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HC OF AN ADMISSIBLE SET[1]

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Abstract. If A is an admissible set, let $HC(A) = \{x \mid x \in A$ and x is hereditarily countable in $A\}$. Then $HC(A)$ is admissible. Corollaries are drawn characterizing the "real parts" of admissible sets and the analytical consequences of admissible set theory.

§1. A transitive set A is admissible if it satisfies pairing, set union, Δ_0-separation and Δ_0-bounding (Platek [7], Barwise [1]). An easy theorem of ZFC (Zermelo-Fraenkel with choice) is that HC, the collection of hereditarily countable sets, is admissible. It is a question of G. Sacks whether this fact is derivable from the axioms for admissible sets alone. In this paper we answer this question affirmatively and use this result to identify the sentences in the language of analysis which hold in all admissible sets (satisfying the axiom of infinity).

THEOREM 1. *Let A be admissible and $HC(A) = \{x \in A \mid \langle A, \in\rangle \models$ transitive closure (x) is countable$\}$. Then $HC(A)$ is admissible. In fact, $HC(A)$ is a Σ_1-elementary submodel of $A (HC(A) \prec_{\Sigma_1} A)$.*

PROOF. First we establish the admissibility of $HC(A)$. That is, we wish to show the following in A:

$$(*) \qquad \forall n\, \exists x \in HC\, \varphi(n, x) \to \exists f \in HC\, \forall n\, \varphi(n, f(n))$$

for each Δ_0-formula φ (with parameter from $HC(A)$). We can assume that $\forall n\, \forall x(\varphi(n, x) \to x$ is transitive). Now any transitive $x \in HC(A)$ is "coded" by some $R \in A$, $R \subseteq \omega$ in the sense that $\langle x, \varepsilon\rangle$ is isomorphic to $\langle \omega, \{(n, m) \mid \langle n, m\rangle \in R\}\rangle$. Conversely, if $R \in A$, $R \subseteq \omega$ is *wellfounded*, i.e., $\langle \omega, \{(n, m) \mid \langle n, m\rangle \in R\}\rangle$ is a wellfounded partial ordering, then R codes a transitive $x \in HC(A)$. Thus $(*)$ can be transformed into the equivalent:

$$\forall n\, \exists R\, (R \text{ wellfounded} \wedge \exists T\, \forall m\, P(\bar{R}(m), \bar{T}(m), n))$$
$$\to \exists R\, \exists T\, \forall n((R)_n \text{ wellfounded} \wedge \forall m\, P(\overline{(R)_n}\,(m), \overline{(T_n)}\,(m), n))$$

for an appropriate recursive P (with parameter from $2^\omega \cap A$).

For each n, consider the tree T_n whose nodes are of the from $(r, t; l)$ where

(1) r and t are finite strings of 0's and 1's,

(2) $\forall m < \mathrm{lh}(t) P(\bar{r}(m), \bar{t}(m), n)$,

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(3) l maps $\{(i, j) \mid r(\langle i, j\rangle) = 0\}$ order-preservingly into the A-countable ordinals.

Then we know that $A \models \forall n\, [\exists$ path through $T_n]$ and we want $F \in HC(A)$ such that $\forall n[F(n)$ is a path through $T_n]$. Note that "f is a path through T_n" is a Σ_1 (over A) predicate of f and n, even though T_n need not be a member of A.

By Σ_1-bounding in A, there is a set $Z \in A$ of paths through the various T_n's such that $\forall n\, \exists f \in Z\, [f$ is a path through $T_n]$. Let $T'_n = \{\tau \in T_n \mid \tau$ can be extended to path in $Z\}$. Then $\{T'_n\}_{n \in \omega} \in A$ and every node in T'_n has an extension in T'_n. It is now easy to find the desired F; simply let $F(n) =$ leftmost path through T'_n.

In [2], Barwise and Fisher prove the Lévy-Shoenfield Absoluteness Theorem in ZF (without AC). Although this Absoluteness Theorem is not true in arbitrary admissible sets, we can use the techniques of [2] to establish the second part of Theorem 1.

The following version of the Skolem normal form is quoted in [2] and provable from the axioms for admissible sets: Let $\mathscr{L}$ be the language of set theory and φ a sentence of $\mathscr{L}$. Then there is an expansion $\mathscr{L}'$ of $\mathscr{L}$ with new relation symbols $R_1, \ldots, R_k$ and an $\forall\exists$-sentence φ' of $\mathscr{L}'$ such that

(i) Every model M of φ can be expanded to a model $\langle M, R_1, \ldots, R_k\rangle$ of φ'.

(ii) The reduct of any model of φ' to $\mathscr{L}$ is a model of φ.

LEMMA. *Let A be an admissible set, T a transitive set belonging to A and suppose $\langle T, \varepsilon\rangle \models \varphi$. Then there is a transitive $T' \in HC(A)$ such that $\langle T', \varepsilon\rangle \models \varphi$.*

PROOF OF LEMMA. We can assume that the axiom of extensionality is a logical consequence of φ. By Skolem normal form, choose an $\forall\exists$-sentence φ' so as to satisfy (i) and (ii) above and $R_1^T, \ldots, R_k^T$ so that $\underline{T} = \langle T, \varepsilon, R_1^T, \ldots, R_k^T\rangle \models \varphi'$. Write $\varphi' = \forall x_1 \cdots \forall x_n\, \psi(x_1, \ldots, x_n)$ where ψ is existential.

Consider $\mathscr{P} = \{\langle \underline{S}, f\rangle \mid \underline{S} = \langle S, E, R_1^S, \ldots, R_k^S\rangle$ is a structure for $\mathscr{L}'$, S is a finite subset of ω, $f : S \to T \cap$ ordinals and for some isomorphism $j : \underline{S} \to \underline{T}$, $f(s) = \mathrm{rank}(j(s))$ for all $s \in S\}$. For $\langle \underline{S}_1, f_1\rangle$ and $\langle \underline{S}_2, f_2\rangle$ in $\mathscr{P}$, define $\langle \underline{S}_1, f_1\rangle \succ \langle \underline{S}_2, f_2\rangle$ if:

(a) $\underline{S}_1 \subseteq \underline{S}_2$ as structures.

(b) For all $a_1, \ldots, a_n \in S_1$, $\underline{S}_2 \models \psi(a_1, \ldots, a_n)$.

(c) $f_2 \mid S_1 = f_1$.

As $\underline{T} \models \varphi'$, we have $\langle \underline{S}_1, f_1\rangle \in \mathscr{P}$ implies there is $\langle \underline{S}_2, f_2\rangle \in \mathscr{P}$ such that $\langle \underline{S}_1, f_1\rangle \succ \langle \underline{S}_2, f_2\rangle$.

Now $\mathscr{P} \in A$ and $A \models$ "$\mathscr{P}$ can be wellordered." Let $< \in A$ be a wellordering of $\mathscr{P}$. Define:

$\langle \underline{S}_0, f_0\rangle = \varnothing$.

$\langle \underline{S}_{n+1}, f_{n+1}\rangle = <$ $-$least $\langle \underline{S}, f\rangle$ such that $\langle \underline{S}_n, f_n\rangle \succ \langle \underline{S}, f\rangle$. If $\bigcup_n \langle \underline{S}_n, f_n\rangle = \langle \underline{S}, f\rangle$ then $\underline{S}$ is a wellfounded structure satisfying the axiom of extensionality, $\underline{S} \in A$ and $A \models$ "$\underline{S}$ is countable." Let $\underline{T}' =$ transitive collapse of $\underline{S}$, $\underline{T}' = \langle T', \varepsilon, R_1^{T'}, \ldots, R_k^{T'}\rangle$. Then $\langle T', \varepsilon\rangle \models \varphi$, $T' \in HC(A)$. This completes the proof of the lemma.

Just as in [2] (see Theorem 2a) the above lemma can be relativized to a parameter $p \in HC(A)$. (We omit the details.) We now show that $HC(A) \prec_{\Sigma_1} A$. Suppose φ is Σ_1 with parameter $p \in HC(A)$ and $A \models \varphi$. Then $\langle T, \varepsilon\rangle \models \varphi$ for some transitive $T \in A$, $p \in T$. By the relativized version of the lemma, $\langle T', \varepsilon\rangle \models \varphi$ for some transitive $T' \in HC(A)$, $p \in T'$. But as Σ_1-formulas persist, $HC(A) \models \varphi$. $\dashv$

REMARKS. (1) We have not really used the admissibility of A in the proof of the second part of Theorem 1, only the fact that A is primitive-recursively closed. Thus: A primitive-recursively closed $\to HC(A) \prec_{\Sigma_1} A$.

(2) Note that $HC(HC(A)) = HC(A)$. Thus any locally countable admissible set is of the form $HC(A)$, A admissible.

(3) The obvious strengthenings of Theorem 1 are false. In fact, Lévy and Feferman (see [6, Theorem 8]) have constructed a model M of ZF in which $\aleph_1 = \aleph_\omega^L$. Then the function $f(n) = \aleph_n^L$ in this model shows that $HC(M)$ is not Σ_2-admissible. In addition, one can use forcing to construct a model N of ZF in which $\aleph_2 = \aleph_\omega^L$ and $\{\aleph_n^L \mid n \in \omega\}$ is Σ_1 over $H_{\aleph_2}(N)$ $(= \{x \mid N \models$ the transitive closure (x) has cardinality $\leq \aleph_1\})$. Then $H_{\aleph_2}(N)$ is not admissible.

§2. The real part of an admissible set. If A is admissible then $2^\omega \cap A$ is called the *real part* of A. The real part of an admissible set is closed under join and "hyperarithmetic in" (and more!). In the reverse direction, if $X \subseteq 2^\omega$ is closed under join and "hyperarithmetic in", let $A_X = \{x \mid \langle T \subset (\{x\}), \varepsilon\rangle$ is isomorphic to $\langle \omega, \{(n, m) \mid \langle n, m\rangle \in R\}\rangle$ for some $R \in X\}$. Then $A_X \cap 2^\omega = X$ and if $X = A \cap 2^\omega$ for some admissible A, then $A_X = HC(A)$. So by Theorem 1, we have:

Fact. X is the real part of an admissible set if and only if A_X is admissible.

We now proceed to translate the admissibility of A_X into a choice principle about X. Recall that we are assuming that X is closed under join and "hyperarithmetic in" throughout.

Pairing, set union and Δ_0-separation are automatic for A_X as X is closed under Turing jump. A typical instance of Δ_0-bounding for A_X looks like:

$$\forall n \in \omega\ \exists y \in A_X\ \varphi(n, y) \to \exists z \in A_X\ \forall n\ \exists y \in z \varphi(n, y),$$

where φ is a Δ_0-formula. We have chosen the domain of φ to be ω without loss of generality as every member of A_X is countable in A_X. For the same reason, we can replace the above by:

$$\forall n\ \exists y \in A_X\ \varphi(n, y) \to \exists f \in A_X\ \forall n\ \varphi(n, f(n)), \qquad \varphi \text{ a } \Delta_0\text{-formula}.$$

As in the proof of Theorem 1, we can transform the preceding into: $\forall n\ \exists R \in X$ [R is wellfounded $\wedge\ P(n, R)$] $\to \exists S \in X\ \forall n$ [$(S)_n$ is wellfounded $\wedge\ P(n, (S)_n)$], P arithmetic (with parameters in X) where $(S)_n = \{m \mid \langle n, m\rangle \in S\}$. The above is an instance of the more general:

$$(*) \qquad \forall n\ \exists R \in X[Q(n, R)] \to \exists S \in X\ \forall n[Q(n, (S)_n)], \qquad Q \text{ a } \Pi_1^1\text{-predicate (with parameters in } X).$$

Note that here Q is Π_1^1 (in the real world!) and not necessarily Π_1^1 over X. For this reason, $(*)$ is called *External* Π_1^1-*AC* (Ext Π_1^1-AC). We have shown that $X \models$ Ext Π_1^1-AC implies A_X is admissible.

Conversely, suppose that A_X is admissible. Then any Π_1^1-predicate $Q(n, R)$ (with parameter in X) restricted to X is Σ_1 over A_X (see [9, Proposition 2.5]). An application of Σ_1- bounding in A_X then yields $(*)$. So we have:

THEOREM 2. *X is the real part of an admissible set iff $X \models$* Ext Π_1^1-AC.

DEFINTION. X is a *β-model* if $X \prec_{\Sigma_1^1} 2^\omega$, i.e. $X \models \varphi \leftrightarrow 2^\omega \models \varphi$ for $\Sigma_1^1\ \varphi$ with parameters from X.

This is equivalent to: If a linear ordering is not wellfounded and belongs to X, then it has a descending chain in X. Any X closed under hyperjump is a β-model.

Let $WO = \{R \mid \{(n, m) \mid \langle n, m\rangle \in R\}$ is a wellordering$\}$. WO is a complete Π^1_1-set. Clearly if X is a β-model, then $X \cap WO$ is Π^1_1 over X. Leo Harrington has recently shown that the converse is not true.

DEFINTION. X is Π^1_1-strong if $X \cap WO$ is Π^1_1 over X (with parameters from X).

PROPOSITION 3. *Suppose X is Π^1_1-strong. If P is Π^1_1 (with parameters from X), then $P \cap X$ is Π^1_1 over X (with parameters).*

PROOF. If P is Π^1_1 (with parameters from X), then $P \cap X$ is many-one reducible (via a recursive function) to $WO \cap X$, so $P \cap X$ is Π^1_1 over X. $\dashv$

Question. If X is Π^1_1-strong and $Q \subseteq X$ is Π^1_1 over X, then does $Q = P \cap X$ where P is Π^1_1 (with parameters from X)?

DEFINTION. Σ^1_n-AC is the axiom scheme

$$\forall n\, \exists R\, P(n, R) \to \exists S\, \forall n\, P(n, (S)_n), \qquad P\ \Sigma^1_n \text{ with parameters.}$$

Ext Π^1_1-AC implies Σ^1_1-AC, so by Theorem 2, Σ^1_1-AC is true in every admissible set. This last fact is due to J. Steel [10].

PROPOSITION 4. *If X is the real part of an admissible set and is Π^1_1-strong, then X is a β-model.*

PROOF. If X is not a β-model, let $\langle L, <_L\rangle \in X$ be a linear ordering which has a descending chain but none in X. Let L^* = the largest wellordered initial segment of L. Then L^* is Π^1_1, so by hypothesis, L^* is Π^1_1 over X. Let $X = A \cap 2^\omega$, A admissible. Then L^* is Σ_1 over A (since it is Π^1_1) and Π_1 over A (since it is Π^1_1 over $A \cap 2^\omega$). Since $A \models \Delta_1$-separation, $L^* \in A \cap 2^\omega = X$.

Now $X \models \Sigma^1_1$-AC. Using this, it is easy to construct a descending chain through L in X. (Apply Σ^1_1-AC to the statement $\forall a \exists b(a \in L - L^* \to b \in L - L^*, b <_L a)$.) $\dashv$

Assume X is Π^1_1-strong. Then $X \models$ Ext Π^1_1-AC iff $X \models \Sigma^1_2$-AC. We have therefore established half of the following result:

THEOREM 5. *X is the real part of an admissible set iff:*

(1) *X is Π^1_1-strong, $X \models \Sigma^1_2$-AC or*

(2) *X is not Π^1_1-strong, $X \models \Sigma^1_1$-AC.*

PROOF. In light of earlier remarks, it suffices to prove:

Claim. $X \models \Sigma^1_1$-AC, $X \not\models$ Ext Π^1_1-AC $\to$ X is Π^1_1-strong.

First we prove:

LEMMA. *Suppose $\langle L, <_L\rangle \in X$ is a linear ordering. Let $WF(L)$ = the largest wellordered initial segment of L. If $WF(L)$ is Π^1_1 over X but not an element of X, then X is Π^1_1-strong.*

PROOF OF LEMMA. Suppose $\langle M, <_M\rangle \in X$ is a linear ordering. which is not a wellordering but $X \models \langle M, <_M\rangle$ is a wellordering. Then $a \in WF(L) \to X \models$ there is an isomorphism of L_a onto an initial segment of $\langle M, <_M\rangle$, where $L_a = \{b \in L \mid b \leq_L a\}$. Since $WF(L) \notin X$ and $X \models \Delta^1_1$-CA, there is $a \in L\text{-}WF(L)$ such that $X \models$ there is an isomorphism of L_a into an initial segment of $\langle M, <_M\rangle$. Therefore, for $\langle M, <_M\rangle \in X$: $\langle M, <_M\rangle$ is a wellordering $\leftrightarrow X \models$ "$\langle M, <_M\rangle$ is a wellordering $\wedge$ $\forall a \in M\, \forall f$ [(f an isomorphism of M_a onto an initial segment of L) $\to f(a) \in WF(L)$]." This shows that $WO \cap X$ is Π^1_1 over X. $\dashv$

PROOF OF CLAIM. Let $P(n, r)$ be a counterexample to Ext Π^1_1-AC. That is, $P(n, R)$ is Π^1_1 (with parameter from X), $\forall n\, \exists R \in X\, P(n, R)$ but $\sim \exists S \in X\, \forall n\, P(n, (S)_n)$.

Let f be a recursive function such that:

(1) $\forall n$, $P(n, R)$ iff $\{f(n)\}^R$ is a wellordering.

(2) $\forall n \forall R$ $\{f(n)\}^R$ is a linear ordering.

We can assume that X is not closed under hyperjump, so let $\langle L, <_L \rangle \in X$ be a linear ordering, $WF(L) \notin X$. For each n and $a \in L$ consider the tree T_n^a whose nodes are pairs (τ, l) where:

(i) τ is a finite string of 0's and 1's.

(ii) l is an order-preserving map from the finite linear ordering $\{f(n)\}^{\tau}_{\mathrm{lh}(\tau)}$ into the $<_L$-predecessors of a.

Then we have:

(a) $a \in L - WF(L) \to \forall n$ T_n^a has a path in X.

(b) $a \in WF(L) \to \exists n$ T_n^a has no path in X.

(a) holds because $\forall n \exists R \in X\, P(n, R)$. (b) holds because $\sim \exists S \in X \forall n\, P(n, (S)_n)$ and Σ_1^1-AC. Now by Σ_1^1-AC,

$$a \in WF(L) \leftrightarrow \forall S \in X \exists n [(S)_n \text{ is not a path through } T_n^a]$$

and we are then done by the lemma. $\dashv$

In case we only consider models of Σ_1^1-DC, we can replace "Π_1^1-strong" by the more natural "β-model" in Theorem 5:

DEFINITION. Σ_n^1-*DC* is the axiom scheme

$$\forall R \exists S\, P(R, S) \to \exists S \forall n\, P((S)_n, (S)_{n+1}) \qquad P\ \Sigma_n^1 \text{ with parameters.}$$

Σ_n^1-*BI* is the axiom scheme $\forall R[R$ a wellordering $\wedge$ $\forall n$ $(P(n) \to n \in \text{Field}(R))$ $\wedge$ $\exists n P(n) \to \exists n(P(n) \wedge \forall m <_R n \sim P(m))]$, P Σ_n^1 with parameters.

Σ_n^1-BI says that a Σ_n^1-subset of a wellordering has a least element (in the sense of the wellordering). BI stands for "Bar Induction." (For more information on bar induction, see [3], [4] and [5].)

LEMMA 6 (HOWARD AND KREISEL [5]). Σ_1^1-*DC* $\leftrightarrow$ Σ_1^1-*BI*.

PROOF. ($\to$) Let R be a wellordering and $P(n) \leftrightarrow \exists S$ $\forall m Q(\bar{S}(m), n)$ a Σ_1^1-predicate contained in the field of R. If P has no $<_R$-least element, then

$$\forall n \forall S \exists S' \exists n' [n' <_R n \wedge (\forall m Q(\bar{S}(m), n) \to \forall m Q(\bar{S}'(m), n'))].$$

By Σ_1^1-DC, R is not wellordered.

($\leftarrow$) Assume $\forall R \exists S \exists T \forall n$ $Q(\bar{R}(n), \bar{S}(n), \bar{T}(n))$, Q recursive. Consider the tree whose nodes are of the form $(r_0, \ldots, r_m; t_1, \ldots, t_m)$ where

(1) Each r_i is a finite string of 0's and 1's.

(2) $\forall n \leq m$ $Q(\bar{r}_i(n), \bar{r}_{i+1}(n), \bar{t}_i(n))$.

$(r_0', \ldots, r_n'; t_1', \ldots, t_n')$ extends the above node if $n > m$ and r_i' extends r_i, t_i' extends t_i for each $i \leq m$. A path through this tree yields the conclusion of Σ_1^1-DC that we want. Let $<$ be the Kleene-Brouwer ordering of this tree. It is enough to show that $<$ is not a wellordering. By Σ_1^1-BI, it is enough to exhibit a Σ_1^1-subset of the tree with no $<$ -least element. But simply consider

$$\{(r_0, \ldots, r_m; t_1, \ldots, t_m) \,|\, \exists R_0 \cdots \exists R_m \exists T_1 \cdots \exists T_m \forall n \forall i < m Q(\bar{R}_i(n), \bar{R}_{i+1}(n), \bar{T}_i(n))\}. \quad \dashv$$

THEOREM 7. *X is the real part of an admissible set satisfying* Σ_1^1-*DC iff*

(1) *X is a* β-*model of* Σ_2^1-*AC* ($= \Sigma_2^1$-*DC*) *or*

(2) *X is a non-*β-*model of* Σ_1^1-*DC.*

PROOF. Note that Σ_2^1-AC$\to \Sigma_2^1$-DC $\to \Sigma_1^1$-DC.

By Theorem 5, it is enough to show that a Π_1^1-strong model of Σ_1^1-DC is a β-model. Suppose $X \models \Sigma_1^1$-DC is Π_1^1-strong. If X is not a β-model, then there is a linear ordering $L \in X$ such that $X \models L$ is a wellordering, but $WF(L) \neq L$. $WF(L)$ is Π_1^1, hence by Π_1^1-strength $WF(L)$ is Π_1^1 over X. But then L-$WF(L)$ contradicts Σ_1^1-BI which holds in X by Lemma 6. $\dashv$

Theorem 5 can be used to characterize which statements in the language of analysis are true in every admissible set. We begin with a lemma which was pointed out by Leo Harrington:

LEMMA 8. *Let $X \models S$ where S is a recursive set of sentences. Then there is a $Y \models S$ which is not Π_1^1-strong.*

PROOF. We can assume that X is countable. Consider the theory $T = ZF^- + \exists X(X \models S) + \forall x(x \text{ finite ordinal} \to \bigvee_{n<\omega} x = \underline{n})$ which is a member of the admissible fragment $L_{\omega_1^{CK}}$. Then $HC \models T$. By the theorem on pinning down ordinals (see [1, Theorem 7.5, p. 107]), T has a model M such that the ordinal ω_1^{CK} is not represented in M; i.e., no partial ordering in M has ordinal rank ω_1^{CK}.

In particular, Kleene's $\mathcal{O} \notin M$ because otherwise $\langle \mathcal{O}, <_{\mathcal{O}}\rangle$ represents ω_1^{CK}.

As $M \models T$, there is $Y \in M$, $Y \models S$. Then Y is not Π_1^1-strong as otherwise $\mathcal{O}$ is Π_1^1-definable over Y and hence an element of M. $\dashv$

Theorem 5 and the lemma immediately yield:

THEOREM 9 (HARRINGTON). *If φ is a sentence of analysis true in some ω-model of Σ_1^1-AC, then φ is true in some admissible set.*

COROLLARY 10. *The ω-consequences (consequences in ω-logic) of the axioms for admissible sets (together with the axiom of infinity) in the language of analysis are precisely the ω-consequences of Σ_1^1-AC.*

In [8], G. Sacks has characterized the 1-sections of normal finite type objects as the real parts of admissible sets satisfying Σ_1-DC. *Σ_1-DC* is the scheme:

$$\forall x\, \exists y\, \varphi(x, y) \to \exists f[\operatorname{dom} f = \omega \wedge \forall n\, \varphi(f(n), f(n+1))]$$

φ a Σ_1-formula with parameters.

The above results can be used to characterize those sentences of analysis true in every 1-section.

LEMMA 11. *Suppose A is a locally countable admissible set of ordinal height α. Suppose $X = A \cap 2^\omega$ satisfies Σ_1^1-DC. Then either $A \models \Sigma_1$-DC or there is a wellordering (of integers) of order type α definable over X.*

PROOF. If X is a β-model, then Σ_1-DC for A is equivalent to Σ_2^1-DC for X; but by Theorem 7, $X \models \Sigma_2^1$-AC and in general Σ_2^1-AC $\to \Sigma_2^1$-DC. So X a β-model $\to$ $A \models \Sigma_1$-DC.

Now suppose X is not a β-model and let $\langle L, <_L\rangle \in X$ be a linear ordering such that $X \models$ "$\langle L, <_L\rangle$ is a wellordering" and $WF(L) \notin X$. Consider an instance of Σ_1-DC:

$$(*) \qquad \forall x\, \exists y\, \varphi(x, y) \to \exists f[\operatorname{dom} f = \omega \wedge \forall n\, \varphi(f(n), f(n+1))].$$

By pairing, it suffices to treat the case where φ is Δ_0. The hypothesis of $(*)$ can be transformed into the equivalent:

$$\forall R\, \exists S[R \text{ wellfounded} \to (S \text{ wellfounded} \wedge Q(R, S))]$$

where Q is an appropriate arithemtic predicate and "R wellfounded" abbreviates "$\{(n, m) \mid \langle n, m\rangle \in R\}$ is a wellfounded partial ordering." Finally, transform this into the equivalent:

$$\forall R\, \exists S\, \exists T[R \text{ wellfounded} \to (S \text{ wellfounded} \wedge \forall n\, P(\bar{R}(n), \bar{S}(n), \bar{T}(n)))]$$

where P is recursive. The conclusion of $(*)$ is now equivalent to:

$$\exists R\, \exists T\, \forall m[(R)_m \text{ is wellfounded} \wedge \forall n P(\overline{(R)_m}(n), \overline{(R)_{m+1}}\,(n), \overline{(T)_m}(n))].$$

For each $a \in L$, consider the tree T_a whose nodes are of the form $(z_0, \dots, z_m; t_1, \dots, t_m; f)$ where

(1) Each z_i is a finite string of 0's and 1's.

(2) $\forall n \leq m\, \forall i < m\ P\,(\bar{z}_i(n), \overline{z_{i+1}}(n), \bar{t}_i(n))$.

(3) For all $i \leq m$, $(f)_i$ is a function mapping $\{(j, k) \mid z_i(\langle j, k\rangle) = 0\}$ order-preservingly into the $<_L$-predecessors of a.

$(z_0', \dots, z_n'; t_1', \dots, t_n'; f')$ extends $(z_0, \dots, z_m; t_1, \dots, t_m; f)$ if $n > m$ and $\forall i \leq m$ (z_i' extends z_i, t_i' extends t_i, $(f')_i$ extends $(f)_i$). Thus if $a \in WF(L)$, then a path through T_a yields the conclusion of $(*)$. For each a, define

$$\begin{aligned} S_a = \{(z_0, \dots, z_m; t_1, \dots, t_m; f) \in T_a \mid \exists R\, \exists T\, \exists F\, \forall i \leq m[&(R)_i \text{ extends } z_i \wedge \\ (T)_i \text{ extends } t_i \wedge F \text{ extends } f \wedge \forall n P((\bar{R}_i(n), \bar{R}_{i+1}(n), \overline{(T)_i}(n)) \\ \wedge (F)_i \text{ maps } \{(j, k) \mid \langle j, k\rangle \in (R)_i\} \text{ order-preservingly into} \\ \text{the } <_L\text{-predecessors of a}]\}. \end{aligned}$$

The assumption of $(*)$ implies that if $a \in WF(L)$, $\tau \in S_a$, then $\exists\, b \in WF(L)$ [τ has a proper extension in $S_b\}$. If for some $a \in WF(L)$ there is no $b \in WF(L)$ such that $\forall \tau\, [\tau \in S_a \to \tau$ has a proper extension in $S_b]$, then define $WF(L)$ in X by:

$$b \in WF(L) \leftrightarrow \exists \tau \in S_a[\tau \text{ has no proper extension in } S_b],$$

so in this case we are done (clearly $\langle WF(L), <_L\rangle$ has order type α).

Otherwise, define

$$g(a) = \begin{cases} <_L\text{-least } b \text{ s.t. } \forall \tau\, [\tau \in S_a \to \tau \text{ has a proper extension in } S_b], & \\ \qquad\qquad\qquad\qquad \text{if such a } b \text{ exists,} & \\ 0 \qquad\qquad\qquad\qquad \text{otherwise.} & \end{cases}$$

Then g: $WF(L) \to WF(L)$ and $g: L \to L \cup \{0\}$ is definable over X. Let $a_0 \in WF(L)$ be such that $S_{a_0} \neq \varnothing$ and consider the sequence $a_0, g(a_0), g(g(a_0)), \dots$. If this sequence runs unboundedly through $WF(L)$, then $WF(L)$ is again definable over X. Otherwise, let $a \in WF(L)$ be the supremum of this sequence. We have: $\tau \in S_{<a} \to \tau$ has a proper extension in $S_{<a}$, where $S_{<a} = \bigcup_{b <_L a} S_b$. But then $S_{<a}$ is a Σ_1^1-subset of the Kleene-Brouwer ordering of T_a with no least member. By Σ_1^1-BI, T_a has a path and this demonstrates the conclusion of $(*)$. $\dashv$

THEOREM 12. *Suppose φ is a sentence of analysis true in some ω-model of Σ_1^1-DC. Then φ is true in some admissible set satisfying Σ_1-DC (and hence φ is true in the 1-section of some normal object of finite type).*

PROOF. The proof of Lemma 8 shows that φ is true in some $X \models \Sigma_1^1$-DC which cannot define a wellordering of integers of order type ω_1^{CK}. By Theorem 5, A_X is admissible and by Lemma 11, $A_X \models \Sigma_1$-DC. $\dashv$

Questions. (1) Is there a locally countable admissible A such that $A \cap 2^\omega \models \Sigma^1_1$-DC but $A \not\models \Sigma_1$-DC?

(2) Is there a nice characterization (as in Theorem 7) of the real parts of admissible sets satisfying Σ_1-DC?

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