Forcing when there are large cardinals: an introduction

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My aim in this short article is to describe two ideas in the context of large cardinal forcing which may not be familiar to those who have not worked in this area. The first idea addresses the problem of showing that large cardinals are consistent with interesting properties, such as a failure of the GCH, and the second idea uses large cardinals to obtain interesting properties for singular cardinals. The first idea originates with work of Silver and the latter with work of Prikry (see chapter 21 of [7]).

Section 1: What are large cardinals?

We begin with the smallest of large cardinals, the (strongly) inaccessibles.
κ is *inaccessible* iff:
$\kappa > \aleph_0$
κ is regular
$\lambda < \kappa \rightarrow 2^\lambda < \kappa$

Inaccessibles qualify as "large" as their existence cannot be proved in ZFC. If κ is inaccessible then V_κ is a model of ZFC. Thus proving the existence of inaccessibles within ZFC entails proving the consistency of ZFC within ZFC, an impossibility by Gödel's second incompleteness theorem.

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Still larger are the measurable cardinals.

κ is *measurable* iff:
$\kappa > \aleph_0$
$\exists$ nonprincipal, κ-complete ultrafilter on κ

This is a perfectly good definition, but proving things about measurable cardinals often demands an alternative, equivalent definition, phrased in terms of (elementary) embeddings.

Embeddings

Let V denote the universe of sets and M an inner model (i.e., a transitive proper class that satisfies the axioms of ZFC).

A definable $j : V \to M$ is an *embedding* iff:
j is not the identity
j preserves the truth of formulas with parameters

The *Critical point* of j is the least ordinal κ such that $j(\kappa) \neq \kappa$. It is easy to show that such a κ must exist and is an uncountable cardinal. Many large cardinal notions are defined in the following way: κ is "large" iff κ is the critical point of an embedding $j : V \to M$ where M is "large", i.e., where M is "close" to V. For example:

κ is λ-*hypermeasurable* iff $H(\lambda) \subseteq M$

κ is λ-*supercompact* iff $M^\lambda \subseteq M$.

Then we have: Measurable = κ^+-hypermeasurable = κ-supercompact.

A remarkable fact due to Kunen (see chapter 17 of [7]) is that we cannot have $j : V \to V$, i.e., M cannot be equal to V. Equivalently, no single embedding can witness λ-hypermeasurability for all λ simultaneously. However (as far as we know) there could be a cardinal κ which is λ-hypermeasurable for all λ, where λ-hypermeasurability is witnessed by different embeddings for different λ. Such cardinals are said to be *strong*.

Section 2: Forcings that preserve large cardinals

Question: Suppose that κ is a large cardinal and G is P-generic over V. Is κ still a large cardinal in $V[G]$?

Silver provided a useful technique, his *lifting method*, which can often be used to give a positive answer to this question. Suppose that $j : V \to M$ and G is P-generic over V. Let P^* denote $j(P)$

Goal: Find a G^* which is P^*-generic over M such that $j[G] \subseteq G^*$.

If this can be achieved, then $j : V \to M$ lifts to $j^* : V[G] \to M[G^*]$, defined by: $j^*(\sigma^G) = j(\sigma)^{G^*}$ (for arbitrary P-names σ). Using the property $j[G] \subseteq G^*$ we have:

$V[G] \models \varphi(\sigma_1^G, \ldots, \sigma_n^G)$ iff
$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ for some $p \in G$ only if
$p^* \Vdash \varphi(j(\sigma_1), \ldots, j(\sigma_n))$ for some $p^* \in G^*$ iff
$M[G^*] \models \varphi(j(\sigma_1)^{G^*}, \ldots, j(\sigma_n)^{G^*})$ iff
$M[G^*] \models \varphi(j^*(\sigma_1^G), \ldots, j^*(\sigma_n^G))$.

All of the above are equivalences, as we can apply the same argument to $\sim \varphi$. This implies that j^* is well-defined and elementary.

Now if G^* can be found in $V[G]$ then j^* is definable in $V[G]$ and therefore κ is still measurable (and maybe more) in $V[G]$. This completes the description of Silver's lifting method.

We now describe a specific application of Silver's method, to prove the following theorem of Woodin.

Theorem 1 *Suppose that κ is κ^{++}-hypermeasurable. Then in a forcing extension, κ is still measurable and $2^\kappa = \kappa^{++}$.*

The proof below is due to Katie Thompson and myself [4].

Step 1. Choose a forcing to make GCH fail at kappa.

The obvious choice here is Cohen(κ, κ^{++}), the forcing that adds κ^{++}-many κ-Cohen sets. A condition in this forcing is a partial function of size $< \kappa$ from $\kappa \times \kappa^{++}$ to 2.

But surprisingly, for the purpose of applying Silver's lifting method there is a better choice: Sacks(κ, κ^{++}), the forcing that adds κ^{++}-many κ-*Sacks* subsets of κ with conditions of size κ. I'll give a precise definition of this forcing later, when we reach the point of needing it.

Step 2. "Prepare" below κ.

We illustrate the need for "preparation" using just κ-Cohen forcing: Suppose that $C \subseteq \kappa$ is κ-Cohen generic. We want to lift $j : V \to M$ to $j^* : V[C] \to M[C^*]$, where C^* is $j(\kappa)$-Cohen generic over M. We also want that G^*, the generic determined by C^* contains the image under j of G, the generic determined by C. But in this simple case, j is the identity on G and therefore we are asking for G^* to contain G, or equivalently, for C to be an initial segment of C^*.

But this is impossible! C does not belong to V so surely it does not belong to M. But as C^* is $j(\kappa)$-Cohen generic over M, all proper initial segments of C^* must belong to M.

To solve this problem, we need that the forcing to add C^* be defined not in M, but in a model that already has C.

Solution: Force not just at κ, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$$

where $P(\alpha)$ denotes α-Cohen forcing.

Suppose that we do this, and let $C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)$ denote the resulting P-generic.

Now we want to lift $j : V \to M$ to

$j^* : V[C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)] \to$
$M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))]$,

where the β_i's are the inaccessibles of M between κ and $j(\kappa)$.

To find the C^*'s, we start off as follows:

Set $C^*(\alpha) = C(\alpha)$ for $\alpha < \kappa$
Set $C^*(\kappa) = C(\kappa)$
Take $\langle C^*(\beta) \mid \kappa < \beta < j(\kappa)\rangle$ to be any generic. With mild assumptions about j, such generics exist. (Justifying this claim here would be too distracting; the argument can be found in [4].)

Finally we come to the "last lift": Take $C^*(j(\kappa))$ to be any generic for $j(\kappa)$-Cohen forcing of $M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$ which contains the condition $C(\kappa) = C^*(\kappa)$. (As in the previous paragraph, such generics exist.)

Step 3: Make the above work with κ-Cohen forcing replaced by some forcing that kills the GCH at κ.

Here is the problem:

For inaccessible $\alpha \leq \kappa$ replace α-Cohen by Cohen(α, α^{++}), the obvious forcing to make the GCH fail at α. All goes well until the last lift: we *can* choose $C^*(\gamma)$ for all M-inaccessible $\gamma < j(\kappa)$ and lift $j : V \to M$ to

$j' : V[C(\alpha_0) * C(\alpha_1) * \cdots] \to$
$M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$

We then need to find a generic for the Cohen$(j(\kappa), j(\kappa^{++}))$-forcing of $M[C^*(\alpha_0) * C^*(\alpha_1) * \cdots * C^*(\kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots]$ which contains $j'[C(\kappa)]$. But Cohen$(j(\kappa), j(\kappa^{++}))$ is a very big forcing (it may have no generic; we may have to force one!) and $j'[C(\kappa)]$ is a very complicated set of conditions in this forcing (it is not easy to force a generic that contains it!).

Here is the solution: Use Sacks(κ, κ^{++}) instead of Cohen(κ, κ^{++}). Then we don't have to build a generic $S^*(j(\kappa))$ for Sacks$(j(\kappa), j(\kappa^{++}))$ because $j'[S(\kappa)]$ builds one for us!

We illustrate this with the simpler forcing κ-Sacks: A condition is a perfect κ-tree with a closed unbounded set of splitting levels, i.e., a subtree of $2^{<\kappa}$ with no terminal nodes closed under increasing sequences of length less than κ with the property that for some CUB subset C of κ, all nodes with length in C are splitting nodes. If G is generic for κ-Sacks then the intersection of the κ-trees in G gives us a function $g : \kappa \to 2$.

For our present purposes, the key fact is the following.

Lemma 2 *(Tuning Fork Lemma) Suppose that $j : V' \to M'$ has critical point κ, g is κ-Sacks generic over V', M' is included in $V'[g]$ and g belongs to M'. Then in $V'[g]$ there are exactly two generics h_0, h_1 for the $j(\kappa)$-Sacks of M' extending g; moreover h_0 and h_1 disagree at κ.*

For the proof, see [4]. The idea is that unlike κ-Cohen forcing, κ-Sacks forcing has a (weak) form of closure, known as κ-fusion, which enables one to show that the image under j of the κ-Sacks generic g *almost* generates a $j(\kappa)$-Sacks generic; the only missing bit of information is what the value of the $j(\kappa)$-Sacks generic should be at κ.

A similar result holds for Sacks(κ, κ^{++}), thereby solving the problem of the "last lift" and completing the proof of Theorem 1.

I mention some other applications of the "tuning fork" method:

(with Magidor [3]) Assume GCH, let κ be measurable and let α be any cardinal at most κ^{++}. Then there is a cofinality-preserving forcing extension in which there are exactly α-many normal measures on κ.

(with Dobrinen [2]) Assume GCH and let κ be λ-hypermeasurable where λ is weakly compact and greater than κ. Then there is a forcing extension in which κ is still measurable and the tree property holds at κ^{++}.

(with Zdomskyy [5]) Assume GCH and let κ be κ^{++}-hypermeasurable. Then there is a cofinality-preserving forcing extension in which κ is still measurable and the symmetric group on κ has cofinality κ^{++}.

Section 2. Forcings which use large cardinals: the SCH

The *Singular cardinal hypothesis (SCH)* is the following statement:

If $2^{\text{cof }(\kappa)} < \kappa$ then $\kappa^{\text{cof }(\kappa)} = \kappa^+$

The SCH implies that the GCH holds at all singular strong limit cardinals.

Theorem 3 *(Prikry) Suppose that κ is measurable and the GCH fails at κ. Then in a forcing extension, κ is still a strong limit cardinal where the GCH fails, but now κ has cofinality ω. In particular, the SCH fails in this forcing extension.*

Prikry proved this using what we now call Prikry forcing, a forcing that preserves cardinals, adds no new bounded subsets of κ but adds an ω-sequence cofinal in κ. To describe this forcing, fix a κ-complete nonprincipal ultrafilter U on κ. We can also assume that U is normal, i.e., if A_i, $i < \kappa$ belong to U then so does their diagonal intersection $\triangle_{i<\kappa} A_i = \{\alpha < \kappa \mid \alpha$ belongs to A_i for each $i < \alpha\}$. A consequence of normality that we will use later is that if $h : [\kappa]^{<\omega} \to F$ is a function from the set $[\kappa]^{<\omega}$ of finite subsets of κ into the finite set F, then some $A \in U$ is *homogeneous for* h, i.e., h is constant on $[\kappa]^n$, the set of cardinality n subsets of A, for each finite n.

A condition in Prikry forcing (for the ultrafilter U) is a pair (s, A) where s is a finite subset of κ and A belongs to U. The condition (t, B) extends the condition (s, A) iff:

t end-extends s
B is a subset of A
$t \setminus s$ is contained in A

Facts: (a) If G is P-generic then $\bigcup\{s \mid (s, A) \in G$ for some $A\}$ is an ω-sequence cofinal in κ.
(b) P is κ^+-cc. This is because if $(s, A), (t, B)$ are conditions and $s = t$ then (s, A) and (t, B) are compatible.

The main lemma about Prikry forcing is the following. We say that (t, B) is a *direct extension* of (s, A) iff $s = t$ and B is a subset of A.

Lemma 4 (The Prikry property) *For σ a sentence of the forcing language, every condition has a direct extension which decides σ (i.e., either forces σ or forces $\sim \sigma$).*

Proof. Suppose that (s, A) is a condition and define $h : [A]^{<\omega} \to 2$ as follows:

$h(t) = 1$ iff $(s \cup t, B) \Vdash \sigma$ for some B
$h(t) = 0$ otherwise.

As U is normal there is $A^* \in U$ which is homogeneous for h, i.e., for each n and $t_1, t_2 \in [A^*]^n$ (the set of size n subsets of A^*), $h(t_1) = h(t_2)$. Then (s, A^*) decides σ: Otherwise there would be $(s \cup t_1, B_1)$, $(s \cup t_2, B_2)$ extending (s, A^*) which force σ, $\sim \sigma$, respectively. We can assume that for some n, both t_1 and t_2 belong to $[A^*]^n$. But then $h(t_1) = 0$, $h(t_2) = 1$, contradicting the homogeneity of A^*. $\square$

Corollary 5 *P does not add new bounded subsets of κ.*

Proof. Suppose $(s, A) \Vdash \dot{a}$ is a subset of λ, where λ is less than κ. Set $(s, A_0) = (s, A)$ and using the Prikry property choose a direct extension (s, A_1) of (s, A_0) which decides "$0 \in \dot{a}$". Then choose a direct extension (s, A_2) of (s, A_1) which decides "$1 \in \dot{a}$", etc. After λ steps we have a direct extension (s, A_λ) of (s, A) which decides which ordinals less than λ belong to $\dot{a}$, and therefore forces $\dot{a}$ to belong to the ground model. $\square$

In summary: If G is P-generic then κ has cofinality ω in $V[G]$ and V, $V[G]$ have the same cardinals and bounded subsets of κ. In particular, if GCH fails at κ in V, then in $V[G]$, κ is a singular strong limit cardinal where the GCH fails.

Down to $\aleph_\omega$

Theorem 3 provides a counterexample to the SCH at a large singular cardinal; can one have a counterexample at $\aleph_\omega$? I.e., can $\aleph_\omega$ be a strong limit cardinal where the GCH fails? Magidor answered this positively.

Theorem 6 *(Magidor) Suppose that κ is measurable. Then there is a forcing extension in which κ equals $\aleph_\omega$. Moreover, κ remains a strong limit cardinal and cardinals above κ are preserved.*

By starting with a measurable cardinal where the GCH fails, this theorem gives us a failure of the SCH at $\aleph_\omega$.

For the proof of Theorem 6, we have to mix Prikry forcing with *Lévy collapses.* The latter are defined as follows.

Suppose that $\alpha < \beta$ are inaccessible. A condition $p \in \text{Lévy}(\alpha, \beta)$ is a partial function of size $< \alpha$ from $\alpha \times \beta$ to β such that $p(\alpha_0, \beta_0) < \beta_0$ for each (α_0, β_0)

in the domain of p. A generic for Lévy(α, β) adds a function from α onto β_0 for each $\beta_0 < \beta$ and therefore ensures that β is at most α^+. The forcing is α-closed and therefore preserves cardinals up to α. Using the fact that β is inaccessible it can be shown that Lévy(α, β) has the β-cc and therefore β becomes α^+ and all cardinals from β up are preserved. Thus Lévy(α, β) is a forcing that makes β into α^+ and otherwise preserves cardinals.

Collapsing Prikry forcing: 1st try

We would like to define a forcing, *collapsing Prikry forcing*, which adds an ω-sequence $\alpha_0 < \alpha_1 < \cdots$ cofinal in κ and also collapses all but finitely many cardinals between adjacent elements of this ω-sequence. The net effect will be that except for cardinals below α_0, the ordertype of the cardinals that remain is just ω. By collapsing α_0 to ω we have then made κ into $\aleph_\omega$.

Here is our first attempt at doing this. Fix a normal measure U on κ. A condition is of the form $((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible
p_i belongs to Lévy(α_i, α_{i+1}) for $i < n - 1$
p_{n-1} belongs to Lévy(α_{n-1}, κ)
A belongs to U

To extend: Strengthen the p_i's, increase n, shrink A and take the new α's from the old A.

Unfortunately, this forcing collapses κ to ω (the p_i's are running wild)! The solution will be to "control" the p_i's using an embedding $j : V \to M$ associated with the normal measure U.

Collapsing Prikry forcing: 2nd try

Let $j : V \to M$ witness that κ is measurable and choose U to be the normal measure $\{A \mid \kappa \in j(A)\}$. We control the p_i's using:

The guiding generic. Choose G in V to be generic over M for Lévy$(\kappa^+, j(\kappa))$ of M.

With mild assumptions on j, such a guiding generic can be shown to exist. Now define a condition to be of the form $((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F)$ where:

$\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessible
p_i belongs to Lévy$(\alpha_i^+, \alpha_{i+1})$ for $i < n-1$
p_{n-1} belongs to Lévy(α_{n-1}^+, κ)
A belongs to U
F is a function with domain A such that $F(\alpha)$ belongs to Lévy(α^+, κ) for each inaccessible α in A
$j(F)(\kappa)$ belongs to G

An extension of
$p = ((\alpha_0, p_0), (\alpha_1, p_1), \ldots, (\alpha_{n-1}, p_{n-1}), A, F)$
is of the form
$p^* = ((\alpha_0^*, p_0^*), (\alpha_1^*, p_1^*), \ldots, (\alpha_{n^*-1}^*, p_{n^*-1}^*), A^*, F^*)$ where:

n^* is at least n
$\alpha_i^* = \alpha_i$ and p_i^* extends p_i for $i < n$
p_j^* extends $F(\alpha_j^*)$ for $j \geq n$
A^* is contained in A
$F^*(\alpha)$ extends $F(\alpha)$ for each $\alpha \in A^*$.

p^* is a *direct extension* of p if in addition $n^* = n$.

A generic produces a Prikry sequence $\alpha_0 < \alpha_1 < \cdots$ in κ together with Lévy collapses $g_0, g_1, \ldots$ where g_i ensures $\alpha_{i+1} = \alpha_i^{++}$. So after collapsing α_0, we see that κ is at most $\aleph_\omega$. The forcing is κ^+-cc. But why isn't κ collapsed? As in ordinary Prikry forcing, we need the Prikry property:

The Prikry property: For σ a sentence of the forcing language, every condition has a direct extension which decides σ.

Using this, one gets: Any bounded subset of κ belongs to $V[g_0, g_1, \ldots, g_n]$ for some n, and therefore κ remains a cardinal. So Prikry collapse forcing makes κ into $\aleph_\omega$ and preserves cardinals above κ. Now start with κ measurable and GCH failing at κ. Then Prikry Collapse forcing makes κ into $\aleph_\omega$ with $\aleph_\omega$ strong limit and GCH failing at $\aleph_\omega$. This is the desired failure of the SCH at $\aleph_\omega$.

Two further applications of Prikry collapse forcing

(with Katie Thompson) Relative to a κ which is κ^{++}-hypermeasurable it is consistent that $\aleph_\omega$ is a strong limit cardinal and there is no universal graph of size $\aleph_{\omega+1}$.

(with Ajdin Halilović) Relative to a weakly compact hypermeasurable it is consistent that the tree property holds at $\aleph_{\omega+2}$.

Some open questions

The field of large cardinal forcing is full of interesting open problems. Below is a random and incomplete list.

1. Consider various cardinal characteristics of the continuum (almost-disjointness number, bounding number, dominating number, splitting number, ...) How do these behave at a large cardinal?

2. Is it consistent that a strongly compact cardinal have a unique normal measure?

3. Is it consistent with a supercompact cardinal for $H(\kappa^+)$ to have a definable wellordering for every uncountable κ?

4. (SCH-type problems): What are the possibilities for the function $n \mapsto 2^{\aleph_n}$ for $n \leq \omega$?

5. Is it consistent that there is no κ-Aronszajn tree for any regular cardinal $\kappa > \omega_1$?

6. Can the nonstationary ideal on ω_1 be saturated with CH?

7. Can $\aleph_\omega$ be Jonsson?

For the definitions of “Arsonszajn tree”, “saturated ideal” and “Jonsson cardinal” we refer the reader to [7], pages 116, 409 and 305, respectively.

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