Forcing when there are Large Cardinals

Summary:

1. What are large cardinals?

2. Forcings which preserve large cardinals (failure of GCH at a measurable)

3. Forcings which destroy large cardinals, but do something interesting (Singular Cardinal Hypothesis)

4. Some open questions

What are large cardinals?

κ is *inaccessible* iff:
$\kappa > \aleph_0$
κ is regular
$\lambda < \kappa \to 2^\lambda < \kappa$

κ inaccessible implies V_κ is a model of ZFC

κ is *measurable* iff:
$\kappa > \aleph_0$
$\exists$ nonprincipal, κ-complete ultrafilter on κ

What are large cardinals?

Embeddings:

V = universe of all sets
M an inner model (transitive class satisfying ZFC, containing Ord)

$j : V \longrightarrow M$ is an *embedding* iff:
j is not the identity
j preserves the truth of formulas with parameters

Critical point of j is the least κ, $j(\kappa) \neq \kappa$

Idea: κ is "large" iff κ is the critical point of an embedding
$j : V \longrightarrow M$ where M is "large"

What are large cardinals?

Suppose that κ is the critical point of $j : V \to M$

κ is λ-hypermeasurable iff $H(\lambda) \subseteq M$

κ is λ-supercompact iff $M^\lambda \subseteq M$

Fact: Measurable $= \kappa^+$-hypermeasurable $= \kappa$-supercompact.

Kunen: No $j : V \to M$ witnesses λ-hypermeasurability for all λ, i.e., M cannot equal V

However: κ could be λ-hypermeasurable for all λ (i.e., the critical point of embeddings with arbitrary degrees of hypermeasurability)

Forcings that preserve large cardinals

Question: Suppose κ is a large cardinal and G is P-generic over V. Is κ still a large cardinal in $V[G]$?

Lifting method (Silver):

Given $j : V \to M$ and G which is P-generic over V

Let P^* be $j(P)$

Goal: Find a G^* which is P^*-generic over M such that $j[G] \subseteq G^*$

Then $j : V \to M$ lifts to $j^* : V[G] \to M[G^*]$, defined by $j^*(\sigma^G) = j(\sigma)^{G^*}$

If G^* belongs to $V[G]$ then κ is still measurable (and maybe more) in $V[G]$

Forcings that preserve large cardinals

An example: Making GCH fail at a measurable cardinal

Theorem

Suppose that κ is κ^{++}-hypermeasurable. Then in a forcing extension, κ is still measurable and $2^\kappa = \kappa^{++}$.

Theorem is due to Woodin; the proof below is due to Katie Thompson and myself.

Step 1. Choose a forcing to make GCH fail at kappa.

Obvious choice: Cohen(κ, κ^{++})
Adds κ^{++}-many κ-Cohen sets
Conditions are partial functions of size $< \kappa$ from $\kappa \times \kappa^{++}$ to 2

Better choice: Sacks(κ, κ^{++})
Adds κ^{++}-many κ-Sacks subsets of κ (defined later)

Forcings that preserve large cardinals

Step 2: Prepare below κ

Here is the problem (illustrated using just κ-Cohen forcing):
Suppose that $C \subseteq \kappa$ is κ-Cohen generic
Want to lift $j : V \to M$ to $j^* : V[C] \to M[C^*]$
Need to find C^* which is $j(\kappa)$-Cohen generic over M and "extends" C, i.e., such that $C = C^* \cap \kappa$
Impossible! C does not belong to M!
Need the forcing to add C^* to be defined not in M but in a model that already has C

Solution: Force not just at κ, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$$

where $P(\alpha)$ denotes α-Cohen forcing.
Let $C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)$ denote the P-generic

Forcings that preserve large cardinals

Now we want to lift $j : V \to M$ to

$j^* : V[C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)] \to$
$M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))]$

where the β_i's are the inaccessibles of M between κ and $j(\kappa)$.

To find the C^*'s:

Set $C^*(\alpha) = C(\alpha)$ for $\alpha < \kappa$
Set $C^*(\kappa) = C(\kappa)$
Take $\langle C^*(\beta) \mid \kappa < \beta < j(\kappa)\rangle$ to be any generic (they exist)

Last lift: Take $C^*(j(\kappa))$ to be any generic for $j(\kappa)$-Cohen forcing of $M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$ containing the condition $C(\kappa) = C^*(\kappa)$ (such generics exist).

Forcings that preserve large cardinals

Step 3: Make this work with κ-Cohen forcing replaced by some forcing that kills the GCH at κ

Here is the problem:

For inaccessible $\alpha \leq \kappa$ replace α-Cohen by Cohen(α, α^{++})
All goes well until the last lift: we *can* choose $C^*(\gamma)$ for all M-inaccessible $\gamma < j(\kappa)$ and lift $j : V \longrightarrow M$ to
$j' : V[C(\alpha_0) * C(\alpha_1) * \cdots] \longrightarrow$
$M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$
We then need to find a generic for the Cohen$(j(\kappa), j(\kappa^{++}))$-forcing of $M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$ which contains $j'[C(\kappa)]$.
But Cohen$(j(\kappa), j(\kappa^{++}))$ is a very big forcing (it may have no generic; we may have to force one!) and $j'[C(\kappa)]$ is a very complicated set of conditions in this forcing (it is not easy to force a generic that contains it!)

Forcings that preserve large cardinals

Here is the solution: Use Sacks(κ, κ^{++}) instead of Cohen(κ, κ^{++}) Then we don't have to build a generic $S^*(j(\kappa))$ for Sacks($j(\kappa), j(\kappa^{++})$) because $j'[S(\kappa)]$ builds one for us!

Illustrate with κ-Sacks: A condition is a perfect κ-tree with a closed unbounded set of splitting levels. If G is generic then the intersection of the κ-trees in G gives us a function $g : \kappa \to 2$.

Lemma

(Tuning Fork Lemma) Suppose that $j : V' \to M'$ has critical point κ, g is κ-Sacks generic over V', M' is included in $V'[g]$ and g belongs to M'. Then in $V'[g]$ there are exactly two generics h_0, h_1 for the $j(\kappa)$-Sacks of M' extending g; moreover $h_0(\kappa) = 0$ and $h_1(\kappa) = 1$.

A similar result holds for Sacks(κ, κ^{++}), thereby solving the problem of the "last lift".

Forcings that preserve large cardinals

Some other applications:

(with Magidor) Assume GCH, let κ be measurable and let α be any cardinal at most κ^{++}. Then there is a cofinality-preserving forcing extension in which there are exactly α-many normal measures on κ.

(with Dobrinen) Assume GCH and let κ be λ-hypermeasurable where λ is weakly compact and greater than κ. Then there is a forcing extension in which κ is still measurable and the tree property holds at κ^{++}.

(with Zdomskyy) Assume GCH and let κ be κ^{++}-hypermeasurable. Then there is a cofinality-preserving forcing extension in which κ is still measurable and the symmetric group on κ has cofinality κ^{++}.

Forcings which use large cardinals: The SCH

Singular cardinal hypothesis (SCH):
If $2^{\text{cof}(\kappa)} < \kappa$ then $\kappa^{\text{cof}(\kappa)} = \kappa^+$
SCH $\Rightarrow$ GCH holds at singular strong limit cardinals

Theorem

(Prikry) Suppose that κ is measurable and the GCH fails at κ. Then in a forcing extension, κ is still a strong limit cardinal where the GCH fails, but now κ has cofinality ω. In particular, the SCH fails in this forcing extension.

Prikry forcing: A forcing that preserves cardinals, adds no new bounded subsets of κ but adds an ω-sequence cofinal in κ

Forcings which use large cardinals: The SCH

Conditions in Prikry forcing:

Fix a normal measure U on κ. A condition is a pair (s, A) where s is a finite subset of κ and A belongs to U.

Extension in Prikry forcing:

(t, B) extends (s, A) iff
t end-extends s
B is a subset of A
$t \setminus s$ is contained in A

Facts: (a) If G is P-generic then $\bigcup\{s \mid (s, A) \in G \text{ for some } A\}$ is an ω-sequence cofinal in κ.
(b) P is κ^+-cc: If $(s, A), (t, B)$ are conditions and $s = t$ then (s, A) and (t, B) are compatible.

Forcings which use large cardinals: The SCH

The main lemma about Prikry forcing is the following. We say that (t, B) is a *direct extension* of (s, A) iff $s = t$ and B is a subset of A.

Lemma (The Prikry property)

For σ a sentence of the forcing language, every condition has a direct extension which decides σ (i.e., either forces σ or $\sim \sigma$).

Forcings which use large cardinals: The SCH

Lemma (The Prikry property)

For σ a sentence of the forcing language, every condition has a direct extension which decides σ (i.e., either forces σ or $\sim \sigma$).

Proof. Suppose that (s, A) is a condition and define $h : [A]^{<\omega} \to 2$ as follows:

$h(t) = 1$ iff $(s \cup t, B) \Vdash \sigma$ for some B
$h(t) = 0$ otherwise.

As U is normal there is $A^* \in U$ which is *homogeneous* for h: For each n and t_1, $t_2 \in [A^*]^n$, $h(t_1) = h(t_2)$. Then (s, A^*) decides σ: Otherwise there would be $(s \cup t_1, B_1)$, $(s \cup t_2, B_2)$ extending (s, A^*) which force σ, $\sim \sigma$, respectively. We can assume that for some n, both t_1 and t_2 belong to $[A^*]^n$. But then $h(t_1) = 1$, $h(t_2) = 0$, contradicting homogeneity. $\square$

Forcings which use large cardinals: The SCH

Corollary: P does not add new bounded subsets of κ.

Proof. Suppose $(s, A) \Vdash \dot{a}$ is a subset of λ, where λ is less than κ. Set $(s, A_0) = (s, A)$ and using the Prirky property choose a direct extension (s, A_1) of (s, A_0) which decides "$0 \in \dot{a}$". Then choose a direct extension (s, A_2) of (s, A_1) which decides "$1 \in \dot{a}$", etc. After λ steps we have a direct extension (s, A_λ) of (s, A) which decides which ordinals less than λ belong to $\dot{a}$, and therefore forces $\dot{a}$ to belong to the ground model. $\square$

In summary: If G is P-generic then κ has cofinality ω in $V[G]$ and V, $V[G]$ have the same cardinals and bounded subsets of κ. In particular, if GCH fails at κ in V, then in $V[G]$, κ is a singular strong limit cardinal where the GCH fails.

Forcings which use large cardinals: The SCH

An improvement: Model where $\aleph_\omega$ is strong limit and the GCH fails at $\aleph_\omega$

Theorem

(Magidor) Suppose that κ is measurable. Then there is a forcing extension in which κ equals $\aleph_\omega$.

For the proof, mix Prikry forcing with *Lévy collapses*:

Suppose that $\alpha < \beta$ are regular. Then Lévy(α, β) is a forcing that makes β into α^+ and otherwise preserves cardinals:

$p \in$ Lévy(α, β) iff p is partial function of size $< \alpha$ from $\alpha \times \beta$ to β such that $p(\alpha_0, \beta_0) < \beta_0$ for each (α_0, β_0) in the domain of p.

Forcings which use large cardinals: The SCH

Collapsing Prikry forcing: 1st try

Fix a normal measure U on κ. A condition is of the form $((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible
p_i belongs to Lévy(α_i, α_{i+1}) for $i < n-1$
p_{n-1} belongs to Lévy(α_{n-1}, κ)
A belongs to U

To extend: Strengthen the p_i's, increase n, shrink A and take the new α's from the old A

Problem: This collapses κ to ω (the p_i's are running wild!)

Solution: Control the p_i's on a measure one set

Forcings which use large cardinals: The SCH

Collapsing Prikry forcing: 2nd try

Let $j : V \to M$ witness that κ is measurable and choose U to be the normal measure $\{A \mid \kappa \in j(A)\}$

Guiding generic: Choose G in V to be generic over M for Lévy$(\kappa^+, j(\kappa))$ of M (this is possible)

Now define a condition to be of the form $((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible

p_i belongs to Lévy$(\alpha_i^+, \alpha_{i+1})$ for $i < n - 1$

p_{n-1} belongs to Lévy(α_{n-1}^+, κ)

A belongs to U

F is a function with domain A such that $F(\alpha)$ belongs to Lévy(α^+, κ) for each inaccessible α in A

$j(F)(\kappa)$ belongs to G

Forcings which use large cardinals: The SCH

An extension of
$p = ((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F)$
is of the form
$p^* = ((\alpha_0^*, p_0^*), (\alpha_1^*, p_1^*), \ldots, (\alpha_{n^*-1}^*, p_{n^*-1}^*), A^*, F^*)$ where:

n^* is at least n
$\alpha_i^* = \alpha_i$ and p_i^* extends p_i for $i < n$
p_j^* extends $F(\alpha_j^*)$ for $j \geq n$
A^* is contained in A
$F^*(\alpha)$ extends $F(\alpha)$ for each $\alpha \in A^*$

p^* is a *direct extension* of p if in addition $n^* = n$

A generic produces a Prikry sequence $\alpha_0 < \alpha_1 < \cdots$ in κ together with Lévy collapses $g_0, g_1, \ldots$ where g_i ensures $\alpha_{i+1} = \alpha_i^{++}$. So after collapsing α_0, we see that κ is at most $\aleph_\omega$.
The forcing is κ^+-cc. But why isn't κ collapsed?

Forcings which use large cardinals: The SCH

The Prikry property: For σ a sentence of the forcing language, every condition has a direct extension which decides σ.

Using this, one gets: Any bounded subset of κ belongs to $V[g_0, g_1, \ldots, g_n]$ for some n, and therefore κ remains a cardinal

Summary: Prikry Collapse forcing makes κ into $\aleph_\omega$ and preserves cardinals above κ.

Now start with κ measurable and GCH failing at κ.
Then Prikry Collapse forcing makes κ into $\aleph_\omega$ with $\aleph_\omega$ strong limit, GCH failing at $\aleph_\omega$ (Strong failure of the SCH)

Open Questions

1. Preserving large cardinals

Consider various cardinal characteristics of the continuum (almost-disjointness number, bounding number, dominating number, splitting number, ...)

How do these behave at a large cardinal?

Is it consistent that a strongly compact cardinal have a unique normal measure?

Is it consistent with a supercompact cardinal for $H(\kappa^+)$ to have a definable wellordering for every uncountable κ?

Open Questions

2. Using large cardinals

(SCH-type problems): What are the possibilities for the function $n \mapsto 2^{\aleph_n}$ for $n \leq \omega$?

Is it consistent that there is no κ-Aronszajn tree for any regular cardinal $\kappa > \omega_1$?

Is it consistent to have stationary reflection at the successor of each singular cardinal?

Can the nonstationary ideal on ω_1 be saturated with CH?

Can $\aleph_\omega$ be Jonsson?