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A GUIDE TO "CODING THE UNIVERSE" BY BELLER, JENSEN, WELCH

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In the wake of Silver's breakthrough on the Singular Cardinals Problem (Silver [74]) followed one of the landmark results in set theory, Jensen's Covering Lemma (Devlin-Jensen [74]): If $0^\#$ does not exist then for every uncountable $x \subseteq \mathrm{ORD}$ there exists a constructible $Y \supseteq X$, $\mathrm{card}(Y) = \mathrm{card}(X)$. Thus it is fair to say that in the absence of large cardinals, V is "close to L".

It is natural to ask, as did Solovay, if we can fairly interpret the phrase "close to L" to mean "generic over L". For example, if $V = L[a]$, $a \subseteq \omega$ and if $0^\#$ does not exist then is V $\mathscr{P}$-generic over L for some partial ordering $\mathscr{P} \in L$? Notice that an affirmative answer implies that in the absence of $0^\#$, no real can "code" a proper class of information.

Jensen's Coding Theorem provides a negative answer to Solovay's question, in a striking way: Any class can be "coded" by a real without introducing $0^\#$. More precisely, if $A \subseteq \mathrm{ORD}$ then there is a forcing $\mathscr{P}$ definable over $\langle L[A], A\rangle$ such that $\mathscr{P} \Vdash V = L[a]$, $a \subseteq \omega$, A is definable from a. Moreover if $0^\# \notin L[A]$ then $\mathscr{P} \Vdash 0^\#$ does not exist. Now as any $M \models \mathrm{ZFC}$ can be generically extended to a model of the form $L[A]$ (without introducing $0^\#$) we obtain: For any $\langle M, A\rangle \models \mathrm{ZFC}$ (that is, $M \models \mathrm{ZFC}$ and M obeys Replacement for formulas mentioning A as a predicate) there is an $\langle M, A\rangle$-definable forcing $\mathscr{P}$ such that $\mathscr{P} \Vdash V = L[a]$, $a \subseteq \omega$, $\langle M, A\rangle$ is definable from a. Moreover if $0^\# \notin M$ then $\mathscr{P} \Vdash 0^\#$ does not exist.

The book ***Coding the Universe*** by Beller, Jensen, Welch appeared in 1982 (London Mathematical Society Lecture Note Series No. 47) and provides the first published proof of this result, as well as some of its applications. It is safe to say that the proof of the Coding Theorem is one of the hardest in all of set theory. The technical considerations are extremely elaborate and the proof draws heavily on Jensen's profound fine structure theory. In light of this the authors must be congratulated for putting this work into published form.

Nonetheless we feel that it would be helpful to readers of ***Coding the Universe*** to have available an outline of the proof which provides some further explanation and motivation for the many intricate definitions and constructions in the book. It is the purpose of this article to present such an outline and to also describe some

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simplifications that can be made, including a description of how Jensen's split into cases "$0^\# \in M$" and "$0^\# \notin M$" can be eliminated.

In Part I we outline Jensen's proof, in the case where $0^\#$ does not belong to the ground model M. Part II then discusses the book itself, in view of the outline of Part I. At the end of Part II is a list of some misprints. We conclude in Part III with a discussion of simplifications that can be made in Jensen's proof and of an alternative method of dealing with the key distributivity lemma which obviates the need for considering the existence of $0^\#$ in the ground model.

Part I. An outline of Jensen's proof. We are given a model $\langle M, A\rangle$ of ZFC, and our goal is to construct an $\langle M, A\rangle$-definable $\mathscr{P}$ such that $\mathscr{P} \Vdash V = L[a]$, $a \subseteq \omega$, A is definable from a. We assume that $M = L[A]$, $H(\kappa)^M = L_\kappa[A]$ for infinite M-cardinals κ (where $H(\kappa) = \{x \mid \text{card (transitive closure of } x) < \kappa\}$. There is no loss in this assumption as otherwise we can generically extend $\langle M, A\rangle$ to have this property (via an $\langle M, A\rangle$-definable forcing—see Easton [70]).

The construction of $\mathscr{P}$ is based on almost disjoint forcing (Jensen-Solovay [68]). Suppose $B \subseteq \kappa^{++}$. The forcing $\mathbf{R}^B$ codes B by a subset $\tilde{G}$ of κ^+: To each $\xi < \kappa^{++}$ we "canonically" assign $b_\xi \subseteq \kappa^+$ and we arrange that $\xi \in B \leftrightarrow \tilde{G} \cap b_\xi$ is bounded in κ^+ ($\tilde{G}$ is almost disjoint from b_ξ). A condition is a pair $(r, \bar{r})$ where $r\colon [\kappa, |r|) \to 2$, $|r| < \kappa^+$ and $\bar{r} \subseteq \{b_\xi \mid \xi \in B\}$, $\operatorname{card}(\bar{r}) \leq \kappa$. Define $(r_1, \bar{r}_1) \leq (r_2, \bar{r}_2)$ iff $r_1 \supseteq r_2, \bar{r}_1 \supseteq \bar{r}_2$, $b_\xi \in \bar{r}_2 \to \tilde{r}_1 \cap b_\xi \subseteq |r_2|$, where $\tilde{r}_1 = \{\eta \mid r_1(\eta) = 1\}$. Then a generic set G can be identified with a function $G\colon [\kappa, \kappa^+) \to 2$. Moreover (given a mild hypothesis on the b_ξ's) we have $\xi \in B \leftrightarrow b_\xi$ is almost disjoint from $\tilde{G} = \{\eta \mid G(\eta) = 1\}$.

There is a similar type of coding $\mathbf{Q}^B$ when κ^+ is replaced by a limit cardinal λ, $B \subseteq \lambda^+$. We require that $\{b_\xi \mid \xi < \lambda^+\}$ form a "quasi-scale": Each b_ξ is an unbounded subset of λ; and, for all $b \subseteq \lambda$, if $b \cap \delta^+$ is bounded in δ^+ for all $\delta < \lambda$ then $b \cap b_\xi$ is bounded in λ for all sufficiently large $\xi < \lambda^+$. A condition in $\mathbf{Q}^B$ is $r = \langle r_\delta \mid \delta \in \operatorname{Card} \cap \lambda\rangle$ where $r_\delta\colon [\delta, |r_\delta|) \to 2$, $|r_\delta| < \delta^+$ and: for all $\xi < \lambda^+$, if $r \notin L_{\mu_\xi}[B \cap \xi]$ then $\xi \in B \leftrightarrow (\bigcup_\delta \tilde{r}_\delta)$ is almost disjoint from b_ξ (μ_ξ will be defined below). Set $r_1 \leq r_2$ iff $r_1 \supseteq r_2$. A generic set can be identified with a function $G\colon \lambda \to 2$ and (with a mild hypothesis on the b_ξ's): $\xi \in B$ iff b_ξ is almost disjoint from $\tilde{G} = \{\eta \mid G(\eta) = 1\}$.

Now we want in both of these cases that $B \in L[G]$. This is clear provided that the sequence of b_ξ's belongs to $L[G]$. Suppose that $B \subseteq \alpha^+$ has the property:

$$(*) \qquad \xi < \alpha^+ \to L[B \cap \xi] \models \operatorname{Card}(\xi) \leq \alpha.$$

Then (setting $\alpha = \kappa^+$ for $\mathbf{R}^B$, $\alpha = \lambda$ for $\mathbf{Q}^B$) we can choose b_ξ canonically in $L[B \cap \xi]$; indeed $b_\xi \notin L_{\mu_\xi + 1}[B \cap \xi]$ where $\mu_\xi = \text{least } \mu > \sup_{\xi' < \xi} \mu_{\xi'}$ such that $L_\mu[B \cap \xi] \models \mathrm{ZF}^- \wedge \operatorname{Card}(\xi) \leq \alpha$. Now to see that $B \in L[G]$ we argue that we can identify $B \cap \xi, b_\xi$ by a simultaneous induction on $\xi < \alpha^+$.

We shall need one more forcing, for the purpose of arranging $(*)$. Again suppose $B \subseteq \alpha^+$. The forcing $\mathbf{F}^B$ adds $B' \subseteq \alpha^+$ so that B' obeys $(*)$, $B \in L[B']$. For technical reasons we cannot arrange $B \in L[B']$ by requiring $\eta \in B \leftrightarrow 2\eta \in B'$; instead we arrange $\eta \in B$ iff B' is almost disjoint from $u_\eta = \{\langle \eta, \gamma\rangle \mid \gamma < \alpha^+\}$. Thus a condition in $\mathbf{F}^B$ is $r\colon [\alpha, |r|) \to 2$, $|r| < \alpha^+$ such that $\xi \leq |r| \to L[r \upharpoonright \xi, B \cap \alpha] \models \operatorname{Card}(\xi) \leq \alpha$. And, we define $r_1 \leq r_2$ iff $r_1 \supseteq r_2$ and $\eta < |r_2|, \eta \in B \to \sup(\tilde{r}_1 \cap u_\eta) \leq |r_2|$. We can identify a generic set with $G\colon [\alpha, \alpha^+) \to 2$, and then G has the following properties: For $\eta < \alpha^+$, $\eta \in B$ iff $\tilde{G}$ is almost disjoint from u_η, $\tilde{G}$ obeys $(*)$. Thus $\tilde{G}$ serves as the

The forcings $\mathbf{R}^B$, $\mathbf{Q}^B$ and $\mathbf{F}^B$ are the main ingredients in Jensen coding. There are two main properties to verify in order to justify their use:

Extendibility. (a) For $\mathbf{R}^B$ this says that given $(r,\bar{r})$, $\xi_0 \in B$, $\xi_1 \notin B$ and $\eta < \kappa^+$ there exists $(s,\bar{s}) \le (r,\bar{r})$, $|s| \ge \eta$, $b_{\xi_0} \in \bar{s}, b_{\xi_1} \cap (\tilde{s} - \tilde{r}) \ne \emptyset$. This is easily verified given that: $\bar{r} \subseteq \{b_\xi \mid \xi < \kappa^{++}\}$, $\mathrm{Card}(\bar{r}) \le \kappa$, $b_{\xi_0} \notin \bar{r} \to b_{\xi_0}$ is almost disjoint from $\bigcup \bar{r}$.

(b) For $\mathbf{Q}^B$ this says that given r and $\xi < \lambda^+$ there exists $s \le r$, $s \notin L_{\mu_\xi}[B \cap \xi]$. This is nontrivial and is proved by induction on ξ. The successor step is easy given that: $b \in L_{\mu_\xi}[B \cap \xi]$, $b \cap \delta^+$ bounded in δ^+ for all $\delta < \lambda \to b$ is almost disjoint from b_ξ. The limit case requires the existence of a closed unbounded $C \subseteq \xi$ such that $C \cap \xi' \in L_{\mu_{\xi'}}[B \cap \xi']$ for all $\xi' \le \xi$, ordertype$(C) \le \lambda$. Such a C can be obtained from the proof of $\square_\lambda$ in $L[B]$, given that B satisfies (*). Now if λ is singular in $L_{\mu_\xi}[B \cap \xi]$ then we can assume that $\delta_0 = $ ordertype $(C) < \delta \in \mathrm{Card} \cap \lambda$. Given r, choose canonically by induction $r \ge r_1 \ge r_2 \ge \cdots$ such that $r_i \notin L_{\mu_{\xi_i}}[B \cap \xi_i]$ and $r_i \restriction \delta = r \restriction \delta$ for all i, where $C = \{\xi_0, \xi_1, \ldots\}$. Then

$$r_\lambda = \bigcup_{i<\lambda} r_i \in L_{\mu_{\xi_\lambda}}[B \cap \xi_\lambda]$$

for limit λ by the hypothesis on C. Thus $s' = r_{\delta_0} \le r$, $s' \notin L_{\mu_{\xi'}}[B \cap \xi']$ for $\xi' < \xi$, $s' \in L_{\mu_\xi}[B \cap \xi]$. We can now obtain the desired $s \le s'$ by arguing as in the successor case. A similar argument works when λ is inaccessible, this time using a normal sequence $\langle \delta_i \mid i < \lambda \rangle$ cofinal in λ and requiring $r_{i+1} \restriction \delta_i = r_i \restriction \delta_i$.

(c) For $\mathbf{F}^B$ this says that given $r, \eta_0 \in |r| - B$ and $\eta < \alpha^+$ there exists $s \le r$, $|s| \ge \eta$, $\tilde{s} \cap u_{\eta_0} \nsubseteq |r|$. This is easily verified.

The Extendibility properties of these forcings justify the claim that the corresponding generic sets do in fact code B; in (c) we also have the property (*). What remains to be seen is that these forcings preserve cardinals. The essential property to establish for this is distributivity.

DEFINITION. $\mathscr{P}$ is *τ-distributive* if given $p \in \mathscr{P}$ and a sequence $\langle D_i \mid i < \tau \rangle$ of dense open subsets of $\mathscr{P}$ there exists $q \le p$, $q \in \bigcap_i D_i$.

$\mathscr{P}$ is *κ-CC* if any collection of pairwise incompatible elements of $\mathscr{P}$ (that is any *antichain* in $\mathscr{P}$) has cardinality less than κ.

The utility of these properties is that τ-distributive forcings preserve cardinals $\le \tau^+$ and κ-CC forcings preserve cardinals $\ge \kappa$.

Distributivity. (a) $\mathbf{R}^B$ is easily seen to be κ-distributive (in fact "κ-closed") and κ^{++}-CC. So $\mathbf{R}^B$ preserves cardinals.

(b) For $r \in \mathbf{Q}^B$, $r = \langle r_\delta \mid \delta \in \mathrm{Card} \cap \lambda \rangle$ and $\tau \in \mathrm{Card} \cap \lambda$ define $(r)_\tau = \langle r_\delta \mid \tau \le \delta \rangle$ and $(r)^\tau = \langle r_\delta \mid \delta < \tau \rangle$. Then $\mathbf{Q}^B \simeq \mathbf{Q}^B_\tau \times \mathbf{Q}^\tau$ where $\mathbf{Q}^B_\tau = \{(r)_\tau \mid r \in \mathbf{Q}^B\}$ and $\mathbf{Q}^\tau = \{(r)^\tau \mid r \in \mathbf{Q}^B\}$. Now we claim that $\mathbf{Q}^B_\tau$ is τ-distributive: Indeed, suppose $r \in \mathbf{Q}^B$, $\langle D_i \mid i < \tau \rangle$ are dense open on $\mathbf{Q}^B_\tau$ and let $X_0 = $ least $X \prec L_{\lambda^{++}}[B]$ such that $\lambda \subseteq X$ and $r, \langle D_i \mid i < \tau \rangle$, $B \in X$. Then define $X_{i+1} = $ least $X \prec L_{\lambda^{++}}[B]$, $X_i \in X$; $X_\lambda = \bigcup_{i<\lambda} X_i$ for limit $\lambda < \tau$. Define $r_0 = r, r_{i+1} = $ least $r \le r_i$, $r \in D_i$, $r \notin X_i$, $r_\lambda = \inf_{i<\lambda} r_i$ for limit λ. Then $r_i \in L_{\mu_{\xi_i}}[B \cap \xi_i]$ where $\xi_i = X_i \cap \lambda^+$. It is now easy to verify that r_λ is a condition for limit λ, as by induction r_λ "codes" $B \cap \xi_\lambda$ and this is all that we have to check, since $r_\lambda \in L_{\mu_{\xi_\lambda}}[B \cap \xi_\lambda]$.

Now as $\mathbf{Q}^B \simeq \mathbf{Q}^B_\tau \times \mathbf{Q}^\tau$ *and* $\mathbf{Q}^B_\tau$ *is τ-distributive* we can argue that successors of accessible cardinals are preserved: If $\tau < \lambda$ is a successor cardinal then $\mathbf{Q}^\tau$ trivially has the τ^+-CC and so $\mathbf{Q}^B_\tau \times \mathbf{Q}^\tau$ preserves τ^+. If $\tau \le \lambda$ is singular then $\mathbf{Q}^B$ must preserve τ^+ as otherwise τ^+ attains some cofinality $\delta < \tau$ and this is impossible as $\mathbf{Q}^B \simeq \mathbf{Q}^B_{\delta+} \times \mathbf{Q}^{\delta+}$ and $\mathbf{Q}^B_{\delta+}$ is δ-distributive, $\mathbf{Q}^{\delta+}$ is τ-CC.

Finally successors of inaccessibles are dealt with using:

Diagonal Distributivity. Suppose D_τ is dense open in $\mathbf{Q}^B_\tau$ for $\tau \in \text{Card} \cap \lambda$. Then for all $r \in \mathbf{Q}^B$ there exists $s \leq r$ such that $(s)_\tau \in D_\tau$ for all $\tau \in \text{Card} \cap \lambda$.

The proof of Diagonal Distributivity is very similar to the proof of τ-distributivity for $\mathbf{Q}^B_\tau$. There is a similar (more trivial) statement with λ replaced by limit $\tau < \lambda$ and $\mathbf{Q}^B$ replaced by $\mathbf{Q}^\tau$.

(c) The forcing $\mathbf{F}^B$ is τ-distributive. The proof is similar to the τ-distributivity proof for $\mathbf{Q}^B_\tau$. Cardinal preservation now follows as $\mathbf{F}^B$ trivially is α^{++}-CC.

Jensen coding. The idea for coding A $\subseteq$ ORD by a real is to build a forcing $\mathscr{P}$ such that if G is $\mathscr{P}$-generic then:

1) $G \cap \lambda$ is $\mathbf{Q}^{G \cap [\lambda, \lambda^+)}$-generic for limit cardinals λ.

2) $G \cap \kappa^+$ is $\mathbf{R}^{G \cap [\kappa^+, \kappa^{++})} \cap \mathbf{F}^{A \cap \kappa^+}$-generic for successor cardinals κ^+.

We are thinking here of ω as a successor cardinal: $0^+ = \omega$. Clearly if we can arrange 1) and 2) then A is coded by the real $G \cap \omega$; for, using it we can decode to obtain $G \cap \omega_1, G \cap \omega_2, \ldots, G \cap \omega_\omega, \ldots$ and thus all of G. And, by 2) we have that for $\eta < \kappa^+$, $\eta \in A$ iff $G \cap \kappa^+$ is almost disjoint from $u_\eta \cap \kappa^+$. So $G \cap \omega$ codes A.

We must see how to put the $\mathbf{R}^B, \mathbf{Q}^B, \mathbf{F}^B$ forcings together to arrange 1) and 2). The first step to take is to avoid conflict between these three codings; namely, we choose b_ξ for $\kappa^+ \leq \xi < \kappa^{++}$ to be a subset of $u_0 = \{\langle 0, \gamma\rangle \mid \gamma \in \text{ORD}\}$, we choose b_ξ for $\lambda \leq \xi < \lambda^+$, λ limit to be a subset of $u_1 = \{\langle 1, \gamma\rangle \mid \gamma \in \text{ORD}\}$ and in the definition of the $\mathbf{F}^B$ forcing we replace u_η by $u'_\eta = \{\langle 2, \gamma\rangle \mid \gamma \in u_\eta\}$. (Despite this change in $\mathbf{F}^B$ we continue to use the same notation $\mathbf{F}^B$ for the last forcing.)

Now if we were merely coding $A \subseteq \omega_{n+1}$ then it is clear how to proceed: we can simply iterate codings of A into $A_n \subseteq \omega_n$, A_n into $A_{n-1} \subseteq \omega_{n-1}, \ldots, A_1$ into $A_0 \subseteq \omega$. But as soon as we go beyong ω_ω we must perform these successor codings simultaneously, not simply as an iteration.

Let us simply denote $\mathbf{F}^{A \cap \kappa^+}$ by S_κ (to coincide with Jensen's notation). As before we inductively define μ_r for $r \in S_\kappa$ by: $\mu_r =$ least $\mu > \sup\{\mu_{r \restriction \xi} \mid \xi < |r|\}$ such that $L_\mu[r, A \cap \kappa] \models \text{ZF}^- \wedge \text{Card}(|r|) \leq \kappa$. Let $\mathscr{A}_r = L_{\mu_r}[r, A \cap \kappa]$. Then $\mathbf{R}^r$ (κ successor) and $\mathbf{Q}^r$ (κ limit) are the natural analogues of $\mathbf{R}^B$ and $\mathbf{Q}^B$ defined over $\mathscr{A}_r$ for coding r into a subset of κ. In particular we will have "codes" $b_r \subseteq \kappa$ for each $r \in S_\kappa$.

What follows is a reasonable "first-approximation" to the desired forcing $\mathscr{P}$. Note that we have not been explicit about our definition of the "codes" b_r. (We will say more in our discussion of Extendibility below.) In what follows we use the notation $b_r = \{b_{r,0} < b_{r,1} < \cdots\} \subseteq \kappa$ for $r \in S_\kappa$.

A Jensen condition p assigns to each κ (in an initial segment of $\{0\} \cup \{\text{Infinite Cardinals}\} = \text{Card}$) a pair $(p_\kappa, \bar{p}_\kappa)$ such that:

(a) $p_\kappa \in S_\kappa$,

(b) $(p_\kappa, \bar{p}_\kappa) \in \mathbf{R}^{p_{\kappa^+}}$,

(c) $\mathscr{A}_{p_\kappa} \models \kappa$ singular $\rightarrow r = \langle p_\gamma \mid \gamma < \kappa\rangle \in \mathbf{Q}^{p_\kappa}$; that is, $r \in \mathscr{A}_{p_\kappa}$ and for all $\xi < |p_\kappa|$ such that $r \notin \mathscr{A}_{p_\kappa \restriction \xi}$: $\xi \in \tilde{p}_\kappa$ iff $b_{p_\kappa \restriction \xi}$ is almost disjoint from $\bigcup_{\gamma < \kappa} \tilde{p}_\gamma = \tilde{r}$, and

(d) $\mathscr{A}_{p_\kappa} \models \kappa$ inaccessible $\rightarrow r = \langle p_\gamma \mid \gamma < \kappa\rangle \in \mathscr{A}_{p_\kappa}$ and for all $\xi < |p_\kappa|$ such that $r \notin \mathscr{A}_{p_\kappa \restriction \xi}$: $\xi \in \tilde{p}_\kappa$ iff $\{j \mid b_{p_\kappa\ \xi, j} \in \tilde{r}\}$ is nonstationary in $\mathscr{A}_{p_\kappa \restriction \xi}$ ($\tilde{r} = \bigcup_{\gamma < \kappa} \tilde{p}_\gamma$).

We denote the collection of Jensen conditions by $\mathscr{P}$. Also for $s \in S_\kappa$, $\mathscr{P}^s$ denotes the collection of Jensen conditions p, $\mathrm{Dom}(p) = \mathrm{Card} \cap \kappa$ such that $p \cup \{\langle s, \phi \rangle\} \in \mathscr{P}$. Lastly for $\tau \in \mathrm{Card}$: $\mathscr{P}_\tau = \{(p)_\tau = p \restriction [\tau, \infty) \mid p \in \mathscr{P}\}$, $\mathscr{P}^s_\tau = \{(p)_\tau \mid p \in \mathscr{P}^s\}$.

Note the change from "almost disjoint" to "nonstationary" in clause (d). We will explain this in our discussion of Extendibility below. Of course by "nonstationary in $\mathscr{A}_{p_\kappa \restriction \xi}$" we mean "disjoint from a closed, unbounded $C \in \mathscr{A}_{p_\kappa \restriction \xi}$".

Before discussing the sets $b_r, r \in S_\kappa$, we list the key properties of Jensen coding:

Extendibility. Suppose $X \subseteq \mathrm{Card}$ is thin ($X \cap \kappa$ is nonstationary for inaccessible κ). Suppose $\gamma \le \xi_\gamma < \gamma^+$ for $\gamma \in X$. Then for all $p \in \mathscr{P}$ there is $q \le p$ such that $|q_\gamma| \ge \xi_\gamma$ for all $\gamma \in X$.

Distributivity. (i) $\mathscr{P}_\tau = \{(r)_\tau = r \restriction [\tau, \infty) \mid r \in \mathscr{P}\}$ is τ-distributive.

(ii) κ inaccessible $\to \mathscr{P}$ is κ-Diagonally Distributive; that is, if D_τ is dense open on $\mathscr{P}_\tau$ for each $\tau \in X \subseteq \kappa$, X thin $\to \{p \mid (p)_\tau \in D_\tau \text{ for } \tau \in X\}$ is dense.

Factoring. Let τ be an infinite cardinal, let G_τ: $[\tau, \tau^+) \to 2$ denote the generic for $\mathscr{P}_\tau$ and set $\mathscr{P}^{G_\tau} = \bigcup \{\mathscr{P}^r \mid r \subseteq G_\tau\}$. Then $\mathscr{P}$ and $\mathscr{P}_\tau * \mathscr{P}^{G_\tau}$ are equivalent forcings (written $\mathscr{P} \simeq \mathscr{P}_\tau * \mathscr{P}^{G_\tau}$), in the sense that they yield the same generic extensions.

Chain Condition. τ a successor cardinal $\to \mathscr{P}^{G_\tau}$ has the τ^+-CC.

Using these properties we can establish the Coding Theorem. Using factoring, Distributivity (i) and Chain Condition we can show that successors of accessible cardinals are preserved; Distributivity (ii) then handles successors of inaccessibles and thus all cardinals are preserved (as in our discussion of cardinal preservation for $\mathbf{Q}^B$). Replacement holds in $\mathscr{P}$-generic extensions thanks to distributivity (the definability of forcing is an easy consequence of distributivity and factoring). Finally, Extendibility implies that A is indeed coded by a real.

Now as with the building blocks the nontrivial properties to establish are Extendibility and Distributivity.

Extendibility. Clearly it suffices to establish the Extendibility statement for $\mathscr{P}^s$, $s \in S_\kappa$, κ limit (where X, $\langle \xi_\gamma \mid \gamma \in X \rangle \in \mathscr{A}_s$ and X is thin in $\mathscr{A}_s$). It is worthwhile to first consider the special case: κ is singular in $\mathscr{A}_{s \restriction |p|}$, where $p \in \mathscr{P}^s$ is given as in the statement of Extendibility and $|p| = $ least ξ, $p \in \mathscr{A}_{s \restriction \xi}$. A natural approach here is the following: Choose a short cofinal sequence $\langle \kappa_i \mid i < \lambda_\kappa \rangle$ of cardinals below κ and given p, define $p = p_0 \ge p_1 \ge \cdots$ successively by $p_{i+1} = $ least $q \le p_i$ such that $(q)^{\kappa_i} \le (p_i)^{\kappa_i}$ in $\mathscr{A}_{p_{\kappa_i}}$, $|q_\gamma| \ge \xi_\gamma$ for $\gamma \in X \cap \kappa_i$. Thus the idea here is to obtain extendibility for κ by inductively using extendibility for the κ_i's.

We quickly run into trouble with this, however: there is a serious difficulty in guaranteeing that at a limit stage λ, $\{p_i \mid i < \lambda\}$ can be extended to a condition p_λ. The reason is that we cannot assume that $\langle (p_i)^{\kappa_\lambda} \mid i < \lambda \rangle$ belongs to $\mathscr{A}_{p_{\kappa_\lambda}}$ and therefore we must certainly extend p_{κ_λ} in defining $p_{\lambda_{\kappa_\lambda}}$. For such an extension to be reasonable it must be possible to code $p_{\lambda_{\kappa_\lambda}} - p_{\kappa_\lambda}$ into κ_λ (as in (c) of the definition of $\mathscr{P}$) using the codes b_r for $r \subseteq p_{\lambda_{\kappa_\lambda}}$. But the union of the domains of the $(p_i)^{\kappa_\lambda}$, $i < \lambda$, may already include many such b_r, and thus we have little freedom in defining $p_{\lambda_{\kappa_\lambda}}$. In particular we do not know if we can do this in such a way as to have a member of S_{κ_λ}, due to the committments already made by the $(p_i)^{\kappa_\lambda}$'s.

The way out of this is to keep the "coding areas" for the different κ_i's disjoint, and to only extend p_i at stage $i < \lambda$ on the "coding area" for κ_i. For example, by fine structure theory we can canonically assign closed unbounded sets $C_\kappa \subseteq \mathrm{Card}$ to singular κ so that $\kappa' \in \mathrm{Lim}(C_\kappa) \to C_{\kappa'} = C_\kappa \cap \kappa'$. Then when defining $b_r, r \in S_\kappa$, we make sure that $b_r \subseteq \{\langle 1, \langle \lambda_\kappa, \gamma \rangle \rangle \mid \gamma \in \mathrm{ORD}\}$, where $\lambda_\kappa = \mathrm{ordertype}(C_\kappa)$. Thus the "coding areas" for κ' and κ are disjoint if $\kappa' \in \operatorname{Lim}(C_\kappa)$. If κ' is a successor element of C_κ then we at least get that the coding areas for κ' and κ intersect in a bounded subset of κ'; this is good enough as the coding at κ' is unaffected by changes on a bounded subset of κ'.

Now when defining p_{i+1} first we only extend p_i on the coding area for κ_i (where $\kappa_0 < \kappa_1 < \cdots$ enumerates $\operatorname{Lim}(C_k)$), so as to code some $q = p_{i+1_{\kappa_i}} \supseteq p_{\kappa_i}$ such that $|q| \geq \xi_{\kappa_i}$ and $X \cap \kappa_i, \langle \xi_\gamma \mid \gamma \in X \cap \kappa_i \rangle \in \mathscr{A}_q$ (we also insist that $(q, \bar{p}_{\kappa_i}) \leq p(\kappa_i)$). Then all conflicts between these different codings are avoided and we have no trouble defining $p_{\lambda_{\kappa_\lambda}}$ for limit λ. Now we would like to fill in the p_i's to bona fide conditions, incorporating the coding information we have just described. Note however that though $\langle (p_i)^{\kappa_\lambda} \mid i < \lambda \rangle \in \mathscr{A}_{p_{\lambda_{\kappa_\lambda}}}$ we do not necessarily have $\langle (p_i)^{\kappa_\lambda} \mid i < \lambda_\kappa \rangle \in \mathscr{A}_{p_{\lambda_{\kappa_\lambda}}}$, which is necessary if we want $\{p_i \mid i < \lambda_\kappa\}$ to be extendible to a condition. Thus the preceding process is repeated ω times, successively extending the $p_{\lambda_{\kappa_\lambda}}$'s so as to absorb the necessary coding information. We can then fill in to a condition $q \leq p$.

At the end of this process we have: $X \cap \kappa_\lambda, \langle \xi_\gamma \mid \gamma \in X \cap \kappa_\lambda \rangle \in \mathscr{A}_{q_{\kappa_\lambda}}, |q_{\kappa_\lambda}| \geq \xi_{\kappa_\lambda}$ and $\bigcup \{q_{\kappa'} \mid \kappa' < \kappa_\lambda\}$ completely codes q_{κ_λ} (for limit λ). Now Extendibility can be easily finished off as our original approach now works: extend $q = q_0 \geq q_1 \geq \cdots$ successively by induction so that $|q_{i+1_\gamma}| \geq \xi_\gamma$ for $\gamma \in X \cap \kappa_i$. Then $r \leq p$ is as desired, where $r(\gamma) = \bigcup \{q_i(\gamma) \mid i < \lambda_\kappa\}$.

There are two more points worth mentioning here. We only concerned ourselves with possible conflicts between the codings at the different κ_i's. Could the coding at κ_i conflict seriously with the coding at some $\gamma \notin \{\kappa_i \mid i < \lambda_\kappa\}$? This is prevented by arranging that $b_r, r \in S_\kappa$, be included in $\bigcup \{[\kappa_i^+, \kappa_i^{++}) \mid i < \lambda_k\}$; thus if $\gamma \notin \{\kappa_i \mid i < \lambda_\kappa\}$ then the coding at κ or any of the κ_i's affects the coding at γ only on a bounded subset of γ. Also this requirement conveniently guarantees that the codings do not place any restrictions on our definition of q_{κ_λ} for limit λ.

The second point is more crucial. We implicitly assumed when we extended p_{κ_λ} to q_{κ_λ} that the method of coding $q_{\kappa_\lambda} - p_{\kappa_\lambda}$ into κ_λ was the "singular cardinal method"; that is, we assumed that $\mathscr{A}_{p_{\kappa_\lambda}} \models \kappa_\lambda$ is singular. Otherwise we have lost control over the codings at the κ_λ's and may have trouble keeping them from conflicting with each other. Jensen's way of dealing with this is to impose an added restriction on the definition of $\mathscr{P}$:

(e) For κ limit, $\kappa \in \operatorname{Dom}(p)$ let $|p \restriction \kappa| =$ least $\xi(p \restriction \kappa \in \mathscr{A}_{p_\kappa \restriction \xi})$. (Thus $|p \restriction \kappa| \leq |p_\kappa|$.) If $\mathscr{A}_{p_\kappa \restriction |p \restriction \kappa|} = \mathscr{A}_{|p \restriction \kappa|} \models \kappa$ singular, $\operatorname{cof}(\kappa) > \omega$ then there exists a CUB $C \in \mathscr{A}_{|p \restriction \kappa|}$ such that $\delta \in D \to \mathscr{A}_{|p \restriction \delta|} \models \delta$ singular, $D \cap \delta \in \mathscr{A}_{|p \restriction \delta|}$.

This clause guarantees that we can work with possibly a subsequence $\langle \kappa_i^* \mid i < \lambda_\kappa^* \rangle$ of the κ_i's and then know at limit stages λ that $\mathscr{A}_{p_{\kappa_\lambda^*}} \models \kappa_\lambda^*$ is singular. Actually, an inductive argument then shows that we can work with the original κ_i-sequence

We now turn to the case when κ is inaccessible in $\mathscr{A}_{|p|}$. Then the above strategy cannot work as we do not have the singular sequences C_{κ_i} (of distinct ordertypes) to keep the codings disjoint at some CUB collection of cardinals κ_i below κ. Instead, Jensen exploits the inaccessibility of κ to obtain a CUB set of κ_i's less than κ where the coding looks the same as at κ. Now we cannot literally have this property, as our given $p \in \mathscr{P}^s$ may satisfy: $\mathscr{A}_{|p|} = \mathscr{A}_{s \restriction |p|} \models \kappa$ regular, $\mathscr{A}_{p_{\kappa'}} \models \kappa'$ singular for all limit $\kappa' < \kappa$. This leads Jensen to "stratify" $\mathscr{A}_s$ as follows: For $i < \omega$ we set $\mu_s^0 = \sup\{\mu_{s \restriction \xi} \mid \xi$

Also $\mu_s = \mu_s^\omega = \sup\{\mu_s^i \mid i < \omega\}$. Correspondingly let $\mathscr{A}_s^i = L_{\mu_s^i}[s, A \cap \kappa]$, $\mathscr{A}_s = L_{\mu_s}[s, A \cap \kappa]$. With this change in the definition of $\mathscr{A}_s$ we now obtain the following: If $\mathscr{A}_s \vDash \kappa$ regular and $p \in \mathscr{P}^s$, then for all sufficiently large $i < \omega$ there exists CUB $C_i \subseteq \kappa$, $C_i \in \mathscr{A}_s$, such that $\mathscr{A}^i_{|p\restriction\xi|} \vDash \delta$ regular for $\delta \in C_i$. (As before, $\mathscr{A}^i_{|p\restriction\delta|}$ denotes $\mathscr{A}^i_{p_\delta\restriction\xi}$, where $\xi =$ least $\xi', p \restriction \delta \in \mathscr{A}_{\xi'}$.)

This strongly suggests the following change in the coding at inaccessibles: Instead of just having a single code b_r for $r \in S_\kappa$ when $\mathscr{A}_r \vDash \kappa$ inaccessible, we should have ω-many codes $b_r^0, b_r^1, \ldots$, where $b_r^i \in \mathscr{A}_r^{i+1}$. Then the coding should be (see clause (d)): $\xi \in \tilde{p}_\kappa$ iff for sufficiently large $i < \omega$, $\{j \mid b^i_{p_\kappa\restriction\xi,j} \in \tilde{r}\}$ is nonstationary in $\mathscr{A}_{p_\kappa\restriction\xi}$. (As before, $\langle b^i_{s,j} \mid j < \kappa\rangle$ is the increasing enumeration of b_s^i.)

Now let us take a look at extendibility for $\mathscr{P}^s$. First suppose that $\mathscr{A}_{|p|} = \mathscr{A}_s$ ($|p| = |s|$) and that for a CUB $C \in \mathscr{A}_s, \delta \in C \to |p_\delta| = |p \restriction \delta|$. Then there is a CUB $D \in \mathscr{A}_s$ such that $\delta \in D \to |p_\delta| = |p \restriction \delta|, \langle \xi_\gamma \mid \gamma \in X \cap \delta\rangle, D \cap \delta \in \mathscr{A}_{p_\delta}, X \cap \delta \in \mathscr{A}_{p_\delta}$ is thin in $\mathscr{A}_{p_\delta}$. Thus by induction we can successively extend $p \geq p_1 \geq p_2 \geq \cdots$ so that $|p_{i+1_\gamma}| \geq \xi_\gamma$ for $\gamma \in X \cap \alpha_i$, where $\langle \alpha_i \mid i < \kappa\rangle$ is the increasing enumeration of D. There is no difficulty in defining p_λ for limit λ as $\langle p_i \mid i < \lambda\rangle \in \mathscr{A}_{p_{\alpha_\lambda}}$ and $|p \restriction \alpha_\lambda| = |p_{\alpha_\lambda}|$. Finally define $q(\gamma) = \bigcup\{p_i(\gamma) \mid i < \kappa\}$; q is as desired.

Thus the only problem with extendibility is arranging $|p| = |s|$ and the existence of a CUB $C \in \mathscr{A}_{|p|}$ such that $\delta \in C \to |p \restriction \delta| = |p_\delta|$. The latter property cannot be "arranged" but must instead be made part of the definition of $\mathscr{P}$. The final clause in the definition of $\mathscr{P}$ is:

(f) If $\mathscr{A}_{p_\kappa} \vDash \kappa$ inaccessible then there exists CUB $C \in \mathscr{A}_{|p\restriction\kappa|}$ such that $\delta \in C \to |p \restriction \delta| = |p_\delta|, \bar{p}_\delta = \varnothing$.

The need for "$\bar{p}_\delta = \varnothing$" will become clear in a moment. Now suppose $p \in \mathscr{P}^s$ and we want to extend p to $q, |q| = |s|$. This is done by induction on $|s|$. First consider the case $|s| = \xi + 1$, and we can assume that $|p| = \xi$. We can choose $i_0 < \omega$ large enough so that $p \in \mathscr{A}^{i_0}_{s\restriction\xi}$. Now for $i \geq i_0$ we can define CUB sets $D_i \in \mathscr{A}_{s\restriction\xi}$ so that $\delta \in D_i \to |p \restriction \delta| = |p_\delta|, \bar{p}_\delta = \varnothing$ and $b^i_{p_\delta} = b^i_{s\restriction\xi} \cap \delta$. Thus $\delta \in D_i$ implies that δ "reflects" the coding of $s(\xi)$ onto the $b^j_{s\restriction\xi}, j \leq i$ (we assume that $i_1 < i_2 \to D_{i_1} \supseteq D_{i_2}$). Now define $\langle q_i \mid i_0 \leq i\rangle$ inductively by: $q_{i_0} = p$, and q_{i+1} is obtained from q_i by coding $s(\xi)$ onto $\{b^i_{s\restriction\xi,j} \mid j \in D_i\}$. The codes $b^i_{s\restriction\xi}$ are defined so that $b^i_{s\restriction\xi,j} \in [j^+, j^{++})$, so q_{i+1} differs from q_i only on $\bigcup\{[\delta^+, \delta^{++}) \mid \delta \in D_i\}$.

It is easy to see that we can define q_i's in $\mathscr{P}^{s\restriction\xi}$ as above. Now suppose $\gamma \in D = \bigcap_i D_i$. ($\kappa$ may be singular in $\mathscr{A}_s$, so D may be empty!) Then $s(\xi)$ has been coded on $b^i_{p_\gamma}$ for all $i \geq i_0$, so we can define $q_\gamma \supseteq p_\gamma, q_\gamma(|p|_\gamma) = s(\xi)$ at such γ and then $q_\gamma(|p_\gamma|)$ is correctly coded (for all sufficiently large i) on $b^i_{q_\gamma\restriction|p_\gamma|}$. Thus the desired q is defined by: $q(\gamma) = \bigcup\{q_i(\gamma) \mid i \geq i_0\}$ for $\gamma \notin D$, $q(\gamma) = (p_\gamma \bigcup\{\langle |p_\gamma|, s(\xi)\rangle\}, \bar{p}_\gamma)$ for $\gamma \in D$. Now it is important to have that $\gamma \in D \to \bar{p}_\gamma = \varnothing$ in order to know that $(q_\gamma, \bar{p}_\gamma) \leq (p_\gamma, \bar{p}_\gamma)$.

Note that we cannot hope for more than "nonstationary" in clause (d), as we are careful to alter q_i in the definition of q_{i+1} only on $\bigcup\{[\delta^+, \delta^{++}) \mid \delta \in D_i\}$. This enables one to show that these changes do not significantly affect the codings at limit cardinals $\gamma \in D$.

We also point out that clause (e) must be verified for the condition q. This is in some sense the nastiest part of extendibility and requires a deep fine structure lemma. Note that though $\mathscr{A}_{p_\gamma} \vDash \gamma$ is regular for $\gamma \in D$ it may happen that $\mathscr{A}_{q_\gamma} \vDash \gamma$ is singular. Thus clause (e) becomes a problem exactly at the transition from inaccessible to singular. There is a similar verification required in the distributivity argument at inaccessibles.

We will not say much about the case $|s|$ limit. In this case p is extended repeatedly to $p \geq q_0 \geq q_1 \cdots$, where the $|q_i|$'s are cofinal in $|s|$, by induction. In fact the $|q_i|$'s are chosen from a very canonical sequence through $|s|$ in order to facilitate the verification that q_λ is a condition for limit λ; especially, to check clause (e).

Distributivity. The key idea in the proof of the Coding Theorem appears in the distributivity argument, which we now describe. Thus suppose $\tau \in \mathrm{Card}$, $p \in \mathscr{P}_\tau$ and $\langle D_i \mid i < \tau \rangle$ is an $L[A]$-definable sequence of dense open subsets of $\mathscr{P}_\tau$. We want to find $q \leq p$, $q \in \bigcap_i D_i$. If we naively extend p to $p_1 \geq p_2 \geq \cdots$ so that $p_{i+1} \in D_i$ we have the problem of defining p_λ for limit λ: though $p_{i_\gamma} \in S_\gamma$ for $i < \lambda$ it need not be the case that $q = \bigcup \{p_{i_\gamma} \mid i < \lambda\} \in S_\gamma$. The main problem is to guarantee that $\langle p_i \restriction \gamma \mid i < \lambda \rangle \in \mathscr{A}_q$ (q "codes" $\langle p_i \restriction \gamma \mid i < \lambda \rangle$).

The strategy for dealing with these problems is as follows. Any construction of the p_i's, $i < \lambda$, for the purpose of meeting the dense sets D_i will of course be definable over the full ground model $\langle L[A], A \rangle$; but we could equally well work over some (sufficiently) elementary submodel $Y \subseteq \langle L[A], A \rangle$ large enough to contain p and a parameter for defining $\langle D_i \mid i < \tau \rangle$. Now choose Y^γ to be the least such with the property that $\gamma \subseteq Y$. The construction of the p_i's inside b^γ = transitive collapse (Y^γ) yields a sequence $\langle p_i^* \mid i < \lambda \rangle$ with the property that $p_i^* \restriction \gamma = p_i \restriction \gamma$ for all $i < \lambda$.

The essential trick is to arrange that $q = \bigcup \{p_{i_\gamma} \mid i < \lambda\}$ code a generic class for $(\mathscr{P}_\gamma)^{b^\gamma}$ = "$\mathscr{P}_\gamma$ in the sense of b^γ". For then, q (together with $A \cap \gamma$) codes $(A)^{b^\gamma}$ and hence the entire model b^γ. As $\langle p_i^* \mid i < \lambda \rangle$ is definable over b^γ, q codes $\langle p_i^* \mid i < \lambda \rangle$ and hence $\langle p_i \restriction \gamma \mid i < \lambda \rangle$.

Now in actual fact it will not be possible to work with a fixed Y^γ but instead a sequence $\langle Y_i^\gamma \mid i < \tau \rangle$, for $\gamma \geq \tau$. We will then alternate the construction of the Y_i^γ's and the p_i's in such a way that $p_i \in Y_{i+1}^\tau$. We must design our choice of the p_i's so as to guarantee for limit λ that $q_\gamma = \bigcup \{p_{i_\gamma} \mid i < \lambda\}$ codes a generic over b_λ^γ = collapse Y_λ^γ. Let $b = b_\lambda^\gamma$ and $\sigma: b \simeq Y_\lambda^\gamma$.

Now it is clear which generic $G \subseteq b$ we want q_γ to code: For any b-cardinal $\delta^* \geq \gamma$ let $G_{\delta^*} = \bigcup \{p_{i\delta^*}^* \mid i < \lambda\}$, where $p_i^* = \sigma^{-1}(p_i)$ (thus $G_\gamma = q_\gamma$). Then we want q_γ to code $G = \bigcup \{G_{\delta^*} \mid \delta^* \in (b - \mathrm{Card}) - \gamma\}$. Now for q_γ to code G we do not actually need full genericity for G over $(\mathscr{P}_\gamma)^b$; thanks to the fact that $|p_i^* \restriction \delta^*| = |p_{i\delta^*}^*|$ for limit b-cardinals δ^* (and hence $p_i^* \restriction \delta^*$ codes $p_{i\delta^*}^*$) it would suffice to have that for successor b-cardinals δ^*: $G \cap \delta^*$ is $\mathscr{P}_\gamma^{p_{i\delta^*}^*}$-generic over $\mathscr{A}_{p_{i\delta^*}^*}^1$. Or, applying σ, we want: If $D \in \mathscr{A}_{p_{i\delta^+}}^1 \cap Y_\lambda^\gamma$ is dense open on $\mathscr{P}_\gamma^{p_{i\delta^+}}$ for some $\delta \geq \gamma$ and $i < \gamma$, then $(p_j)_\gamma^{\delta^+}$ extends an element of D for some $j < \lambda$ (where $(p_j)_\gamma^{\delta^+} = p_j \restriction [\gamma, \delta]$).

Now that we have the genericity requirement in the proper form it is possible to describe Jensen's strategy for meeting it. Assume for the moment that we already know that $\mathscr{P}_\gamma^r$ is γ-distributive in $\mathscr{A}_r^1$ for $r \in S_{\delta^+}$, $\delta \in \mathrm{Card}$. Then we can choose p_{i+1} so that $(p_{i+1})_\gamma^{\delta^+}$ extends an element of D for all dense open D on $\mathscr{P}_\gamma^{p_{i\delta^+}}$, $D \in \mathscr{A}_{p_{i\delta^+}}^1 \cap Y_i^\gamma$ by γ-distributivity. Choose δ to be $\delta_i > \sup(\mathrm{Dom}\, p_i)$. Then at stage λ we have: For all $i < \lambda$, *if* $D \in \mathscr{A}_{p_{i\delta_i^+}}^1 \cap Y_i^\gamma$ is dense open on $\mathscr{P}^{p_{i\delta_i^+}}$ then $(p_j)_\gamma^{\delta_i^+}$ extends an element of D for some $j < \lambda$. Now Jensen also arranges that if $D \in \mathscr{A}_r^1$, $r \in S_{\delta^+}$, is dense open on $\mathscr{P}_\gamma^r$ then for any $\eta > \delta$, $s \in S_{\eta^+}$ we have that $\{p \in \mathscr{P}_\gamma^s \mid (p)^{\delta^+}$ extends an element of $D\}$ is also dense open on $\mathscr{P}_\gamma^s$. Thus in fact we have achieved: For any D

$\in \mathscr{A}^1_{p_{i_{\delta^+}}} \cap Y^\gamma_\lambda$ which is dense open on $\mathscr{P}^{p_{i_{\delta^+}}}_\gamma$ there exists $j < \lambda$ so that $(p_i)^{\delta^+}_\gamma$ extends an element of D. Thus $q_\gamma = \bigcup\{p_{i_\gamma} \mid i < \lambda\}$ codes a generic on $(\mathscr{P}_\gamma)^b$, and we have achieved our goal.

But we have only dealt with one cardinal γ. To prove distributivity we actually need to handle all $\gamma \in \mathrm{Dom}(p_i)$ at stage $i+1$. Suppose $\tau \leq \delta \in \mathrm{Dom}(p_i)$ and that δ is a successor cardinal. An application of the δ-distributivity of $\mathscr{P}^{p_{i_{\delta^+}}}_\delta$ in $\mathscr{A}^1_{p_{i_{\delta^+}}}$ can be used to show that the collection of $r \in \mathscr{P}^{p_{i_{\delta^+}}}_\tau$ with the following property is dense below $(p_i)^{\delta^+}$: For all dense open D on $\mathscr{P}^{p_{i_{\delta^+}}}_\tau$, $D \in \mathscr{A}^1_{p_{i_{\delta^+}}} \cap Y^\delta_i$ either $r \in D$ or for some $\eta < \delta$ $\{q \in \mathscr{P}^{r_{\eta^+}}_\tau \mid q \cup (r)_{\eta^+} \in D\}$ is dense open on $\mathscr{P}^{r_{\eta^+}}_\tau$. A further argument using distributivity and induction establishes the same claim for limit δ. Finally one last argument is needed to show that the collection of $r \in \mathscr{P}_\tau$ such that $(r)^{\delta^+}$ has the above property simultaneously for a thin set of $\delta \in \mathrm{Dom}(p_i)$ is dense below p_i.

We have now arrived at the definition of the auxiliary dense sets $\Sigma^{p_i}_f$. Thus $\mathrm{Dom}(f)$ is thin and for each $\delta \in \mathrm{Dom}(f)$, $f(\delta) \subseteq \mathscr{A}^1_{p_{i_{\delta^+}}}$ has cardinality $\leq \delta$. And $\Sigma^{p_i}_f$ consists of all $p \in \mathscr{P}_\tau$ such that for all $\delta \in \mathrm{Dom}(f)$, all $D \in f(\delta)$ which are dense open on $\mathscr{P}^{p_{i_{\delta^+}}}_\tau$: either $(p)^{\delta^+}$ meets D or for some $\eta < \delta$, $\{q \in \mathscr{P}^{p_{\eta^+}}_\tau \mid q \cup (p)^{\delta^+}_{\eta^+} \in D\}$ is dense open on $\mathscr{P}^{p_{\eta^+}}_\tau$.

Now for the construction of the p_i's: Choose $p_{i+1} \in \Sigma^{p_i}_{f_i}, p_{i+1} \leq p_i$, where $f_i(\delta) = \mathscr{A}^1_{p_{i_{\delta^+}}} \cap Y^\delta_i$ if $\delta \in Y^\delta_i$; $f_i(\delta)$ undefined otherwise. Then $\mathrm{Dom}(f_i)$ is thin for each i. Also choose p_{i+1} so that $\alpha_{i+1} = \bigcup \mathrm{Dom}(p_{i+1}) > \bigcup Y^{\alpha_i}_i$ (where $\alpha_i = \bigcup \mathrm{Dom}(p_i)$).

Now we claim that $q_\gamma = \bigcup \{p_{i_\gamma} \mid i < \lambda\}$ codes a generic for $(\mathscr{P}_\gamma)^{b^\gamma_\lambda}$ for all $\gamma \in \bigcup\{\mathrm{Dom}(p_i) \mid i < \lambda\}$. The point is that any dense set D on $\mathscr{P}^{p_{i_{\delta^+}}}_\gamma$ in Y^γ_i was either met by $(p_{i+1})^{\delta^+}_\gamma$ or was "reduced" to the problem of meeting some dense set on $\mathscr{P}^{p_{i+1_{\eta^+}}}_\gamma$ for some $\eta < \delta$. (This is true for i large enough so that $\gamma \in \mathrm{Dom}(p_i)$.) Thus eventually D must be met by some $(p_j)^{\delta^+}_\gamma, j < \lambda$, else we have an infinite descending sequence $\delta > \eta_0 > \eta_1 > \cdots$ of cardinals!

We should recall that in the above argument we have assumed γ-distributivity of $\mathscr{P}^r_\gamma$ for $r \in S_{\delta^+}$ and $\gamma \leq \delta$. Now a similar argument to the above can be used to establish this, using distributivity for forcings $\mathscr{P}^s_\gamma, s \in S_{\eta^+}$ for $\eta < \delta$. Thus distributivity for $\mathscr{P}^r_\gamma, r \in S_{\gamma^+}$, is established by induction on δ, simultaneously with the assertion that the appropriate Σ^p_f's are dense.

There are some differences between the arguments for the cases where δ is inaccessible as opposed to where δ is singular. In the inaccessible case we run into the troublesome situation of having to verify clause (e), as in Extendibility. Again a key fine structure lemma is required and the argument is somewhat lengthy. In the singular case of uncountable cofinality, Jensen does not directly establish the density of the required Σ^p_f's but instead works with a canonical sequence $\delta_0 < \delta_1 < \cdots$ cofinal in δ and establishes the density of $\Sigma^p_{f'}$, where $f' = f \restriction (\mathrm{Dom}(f) - \{\delta_0, \delta_1, \ldots\})$. This suffices to carry out the distributivity argument. Of course once distributivity is established the original Σ^p_f's can be shown to be dense; there does not appear to be a direct argument for this however.

We now come to an extremely important point concerning Jensen's distributivity argument for $\mathscr{P}^r_\gamma, r \in S_{\delta^+}, \gamma \leq \delta$. In analogy to the distributivity proof for $\mathscr{P}_\gamma$ we arrange that q_γ codes a generic G for $(\mathscr{P}^{r^*}_\gamma)^b$ where $b = b^\gamma_\lambda$ and $\pi\colon b \simeq Y^\gamma_\lambda \prec \mathscr{A}^1_r, r^* = \pi^{-1}(r)$. However, the use of the Σ^p_f's does not actually establish the genericity of G over b but only the genericity of $G \restriction \delta'$ for $\delta' < \delta^* = \pi^{-1}(\delta)$ and the genericity of $G \restriction [\delta^*, \delta^{*+})$. (In the distributivity proof for $\mathscr{P}_\gamma$ the former statement suffices as δ^*

can be thought of as $b \cap \mathrm{ORD}$; thus genericity over b is obtained.) In particular we do not have enough genericity to argue that $b \subseteq b[q_\gamma] = L_{\mu'}[A \cap \gamma, q_\gamma] \models \delta^*$ is cardinal, where $\mu' = b \cap \mathrm{ORD}$. This is a problem, for we need the latter property to show that $\mu' < \mu_{q_\gamma}$ and hence that $b \in \mathscr{A}_{q_\gamma}, q_\gamma \in S_\gamma$.

Jensen deals with this difficulty by altering the definition of S_γ. This alteration is the fundamental reason why the argument splits into two cases, depending upon whether or not $0^\#$ belongs to the ground model $L[A]$. Jensen's requirement for $r \in S_\gamma$ (which succeeds in the $\sim 0^\#$ case) is that not only do we have $L[r \restriction \xi, A \cap \gamma] \models \mathrm{Card}(\xi) \leq \gamma$ for $\xi \leq |r|$, but in fact $L_\eta[r \restriction \xi, A \cap \gamma] \models \mathrm{Card}(\xi) \leq \gamma$ for some $\eta < (\xi^+)^L$. And, by definition, μ_r is large enough so that $\mathscr{A}_r = L_{\mu_r}[r, A \cap \gamma] \models$ "$r \in S_\gamma$". We can now argue that $\mu' < \mu_{q_\gamma}$ in the distributivity proof, as any η such that $L_\eta[q_\gamma, A \cap \gamma] \models \mathrm{Card}(|q_\gamma|) \leq \gamma$ must be at least δ^* (by the genericity of $G \restriction \delta'$ for $\delta' < \delta^*$) and therefore $\mu_{q_\gamma} > \mu'$, as $L_{\mu'} \models \delta^*$ is a cardinal $> \gamma$.

This ends our outline of Jensen's proof, in the $\sim 0^\#$ case. In Part III below we will outline an approach which overcomes the genericity problems discussed above and therefore provides a uniform proof of the Coding Theorem which makes no distinction concerning the existence of $0^\#$ in the ground model.

Part II. A guide to the book. In light of the above outline we can now explain the details of the exposition provided in ***Coding the Universe***. Such an explanation is most greatly needed in Chapters 2 and 3, the heart of the proof.

In Chapter 1 the "building blocks" are discussed rather thoroughly. Most readers with a basic knowledge of forcing should find this chapter very readable. Some important definitions are missing however (though they can sometimes be found in the "Notational Index"): For example, "γ-Distributive", "κ-CC" (defined in Part I of this review), "Predense" ($X \subseteq \mathscr{P}$ is predense if $\{p \in \mathscr{P} \mid p \leq \text{some } q \in X\}$ is dense open) and "ZF^-" ($=$ZF without Power Set). Lemma 1.8 on p. 23 constitutes what I referred to as "Diagonal Distributivity" in Part I and is used to show that successors of inaccessibles are preserved by the coding at limit cardinals. (This same form of distributivity appears again as Theorem 3.2 on p. 74.) The readers should be warned of a gap in the extendibility proof for the limit coding on p. 21, where it is assumed that $I = \{\text{infinite cardinals } < \beta\}$ has ordertype β (see the definition of the η_i's). However, this gap is easy to fill, as if I has ordertype $< \beta$ then β is singular and we can assume that $\varepsilon = \mathrm{ordertype}(C)$ is less then β; so define $\eta_i = \eta_0$ for all $i \leq \varepsilon$, where $\eta_0 < \beta$ is some cardinal greater than ε.

Lastly, we mention that Chapter 1 discusses the "generic codes" b_ξ on p. 12; these are introduced so as to obtain the "persistence" property: If $s \in S_\kappa$, κ a successor cardinal and $G \subseteq \tau^+$ codes a generic for $\mathscr{P}_\tau$, then G also codes a generic for $\mathscr{P}_\tau^s$ (over $\mathscr{A}_s^1$). Another way to say this is that if $X \in \mathscr{A}_s^1$ is predense on $\mathscr{P}_\tau^s$ then X is also predense on $\mathscr{P}_\tau$. This property is useful in the distributivity argument (as we mentioned in Part I), though is not really necessary. (We will say more about this in our discussion of Chapter 3.)

Chapter 2 begins the heart of the proof and includes the definition of $\mathscr{P}$ together with a proof of Extendibility.

In a preliminary step a class of Cohen sets is added to improve the ground model $\langle M, A\rangle$ to a model $\langle M', A'\rangle$ obeying the "global axiom of choice" and $\diamond$ at of the universe.) The Cohen sets are also used to show that large cardinal properties are preserved by the forcing.

On the bottom of p. 27 begins the definition of S_α; clause (1) corresponds to our $\mathbf{F}^{A\cap\alpha^+}$ from Part I. Clause (2) is added for the reason that we discussed in the outline of the Distributivity argument. This is the main use of $\sim 0^\#$; the assumption $\alpha \geq \omega_2$ in (2) is required for the reasons discussed on p. 17.

As we mentioned in Part I, it is necessary to define an infinite sequence of μ_p^i's rather than just a single μ_p in order to properly deal with coding at inaccessibles. Clause (b) on p. 28 is needed to show that μ_p is large enough in the distributivity argument. Clause (c) says that $\mathscr{A}_p^1$ obeys the covering lemma; this is used to prove that μ_p is big enough so that $\mathscr{A}_p$ contains the canonical cofinal sequence C_α when $\mathscr{A}_p \models \alpha$ singular (see Remark (1) on p. 43). Clause (d) is useful because it implies that $\mathscr{A}_p^1$ is $<\alpha$-closed when α is a successor cardinal (see Fact 2.4.4 on p. 34). This in turn is used in the proof of Lemma 2.7. (Note the assertion "$\langle v, |p|\rangle \in \mathscr{P}_\xi$" in the proof of Lemma 1.3.)

Lemmas 2.3, 2.4 are the "fine-structure" lemmas. Lemma 2.3 provides the $\square$-sequences used in establishing extendibility at limit cardinals; clause (v) of Lemma 2.3 and Lemma 2.4 are needed in the verifications of what we referred to as "clause (e)" in Part I. Thus they are needed to verify that the extendibility and distributivity constructions are sufficiently "canonical". Lemma 2.5 is the "generic codes" lemma and, as already explained, is needed for Lemma 2.11.

Note that in the definition of the successor coding $\mathbf{R}^s$ on p. 34 that $A \cap \alpha^+$ is not coded "directly" but instead "almost disjointly" using the V_δ's. (We also did this in defining $\mathbf{F}^{A\cap\alpha^+}$.) The reason is this: Suppose $s \in S_\alpha, \alpha$ inaccessible and $\pi\colon T \simeq M \prec \mathscr{A}_s, s \in M$, T transitive. Suppose also that $M \cap \alpha = \bar{\alpha} < \alpha$ so that $\pi(\bar{\alpha}) = \alpha$. Then we would like to say that $\bar{s} = \pi^{-1}(s)$ belongs to $S_{\bar{\alpha}}$. (This comes up in the extendibility argument at inaccessibles. Also see Lemma 2.12 on p. 48, the Collapsing Lemma.) If we coded A "directly", say $\eta \in A \leftrightarrow s(\eta) = 1$, then we would need to know that $\pi^{-1}(A \cap |s|) = A \cap \pi^{-1}(|s|)$, which puts too strong a restriction on A. Instead Jensen codes A "almost disjointly" and therefore the verification that $\bar{s} \in S_{\bar{\alpha}}$ is easy.

The limit coding is discussed in §2.5. As explained in the discussion of Extendibility in Part I, it is important to define $\rho_{s\beta}^n \in (\beta^+, \beta^{++})$ (on p. 41) rather than (β, β^+); also, on p. 42, $\tilde{\rho}_{si}$ is used to keep the codings at different γ_i^β's disjoint. (Actually the use of $|C_\beta|$ in that definition is not actually necessary. For, $\rho_{s\gamma_i}$ codes the model $\mathscr{A}_{s\gamma_i^+}$ which contains (the collapse of) C_β and satisfies that "(collapse of) β is the largest cardinal". Thus if $\beta' \in C_\beta$ and $s' \in \bar{S}_{\beta'}$, then the codings of s and s' are already disjoint without the insertion of $|C_\beta|$ and $|C_{\beta'}|$, as the $\rho_{s'\gamma_i'}$'s code models which contain (the collapse of) $C_{\beta'}$ and satisfy that "(collapse of) β' is the largest cardinal"; and $|C_{\beta'}| \neq |C_\beta|$.)

It is important to know that the singular sequence C_β belongs to $\mathscr{A}_s$ when $s \in S_\beta, \mathscr{A}_s \models \beta$ is singular. This is asserted in Remark (1) on p. 43, but it should be pointed out that this is a use of clause (c) (in the definition of μ_p^{i+1}) on p. 28. In fact one can assert more: $C_\beta \in L_{\mu_s}[A \cap \beta]$.

We arrive at the definition of $\mathscr{P}_\tau^s$ on pp. 43–44. Clause (ii) on p. 44 corresponds to clause (f) of Part I and clause (iv) to (e) of Part I.

The Collapsing Lemma on p. 48 is used repeatedly in the extendibility proof at inaccessibles; it was used implicitly in the outline of that proof in Part I. The "smoothness" condition $|p \upharpoonright \gamma| = |p_\gamma|$ (for CUB-many γ) is seen to be essential here to argue that $p_{\bar{\alpha}} = \pi^{-1}(s)$ (where $\pi = \pi^n_{s\bar{\alpha}}$).

The Extendibility argument of §2.7 proceeds as was outlined in Part I; one should note the use of clause (iv) (in the definition of $\mathscr{P}^s_\tau$) in Case 2 on p. 53. The singular extendibility proof in Lemma 2.14.3 makes use of $\tilde{\mathscr{P}}^s_\tau$, that "part" of $\mathscr{P}^s_\tau$ which refers only to the coding area for β. As far as I can tell the middle paragraph on p. 55 (lines 8, 9, 10 from the bottom) is irrelevant and can be safely ignored.

The extendibility proof at inaccessibles begins on p. 59 and continues through the end of Chapter 2. The crucial verification is clause (d) on p. 63 (for Case 1). It should be mentioned here that there is a crucial use of "$\dot{p}_{\bar{\alpha}} = \varnothing$" here to verify that $(\dot{q}_{\bar{\alpha}}, q_{\bar{\alpha}}) \leq (\dot{p}_{\bar{\alpha}}, p_{\bar{\alpha}})$ in $\mathbf{R}^{q_{\bar{\alpha}}+}$. This justifies the inclusion of "$\dot{p} = \varnothing$" in clause (ii) of the definition of $\mathscr{P}^s_\tau$. (Note that $\dot{p}_{\bar{\alpha}}$ corresponds to what we called $\bar{p}_{\bar{\alpha}}$ in Part I.)

This concludes our discussion of Chapter 2. There are a number of misprints in this chapter which we list (together with misprints from other chapters) at the end of this part.

The key distributivity proof is to be found in Chapter 3. Distributivity is first established for $\mathscr{P}^s_\tau, s \in S_{\alpha^+}$, and then afterwards (§3.7) for $\mathscr{P}_\tau$. This is because the latter argument depends upon the density of the sets $\Sigma^{\alpha_i^+, p_i}_{f_i}$ (see p. 116) which depends in turn on the distributivity of forcings of the form $\mathscr{P}^s_\tau, s \in S_{\alpha^+}$. (Incidentally $\Sigma^{\alpha^+, p}_f$ when $\mathrm{Dom}(p) = \mathrm{Card} \cap [\tau, \alpha]$ should be defined as $\Sigma^{s,p}_f$, where $s = \varnothing$-member of S_{α^+}.)

There are two forms of distributivity for $\mathscr{P}^s_\tau, s \in S_{\alpha^+}$, when α is inaccessible: ordinary τ-distributivity in $\mathscr{A}^1_s$ (Theorem 3.1) and "diagonal distributivity" in $\mathscr{A}^1_s$ (Theorem 3.2). The latter is needed for cardinal preservation at α^+, as well as to establish the density of Σ_f's. Lemmas 3.4, 3.8 and 3.10 show progressively that $\Sigma^{s,p}_f$ is dense in $\mathscr{P}^s_\tau$ for $p \in \mathscr{P}^s_\tau$ and $f \in \mathbf{F}(p)$, assuming distributivity for $\mathscr{P}^s_\tau$. This is necessary as the argument for the density of $\Sigma^{s,p}_f$ without assuming distributivity (Lemma 3.12 in the inaccessible case) makes use of the density of Σ_g's based at smaller cardinals. Thus distributivity and Σ_f-density are established by a double induction.

Lemma 3.5 is the Factoring property for $\mathscr{P}^s_\tau$ and is in fact a straightforward consequence of: $\mathscr{P}^s_\tau \simeq \mathscr{P}^s_\delta * \mathscr{P}^{G_\delta}_\tau$ for any cardinal $\delta \in (\tau, \alpha)$. (Incidentally, this equivalence holds as $\mathscr{P}^s_\tau$ is isomorphic to the dense subset of $\mathscr{P}_\delta * \mathscr{P}^{G_\delta}_\tau$ consisting of all $((p)_\delta, [(p)^\delta])$ for $p \in \mathscr{P}_\tau$, where $[x]$ for $x \in \mathscr{A}_s$ is the canonical term denoting x. Now Factoring follows from the usual Product Lemma.) In particular the restriction to successor cardinals in (b) is superfluous. We might point out, however, that the δ^+-CC of $\mathscr{P}^{G_\delta}_\tau$ does require that δ is a successor cardinal as well as the property that D^0 is $\mathscr{P}^{p_\delta}_\tau$-generic over $\mathscr{A}^1_{p_\delta}$ for $p \in G_{D^1}$ (where D is $\mathscr{P}^s_\tau$-generic over $\mathscr{A}^1_s, s \in S_\alpha, D^0 = D \cap [\tau, \delta), D^1 = D \cap [\delta, \alpha)$—this is Corollary 3.5.1). The latter is a consequence of the use of the "generic codes" of Lemma 2.5, as was Lemma 2.11.

The reader must be warned that throughout Chapter 3 (and the rest of the book as well) the authors consistently use the word "dense" when they mean "open dense". The only exceptions that I detected are near the definitions of $\mathscr{P}^*_\tau$ on pp. 73 and 115. Of course this is not a serious error; however, many results, such as Theorem 3.2, are false as stated without this change. More importantly, the failure to make the distinction hides the use of Lemma 2.11 in a multitude of places. For example, though Lemma 3.4 is fine as stated provided the Δ_ν's are assumed open dense, Lemma 2.11 is required to assert that Δ is open dense. The openness of Δ is used in the proof of Lemma 3.8 where it is asserted on the last line of p. 81 that Δ^* is dense (which requires the openness of $\Delta_i^*, i < \kappa$). Similarly the fact that $\Sigma_f^{s,p}$ is open is a consequence of Lemma 2.11.

Actually these uses of Lemma 2.11 (and hence the need for generic codes) can be eliminated by observing that the proof of Lemma 3.4 really shows that Δ contains an open dense set. This can be used to show that the Σ_f's contain open dense sets, and this is enough to carry out the distributivity proof. (Alternatively: replace all occurrences of "$\Delta^{(q)_{\gamma^+}}$ is dense in $\mathscr{P}_\tau^{q_{\gamma^+}}$" by "$q_{\gamma^+} \Vdash \Delta^{(G)_{\gamma^+}}$ is dense in $\mathscr{P}_\tau^{G_{\gamma^+}}$", where G denotes the generic for $\mathscr{P}_\tau^s$).

Lemmas 3.6 and 3.7 are the cardinal-preservation and coding lemma. The use of Lemma 2.11 in the proof of 3.6 is eliminable as we can assume that Δ_ν^p is predense in $\mathscr{P}_\tau^{G_{D^i}}$.

The use of the Σ_f's to establish distributivity is presented in §3.4. As noted in Part I, the original condition p_0 is extended to $p_0 \geq p_1 \geq p_2 \geq \cdots$, where $p_{i+1} \in \Sigma_{f_i}^{s,p_i}$ and $\mathrm{Dom}(f_i)$ is thin. Note that conditions can be lengthened at a thin set of cardinals by clause (d) of the Extension Lemma 2.14. (Extendibility on a set which is not thin is impossible as it would contradict the Collapsing Lemma 2.12.)

The key assertions in the distributivity proof are the genericity claims: Lemmas 3.16 and 3.17. Note that it is not asserted that D' is $(\mathscr{P}_\gamma^{s'})^b$-generic over b; this is the lack of full genericity alluded to in Part I. Accordingly one cannot assert "$L_{\mu'}[A \cap \gamma, p_\gamma] \models \xi$ is a cardinal" in the proof of Lemma 3.22(b) on p. 98, and thus the need for the altered definition of S_α. The bulk of the verification that p is a condition is in checking clause (iv) in the definition of $\mathscr{P}_\tau^{p_\gamma}$ when $\mathscr{A}_{p_\gamma} \models \gamma$ is singular. Full use of Lemma 2.4 is required.

The proof of Theorem 3.2 is along similar lines. The singular case of uncountable cofinality presents one new problem: the analogue of Lemma 3.12 asserting the density of the Σ_f's is not immediately established due to the lack of a CUB subset of α which is disjoint from the domain of f. So instead Jensen works essentially with $f' = f \restriction (\mathrm{Dom}(f) - C_\alpha)$ and argues that this suffices. An important point is the Fact on p. 114 (used in the proof of Lemma 3.40.7), which asserts that $A \cap \gamma$ can recover the collapse of C_α. It is for the proof of this Fact that Jensen defined μ_s^1 so that $\mathscr{A}_s^1$ satisfies the Covering Lemma.

Chapter 3 ends with the distributivity proof for $\mathscr{P}_\tau$. There are no surprises here, given the techniques used in establishing distributivity for $\mathscr{P}_\tau^s$.

All the work is now complete for the proof of the Coding Theorem. The first section of Chapter 4 ties everything together, showing that $\mathscr{P} \Vdash \mathrm{ZFC} \wedge V = L[R]$ for some $R \subseteq \omega$, A is definable from R. Jensen uses the property "G $\mathscr{P}$-generic $\rightarrow G \cap \mathscr{P}^{\alpha^+}$ is $\mathscr{P}^{\alpha^+}$-generic over $\mathscr{A}_{\alpha^+}^1$" to show that $\Vdash_{\Sigma_0}$ is definable. (Actually this is unnecessary: Suppose τ is a successor cardinal, $M \prec \langle L_{\tau^{++}}[A], A \cap \tau^{++}\rangle, \tau \subseteq M$, $\mathrm{card}(M) = \tau$. Let $\pi: T \simeq M$, T transitive. Then $T[D_\tau] \models \mathscr{P}^{D_\tau}$ is τ^+-CC, whenever $D_\tau \subseteq [\tau, (\tau^+)^T)$ is $(\mathscr{P}_\tau)^T$-generic over T. Moreover any predense $X \subseteq \mathscr{P}^{D_\tau}$, $X \in T[D_\tau]$, is also predense on $\mathscr{P}^s$ for any $s \in S_\tau, D_\tau \subseteq s$, as in that case $X \in L_{(\tau^+)^T}[A] \prec L_{\tau^+}[A]$ and there exists $p \in G_{D_\tau}$ such that $L_{(\tau^+)^T}[A] \models X$ is predense in $\mathscr{P}^s$ for all $s \supseteq p_\tau$. Thus by the Truth Lemma we can write: $p \Vdash \phi$ iff for all T as above, rank$(\phi) < \tau, p \in L_\tau[A]$ we have that $p \Vdash \phi$ in $(\mathscr{P})^T$.)

§§4.2 and 4.3 verify the large cardinal preservation properties of $\mathscr{P}$. One must verify that the preliminary forcing which adds a class of Cohen sets does preserve these properties; the proof of that is given in the Appendix.

§4.4 provides an example of a class-generic real in $L[0^\#]$ which is not set-generic (over L). This is a very interesting construction and serves as the motivation for much of Chapter 5. The idea here is to build a real R which generically over L codes a class of Cohen sets. This is done by making R $\mathscr{P}$-generic over $\langle L, A\rangle$, where $A = \varnothing$ and $\mathscr{P}$ is the Jensen coding of A.

The construction here is very much in the style of the distributivity argument in that a sequence $p^0 \geq p^1 \geq \cdots$ is built (with the aid of $0^\#$) so that $p^{i+1} \in \Sigma^{p^i}_{f_i}$ for an appropriate f_i. This time at the limit stage, p^ω meets all constructible predense sets! Roughly speaking, the p^i's are not single conditions but amenable classes such that $p^i \restriction \alpha$ is a condition for each $\alpha \in$ Card. At stage $i+1$, p^{i+1} is chosen so that $p^{i+1} \restriction \nu^+$ "reduces" all predense sets $D \in \mathscr{P}^{p^i_{\nu^+}}$ which can be defined using ordinals $\leq \nu$ and an i-tuple of indiscernibles $> \nu$. An indiscernibility argument shows that the resulting p^{i+1} is amenable. As any predense $D \in \mathscr{P}^{p^\omega_{\nu^+}}$ can be defined in the above way for some i, it follows that p^ω meets all predense $D \in L$. Actually for technical reasons it is necessary to work first with the forcing $\mathscr{P}_\omega$ and then later code the resulting $\mathscr{P}_\omega$-generic $D_\omega \subseteq \omega_1$ into a real.

Chapter 5 further explores the relationship between $0^\#$ and class forcing over L. Three notions of "class-generic over L" are defined and Solovay's conjecture is resurrected in the form "If $0^\# \notin L[a], a \subseteq \omega$, then a is *class*-generic over L".

Beller shows that $0^\#$ itself is not "medium class-generic" over L, leaving "weak class-generic" as an open question. He then makes a study of the genericity properties of reals produced by the method of §4.4; note that the reals built there were only $\mathscr{P}$-generic for predense sets in L, not necessarily for L-definable classes. Beller shows that these reals can be made $\mathscr{P}$-generic for L-definable classes (though they need not be) and must be "medium class-generic".

On p. 168 Beller suggests an approach to the new version of Solovay's conjecture in the form of the question: Assuming $\sim 0^\#$, is every real a set-generic over some (proper) inner model $L[b]$ of $L[a]$? The answer is no, for there can exist a real a and an amenable $A \subseteq$ ORD such that $n \in a \leftrightarrow A_n = \{\alpha \mid \langle n, \alpha\rangle \in A\}$ is definable with parameters in $L[a]$ ($\leftrightarrow A_n$ is Δ_2-definable with parameters in $L[a]$). Then if $a \notin L[b]$ there must exist n such that $n \in a$ yet A_n is not Δ_2-definable with parameters in $L[b]$, and this property persists in set-generic extensions of $L[b]$.

§5.2 is devoted to answering a question of Sacks: If L_β is a countable model of ZF then is there an $a \subseteq \omega$ such that $L_\beta[a] \models$ ZF and $L_\alpha[\alpha] \not\models$ ZF for all $\alpha < \beta$? The answer is yes, and this was shown independently by R. David [82]. David [83] actually proved a stronger result: If A is a Σ_1 class of admissibles such that L-Card $\subseteq A$ then there is an generic real R such that the R-admissibles all belong to A. This is stronger, for it is not difficult to intitially extend L to $L[R_0]$ so that $L[R_0]$-Card $\subseteq \{\beta \mid L_\beta[R_0] \not\models \text{ZF}\}$ (see §5.2.3); then apply David's result over $L[R_0]$.

Finally, §5.3 explores the general question of when $0^\#$ can build generic objects for L-definable forcings. For example, if κ is an indiscernible then there exists a

Cohen subset of κ in $L[0^{\#}]$ iff $\mathrm{cof}(\kappa) = \omega$ in $L[0^{\#}]$. Other sufficient conditions for generic existence in $L[0^{\#}]$ are proved for certain types of class forcings as well.

Chapter 6 presents a very beautiful treatment of fine structure theory, providing a new and extremely natural construction of $\square$-sequences. (The considerations of this chapter are also key to the study of higher-gap morasses.) An important category of maps is defined on p. 203 between structures of the form $L_\nu[a]$, ν p.r. closed, $L_\nu[a] \models$ There is a largest uncountable cardinal, ν not a cardinal in $L[a]$. There the "$\square$-sequence" at $L_\nu[a]$ is just $\{\bigcup \mathrm{Range}(f \upharpoonright \nu') \mid f\colon L_{\nu'}[a'] \to L_\nu[a]$ belongs to the category$\}$ (see the top of p. 210). What follows is a very thorough analysis of these sequences, culminating in Lemma 6.30 which asserts that if an ordinal occurs in a $\square$-sequence then it arises as $\bigcup \mathrm{Range}\,(f \upharpoonright \nu')$ for a highly canonical f. This is the key to showing the coherence property for these sequences. The latter part of the chapter reworks the same theory but for models $L_\nu[a]$ where ν is singular in $L[a]$. This is needed to establish Lemma 2.4, asserting the existing of canonical singular sequences.

The "generic codes" are built in Chapter 7 with the aid of a "quasi-morass", a morass-like structure where the levels are not linear but instead form a tree. The construction of a quasi-morass from the $\square$-sequences of Chapter 6 is exactly like the usual construction of a morass from "coherent" $\square$-sequences.

Jensen's proof of the Coding Theorem without the hypothesis "$\sim 0^{\#}$" appears in Chapter 8. As is pointed out there, the key obstacle to overcome is the use of 2) in the definition of S_α to guarantee that μ_ξ is "large enough". The approach taken in Chapter 8 is to define $\mathscr{A}^1_s$ so as to contain the $\#$ of $(A \cap \alpha) \cup s$. This enables Jensen to obtain $b \in \mathscr{A}^1_{p_\gamma}$ at the key step in the distributivity argument, as $b \in L[A \cap \alpha, p_\gamma]$ and $\mathscr{A}^1_{p_\gamma}$ contains the $\#$ of $(A \cap \alpha) \cup p_\gamma$, it also contains $H(\gamma^+)^{L[A \cap \alpha, p_\gamma]}$ and hence b. Another approach is outlined in Part III below.

The book ends with Chapter 9 which contains results about coding "over $0^{\#}$". Thus Jensen starts with a model M and an iterable elementary embedding $j\colon M \to M$ and then codes M by a real a in such a way that j extends to $\tilde{j}\colon L[a] \to L[a]$. Thus $a^{\#}$ exists; in addition Jensen can control the canonical a-indiscernibles so that they agree eventually with the iteration points of j.

This completes our discussion of the book's exposition. As mentioned earlier there are a number of misprints; we list here some of the more confusing ones (line $-N$ refers to the Nth line from the bottom of the page).

Page 14: γ should be γ^+ (line -4).

Page 17: η should be δ (line -5), and δ should be δ' (line -3).

Page 32: μ^0_p should be α (fifth line of Remark 2).

Page 43: $\in$ should be $\notin$ in Remark 2, both times.

Page 47: α should be α^+ in the statement of Lemma 2.11.

Page 50: Lemma 2.14(d) should require that $\langle \xi_\gamma \mid \gamma \in X \rangle$ belong to $\mathscr{A}_s$.

Page 51: $X_{s\bar{\alpha}}$ should be $X^n_{s\bar{\alpha}}$ and X^n_s should be $X^n_{s\bar{\alpha}}$ (line 7), and $\bar{\alpha} \in X$ should be $\bar{\alpha} \notin X$ (line 13).

Page 54: $\beta' \in C_\beta$ should be $\beta' \notin C_\beta$ (line 3).

Page 56: $q(\beta)$ should be $q(\beta_i)$ (line -8), and δ^* should be δ^+ (line -7).

Page 61: In the line before Claim 2, $\upharpoonright$ should be $\wedge$.

Page 63: The second $\in$ in line -3 should be $\notin$.

Page 69: $s \in S^*_\alpha$ should be $\bar{s} \in S^*_{\bar{\alpha}}$ (line -12), and s_δ should be $\bar{s}_\delta$ (line -11)
Page 77: γ^+ should be δ^+ (line 1).
Page 81: α^*_i should be α^+_i (line -3).
Page 148: p^{iv}_γ should be $p^{iv}_{\gamma^+}$ in the definition of f_{iv}.
Page 159: Delete "L-definable" from the first line of the proof of Lemma 5.3.
Page 203: The domain of f should be $J^{\bar{A}}_{\bar{\beta}}$ in line -2.

Part III. Simplifications and an alternate approach. Jensen's proof can be simplified in a number of ways. Some of these have already been discussed in Part II. We now review these simplifications and also describe an approach different from Chapter 8 for eliminating the "$\sim 0^{\#}$" hypothesis.

a) The requirement that $\mathscr{A}^n_s$ obey the covering lemma (clause (c), p. 28) can be dropped. (This was pointed out on p. 257.) Jensen's uses of it were as follows: First, in defining the limit coding (pp. 42–43) Jensen uses singular sequences C_β from L, and then it is important to know (Remark (1) on p. 43) that $C_\beta \in \mathscr{A}_s$ if $s \in S_\beta, \mathscr{A}_s \models \beta$ is singular. Second, the requirement is used to prove the Fact on p. 114 which is needed to "recover the generic from p_γ" in the distributivity proof at singular cardinals of uncountable cofinality. The way around this is to, instead of the C_β's, use the D_p's from Lemma 2.4 (on p. 31). Thus when coding $p(\xi)$, $\mathscr{A}_p \models \beta$ singular one should use the CUB set $D_{p \restriction \Omega_\beta}$ given by the lemma. The necessary coherence properties are guaranteed by 2.4(iii). And, a different proof of the Fact is: Show by induction on $\xi < \alpha^*$ that $(A \cap \gamma) \cup p_\gamma$ can recover $D' \cap \xi$; then if $\Omega_{\alpha'} =$ least ξ such that $\mathscr{A}'_{D' \cap \xi} \models \alpha'$ is singular, it follows that $D'_{p'} \in \mathscr{A}'_{p'}$, where $p' =$ characteristic function of D' on $[\alpha', \xi)$ (and $\mathscr{A}'_{p'}$ and $D'_{p'}$ are obtained by relativizing the definitions of $\mathscr{A}_p$ and D_p to b). It follows that $D'_{p'} =$ collapse of D_p (where $p \subseteq p_{\lambda_\alpha}$, $|p| = \Omega_\alpha$) belongs to $L_{\mu'}[A \cap \gamma, p_\gamma]$. We are assuming here Claim (i) for $i < \rho$, but this can be proved by induction on i, using the above argument to handle the case of i being a limit ordinal.

b) Lemma 2.5 (the "generic codes") can be omitted. This lemma was used implicitly in the discussion of the Σ_f's, but as we pointed out in Part II this can be avoided by noticing that while Σ_f need no longer be open without Lemma 2.11 (which depends heavily on "generic codes") it still contains a dense open set of conditions. (Also the more minor use of Corollary 3.5.1 in the proof Lemma 4.1 is easily eliminated; see the discussion in Part II.)

Thus the preliminary step of adding the Cohen sets (p. 26), as well as the Appendix, are needed only for preserving large cardinal properties.

c) We now come to the most difficult clause to eliminate: The special requirement on elements of S_α, clause 2) on pp. 27–28 (together with (b) on p. 28). It is desirable to eliminate this clause as it is the key use of "$\sim 0^{\#}$".

Our proposed change is best motivated by reexamining the distributivity argument: Note that a key point was that p_γ coded only a "partially" generic $G \subseteq b$; this is because there was no way to show that $G \cap \alpha'$ generically coded $G \cap [\alpha', \alpha^*)$. We redefine the forcing $\mathscr{P}^s_\tau$ so as to guarantee at this point in the proof that G is fully generic over b.

To guarantee the full genericity of G we redefine $\mathscr{P}^s_\tau$ so that in some sense the Σ_f's are dense "by design" at limit cardinals. Specifically, let $\mathscr{P}^{<s} = \bigcup \{\mathscr{P}^{s \restriction \xi}_\tau \mid \xi < |s|\}$. Then $\mathscr{P}^{<s}_\tau \subseteq \mathscr{A}^0_s$. Also for $s \in S_\alpha$ define $v_s =$ the largest p.r. closed v such that

$L_\nu[A \cap \alpha, s] \models \mu_s^0$ is the largest cardinal (if there is such a $\nu > \mu_s^0$; otherwise $\nu_s = \mu_s^0$). Then we require the following for $p \in \mathscr{P}_\tau^s$, $|p| = |s|$: If $D \in \tilde{\mathscr{A}} = L_{\nu_s}[A \cap \alpha, s]$ is predense on $\mathscr{P}_\tau^{<s}$ then for some $\beta < \alpha$, $D^{(p)_\beta}$ is predense on $\mathscr{P}_\tau^{p_\beta}$ (where $D^{(p)_\beta} = \{q \in \mathscr{P}_\tau^{p_\beta} \mid q \cup (p)_\beta \leq$ some element of $D\}$). Once the τ-distributivity of $\mathscr{P}_\tau^{<s}$ in $\tilde{\mathscr{A}}_s$ is established, it follows that the above holds not just for some $\beta < \alpha$ but for all $\beta < \alpha$ (including limit β. This requirement also implies that $\mathscr{P}^D$ is β^+-CC for $D \in S_\beta^+$, β limit).

Now let us see what this does for us in the distributivity proof. As before we have that p_γ codes G and $G \cap \delta'$ is generic for each b-cardinal $\delta' < \alpha'$. But now the genericity of $G \cap \alpha^*$ follows as any predense $D \in L_{\alpha^+}[G \cap [\alpha', \alpha^*), A' \cap \alpha']$ has been "reduced" by the definition of the conditions; thus the fact that G meets D follows from the genericity of $G \cap \beta'$ for $\beta' < \alpha'$. Now we can argue that in fact G is (fully) generic over b.

Of course genericity over b is not enough to know that μ_{p_γ} is "large enough". We must also know the distributivity of $(\tilde{\mathscr{P}}_\gamma)^b$. Now in the the argument below we actually prove distributivity for $\mathscr{P}_\tau^{<s}$ in $\tilde{\mathscr{A}}_s$ for $s \in S_\alpha$, α limit; so if $|s| = \xi + 1$ we can assume distributivity of $\mathscr{P}_\tau^{<s \restriction \xi}$ and thus we have in the distributivity proof: $\mu_{p_\gamma} > \mu''$ = collapse of $\mu_{s \restriction \xi}^0$. This is sufficient to argue that μ_{p_γ} is large enough. If $|s| = \lambda$ limit then distributivity follows by induction.

The extendibility argument must now be different than Jensen's, as the "pre-density reduction" requirement on conditions adds a new difficulty.

To illustrate the method consider the special case of extending $p \in \mathscr{P}^s$, $|p| = \xi$, to $q \in \mathscr{P}^s$, $|q| = \xi + 1 = |s|$. We assume that $s \in S_\alpha, \mathscr{A}_s \models \alpha$ singular, $\mathrm{cof}(\alpha) > \omega$ and that $\mathscr{P}_\tau^{<s\restriction\xi}$ is τ-distributive in $\tilde{\mathscr{A}}_{s\restriction\xi}$ for $\tau \in \mathrm{Card} \cap \alpha$. The main part of the proof is to show that if $f \in \mathscr{A}_{s\restriction\xi} = \mathscr{A}_\xi$ is such that $f(\gamma) < \gamma^+$ for $\gamma < \alpha$ then p can be extended to $q \in \mathscr{P}^{s\restriction\xi}$ so that $|q_\gamma| \geq f(\gamma)$ for all γ.

This is done by "induction" on f. Specifically, for each $\beta < \mu_{s\restriction\xi} = \mu_\xi$ and $n \in \omega$ consider the $f = f_\beta^n$ defined by $f(\gamma) = H_\beta^n \cap \gamma^+$, where $H_\beta^n = \Sigma_n$ Skolem hull of γ inside $J_\beta[A \cap \alpha, s \restriction \xi]$. Then we show the above statement by induction on (β, n), ordered lexicographically, and simultaneously show the density of $\Sigma_{f_\beta^n}$.

When $n = 1$ we can use induction to successively extend p to $p_1 \geq p_2 \geq \cdots$ so that $q = \bigcup_i p_i$ is as desired. One must choose an "approximation" to $J_\beta[A \cap \alpha, s \restriction \xi]$ and correspondingly an approximation $f_{\beta_0}^1, f_{\beta_1}^1, \ldots$ (β limit) or $f_{\beta'}^1, f_{\beta'}^2, \ldots$ ($\beta = \beta' + 1$) to f; then p_{i+1} is chosen so as to "handle" the ith term f_i of this approximation and also to belong to Σ_{f_i}. The argument that p_λ is a condition for limit λ is like Jensen's, except now we have full genericity of the G coded by p_{λ_γ} and we use the distributivity of the collapse of $\mathscr{P}^{<s\restriction\xi}$.

When $n = k + 1 > 1$ the argument complexifies. Roughly, we want to argue as follows: Let $\gamma_0 < \gamma_1 < \cdots$ be the limit points of the canonical cofinal sequence $D_{s\restriction\Omega_\alpha}$ below α. Let $M_0 = J_{\bar\beta_0}[\bar A_0 \cap \bar\alpha_0, \bar s_0]$ be the transitive collapse of the Σ_k Skolem hull of γ_0 inside $J_\beta[A \cap \alpha, s \restriction \xi]$. We would like to use induction first to build q_0 in $(\mathscr{P}^{s_0})^{M_0}$ to "handle" $f_0 = (f_{\bar\beta_0}^{k+1})^{M_0}$ (Note that $(f_{\bar\beta_0}^{k+1})^{M_0}$ is not the same as the collapse of f_β^{k+1}, as M_0 is not a Σ_{k+1}-elementary submodel.) Of course q_0 cannot actually belong to M_0, but must instead be thought of as the direct limit of a $\Sigma_k(M_0)$-collection of compatible conditions in $(\mathscr{P}^{\bar\alpha_0})^{M_0}$, a "$\Sigma_k$-quasicondition" for short. Now let M_1 denote the model for γ_1 corresponding to M_0; then there is a natural embedding

$$\pi_{01}: M_0 \xrightarrow{\Sigma_\kappa} M_1.$$

Now consider $q_1' = \pi_{01}[q_0]$. We would like to use induction to extend q_1' to q_1 to handle $f_1 = (f_{\beta_1}^{k+1})^{M_1}$ and so that q_1 meets $\Sigma^{q_1'}_{\pi_{01}[f_0]} = \Sigma^{q_1'}_{f_1'}$. Then if we continue this process through all the γ_i's we finally get $q \leq p$, so that q handles f_β^{k+1}. (The fact that we have met the $\Sigma^{q_1'}_{f_1'}$ guarantees that q_λ is a quasicondition and that q is a condition.)

If $\mathscr{A}_s \models \alpha$ inaccessible, $|s| = \xi + 1$, then we can use the extendibility argument of Jensen, as we have the Collapsing Lemma 2.12.

If $|s| =$ limit then we can again follow Jensen's argument except now a second complication arises, caused by the restriction that we have put on the definition of $\mathscr{P}^s_\tau$. If $\mu_s^0 < \nu_s$ then we must arrange when we extend a given p to $q \in \mathscr{P}^s, |q| = |s|$, that q "reduces" all predense $D \in \tilde{\mathscr{A}}_s$. (If $\mu_s^0 = \nu_s$ this is trivial as extendibility for $\mathscr{P}^{<s}$ is enough to know that q reduces predense $D \in \tilde{\mathscr{A}}_s = \mathscr{A}_s^0$.) The argument here depends on the nature of ν_s. Note that we can think of $\mathscr{A}_s^0$ as corresponding to a point in Jensen's quasi-morass (see Chapter 7). Thus we induct on the quasi-morass, following cases much as Jensen does in his "generic codes" construction. However, there is no need for $\diamondsuit$. This construction implicitly contains a proof of the distributivity of $\mathscr{P}^{<s}$ in $\tilde{\mathscr{A}}_s$ as well.

This ends the outline of an alternate approach to the Coding Theorem. The details will appear in our forthcoming paper entitled "Strong Coding".

This also ends this guide which we hope will be of value to current and future readers of ***Coding the Universe***. To be sure, a large investment is required to appreciate the depth of Jensen's theorem, but this is more than compensated for by the beauty of the mathematics.

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