Gödel's Achievements and their Significance for Modern Mathematics

Sy D. Friedman, Institut für Formale Logik, Universität Wien

As we see in the Gödel Lectures, by applying logical reasoning to precisely defined concepts, mathematics leads to remarkable results, both of theoretical and practical importance. But what is the nature of mathematical reasoning? What does it mean for a statement of mathematics to "follow logically" from others? Is it possible that the techniques of mathematics are powerful enough to answer all questions which can be formulated in mathematical terms? And can we be sure that mathematics will not lead us to contradictions?

Surprisingly, Gödel showed that these questions about mathematical reasoning, sometimes called questions of *metamathematics*, can be formulated as questions *within* mathematics itself and be given definitive answers!

To give some idea of how Gödel turned metamathematics into mathematics, consider Aristotelian Logic. Here we have letters A,B,C ..., which stand for statements which can be either true or false, and we have logical connectives, such as AND, OR, NOT and IMPLIES. If someone says to you "If you give me 20 Euro I will not invite your girlfriend for coffee", he is saying A IMPLIES (NOT B), a statement of Aritotelian Logic, where A = "You give me 20 Euro" and B = "I will invite your girlfriend for coffee". Is your friend telling the truth? Well, either you give your friend 20 Euro or you do not, that is, either A is true or false. Similarly, either B is true or false. There are therefore a total of four possible ways of assigning a value T (true) or F (false) to A and B. In only one case is your friend not telling the truth, namely, the case where both A and B are true. So we see that your friend's statement follows logically from the statement NOT (A AND B). In a similar way, we can determine in finitely-many steps if any particular statement of Aristotelian Logic follows logically from finitely-many other statements, by listing all possible ways of assigning T or F to the letters A,B,C, ... which makes the other statements all true, and checking if the given statement in each case also comes out true.

Mathematics requires more than Aristotelian Logic. Indeed even a simple statement like "Everybody loves somebody sometime" requires more. We can express this statement as

ALL p EXISTS q EXISTS t (PERSON(p) AND PERSON(q) AND TIME(t) AND LOVE(p,q,t)) where

ALL = "for all", EXISTS = "there exists", PERSON(x) = "x is a person", TIME(t) = "t is a point in time" and LOVE(p,q,t) = "p loves q at time t".

This kind of logic is called Predicate Logic, and is sufficient to express not only the statement above, but in fact any statement of mathematics. There is also a definition of "follows logically" for Predicate Logic, which instead of simple truth values T, F makes use of interpretations or *models*.

We come now to the key question, crucial for our understanding of mathematical proof: *As for Aristotelian Logic, can we determine if a particular statement of Predicate Logic follows logically from other statements?* If the answer is YES, then this means that with a single method or *algorithm*, we can decide whether or not an arbitrary mathematical statement follows from any given set of axioms. Now there is a special set of axioms for mathematics, the Zermelo-Fraenkel axioms for set theory, which are sufficient to represent the techniques used in mathematics. Thus we may have reduced mathematics to simple calculation: To determine whether or not the Goldbach Conjecture can be proved, we simply apply our universal algorithm to determine whether or not it follows logically from the Zermelo-Fraenkel axioms!

Gödel showed that the answer to our question is "almost" YES, but in fact NO. The idea is the following: Statements of mathematics can be expressed in Predicate Logic, where they can be expressed as a finite sequence of symbols. By coding each symbol by a natural number $0, 1, 2, ...$ we can therefore think of each statement as a finite sequence of natural numbers. And finite sequences of natural numbers can then be coded by single natural numbers: for example, the sequence $(3, 2, 6, 1, 2)$ can be coded as $2^3 3^2 5^6 7^1 11^2$. The result is that each statement of mathematics now has a code number or *Gödel number* in the natural numbers (turning metamathematics into mathematics). A consequence of Gödel's work is that there is a precise definition of what it means for a set of natural numbers to be *recursive*, which means that with an algorithm we can test whether or not a given natural number belongs to the set. Similarly, there is a precise definition of *recursively enumerable*, which means that the elements of the set can be listed by some algorithm. Gödel's fundamental result is this: Suppose that S is a system of axioms, like Zermelo-Fraenkel, which is sufficient to carry out the calculations of elementary arithmetic. Then the set of Gödel numbers of statements of mathematics which follow from the axioms of S is recursively enumerable but not recursive. Applying this to the Zermelo-Fraenkel system, we see that there is an algorithm to list the theorems of mathematics (Gödel's Completeness Theorem), but none to decide whether or not a given statement is provable within mathematics.

This work also implies that mathematics is fundamentally *incomplete*, in the sense that there will always be statements of mathematics which we cannot prove or disprove (Gödel's First Incompleteness Theorem). Otherwise there would be an algorithm to decide whether or not a given statement A is provable within mathematics, as we can list the theorems of mathematics by an algorithm and wait until either the given statement A or its negation (NOT A) appears in this list; in the former case A is a theorem and the latter case it is not. A finer analysis of Gödel's proof results in his Second Incompleteness Theorem, which says that the metamathematical statement that mathematics is free of contradiction, which by Gödel's work can also be expressed mathematically, is not a theorem of mathematics.

Gödel's best-known work, on incompleteness, is negative in character: it tells us what mathematics *cannot* do. Despite its unparalleled significance for the history and foundations of mathematics, its impact on modern mathematics is limited. Indeed, as the Gödel Lectures show, the phenomenal progress of mathematics has not been significantly hindered by Gödel's incompleteness results.

Of greater importance for modern mathematics are Gödel's positive results, expressed by his Completeness Theorem and his later work in set theory. The Completeness Theorem shows that a notion which is on the surface highly abstract, the notion of logical implication, is in fact captured by the much more concrete notion of recursively enumerable set. This is the prototype of a wide variety of completeness results throughout mathematics. Gödel's work in set theory is surely his greatest mathematical contribution, and its impact continues to be felt today. After demonstrating the fundamental incompleteness of mathematics, Gödel provided us with an important proposal for how to overcome it: First, he isolated a particular interpretation of the Zermelo-Fraenkel axioms for mathematics, called the universe L of constructible sets, and provided techniques for determining what is true in this universe. Then he proposed the addition of new axioms of "large infinity" to the axioms of mathematics, suggesting that these axioms may resolve many questions that are not answered otherwise. Subsequent work has verified the correctness of Gödel's proposal, as it has developed universes similar to Gödel's universe of constructible sets which satisfy his axioms of large infinity, and which therefore go a long way toward resolving the failures of completeness exhibited by the usual axioms for mathematics. There is even now the hope that the axioms of large infinity, together with the assumption that the universe of sets resembles Gödel's constructible universe, will be sufficient to answer all meaningful questions of modern mathematics.

Professor Friedman is O.-Univ. Professor für Mathematische Logik at the University of Vienna and director of the Institut für Formale Logik. He moved to Vienna from MIT in 1999, where he was Full Professor of Mathematics, to revive the important Vienna tradition in mathematical logic created by Gödel in the 1930's. His institute offers a wide range of courses in mathematical logic, from Gödel's work through the modern developments in set theory, and has become one of the leading European centers in this field.