \underline{b}) | a \in L\}$ + {$\underline{R}$ weakly preserves k-indiscernibles at (κ,γ) for infinitely many $k | \kappa > \gamma$ in X}, where X is a previously fixed special amenable class containing $I\text{-}\alpha$ for some $\alpha < \aleph_1$.

Also let L be obtained from L' by discarding constants $\underline{a}$ for $a \notin \omega$. We shall establish a completeness theorem for suitable theories in the fragment of $L_{\infty\omega}$ defined by L.

DEFINITION $S \subseteq L_{\infty\omega} \cap L$ is suitable if S is a special amenable class of quantifier-free sentences consistent (in infinitary logic) with T_0.

EXAMPLE If R is a real such that R weakly preserves k-indiscernibles for infinitely many k at (κ,γ) for all $\kappa > \gamma$ in X then any Π^1_2 property true of R can be expressed by a suitable collection of sentences.

The following is the key fact.

MAIN LEMMA If S is suitable then S has a model. Moreover there is a countable ordinal α such that S has a model definable over $L(0^{\#},T)$ whenever $T \subseteq \omega$ is a counting of α.

It will be easy to establish the Theorem, using the proof of the Main Lemma. The proof of the Main Lemma is based on a Henkin-style construction in ω steps. This construction produces an increasing sequence $S_0 \subseteq S_1 \subseteq \ldots$ of suitable classes of sentences so that $S_0 = S$ and whenever ϕ is a quantifier-free sentence of $L_{\infty\omega} \cap L$ then for some n, $\phi \in S_n$ or $\sim\phi \in S_n$. In addition if ϕ is a disjunction $\vee\Phi$ and $\phi \in S_n$ then $\psi \in S_m$ for some $\psi \in \Phi$ and $m \in \omega$. Given these properties we can define $R = \{n \in \omega | \underline{R}(\underline{n}) \in S_m$ for some $m\}$ and then an easy induction shows that for quantifier-free $\phi \in L_{\infty\omega} \cap L$, $\phi \in \bigcup\{S_n | n \in \omega\}$ iff $L(R) \models \phi$ when $\underline{a}$ is interpreted as a and $\underline{R}$ is interpreted as R.

Our strategy for arranging the preceding properties of the S_n's is based upon Jensen's construction of a real R which is cardinal-preserving but not set-generic over L (see section 4.4 of Beller-Jensen-Welch [82]). We shall define a certain countable indiscernible i and then build the S_n's so that whenever $i \leq \kappa \in I, \kappa^*$ = least element of I greater than κ and ϕ is a quantifier-free sentence of $L^L_{\kappa^*\omega} \cap$ Skolem hull $(\kappa \cup \{\kappa,i_1,\ldots,i_n\})$ then for some quantifier-free $\psi \in L^L_{\kappa\omega}$, S_{n+1} contains the sentence $\phi \longleftrightarrow \psi$ (where $i_1,\ldots,i_n$ are distinct indiscernibles greater than κ). It follows that for any quantifier-free $\phi \in L_{\infty\omega} \cap L = L^L_{\infty\omega}$ there exists n,ψ so that $(\phi \longleftrightarrow \psi) \in S_n$ and ψ is a quantifier-free sentence of $L^L_{i\omega}$. In addition a counting of i will be used to provide a method of deciding and choosing disjuncts for quantifier-free sentences of $L^L_{i\omega}$ in ω steps.

Thus the main point is to show that κ-many sentences in $L^L_{\kappa^*\omega}$ can be simultaneously "reduced" to equivalent sentences in $L^L_{\kappa\omega}$, for all $\kappa \in I-i$. This is where weak k-indiscernible preservation is used. We shall also make use of Shelah's Strong Covering property to handle disjunctions.

We now define i. As T_0,S are special amenable we can choose countable $i_1<\ldots<i_n$ from I and terms t_0,s so that for $j < j_1<\ldots<j_m$ in $I, j > i_n$ we have that $T_0 \cap L_j = t_0(i_1,\ldots,i_n,j,j_1,\ldots,j_m)$, $S \cap L_j = s(i_1,\ldots,i_n,j,j_1,\ldots,j_m)$. Then i is the least element of I greater than i_n. Note that i is countable and $i \leq \kappa \in I \longrightarrow \kappa \in X$. Also choose a listing $\phi_0,\phi_1,\ldots$ of $L^L_{i\omega}$ in an ω-sequence.

We can now define the desired sequence $S_0 \subseteq S_1 \subseteq \ldots$ of suitable classes of sentences. Let S_0 equal S. Assuming S_{k-1} has been defined we now define S_k. Each S_k will be defined so as to be special amenable with parameters less than i.

Pick $\kappa \in I$, $\kappa \geq i$ and let κ^* = least element of I greater than κ. As $\kappa,\kappa^* \in X$, T_0 proves that $\underline{R}$ weakly preserves ℓ-indiscernibles at $\underline{(\kappa^*,\kappa)}$ for infinitely many ℓ. Define $\ell \geq k,3$ to be least so that $T'_k(\kappa)$ is consistent where $T'_k(\kappa) = T_0 \cup S_{k-1} \cup \{\underline{R}$ weakly preserves ℓ-indiscernibles at $\underline{(\kappa^*,\kappa)}\}$ and then choose C_k to be the L-least CUB subset of κ such that $T''_k(\kappa)$ is consistent where $T''_k(\kappa) = T_0 \cup S_{k-1} \cup \{\underline{I_\ell \cap C_k}$ is ℓ-indiscernible$^{\underline{R}}\}$ and I_ℓ is defined as in Lemma 1.

Note that by indiscernibility, $i \leq \kappa < \bar{\kappa}$ in $I \longrightarrow C^{\kappa}_k = C^{\bar{\kappa}}_k \cap \kappa$ and hence T''_k is consistent where $T''_k = T_0 \cup S_{k-1} \cup \{\underline{I_\ell \cap C_k \cap \beta}$ is ℓ-indiscernible$^{\underline{R}} | \beta \in \mathrm{ORD}\}$ and $C_k = \cup\{C_k | i \leq \kappa \in I\}$. Also T''_k is special amenable with parameters less than i and $I - i \subseteq C_k$.

Suppose now that ϕ is a quantifier-free sentence of $L^L_{\infty\omega}$ such that for some $\kappa_1 < \ldots < \kappa_\ell$ in $I_\ell \cap C_k$ and term t we have that $\phi = t(\gamma,\kappa_1,\ldots,\kappa_\ell)$ where $\gamma < \kappa_1$ and $\phi \notin L^L_{\kappa_1\omega}$. In this case we say that "ϕ^* is defined". Now let $\kappa^\gamma_1 < \ldots < \kappa^\gamma_\ell$ be the first ℓ elements of $I_\ell \cap C_k$ greater than γ and define $\phi^* = t(\gamma,\kappa^\gamma_1,\ldots,\kappa^\gamma_\ell)$ where $t,\kappa_1,\ldots,\kappa_\ell,\gamma$ as above are chosen to be L-least. Set $\bar{S}_k = \{\phi \longleftrightarrow \phi^* | \phi^*$ is defined$\} \cup S_{k-1}$.

To see that $T_0 \cup \bar{S}_k$ is consistent it suffices to show that T''_k proves every sentence in $\bar{S}_k$. This is a consequence of the following.

<u>CLAIM</u> T''_k proves that $I_\ell \cap C_k$ is strongly ℓ-indiscernible$^{\underline{R}}$: Whenever $\kappa_1 < \ldots < \kappa_\ell, \bar{\kappa}_1 < \ldots < \bar{\kappa}_\ell$ are from $I_\ell \cap C_k$ and γ is less than both κ_1 and $\bar{\kappa}_1$ then for all $\phi(\underline{R},x_0,\ldots,x_\ell)$ we have that $T''_k \vdash \phi(\underline{R},\underline{\gamma},\underline{\kappa_1},\ldots,\underline{\kappa_\ell}) \longleftrightarrow \phi(\underline{R},\underline{\gamma},\underline{\bar{\kappa}_1},\ldots,\underline{\bar{\kappa}_\ell})$.

<u>PROOF OF CLAIM</u> Note that T''_k proves that $\underline{\kappa}$ is inaccessible, whenever $\kappa \in I_\ell \cap C_k$. Argue now in T''_k. Suppose $\phi(R,\gamma,\kappa_1,\ldots,\kappa_\ell) \not\longleftrightarrow \phi(R,\gamma,\bar{\kappa}_1,\ldots\bar{\kappa}_\ell)$. Pick j least so that $\kappa_j \neq \bar{\kappa}_j$ and assume that $\kappa_j < \bar{\kappa}_j$. If $\phi(R,\gamma,\kappa_1,\ldots,\kappa_j,\bar{\kappa}_{j+1},\ldots,\bar{\kappa}_\ell) \longleftrightarrow \phi(R,\gamma,\bar{\kappa}_1,\ldots,\bar{\kappa}_\ell)$ then we can replace $\bar{\kappa}_1,\ldots,\bar{\kappa}_\ell$ by $\kappa_1,\ldots,\kappa_j,\bar{\kappa}_{j+1},\ldots,\bar{\kappa}_\ell$, thereby increasing the first place at which the sequence differs from $\kappa_1,\ldots,\kappa_\ell$. Continuing in this way if necessary we see that we can in fact assume that $\kappa_1,\ldots,\kappa_\ell$ differs from $\bar{\kappa}_1,\ldots,\bar{\kappa}_\ell$

only at the j^{th} place $\kappa_j < \bar{\kappa}_j$.

Now we can also assume that $I_\ell \cap C_k$ is unbounded in $\kappa_{j+1} \leq \infty$ as by indiscernibility if $\phi(R,\gamma,\kappa_1,\ldots,\kappa_\ell) \not\longleftrightarrow \phi(R,\gamma,\kappa_1,\ldots,\kappa_{j-1},\bar{\kappa}_j,\kappa_{j+1},\ldots,\kappa_\ell)$ for some $\gamma < \kappa_1$ then the same is true if $\kappa_{j+1},\ldots,\kappa_\ell$ is replaced by any larger $(\ell\text{-}j)$-tuple from $I_\ell \cap C_k$. (Also note that $I_\ell \cap C_k$ contains many of its limit points as it contains I-i.) Now list the elements of $I_\ell \cap C_k$ between κ_{j-1} and κ_{j+1} as $\delta_0 < \delta_1 < \ldots$ and for $\alpha < \beta < \kappa_{j+1}$ define $f(\alpha,\beta) =$ least $\bar{\gamma} < \kappa_1$ such that $\phi(R,\bar{\gamma},\kappa_1,\ldots,\kappa_{j-1},\delta_\alpha,\kappa_{j+1},\ldots,\kappa_\ell)$ $\not\longleftrightarrow \phi(R,\bar{\gamma},\kappa_1,\ldots,\kappa_{j-1},\delta_\beta,\kappa_{j+1},\ldots,\kappa_\ell)$. This is possible by indiscernibility. But by the Erdős-Rado Theorem there must be a homogeneous set for the partition f of cardinality ≥ 3, which is impossible. This proves the Claim.

The consistency of $T_0 \cup \bar{S}_k$ is thus established. We now take steps to handle disjunctions by introducing a refinement of the above construction. Choose $\kappa \in I$, $i \leq \kappa$ and let $H_k(\kappa) =$ Skolem hull$(\kappa \cup \{\kappa,\kappa_1,\ldots,\kappa_k\}) \cap L_{\kappa_1}$ where $\kappa < \kappa_1 < \ldots < \kappa_k$ belong to I. Note that $H_k(\kappa)$ is defined independently of the choice of $\kappa_1 < \ldots < \kappa_k$. Now as T_0 proves Shelah's Strong Covering property (see Shelah [82]) we can choose $M_k(\kappa) \in L_{\kappa_1}$ to be L-least so that $H_k(\kappa) \subseteq M_k(\kappa)$, $M_k(\kappa)$ has L-cardinality κ and $T_0 \cup S_{k-1}$ is consistent with the sentence: $M_k(\kappa) = N \cap L$ where $\underline{R} \in N$ and N is a $\sum_1$-elementary submodel of $L(\underline{R})$. Let $T_k'(\kappa)$ be obtained by adding this sentence to $T_0 \cup S_{k-1}$.

Now pick $\gamma < \kappa$, a term t and $\kappa < \kappa_1 < \ldots < \kappa_\ell$ from I so that $M_k(\kappa) = t(\gamma,\kappa,\kappa_1,\ldots,\kappa_\ell)$. As $\kappa,\kappa^+ \in X$ we know that T_0 proves that $\underline{R}$ weakly preserves m-indiscernibles at $\underline{(\kappa^+,\kappa)}$ for some $m > \ell$. So choose $m > \ell$ so that $T_k''(\kappa)$ is consistent where $T_k''(\kappa) = T_k'(\kappa) \cup \{\underline{R}$ weakly preserves m-indiscernibles at $\underline{(\kappa^+,\kappa)}\}$. Then let $C_k^\kappa \subseteq \kappa^+$ be L-least so that C_k^κ is CUB in (κ^+,κ) and $T_k'''(\kappa)$ is consistent where $T_k'''(\kappa) = T_k''(\kappa) \cup \{\underline{I_m \cap C_k^\kappa}$ is m-indiscernible$^{\underline{R}}\}$.

Note that we can choose $\kappa < \kappa_1 < \ldots < \kappa_\ell$ in I so that in fact $\kappa,\kappa_1,\ldots,\kappa_\ell \in I_m \cap C_k^\kappa$ since C_k^κ is CUB in (κ^+,κ) and $I \subseteq I_m$. So for any $\gamma < \bar{\kappa} \in I_m \cap C_k^\kappa \cap \kappa$ we have that $T_k'''(\kappa)$ proves that "$M_k(\bar{\kappa}) = N \cap L$ where $\underline{R} \in N$ and N is a $\sum_1$-elementary submodel of $L(\underline{R})$", where $M_k(\bar{\kappa}) = t(\gamma,\bar{\kappa},\kappa_1,\ldots,\kappa_\ell)$. Moreover by indiscernibility we can choose t,γ,ℓ,m independently of $\kappa \in$ I-i and obtain the consistency of $T_k = T_0 \cup S_{k-1} \cup$ $\{t(\gamma,\underline{\kappa},\underline{\kappa_1},\ldots,\kappa_\ell) = N \cap L$ where $\underline{R} \in N$ and N is a $\sum_1$-elementary submodel of $L(\underline{R}) \mid \kappa < \kappa_1 < \ldots < \kappa_\ell$ belong to $I_m \cap C_k\}$ where $C_k = \cup\{C_k^\kappa \cap \kappa \mid \kappa \in \text{I-i}\}$.

The point of T_k is that we can now "shrink" disjunctions in a strong sense: Define "ϕ^* is defined" as before using the above definition of C_k and replacing ℓ by m as defined above. In this way we obtain $\bar{S}_k =$ $\{\phi \longleftrightarrow \phi^* \mid \phi^*$ is defined$\}$ and $T_k \cup \bar{S}_k$ is consistent. But now we can also consistently adjoin the sentences $\{\vee\Phi \longleftrightarrow \vee(\Phi \cap M_k(\kappa)) \mid \vee\Phi \in M_k(\kappa) =$

$t(\gamma,\kappa,\kappa_1,\ldots,\kappa_\ell), \kappa < \kappa_1 < \ldots < \kappa_\ell$ in $I_m \cap C_k\}$. Let $\bar{\bar{S}}_k$ denote $\bar{S}_k$ together with the above sentences. So $T_0 \cup \bar{\bar{S}}_k$ is consistent.

To obtain S_k from $\bar{\bar{S}}_k$ we must consider the sentence ϕ_{k-1} from $L^L_{i\omega}$ and add it to $\bar{\bar{S}}_k$ if $T_0 \cup \bar{\bar{S}}_k \cup \{\phi_{k-1}\}$ is consistent; otherwise add $\sim\phi_{k-1}$. Also in the former case if $\phi_{k-1} = \vee\Phi$ then choose the L-least $\psi \in \Phi$ so that $T_0 \cup \bar{\bar{S}}_k \cup \{\psi\}$ is consistent and add ψ to $\bar{\bar{S}}_k$. Finally S_k consists of all quantifier-free sentences of $L^L_{\infty\omega}$ which are logical consequences of T_0 together with the resulting class of sentences. This completes the definition of S_k.

We now verify the desired properties of the S_k's. First we show that for any quantifier-free $\phi \in L^L_{\infty\omega}$ there exists k such that $T_0 \cup S_k \vdash \phi$ or $T_0 \cup S_k \vdash \sim\phi$. We do this by induction on the least $\kappa_\phi \in I-i$ such that $\phi \in L^L_{\kappa_\phi\omega}$: If $\kappa_\phi = i$ then $\phi = \phi_k$ for some k and thus by construction either $\phi \in S_{k+1}$ or $\sim\phi \in S_{k+1}$. Otherwise for some k $\phi = t(\gamma,\kappa_1,\ldots,\kappa_k)$ where $\gamma < \kappa_1 < \kappa_\phi$ and $i \le \kappa_1 < \ldots < \kappa_k$ belong to $I-i$. Now $\kappa_1,\ldots,\kappa_k$ belong to $I_m \cap C_k$ for all m so by construction S_k contains the sentence $\phi \longleftrightarrow \phi^*$ where $\phi^* = t(\gamma,\kappa_1^\gamma,\ldots,\kappa_k^\gamma)$ and $\kappa_1^\gamma < \ldots < \kappa_k^\gamma$ are less than κ_1. Thus $\kappa_{\phi^*} \le \kappa_1$ and so by induction $T_0 \cup S_\ell \vdash \phi^*$ or $T_0 \cup S_\ell \vdash \sim\phi^*$ for some . For $\ell \ge k$ we have $T_0 \cup S_\ell \vdash \phi$ or $T_0 \cup S_\ell \vdash \sim\phi$.

We also argue that if $\phi = \vee\Phi$ and $T_0 \cup S_k \vdash \phi$ then $T_0 \cup S_\ell \vdash \psi$ for some $\psi \in \Phi$, $\ell \in \omega$. Again this is shown by induction on κ_ϕ. If $\kappa_\phi = i$ then this follows directly from the construction. Otherwise choose k so that $T_0 \cup S_k \vdash \phi$ and $\phi = t(\gamma,\kappa,\kappa_1,\ldots,\kappa_k)$ where $\gamma < \kappa < \kappa_\phi$ and $i \le \kappa < \kappa_1 < \ldots < \kappa_k$ belong to $I-i$. Thus S_k contains the sentence $\phi \longleftrightarrow \phi^*$ where $\phi^* = t(\gamma,\kappa^\gamma,\kappa_1^\gamma,\ldots,\kappa_k^\gamma)$ and $\kappa^\gamma < \kappa_1^\gamma < \ldots < \kappa_k^\gamma$ in $I_m \cap C_k$ are less than κ_1. Write $\phi^* = \vee\Phi^*$. Now by construction $T_0 \cup S_k$ proves $\phi^* \longleftrightarrow \vee(\Phi^* \cap M_k(\kappa^\gamma))$ as $\phi^* \in H_k(\kappa^\gamma) \subseteq M_k(\kappa^\gamma)$. By induction $T_0 \cup S_{k'} \vdash \psi^*$ for some $\psi^* \in \Phi^* \cap M_k(\kappa^\gamma)$ and some $k' \ge k$. Now we can write $\psi^* = \bar{t}(\bar{\gamma},\kappa_1^\gamma,\ldots,\kappa_m^\gamma)$ for some $\bar{\gamma} < \kappa_1^\gamma$ and $\bar{t}$ since $M_k(\kappa_1^\gamma) \subseteq$ Skolem hull of $(\kappa^\gamma{}_1 \cup \{\kappa_1^\gamma,\ldots,\kappa_m^\gamma\})$. (Recall the requirement $m > \ell$ in the construction.) Finally we see that S_k contains the sentence $\psi^* \longleftrightarrow \psi$ where $\psi = \bar{t}(\bar{\gamma},\kappa_1,\ldots,\kappa_m)$ and $\psi \in \Phi$. Thus $T_0 \cup S_{k'} \vdash \psi$ for some $\psi \in \Phi$.

We can now complete the proof of the Main Lemma. Define $R = \{n \mid T_0 \cup S_m \vdash \underline{R}(\underline{n})$ for some $m\}$. We show by induction on quantifier-free $\phi \in L^L_{\infty\omega}$ that $L(R) \models \phi$ iff $T_0 \cup S_m \vdash \phi$ for some m. If ϕ is atomic then ϕ is of the form $\underline{n} \in \underline{m}, \underline{n} = \underline{m}$ or $\underline{R}(\underline{n})$ and so the result is clear by the definition of R (and the consistency of $T_0 \cup S_m$ for all m). If $\phi = \sim\psi$ then the result follows by induction from the consistency of $T_0 \cup S_m$ for each m and the completeness of $\cup\{T_0 \cup S_m \mid m \in \omega\}$. If $\phi = \vee\Phi$ then the result follows by induction and the fact demonstrated in the preceding paragraph. We have shown that $S = S_0$ has a model. Also note that there is such a model definable over $L(0^{\#},T)$ whenever $T \subseteq \omega$ is a counting of i. This completes the proof of the Main Lemma.

Finally we use the argument for the Main Lemma to prove the Theorem. Choose a real $R \in A$ which is nonconstructible and which weakly preserves k-indiscernibles for infinitely many k. By Lemmas 1-3 choose a special amenable class X containing I-α for some countable α so that R weakly preserves k-indiscernibles at (κ,γ) for infinitely many k, whenever $\kappa > \gamma$ belong to X. We consider the suitable class S of sentences which express "$\underline{R} \in A, \underline{R}$ is nonconstructible". Now given a counting T of a particular countable ordinal α, the proof of the Main Lemma provides the construction of a real R in $L(0^{\#},T)$ such that $L(R)$ is a model of S. However note that as $T_0 \cup S \vdash \underline{R}$ is nonconstructible there is the freedom at any stage k of the construction to consistently adjoin either of the sentences "$\underline{R}(\underline{n})$", "$\sim\underline{R}(\underline{n})$" to S_k, for some $n \in \omega$. Thus in this way it is easy to build a perfect binary tree to possible constructions, any branch through which yields a distinct model of S and hnece a distinct element of A. The collection of all reals produced in this way constitutes a perfect closed subset of A. This completes the proof of the Theorem.

POSTSCRIPT 1) There is a nodification of the condition "R weakly preserves k-indiscernibles for infinitely many k" which can be substituted into the statement of our Theorem. For $A,Y \subseteq \aleph_1$ we say that Y indiscernibly defines A if for some ℓ, Y is ℓ-indiscernible and for some term t, some $\gamma < \aleph_1$: $A \cap y_1 = t(\gamma,y_1,\ldots,y_\ell)$ for all $y_1<\ldots<y_\ell$ in Y, $\gamma < y_1$. Thus if $A \in L$ then I_ℓ indiscernibly defines A for sufficiently large ℓ. Now the new condition on R is: $(\star)$ For infinitely many k, X k-indiscernible, $X \in L \longrightarrow$ there exists $A \subseteq \aleph_1, A \in L$ such that $X \cap Y$ is k-indiscernible^R whenever $Y \subseteq \aleph_1, Y \in L$ and Y indiscernibly defines A.

2) The advantage of the preceding property $(\star)$ is that we can show: If R Jensen codes an amenable class then R satisfies $(\star)$. Thus any amenable forcing for producing a nonconstructible Π^1_2 singleton must be somewhat different than Jensen coding.

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