Isomorphism on Hyp

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Abstract

We show that isomorphism is not a complete Σ^1_1 equivalence relation even when restricted to the hyperarithmetic reals: If E_1 denotes the Σ^1_1 (even Δ^1_1) equivalence relation of [4]then for no Hyp function f do we have xEy iff $f(x)$ is isomorphic to $f(y)$ for all Hyp reals x, y. As a corollary to the proof we provide for each computable limit ordinal α a hyperarithmetic reduction of $\equiv_\alpha$ (elementary-equivalence for sentences of quantifier-rank less than α) on arbitrary countable structures to isomorphism on countable structures of Scott rank at most α.

In classical descriptive set theory, analytic equivalence relations (i.e., Σ^1_1 equivalence relations with parameters) are compared under the relation of Borel reducibility (see [3]). An important subclass of the Σ^1_1 equivalence relations is the class of isomorphism relations, i.e., the restrictions of the isomorphism relation on countable structures (viewed as an equivalence relation on reals coding such structures) to the models of a sentence of the infinitary logic $L_{\omega_1\omega}$. Scott's Theorem implies that the equivalence classes of any isomorphism relation are Borel, and therefore no isomorphism relation can be complete (under Borel reducibility) within the class of Σ^1_1 equivalence relations as a whole, as some of these have non-Borel equivalence classes.

The picture is different in the computable setting. It is shown in [1] that isomorphism on computable structures (viewed as an equivalence relation

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on natural numbers coding such structures), indeed on computable trees, is complete for Σ^1_1 equivalence relations under the natural analogue of Borel-reducibility for equivalence relations on numbers: E is reducible to F iff for some hyperarithmetic $f : \mathbb{N} \to \mathbb{N}$, $E(m, n)$ iff $F(f(m), f(n))$ for all m, n.

In [2] we surveyed the situation for classes of structures intermediate between the class of computable structures and the class of arbitrary countable structures. But one important case was not treated in that paper, the class of hyperarithmetic structures. The purpose of the present paper is to fill that gap.

By a Σ^1_1 *equivalence relation* we mean an equivalence relation on the reals which is Σ^1_1 definable without parameters (equivalently, with a hyperarithmetic real parameter). By a *Hyp function* from reals to reals we mean a function which is Δ^1_1 (equivalently Σ^1_1) definable without parameters. (Hyp stands for Hyperarithmetic, which equals Δ^1_1.) A Σ^1_1 equivalence relation F is *complete on Hyp* if for any Σ^1_1 equivalence relation E there is a Hyp function f such that for Hyp reals x, y: xEy iff $f(x)Ff(y)$. In [2] the question of whether isomorphism is complete on Hyp was left open. The method of [1] showing that it is complete on the class of computable structures does not seem to work due to the absence of a Hyp enumeration of all Hyp reals, and the use of Scott's Isomorphism Theorem to show that it is incomplete on the class of arbitrary countable structures does not work either, as if the countable structure $\mathcal{A}$ has a Hyp code there need not be a Borel set B with Hyp code which agrees on Hyp with the set of codes for structures isomorphic to $\mathcal{A}$.

The solution comes from a deeper look at descriptive set theory and infinitary logic.

Theorem 1 *Isomorphism is not complete on Hyp: There is a Σ^1_1 (even Δ^1_1) equivalence relation E such that for no Hyp function f do we have xEy iff $f(x), f(y)$ code isomorphic structures (on ω) for all Hyp reals x, y.*

The Relation E_1

For $x : \omega \times \omega \to 2$ and $n \in \omega$ define $(x)_n : \omega \to 2$ by $(x)_n(i) = x(n, i)$. The equivalence relation E_1 is defined by:

$$x\ E_1 y \text{ iff } (x)_n = (y)_n \text{ for large enough } n.$$

E_1 is a Hyp equivalence relation. It was introduced in [4], where it was shown that there is no Borel reduction of E_1 to isomorphism on countable structures (or even to any orbit equivalence relation determined by a Borel action of a Polish group). First we show:

Theorem 2 *Suppose that α is a limit of admissible ordinals. Then E_1 is not Hyp-reducible to $\simeq$ (isomorphism) on L_α: There is no total Hyp function f such that for x, y in L_α, $x\ E_1\ y$ iff $f(x), f(y)$ code isomorphic structures on ω.*

Proof. Suppose f were a Hyp-reduction of E_1 to $\simeq$ on L_α. For structures $\mathcal{A}$, $\mathcal{B}$ on ω define: $\mathcal{A} \simeq_n \mathcal{B}$ iff $\mathcal{A}$, $\mathcal{B}$ are isomorphic via an isomorphism which fixes $0, 1, \ldots, n-1$.

Also write $x\ E_1^{n,k}\ y$ iff $(x)_i = (y)_i$ for $i \geq n$ and $(x)_i \restriction k = (y)_i \restriction k$ for $i < n$.

Claim. Suppose that $g : \omega \times \omega \to 2$ is Cohen-generic over $L_{\omega_1^{ck}}$. Then for each m, n there is a k so that if $h : \omega \times \omega \to 2$ is Cohen-generic over $L_{\omega_1^{ck}}$ and $g\ E_1^{n,k}\ h$ then $f(g) \simeq_m f(h)$.

Proof. For any $x : n \times \omega \to 2$ let g^x be defined to agree with g on $(\omega \setminus n) \times \omega$ and to agree with x on $n \times \omega$. Also let $x_0 : n \times \omega \to 2$ take the constant value 0. Now note that $x = g \restriction n \times \omega$ is Cohen-generic over $L_{\omega_1^{ck}}[g \restriction (\omega \setminus n) \times \omega]$ and let k be large enough so that the condition $g \restriction n \times k$ on x forces that $f(g^x), f(g^{x_0})$ are isomorphic via an isomorphism sending $(0, 1, \ldots, m-1)$ to $\vec{k} = (k_0, k_1, \ldots, k_{n-1})$ for some fixed $\vec{k}$. If $h : \omega \times \omega \to 2$ is Cohen-generic over $L_{\omega_1^{ck}}$ and $g\ E_1^{n,k}\ h$ then $f(g), f(h)$ are both isomorphic to $f(g^{x_0})$ via an isomorphism sending $(0, 1, \ldots, m-1)$ to $\vec{k}$ and therefore $f(g) \simeq_m f(h)$. $\square$ (*Claim*)

Now inductively build sequences $((g^n, j_n) \mid n \in \omega)$ and $(\pi_n \mid 0 < n \in \omega)$ as follows (where the $g^n : \omega \times \omega \to 2$ are Cohen-generic over $L_{\omega_1^{ck}}$, $0 < j_0 < j_1 < \cdots$ are natural numbers and π_n is an isomorphism of $f(g^{n-1})$ onto $f(g^n)$). Fix an enumeration $(D_n \mid n \in \omega)$ in L_α of the dense sets for Cohen forcing which are definable over $L_{\omega_1^{ck}}$. Let $g^0 : \omega \times \omega \to 2$ be an arbitrary element of L_α which is Cohen-generic over $L_{\omega_1^{ck}}$ and set $j_0 = 1$. Suppose that g^n, j_n have been defined (also π_n if $n > 0$). To obtain g^{n+1} first apply the Claim to produce $k_n \geq j_n$ so that if h is Cohen-generic over $L_{\omega_1^{ck}}$ and $g^n E_1^{j_n,k_n} h$ then $f(g^n) \simeq_{l_n} f(h)$, where l_n is greater than the images and pre-images of the numbers less than n under the composition $\pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_0$ (if $n = 0$ set $l_n = 0$). Then choose j_{n+1} large enough so that some Cohen condition contained in $j_{n+1} \times j_{n+1}$, extending $g^n \restriction j_n \times k_n$ and satisfied by g^n belongs to the dense set D_n. Let g^{n+1} be g^n except at the pair (j_{n+1}, j_{n+1}) where its value is different from the value given by g^n. Finally, let π_{n+1} be an isomorphism witnessing $f(g^n) \simeq_{l_n} f(g^{n+1})$.

The resulting sequences have the following properties:

1. $f(g^n) \simeq_{l_n} f(g^{n+1})$ where the l_n's go to infinity. (thus the compositions $\pi_n \circ \pi_{n-1} \circ \cdots \circ \pi_0$ converge to a bijection).
2. The j_n's and k_n's increase to infinity (so the g^n's converge).
3. g^n, g^{n+1} agree on $j_{n+1} \times j_{n+1}$ but $(g^n)_{j_{n+1}}, (g^{n+1})_{j_{n+1}}$ differ somewhere.
4. $g =$ the limit of the g^n's is Cohen-generic over $L_{\omega_1^{ck}}$.

Then g is not E_1-equivalent to g^0 by 3. Now recall our assumption that α is a limit of admissibles. This implies that wellfoundedness is absolute to L_α (i.e. any tree in L_α that is illfounded is also illfounded in L_α) and from this it follows that any two structures which are countable in L_α and isomorphic are also isomorphic in L_α (build a tree of partial isomorphisms). It now follows that the sequence of g^n's can be built in L_α. Using 1 and 4, $f(g^0) \simeq f(g)$. But this contradicts the assumption that f is a reduction of E_1 to $\simeq$ on L_α. □

Now to prove Theorem 1 we modify the above argument as follows. Suppose that f were a Hyp reduction of E_1 to isomorphism on Hyp and choose a large enough computable ordinal α so that the code for f belongs to L_α. Fix a Hyp $g^0 : \omega \times \omega \to 2$ which is Cohen-generic over L_α and belongs to L_β where β is also computable. We would like to build sequences $((g^n, j^n) \mid n \in \omega)$ and $(\pi_n \mid 0 < n \in \omega)$ as above which are Hyp, as this will then yield the desired contradiction. This is possible provided there is a computable bound on the Scott ranks of all of the relevant structures $f(g^n)$, because if γ is a computable ordinal then the isomorphism relation on structures of Scott rank at most γ is Hyp.

Note that if $g, h : \omega \times \omega \to 2$ and gE_1h fails then $f(g), f(h)$ are non-isomorphic Hyp structures and therefore for some computable ordinal γ, $f(g) \not\equiv_\gamma f(h)$ (where $\equiv_\gamma$ is elementary equivalence for sentences of quantifier-rank less than γ). Now the set of pairs (g, h) in L_β such that gE_1h fails is a Hyp set (i.e., it belongs to $L_{\omega_1^{ck}}$) and therefore there is some fixed computable ordinal γ such that $f(g) \not\equiv_\gamma f(h)$ for all such pairs (g, h).

Lemma 3 *Suppose that α is a nonzero computable ordinal.*
Then there is a Hyp function $\mathcal{A} \mapsto \mathcal{A}^*$ from countable relational structures $\mathcal{A}$ to countable structures $\mathcal{A}^*$ such that:
(a) $\mathcal{A} \simeq \mathcal{B} \to \mathcal{A}^* \simeq \mathcal{B}^*$.
(b) $\mathcal{A}^* \equiv_\alpha \mathcal{B}^* \to \mathcal{A} \equiv_\alpha \mathcal{B}$.
(c) For each $\mathcal{A}$, $\mathcal{A}^*$ has Scott rank at most α.

Proof. We define $\mathcal{A}^*$ as follows:

a. An element of A^* is an $\equiv_\alpha$ class $[a_1, \ldots, a_n]$ of a tuple $(a_1, \ldots, a_n)$ from A (where two tuples are equivalent under $\equiv_\alpha$ iff they have the same length and satisfy the same formulas in $\mathcal{A}$ of quantifier-rank less than α).
b. $R^{\mathcal{A}^*}([a_1, \ldots, a_n])$ iff $R^{\mathcal{A}}(a_1, \ldots, a_n)$. (Thus the n-ary predicate $R^{\mathcal{A}}$ becomes the unary predicate $R^{\mathcal{A}^*}$. Note that $R^{\mathcal{A}^*}$ is well-defined.)
c. $[a_1, \ldots, a_n] * [b_1, \ldots, b_m] = [a_1, \ldots, a_n, b_1, \ldots, b_m]$ (i.e. we add a new binary concatenation function $*$).
d. $L_m([a_1, \ldots, a_n])$ iff $m = n$ (we add ω-many new unary predicates L_m, $m \in \omega$).

Claim 1. If $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then $\mathcal{A}^*$ is isomorphic to $\mathcal{B}^*$. (This is clear.)

Claim 2. If $\mathcal{A}^* \equiv_\alpha \mathcal{B}^*$ then $\mathcal{A} \equiv_\alpha \mathcal{B}$.

Proof. By induction on $\varphi = \varphi(x_1, \ldots, x_n)$ we show there is a formula $\varphi^* = \varphi^*(x^*)$ with the same quantifier-rank as φ such that $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ iff $\mathcal{A}^* \models \varphi^*([a_1, \ldots, a_n])$. For atomic $\varphi = R(x_1, \ldots, x_n)$ we may take φ^* to be $R(x^*)$. And $(\sim \varphi)^* = \sim \varphi^*$, $(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$.

If φ is $\exists y \psi(x_1, \ldots, x_n, y)$ then take φ^* to be $\exists y^*(\psi^*(x^* * y^*)$ and $L_1(y^*))$. We have:

$\mathcal{A} \vDash \exists y \psi(a_1, \ldots, a_n, y)$ iff
$\mathcal{A} \vDash \psi(a_1, \ldots, a_n, a_{n+1})$ for some a_{n+1} iff
$\mathcal{A}^* \vDash \psi^*([a_1, \ldots, a_n, a_{n+1}])$ for some a_{n+1} iff
$\mathcal{A}^* \vDash \psi^*([a_1, \ldots, a_n] * [a_{n+1}])$ and $L_1([a_{n+1}])$ for some a_{n+1} iff
$\mathcal{A}^* \vDash \exists y^*(\varphi^*([a_1, \ldots, a_n] * y^*)$ and $L_1(y^*))$. □ (*Claim 2.*)

Claim 3. Suppose that $(\mathcal{A}^*, [\vec{a}_1], \ldots, [\vec{a}_n]) \equiv_\alpha (\mathcal{A}^*, [\vec{b}_1], \ldots, [\vec{b}_n])$. Then $[\vec{a}_i] = [\vec{b}_i]$ for each i (and therefore $\mathcal{A}^*$ has Scott rank at most α).

Proof. The hypothesis implies that $(\mathcal{A}^*, [\vec{a}_i]) \equiv_\alpha (\mathcal{A}^*, [\vec{b}_i])$ for each i. And it is enough to show that $\vec{a}_i \equiv_\alpha \vec{b}_i$ for each i. If φ has quantifier-rank less than α then $\mathcal{A} \vDash \varphi(\vec{a}_i)$ iff $\mathcal{A}^* \vDash \varphi^*([\vec{a}_i])$ iff $\mathcal{A}^* \vDash \varphi^*([\vec{b}_i])$ iff $\mathcal{A} \vDash \varphi(\vec{b}_i)$. So as φ^* also has rank less than α we are done. □ (*Lemma 3*)

Now recall that we have computable ordinals $\beta < \gamma$ such that for g, h in L_β, gE_1h iff $f(g) \equiv_\gamma f(h)$. Applying the Lemma when α is equal to γ we obtain for g, h in L_β:

$gE_1h \rightarrow$
$f(g) \simeq f(h) \rightarrow$
$f(g)^* \simeq f(h)^*$

and

$f(g)^* \simeq f(h)^* \rightarrow$
$f(g)^* \equiv_\gamma f(h)^* \rightarrow$
$f(g) \equiv_\gamma f(h) \rightarrow$
gE_1h

and therefore have a Hyp reduction of E_1 on L_β to isomorphism on Hyp whose range consists of structures of Scott rank bounded by a fixed computable ordinal. As explained above, this allows us to repeat the proof of Theorem 2 to reach a contradiction from the assumption of a Hyp reduction E_1 to isomorphism on Hyp. This completes the proof of Theorem 1.

As a corollary to the proof of Lemma 3 we also obtain the following, which may be of independent interest.

Theorem 4 *For each computable limit ordinal α there is a Hyp reduction of the equivalence relation $\equiv_\alpha$ on countable structures to isomorphism on countable structures of Scott rank at most α.*

Proof. For each countable structure $\mathcal{A}$ and $\beta < \alpha$ let $\mathcal{A}^*_\beta$ be the structure of Scott rank at most β defined in the proof of Lemma 3. Now form $\mathcal{A}^*$ by taking the union of disjoint copies of the structures $\mathcal{A}^*_\beta$, $\beta < \alpha$, expanded with the quasiorder $x_{\beta_0} \leq x_{\beta_1}$ iff $\beta_0 \leq \beta_1$ when x_{β_i} belongs to the copy of $\mathcal{A}^*_{\beta_i}$. Thus $\mathcal{A}^*$ consists of the structures $\mathcal{A}^*_\beta$, $\beta < \alpha$, ordered in ordertype α.

If $\mathcal{A} \equiv_\alpha \mathcal{B}$, then for each $\beta < \alpha$, $\mathcal{A}^*_\beta$ is isomorphic to $\mathcal{B}^*_\beta$ via the isomorphism which sends $[a_1, \ldots, a_n]$ to $[b_1, \ldots, b_n]$, where $[b_1, \ldots, b_n]$ satisfies the same formulas of quanitfier rank at most β as $[a_1, \ldots, a_n]$ (the fact that $\mathcal{A} \equiv_\alpha \mathcal{B}$ implies that there is such a unique $[b_1, \ldots, b_n]$). It follows that $\mathcal{A}^*$ is isomorphic to $\mathcal{B}^*$. Conversely, if $\mathcal{A}^*$ is isomorphic to $\mathcal{B}^*$ then *Claim 2* of the proof of Lemma 3 implies that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of quantifier rank less than α and therefore $\mathcal{A} \equiv_\alpha \mathcal{B}$.

Finally, note that *Claim 3* of the proof of Lemma 3 implies that $\mathcal{A}^*$ has Scott rank at most α. $\square$

Remarks. The results of this paper relativise in the natural way: For any real parameter x, no reduction to isomorphism of E_1 restricted to the reals Hyp in x is Hyp in x. From this one can infer the Kechris-Louveau result that there is no Borel reduction of the entire E_1 to isomorphism on countable structures.

Question. Suppose that E is a Σ^1_1 equivalence relation and E_1 is not Hyp-reducible to E on Hyp. Then is E Hyp-reducible to isomorphism on Hyp?

References

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