The Current State of the Foundations of Set Theory

Gödel's work on incompleteness still casts a long shadow on the foundations of set theory:

Gödel's First Incompleteness: There is no complete system of axioms for mathematics: for any system, there will be a statement that can neither be proved nor disproved using the axioms of that system.

However there is a system of axioms, called ZFC and formulated in the language of set theory, which does a pretty good job: it seems strong enough to answer about 90% of the statements of mathematical interest.

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ZFC however does a very bad job for set theory itself: most of the interesting statements of abstract set theory can't be answered using just ZFC; the most famous example is:

The Continuum Hypothesis (CH): If X, Y are uncountable sets of real numbers then there is a bijection between X and Y.

Gödel: ZFC does not refute CH, i.e. ZFC + CH is consistent.

Cohen: ZFC does not prove CH, i.e., ZFC $+ \sim$ CH is consistent.

We say that CH is *undecidable* in ZFC.

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Here is another example:

The projective sets of reals are defined as follows:

i. Open sets are projective.
ii. The complement of a projective set is projective.
iii. If f is a continuous function and X is projective then so is $f[X]$, the image of X under f.

Projective Measurability (PM): All projective sets are Lebesgue measurable.

Gödel: ZFC does not prove PM.
Solovay: ZFC does not refute PM.

However there is an important difference between these two examples, CH and PM:

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When we say that ZFC cannot prove or refute something, we are of course assuming that ZFC is a consistent theory!
Otherwise ZFC proves a contradiction and from a contradiction we can derive anything at all.

So the CH example is really the following statement:

Assuming ZFC is consistent, ZFC does not refute CH.
Assuming ZFC is consistent, ZFC does not prove CH.

But the PM example is actually the following:

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Assuming ZFC is consistent, ZFC does not prove PM.
Assuming that the theory (ZFC + There is an *inaccessible infinity*) is consistent, ZFC does not refute PM.

And we cannot get rid of inaccessible infinities, because we have a converse:

Shelah: If ZFC does not refute PM (i.e., if ZFC + PM is consistent) then (ZFC + There is an inaccessible infinity) is consistent!

Axioms of Infinity (Large Cardinal Axioms)

What is an inaccessible infinity (inaccessible cardinal)?

First note the following obvious facts:

i. If A is a finite set then so is $\mathcal{P}(A)$, the set of subsets of A (the *power set* of A).
ii. If A is a finite set and for each element a of A, B_a is a finite set then the union of the B_a's is also finite.

Therefore we can say that the size (cardinality) of the set of natural numbers is *inaccessible*, as it cannot be reached using only finite sets.

We say that an uncountable set has *inaccessible* size (cardinality) if it cannot be reached using sets of smaller size in a similar way.

Axioms of Infinity (Large Cardinal Axioms)

Can we prove that inaccessible cardinals exist? We cannot:
The theory ZFC + There is an inaccessible cardinal is strong enough to prove that ZFC is consistent. But:

Gödel's Second Incompleteness: (Assuming ZFC is consistent) ZFC cannot prove that ZFC is consistent.

So in ZFC one cannot prove that inaccessible cardinals exist.

The result

If (ZFC + There is an inaccessible cardinal) is consistent
then so is (ZFC + PM)

is an example of a *Consistency Upper Bound* result. It establishes the consistency of ZFC together with a statement of interest, in this case PM, assuming the consistency of ZFC together with the existence of a large infinity, in this case an inaccessible cardinal.

But this is just the beginning. A huge number of statements in set theory have been shown to be consistent with ZFC in this way, using various kinds of large cardinals. Here is a brief list of some of these large cardinal notions:

The Modern Meta-Mathematics of Set Theory

Inaccessible
Mahlo
Weakly compact
Ramsey
Measurable
Hypermeasurable
Woodin
Superstrong
Hyperstrong
n-Superstrong
ω-Superstrong

The above notions of infinity get stronger and stronger (as you go down the list) and go all the way "to the edge of inconsistency": the natural extension to $\omega + 1$-Superstrong is inconsistent!

The Modern Meta-Mathematics of Set Theory

Now the result

If (ZFC + PM) is consistent
then so is (ZFC + There is an inaccessible cardinal)

is an example of a *Consistency Lower Bound* result.
It shows that a certain large infinity is required for establishing the consistency with ZFC of a statement of interest.

With PM we have the ideal situation:

$$\text{Con}(\text{ZFC} + \text{Inaccessible}) \rightarrow \text{Con}(\text{ZFC} + \text{PM}) \rightarrow \text{Con}(\text{ZFC} + \text{Inaccessible})$$

so we have exactly "measured" the *consistency strength* of PM.

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More often, however, we just get upper and lower bounds which don't match; for example, if PFA stands for the Proper Forcing Axiom we have:

Con(ZFC + Supercompact) $\rightarrow$ Con(ZFC + PFA) $\rightarrow$ Con(ZFC + Woodin)

It is conjectured that Con(ZFC + PFA) $\rightarrow$ Con(ZFC + Supercompact), but this remains open.

To summarise: Large cardinals provide the tools needed for establishing the consistency of statements in set theory (Consistency Upper Bounds). We have made some progress toward showing that large cardinals are necessary for such consistency results (Consistency Lower Bounds), but techniques for obtaining the consistency of more than Woodin cardinals are still missing.

A Big Question

The incompleteness of the ZFC axioms is obviously of great importance for set theory; indeed most of the interesting questions of abstract set theory are undecidable in ZFC.

Question: Does the incompleteness of ZFC matter for "real mathematics"?

The answer naturally depends on what is meant by "real mathematics".

Consider three examples:

A Big Question

The Borel Conjecture: Strong measure 0 sets are countable.

A set of reals X has strong measure 0 if it can be covered by a union of intervals I_n whose lengths decrease to 0 arbitrarily fast.

The Whitehead Problem: If G is a Whitehead group (i.e. G is Abelian and $Ext^1(G, \mathbb{Z}) = 0$) then must G be free?

The Kaplansky Conjecture: Any algebraic homomorphism from $C(X)$, X compact Hausdorff, into another Banach algebra is continuous.

These questions were raised by "real" mathematicians (not logicians!). They are all undecidable in ZFC.

[Sy: Tell your Kaplansky story.]

A Big Question

The Borel, Whitehead and Kaplansky problems concern large objects (uncountable sets of reals, uncountable groups, “wild” algebra homomorphisms).

Can we avoid undecidability if we stick to “countable” mathematics?

Not really.

PM (Projective Measurability) is expressible in countable mathematics (by “coding” projective sets of real numbers by single real numbers), and PM is something that mathematicians, not just logicians, might care about.

A Big Question

Can we avoid undecidability if we stick to "finite" mathematics?

Recall that we have 2 forms of undecidability:

CH-style: Undecidability assuming only Con(ZFC)
PM-style: Undecidability assuming more than Con(ZFC)

Good news! Statements of finite mathematics seem to be immune from CH-style undecidability.

However PM-style undecidability is unavoidable for logicians:

A Big Question

Matijasevic: Let S be any sentence of set theory. Then there is a polynomial $p(x_1, \ldots, x_n)$ with integer coefficients such that provably in ZFC, $p(x_1, \ldots, x_n)$ has no solution in integers if and only if ZFC + S is consistent.

For example, the consistency of ZFC + There is a supercompact cardinal is equivalent to the unsolvability of some Diophantine equation.

It doesn't get more "finite" than that! But the polynommials we get from Matijasevic are ridiculously big as well as mathematically uninteresting; this is a logicians' trick!

A Big Question

For a long time logicians assumed that the only statements of finite mathematics which fall victim to undecidability are the ones created using logicians' tricks. Any "natural" statement of finite *mathematics* (as opposed to logic!) should be decidable in ZFC.

But that was before Paris-Harrington.

Even though Paris and Harrington are logicians, they discovered a remarkable statement of finite *mathematics* which one might have expected a non-logician to discover.

Now in fact the Paris-Harrington statement *is* provable in ZFC; but it not provable in ZFC without the axiom that says that infinite sets exist, and this is still very shocking for logicians.

Paris-Harrington

Ramsey's Theorem tells us that if we write $[\mathbb{N}]^k$ for the set of k-element subsets of $\mathbb{N}$ then whenever we write $[\mathbb{N}]^k = P_1 \cup P_2$ there is an infinite $H \subseteq \mathbb{N}$ such that $[H]^k \subseteq P_1$ or $[H]^k \subseteq P_2$.

The Finite Ramsey Theorem says that if we don't insist that H be infinite but only of some desired finite size L, we can work with $[\{1, 2, \ldots, M\}]^k$ instead of the full $[\mathbb{N}]^k$ as long as M is large enough in comparison to L.

Paris-Harrington imposes one extra innocent-looking requirement on the set H: It should have more elements than its least element. So $\{2, 5, 7\}$ is OK but $\{4, 5, 7, 12\}$ is not.

Paris-Harrington: Finite Ramsey holds with the extra requirement that H have more elements than its least element. But this is not provable in ZFC without the axiom of infinity!

Should we worry about Paris-Harrington?

Paris-Harrington is not enough to convince us that finite marthematics really falls victim to undecidability: indeed the PH Theorem *is* provable in a small subtheory of ZFC.

Moreover unlike the Borel, Kaplansky and Whitehead problems, Paris-Harrington was manufactured by logicians.

There have been further examples of undecidability in finite mathematics, but so far they have either been manufactured by logicians or are decidable in ZFC. So for now it is reasonable to assume that ZFC is indeed adequate for answering “natural” questions of finite mathematics and the only worries concern the decidability of properties of infinite objects.

But what should we do about undecidability in infinite mathematics?

Option 1: Ignore undecidability!

For the mathematician this means crossing one's fingers that it won't come up in one's own work. For the set-theorist this means celebrating the chaos of a multitude of different interpretations of set theory.

Option 2: Take steps to avoid undecidability.

Working in finite mathematics is still very safe. Countable mathematics is more dangerous but nearly all examples of undecidability in countable mathematics involve "coding" simple uncountable objects by countable ones, a rare occurence in mathematics. Working in uncountable mathematics has become very risky and unfortunately logicians offer few guarantees.

What should we do about undecidability?

Option 3: Learn to love undecidability.

This requires learning set theory, something mathematicians rarely have the time or desire to do.

Option 4: Strengthen the ZFC axioms.

My choice!

Stronger axiom systems leave fewer statements undecided.

But how do we strengthen ZFC?

Strengthening ZFC: Truth and Evidence

Set-theorists don't want to add new axioms unless they are *true*

But what do we take as evidence for the *truth* of a new axiom of set theory?

Currently there are three forms of such evidence, corresponding to the three distinct roles that set theory plays.

Practice-Based Evidence

Here we focus on the value of a new axiom for the development of set theory as a branch of mathematics. Some examples:

Gödel's $V = L$

Large cardinal axioms like "There is a supercompact cardinal"

Forcing axioms like PFA (Proper Forcing Axiom)

Determinacy axioms like PD (Projective Determinacy)

Cardinal characteristic axioms like "The Cichon Diagram is strict"

Each of these axioms has inspired deep and beautiful set theory.

But they can't all be true! $V = L$ contradicts the others and PFA contradicts the strictness of the Cichon Diagram.

What are we going to do about this?

Answer: Combine this with other forms of evidence!

Foundational Evidence and the Independence Project

Set theory's original and most important role was to provide a foundation for mathematics.

This has been very successful, except for the severe problem of independence: ZFC is just too weak to resolve questions like the Borel Conjecture, the Whitehead Problem and the Kaplansky Conjecture.

The Independence Project (Grant proposal submitted)

First systematic study of independence across mathematics.

Key question: Are particular axioms most effective for resolving independence across mathematics as a whole?

If so, this provides *foundational evidence* for such axioms.

Foundational Evidence and the Independence Project

Axioms with foundational evidence:

Gödel's $V = L$

Forcings axioms like PFA

In particular, both resolve the Borel, Whitehead and Kaplansky problems (in different ways)

So far, the other practice-based axioms, Large cardinal axioms, Determinacy axioms and Cardinal characteristic axioms, have not had any impact on mathematics outside of set theory; but the first two do not contradict PFA, so PFA $+$ Large cardinal axioms (which imply Determinacy axioms) has both practice-based and foundational support.

Intrinsic Evidence and the Hyperuniverse Programme

Set theory is also a study of the *set concept*.

An intrinsic feature of the set concept is the *maximality* of the universe of sets.

The *Hyperuniverse Programme (HP)* is a programme for extracting consequences of this maximality feature.

There is therefore *intrinsic evidence* for the axioms that arise from the *HP*.

The programme is new, but preliminary indications are that the axioms arising in the *HP* contradict $V = L$ and PFA, but are compatible with Large cardinal axioms and imply that CH is very false (the continuum is very large)

The strongest possible evidence

If there is practice-based, foundational and intrinsic evidence for an axiom then we can make a strong case for its truth and add it to ZFC.

Unfortunately this still leaves the size of the continuum undecided; if $\mathfrak{c}$ is the size of the continuum, then:

Intrinsic evidence: $\mathfrak{c}$ is very large

Foundational evidence: $\mathfrak{c}$ is $\aleph_2$

Practice-based evidence: $\mathfrak{c}$ can be anything

However, at least "not CH" and PD do well based on all three forms of evidence.

The strongest possible evidence

Can we therefore say that “not CH” and PD are “true”?

Perhaps, but we first need a better understanding of both foundational and intrinsic truth, obtained through the further development of the *Independence Project* and the *Hyperuniverse Programme*.

It will be very interesting to see how things turn out.

Thanks for listening.