Genericity and Large Cardinals

Sy D. Friedman

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A result of great significance in the theory of forcing is the following.

Theorem 1 *(Corollary to Jensen's Coding Theorem) There is an L-definable class forcing P such that if G is P-generic over L then:*

(a) $\langle L[G], G\rangle$ is a model of ZFC, and cofinalities are the same in L as in $L[G]$.
(b) Some real in $L[G]$ is not *set-generic over L.*

A natural question to ask is whether this result has an analogue in the context of large cardinals. The purpose of this article is to provide such an analogue, taking into account difficulties raised by the existence of Woodin cardinals.

To describe the latter difficulties we consider the forcing P, described as follows. Let δ be inaccessible and consider the language $\mathcal{L}(\delta)$:
(a) $n \in \mathbf{R}$ belongs to $\mathcal{L}(\delta)$, where $n \in \omega$ and $\mathbf{R}$ denotes a real.
(b) $\varphi \in \mathcal{L}(\delta) \to \sim \varphi \in \mathcal{L}(\delta)$.
(c) $\Phi \subseteq \mathcal{L}(\delta)$, Card $\Phi < \delta \to \wedge\Phi \in \mathcal{L}(\delta)$.

Of course $\wedge\Phi$ is to be interpreted as the conjunction of the sentences in Φ. A set of sentences $\Phi \subseteq \mathcal{L}(\delta)$ is *consistent* iff in some (set-generic) extension of V, some real R satisfies each sentence in Φ. A single sentence $\varphi \in \mathcal{L}(\delta)$ is *consistent* iff $\{\varphi\}$ is consistent. We endow $\mathcal{L}(\delta)$ with the ordering: $\varphi \leq \psi$ iff $\wedge\{\varphi, \sim \psi\}$ is not consistent. Then P is the pre-ordering $(\mathcal{L}^+(\delta), \leq)$ where $\mathcal{L}^+(\delta) = \{\varphi \in \mathcal{L}(\delta) \mid \varphi \text{ is consistent}\}$.

In a weak sense, every real outside of V is P-generic over V: Let R be a real and let $G(R)$ be $\{\varphi \in \mathcal{L}^+(\delta) \mid R \text{ satisfies } \varphi\}$.

Lemma 2 *(a) φ, $\psi \in G(R) \to \varphi$, ψ are compatible in $P = \langle \mathcal{L}^+(\delta), \leq \rangle$.*
(b) $\varphi \leq \psi$, $\varphi \in G(R) \to \psi \in G(R)$.
(c) Suppose that $A \subseteq \mathcal{L}^+(\delta)$ is predense (i.e., every $\varphi \in \mathcal{L}^+(\delta)$ is compatible with some element of A). If Card $A < \delta$ then $G(R) \cap A \neq \emptyset$.

Proof. (a) and (b) are clear. For (c), note that as Card $A < \delta$ we may form the sentence $\varphi = \wedge\{\sim \psi \mid \psi \in A\} \in \mathcal{L}(\delta)$. If $G(R) \cap A = \emptyset$ then R satisfies φ and hence φ is an element of $\mathcal{L}^+(\delta)$ incompatible with each element of A. This contradicts our assumption that A is predense. $\square$

Of course full P-genericity over V would require that (c) hold without the assumption Card $A < \delta$. If P is δ-cc (i.e., antichains in P have cardinality $< \delta$) then we do achieve full P-genericity, as this cardinality assumption becomes superfluous. We next show how to modify P to a δ-cc forcing, following an idea of Woodin.

Definition. Suppose that $A \subseteq V_\delta$ and $\kappa < \delta$. Then κ is *A-strong below δ* iff for all $\alpha < \delta$ there is an elementary embedding $j : V \to M$ with critical point κ such that $\alpha < j(\kappa)$ and $A \cap V_\alpha = j(A) \cap V_\alpha$.

For any $A \subseteq V_\delta$ in V and $\kappa < \delta$ let $T(\kappa, A)$ consist of all sentences

$$\wedge\{\sim \varphi \mid \varphi \in A \cap V_\kappa\} \to \wedge\{\sim \varphi \mid \varphi \in A \cap V_\alpha\}$$

as α varies over the ordinals less than δ. Now suppose that R is a real (outside of V) and κ is A-strong below δ in $V[R]$. Then $T(\kappa, A)$ is contained in $G(R)$ and hence $T(\kappa, A)$ is consistent. More generally, suppose that R *preserves A-strength below δ over V* for every $A \subseteq V_\delta$ in V, in the sense that whenever $\kappa < \delta$ and κ is A-strong below δ in V, then κ is A-strong below δ in $V[R]$. Then $T = \bigcup\{T(\kappa, A) \mid A \subseteq V_\delta,\ A \in V,\ \kappa \text{ is } A\text{-strong below } \delta\}$ is contained in $G(R)$ and hence T is consistent. Let $P_T = \langle \mathcal{L}^+_T(\delta), \leq_T \rangle$ where $\mathcal{L}^+_T(\delta) = \{\varphi \in \mathcal{L}(\delta) \mid T \cup \{\varphi\} \text{ is consistent}\}$ and $\varphi \leq_T \psi$ iff $T \cup \{\varphi, \sim \psi\}$ is not consistent.

Claim. Suppose that for every $A \subseteq V_\delta$ there is $\kappa < \delta$ such that κ is A-strong below δ. Then P_T is δ-cc.

Proof. Suppose that $A \subseteq \mathcal{L}^+_T(\delta)$ is predense in P_T and choose $\kappa < \delta$, κ A-strong below δ. We assert that $A \cap V_\kappa$ is predense in P_T: If not, then some $\psi \in P_T$ is P_T-incompatible with each $\varphi \in A \cap V_\kappa$; but as $T(\kappa, A) \subseteq T$, we then have that ψ is P_T-incompatible with every $\varphi \in A$, contradicting the predensity of A. It follows that P_T has no antichain of cardinality δ. $\Box$

Definition. δ is a *Woodin cardinal* if for every $A \subseteq V_\delta$ there is $\kappa < \delta$ such that κ is A-strong below δ.

We have shown:

Theorem 3 *(Woodin) Suppose that R is a real, V is an inner model, δ is a Woodin cardinal in V and R preserves A-strength below δ over V for every $A \subseteq V_\delta$ in V. Then R is set-generic over V.*

The previous result would appear to raise a serious obstacle to extending Jensen's Theorem past the level of a Woodin cardinal. Fortunately, the notion of Woodin cardinal has an alternative definition, which can be used to overcome this obstacle. Let C be a CUB subset of κ and for α in C, let α^+_C denote the C-successor to α. We say that κ is *C-strong* iff there is an elementary embedding $j : V \to M$ with critical point κ such that all subsets of $\kappa^+_{j(C)}$ belong to M. Then δ is Woodin iff for every CUB subset C of δ there is a $\kappa < \delta$ in Lim C which is $C \cap \kappa$-strong. (See [3].) We can additionally require that some j witnessing the $C \cap \kappa$-strength of κ satisfy $\kappa^+_{j(C)} < \delta$, and that the set of such κ be stationary in δ.

Using this second definition of Woodinness we establish the following large-cardinal analogue of Theorem 1.

Theorem 4 *Suppose that V is an "L-like" model. There is a V-definable class-forcing P such that if G is P-generic over V then:*
(a) $\langle V[G], G\rangle$ is a model of ZFC, and cofinalities are the same in V as in $V[G]$.
(b) If κ is Woodin in V then κ is Woodin in $V[G]$.
(c) Some real in $V[G]$ is not *set-generic over V.*

This result is proved by constructing a class-forcing which "preserves" a witness in V to the second definition of Woodinness. Witnesses to the first definition of Woodinness in $V[G]$ cannot be definable in V, by Theorem 3.

We next clarify the above hypothesis on V.

Condensation, $\square$ and Extenders

"L-like" models obey suitable forms of Gödel's Condensation Principle and Jensen's $\square$ Principle. As essentially the only known examples of such models are in fact models built from extenders, we begin with a definition of *good extender model*.

An inner model M is *rigid* if there is no elementary embedding from M to itself, other than the identity. Extender models arise naturally when one attempts to construct an "L-like" rigid model.

Suppose that L is not rigid and $j : L \to L$ (i.e., j is a nontrivial elementary embedding from L to L). We may hope to move one step closer to rigidity by replacing L by $L[j \restriction L_\alpha]$, where α is least so that $j \restriction L_\alpha \notin L$. A useful fact is that α is the ordinal $(\kappa^+)^L$, where κ is the critical point of j.

The function $j \restriction L_\alpha$, where $\alpha = (\kappa^+)^L$ is called the *extender derived from* j. Thus one hopes to successively add extenders until the process converges upon a model that is either rigid or contains the extender derived from some embedding of it to itself. In the latter case this model has a "superstrong cardinal", a property much stronger than Woodinness.

The models that arise in this construction are called *extender models*.

Definition. An *extender sequence* is a sequence $E = \langle E_\nu \mid \nu \in \mathrm{ORD}\rangle$ such that for all ν, E_ν is either empty or:

$$E_\nu : L^E_{\kappa^+} \to L^E_\nu$$

is cofinal and Σ_1-elementary, where κ is the critical point of E_ν, κ^+ denotes κ^+ of L^E_ν and for any η, L^E_η denotes the structure $\langle L_\eta[E], E \restriction \eta\rangle$.

Definition. An *extender model* is a model $L^E = \langle L[E], E\rangle$ where E is an extender sequence. An *initial segment* of L^E is a structure of the form $L^E_{\leq\alpha} = \langle L^E_\alpha, E_\alpha\rangle$, $\alpha \in \mathrm{ORD}$.

We cannot expect extender models to obey the following analogue of the strong form of condensation which holds in L: If H is Σ_1-elementary in $L^E_{\leq\alpha}$ then H is isomorphic to an initial segment of L^E. Indeed this fails whenever L^E contains a measurable cardinal. However one can have the weaker form of condensation stated next. For $0 < n < \omega$, the Σ_n *projectum* of $L^E_{\leq\alpha}$ denotes the least ordinal γ such that for some $x \in L^E_{\leq\alpha}$, $L^E_{\leq\alpha}$ is the Σ_n Skolem hull in itself of $\gamma \cup \{x\}$.

Condensation. (a) Suppose that κ is a cardinal of L^E, κ is the Σ_1 projectum of $L^E_{\leq\alpha}$, x belongs to L^E_α and $L^E_{\leq\alpha}$ is the Σ_1 Skolem hull in itself of $\kappa\cup\{x\}$. For $\gamma < \kappa$ let $H(\gamma, x)$ denote the Σ_1 Skolem hull of $\gamma\cup\{x\}$ in $L^E_{\leq\alpha}$ and $\overline{H}(\gamma, x)$ its transitive collapse. Then for sufficiently large $\gamma < \kappa$, if γ is the Σ_1 projectum of $\overline{H}(\gamma, x)$ then $\overline{H}(\gamma, x)$ is an initial segment of L^E. (b) If $\gamma < \kappa$ are cardinals of L^E, $0 < n \in \omega$ and H is the Σ_n Skolem hull of γ in $L^E_{\leq\kappa}$ then the transitive collapse of H is an initial segment of L^E.

For an uncountable L^E-cardinal κ, the set of γ less than κ such that γ equals the Σ_1 projectum of $\overline{H}(\gamma, x)$ is a CUB subset of κ (containing all uncountable cardinals less than κ). Thus Condensation implies GCH via the *Gödel property*: If $x \subseteq \kappa$ and x belongs to L^E, then x belongs to L^E_α for some α less than κ^+ of L^E.

Good extender models also obey a suitable form of Jensen's $\Box$ Principle. A *good* $\Box$*-sequence at singular cardinals* for L^E is an L^E-definable sequence $\langle C_\alpha \mid \alpha$ a singular cardinal of $L^E\rangle$ such that for each singular cardinal α of L^E:

1. C_α is CUB in α of ordertype less than α.
2. If $\bar\alpha$ is a limit point of C_α then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.
3. C_α is definable over $L^E_{\leq\beta(\alpha)}$ via a definition independent of α, where $\beta(\alpha)$ is the least ordinal β such that α is singular in $L^E_{\beta+1}$.
4. Suppose that $\beta \leq \beta(\alpha)$, $x \in L^E_\beta$ and $L^E_{\leq\beta}$ is the Σ_1 Skolem hull in itself of $\alpha \cup \{x\}$. If unboundedly many $\bar\alpha < \alpha$ satisfy $\bar\alpha = \alpha \cap$ the Σ_1 Skolem hull of $\bar\alpha \cup \{x\}$ in $L^E_{\leq\beta}$ then sufficiently large elements of C_α have this property.

In summary, an extender model is *good* iff it obeys Condensation and possesses a good $\Box$-sequence at singular cardinals.

By combining work of [2], [4] and [5], we have:

Fact. If there is a Woodin limit of Woodin cardinals then there is a good extender model with a Woodin limit of Woodin cardinals.

An *L-like model* is a model with the above goodness properties, but which is not necessarily built from extenders. Such a model is of the form $L^A = L[\langle A_\alpha \mid \alpha \in \text{ORD}\rangle]$, where the structure $L^A_{\leq\alpha} = \langle L^A_\alpha, A_\alpha\rangle$ is amenable for each α, such that Condensation and $\Box$ at Singulars hold precisely as above, with E replaced everywhere by A. By relativising the above Fact to a real R, we obtain L-like models containing R with Woodin cardinals.

Suppose that L^A is L-like. The *extender E derived from the embedding* $j : L^A \to M$ is the restriction of j to L^A_α, where $\alpha = \kappa^+$ of L^A and κ is the critical point of j. We also write $\kappa(E) = \kappa$. An *extender in* L^A is an extender derived from some embedding $L^A \to M$ which belongs to L^A. Let E be an extender in L^A as above and let ν be the supremum of the range of E on α. Suppose that $\kappa^+ \le \sigma \le E(\kappa)$. Then we can form a new extender $F = E \downarrow \sigma$ as follows: Let $\pi : H(\sigma) \simeq L^{\bar{A}}_{\bar{\nu}}$ where $H(\sigma)$ is the Σ_1 Skolem hull of $\sigma \cup \text{Range}\,(E)$ in L^A_ν. Then $F : L^A_\alpha \to L^{\bar{A}}_{\bar{\nu}}$ is the composition πE. Clearly F is cofinal and Σ_1 elementary, and κ is the critical point of F. The *true length* of E is the least σ such that $E \downarrow \sigma = E$. Note that if $\sigma = E(\kappa)$ then $E \downarrow \sigma = E$, so true length is always defined. For a set T of extenders, we define $T \downarrow \sigma$ to be the set of all $E \downarrow \sigma$, $E \in T$.

If E is an extender in L^A derived from some $j : L^A \to M$, then there is a canonical extension E^* of E to L^A (possibly differing from j): Let κ be the critical point of E and consider $U = \{(f, a) \mid f : L^A_\kappa \to L^A,\ a \in L^A_{E(\kappa)}\}$. Set $(f, a) =^* (g, b)$ iff $(a, b) \in E(\{(u, v) \mid f(u) = g(v)\})$ and $(f, a) \in^* (g, b)$ iff $(a, b) \in E(\{(u, v) \mid f(u) \in g(v)\})$. Then $\text{Ult}(L^A, E) = (U/ =^*, \in^*)$ is well-founded and set-like, so we identify it with its transitive collapse. The desired extension E^* of E is defined by $E^*(x) = [f_x, 0]$ where f_x is the constant function with value x and $[f, a]$ denotes the $=^*$ equivalence class of (f, a). A useful fact is: $[f, a] = E^*(f)(a)$. In the sequel we shall identify E with E^*, and therefore write $E(x)$ instead of $E^*(x)$ for arbitrary elements x of L^A.

Class Forcing in the Presence of Woodin Cardinals

We prove Theorem 4. Suppose that $V = L^A$ is an L-like model and fix a good $\Box$-sequence at singular cardinals $\langle C_\alpha \mid \alpha \text{ a singular cardinal}\rangle$.

For a cardinal α we define an *α-extender* to be an extender E (derived from some embedding $V \to M$) of true length α such that all bounded subsets of α belong to $\text{Ult}_E = \text{Ult}(V, E)$, and A agrees with $E[A]$ below α (where $E[A]$ denotes $\bigcup\{E(A \restriction \alpha) \mid \alpha \in \text{ORD}\}$). We write $\alpha(E) = \alpha$. An extender is *tight* iff it is an α-extender for some cardinal α and its critical point is not Woodin.

Steering Ordinals. Fix an uncountable cardinal α. By induction on η in $[\alpha,\alpha^+)$ we define ordinals $\mu^{<\eta}$, μ^η_k, $k\in\omega$ and μ^η as follows: $\mu^{<\alpha}=\alpha$, and for $\eta>\alpha$, $\mu^{<\eta}$ is the supremum of the $\mu^{\eta'}$, $\alpha\le\eta'<\eta$. We define μ^η_0 to be the least μ greater than $\mu^{<\eta}$ such that μ is a multiple of α and α is the largest cardinal of L^A_μ. $\mu^\eta_k=\mu^\eta_0+\alpha\cdot k$ for $k\in\omega$ and $\mu^\eta=\mu^\eta_0+\alpha\cdot\omega$.

Canonical CUB Sets. Suppose that α is an uncountable limit cardinal, $\eta\in[\alpha,\alpha^+)$ is a multiple of α and $k\in\omega$. We define the canonical CUB subset $C^{\eta,k}_\alpha$ of α to be $\{\bar\alpha<\alpha\mid\bar\alpha=\alpha\cap$ the Σ_1 Skolem hull of $\bar\alpha\cup\{\eta\}$ in $L^A_{\le\mu^\eta_k}\}$ if this set is unbounded in α: otherwise we take $C^{\eta,k}_\alpha$ to be C_α. The canonical CUB subsets of α carry the natural ordering: $C^{\eta_0,k_0}_\alpha\le C^{\eta_1,k_1}_\alpha$ iff $\eta_0<\eta_1$ or ($\eta_0=\eta_1$ and $k_0\le k_1$). If this holds, then a final segment of C^{η_1,k_1}_α is contained in C^{η_0,k_0}_α (using property 4 of the good $\Box$-sequence $\langle C_\alpha\mid\alpha$ a singular cardinal$\rangle$ when C^{η_1,k_1}_α equals C_α).

We consider the following class T of tight extenders. By induction on the uncountable cardinal α define E_α and D_α as follows. For α a limit cardinal, D_α is the least canonical CUB subset D of α, if it exists, such that $D\ne C_\alpha$ and for some $\alpha_0<\alpha$, no E_β, $\alpha_0<\beta<\alpha$ witnesses the $D\cap\kappa(E_\beta)$-strength of $\kappa(E_\beta)$. For α a successor cardinal, E_α is the L^A-least tight extender E, if it exists, such that $\alpha(E)=\alpha$ and:
1. $\kappa(E)<\beta<\alpha$, E_β defined $\to\kappa(E)<\kappa(E_\beta)$.
2. E witnesses the $D_{\kappa(E)}$-strength of $\kappa(E)$.

Let T be the class of all E_α, α an uncountable successor cardinal, as defined above. Then we claim that the Woodinness of each Woodin cardinal δ is witnessed by extenders in T (via the second definition of Woodinness). If not, then let C be the least canonical CUB subset of δ such that for some $\delta_0<\delta$, no E_β, $\delta_0<\beta<\delta$ witnesses the $C\cap\kappa(E_\beta)$-strength of $\kappa(E_\beta)$. Clearly C exists by the failure of T to witness the Woodinness of δ and the fact that any CUB subset of δ contains a final segment of a canonical one. By Condensation (a), $\{\alpha\mid\alpha<\delta,\ \alpha<\beta<\delta\to\kappa(E_\beta)\ge\alpha$ (when E_β is defined) and $C\cap\alpha=D_\alpha\}$ contains a CUB set. As δ is Woodin, this CUB set contains a $\kappa>\delta_0$ which is $C\cap\kappa$-strong (via an extender preserving A), and clearly the least such κ is not Woodin (as witnessed by $C \cap \kappa$). Moreover $\kappa < \beta < \delta \to \kappa \neq \kappa(E_\beta)$ (when E_β is defined), else E_β witnesses the D_κ-strength of $\kappa = \kappa(E_\beta)$, contrary to the choice of C. Let E be tight and witness the $C \cap \kappa$-strength of κ. Then E is a candidate for $E_{\alpha(E)}$, which therefore is defined and witnesses the $C \cap \kappa$-strength of κ, contradicting the choice of C and δ_0.

Note that T is *uniform* in the sense that $E \in T \to T$ and $E[T]$ ($= T$ as defined in Ult_E) have the same extenders of true length less than $\alpha(E)$, and is *nested* in the sense that $E_0 \neq E_1$ in T, $\kappa(E_0) \leq \kappa(E_1) \to$ either $\kappa(E_0) < \alpha(E_0) < \kappa(E_1) < \alpha(E_1)$ or $\kappa(E_0) < \kappa(E_1) < \alpha(E_1) < \alpha(E_0)$.

If α is a cardinal then α is *overlapped* by the tight extender E iff $\kappa(E) < \alpha < \alpha(E)$. For each α there are at most finitely many $E \in T$ which overlap α, as T is nested. If E overlaps α then we define α^+_E to be $\bigcup\{E(f)(\alpha) \mid f : \kappa(E) \to \kappa(E),\ f(\gamma) < \gamma^+ \text{ for each } \gamma < \kappa(E)\}$, an ordinal less than α^+, and $\alpha^*_E = \bigcup\{E(f)(\alpha) \mid f : \kappa(E) \to L^A_{\kappa(E)},\ f(\gamma) \text{ a subset of } [\gamma^+, \gamma^{++}) \text{ of cardinality} \leq \gamma \text{ for each } \gamma < \kappa(E)\}$, a subset of α^{++} of cardinality α. We say that α is *overlapped by T* iff α is overlapped by some $E \in T$. (Note: Although α^+_C was already defined for a CUB set C, there is little danger of confusion with the notation α^+_E for an extender E.)

For α an uncountable limit cardinal, let C^T_α denote the set of cardinals $\bar\alpha$ less than α which are overlapped by the same extenders in T as α; using the nestedness of T, this is a CUB subset of α whose successor elements are successor cardinals. Note that as $T \cap L^A_\alpha$ is definable over L^A_α, a final segment of $C^{\alpha,0}_\alpha$ is contained in C^T_α, unless α is singular and $C^{\alpha,0}_\alpha = C_\alpha$. In the latter case we redefine C_α by replacing the current C_α by $C_\alpha \cap C^T_\alpha$, if this is unbounded in α, and otherwise by the L^A-least unbounded subset of C^T_α of ordertype ω consisting of successor cardinals. This new definition of C_α does not alter our above definition of T, satisfies the goodness properties 1-3 and has the additional property that a final segment of C_α is contained in C^T_α for each singular cardinal α. (Goodness property 4 is not needed in the special case $C^{\alpha,0}_\alpha = C_\alpha$.)

Coding Apparatus. Fix an uncountable cardinal α. For $\eta \in [\alpha, \alpha^+)$ the coding structure $\mathcal{A}^\eta$ is defined to be $L^A_{\leq\mu^{\eta_0+\eta}}$, where η_0 is least so that E_α, if defined, belongs to $L^A_{\mu^{\eta_0}}$. For $\eta \in [\alpha, \alpha^+)$ a multiple of α and $i < \alpha$ set $H^\eta(i)$ = the Σ_1 Skolem Hull of $i \cup \{\eta\}$ in $\mathcal{A}^\eta$ and $f^\eta(i)$ = the ordertype of $H^\eta(i) \cap \mathrm{ORD}$. For α a successor cardinal: $B^\eta = \{i < \alpha \mid i = H^\eta(i) \cap \alpha\}$, b^η = Range $f^\eta \restriction B^\eta$ and for $\alpha \leq \eta = \bar{\eta} + \delta$, where $\bar{\eta}$ is a multiple of α and $\delta < \alpha$, $b^\eta = \{\gamma + \delta \mid \gamma \in b^{\bar{\eta}}\}$.

For an uncountable limit cardinal α, $\eta \in [\alpha, \alpha^+)$ a multiple of α and $k \in \omega$ we define the coding domain $B^{\eta,k}_\alpha$: If D_α is of the form $C^{\eta_0,k_0}_\alpha < C^{\eta,k}_\alpha$ then $B^{\eta,k}_\alpha$ consists of all $(\bar{\alpha}^+_{D_\alpha})^+_{C^T_\alpha}$, $\bar{\alpha} \in C^{\eta,k}_\alpha$. Otherwise $B^{\eta,k}_\alpha$ consists of all $\bar{\alpha}^+_{C^T_\alpha}$, $\bar{\alpha} \in C^{\eta,k}_\alpha$. Using the fact that D_α is canonical, it follows that if $\eta_0 < \eta_1$ or ($\eta_0 = \eta_1$ and $k_0 < k_1$) then a final segment of B^{η_1,k_1}_α is contained in B^{η_0,k_0}_α.

Strings. Strings at an infinite cardinal α are functions $s : |s| \to 2$, where $\alpha \leq |s| < \alpha^+$, $|s|$ is a multiple of α, s belongs to $\mathcal{A}^{|s|}$ and for each η, $\alpha \leq \eta < |s|$, either $s \restriction \eta$ belongs to $\mathcal{A}^\eta$ or $s(\eta) = 0$. We write μ^s, $\mu^{<s}$, $\mathcal{A}^s$, $\mathcal{A}^{<s}$, ... for μ^η, $\mu^{<\eta}$, $\mathcal{A}^\eta$, $\mathcal{A}^{<\eta}$, ... where $\eta = |s|$. Let S_α denote the collection of strings at α.

A Partition of the Ordinals. Let B, C and D denote the classes of ordinals congruent to 0, 1 and 2 mod 3, respectively. For any ordinal α and $X = B$, C or D we write α^X for the α-th element of X, when X is listed in increasing order. For S a set of ordinals, $S^X = \{\alpha^X \mid \alpha \in S\}$.

The Successor Coding. Suppose α is an infinite cardinal, $s \in S_{\alpha^+}$. R^s consists of all pairs (t, t^*) where t belongs to S_α and t^* is a subset of $[\alpha^+, |s|)$ of cardinality at most α. Write $t^{*,i} = \{\eta \in t^* \mid s(\eta) = i\}$. (The ordering of R^s is not specified here, but is embedded into our later definition of extension for the class P of forcing conditions.)

We come next to the definition of the limit coding, which makes use of "coding delays".

Limit Precoding. Suppose that α is an uncountable limit cardinal and s belongs to S_α. Let k be least so that s belongs to $L^A_{\mu^s_k}$. Write $\widetilde{\mathcal{A}}^s = L^A_{\mu^s_k}$. Now X *precodes* s if X is the Σ_1 theory of $\tilde{\mathcal{A}}^s$ with parameters from $\alpha \cup \{s\}$, viewed as a subset of α.

Limit Coding. Suppose $s \in S_\alpha$, α is an uncountable limit cardinal and $p = \langle (p_\beta, p^*_\beta) \mid \beta \in \mathrm{Card} \cap \alpha \rangle$, where $p_\beta \in S_\beta$ for each $\beta \in \mathrm{Card} \cap \alpha$ and

Card denotes the class of infinite cardinals. We wish to define "p codes s". First we define a sequence $\langle s_\gamma \mid \gamma \leq \gamma_0 \rangle$ of elements of S_α as follows. Let $s_0 = \emptyset$. For limit $\gamma \leq \gamma_0$, $s_\gamma = \bigcup \{ s_\delta \mid \delta < \gamma \}$. Now suppose s_γ is defined. Then for $\beta \in$ Card $\cap\, \alpha$ consider $f_p^{s_\gamma}(\beta) =$ least $\delta \geq f^{s_\gamma}(\beta)$ such that $p_\beta(\delta^C) = 1$; if the latter is defined, then also define $X_\beta \subseteq \beta$ by: $\xi \in X_\beta$ iff $p_\beta((f_p^{s_\gamma}(\beta) + 1 + \xi)^C) = 1$. Now set $\gamma_0 = \gamma$ unless there is an $\eta > |s_\gamma|$ and $k \in \omega$ such that for some final segment B of $B_\alpha^{\eta,k}$, $f_p^{s_\gamma}$ is defined on B, $f_p^{s_\gamma} \restriction B \in \mathcal{A}^\eta$ and for some $X \subseteq \alpha$ in $\mathcal{A}^\eta$, $X_\beta = X \cap \beta$ for $\beta \in B$. There can be at most one such X, using the fact that if $\eta_0 < \eta_1$ or ($\eta_0 = \eta_1$ and $k_0 < k_1$) then a final segment of $B_\alpha^{\eta_1,k_1}$ is contained in $B_\alpha^{\eta_0,k_0}$. If Even $(X) = \{\xi \mid 2\xi \in X\}$ precodes an element t of S_α extending s_γ of length η then set $s_{\gamma+1} = t$. Otherwise let $s_{\gamma+1}$ be $s_\gamma * \vec{0}$, with $\vec{0}$ of length $\eta - |s_\gamma|$. (The notation $s_{\gamma+1} = s_\gamma * \vec{0}$ means that $s_{\gamma+1}$ extends s_γ and $s_{\gamma+1}(\eta) = 0$ for $|s_\gamma| \leq \eta < |s_{\gamma+1}|$.) Now *$p$ codes s* iff $s = s_\gamma$ for some $\gamma \leq \gamma_0$.

A real *preserves* the extender E iff the canonical embedding $V \to \mathrm{Ult}_E$ extends to an elementary embedding $V[R] \to \mathrm{Ult}_E[R]$. We show that there is a definable ZFC-preserving class forcing which adds a non set-generic, cofinality-preserving real R preserving the extenders in T. Moreover, for δ inaccessible in V, every CUB subset of δ in $V[R]$ contains a CUB subset in V. It follows that Woodinness is preserved by R.

We are about to define P, the class of forcing conditions. To ensure that extenders in T are preserved, we impose a strong Preservation Requirement on conditions in P. To accomodate this Requirement, we must use a special notion of extension, in which values not "fixed" by a condition are allowed to change when the condition is extended. However, making use of the fact that the critical points of extenders in T are not Woodin, we can demand that values in the interval $[\alpha, \alpha^+)$ will not change if the condition "recognizes" that the critical points of all extenders in T overlapping α are non-Woodin. This restriction is needed to show that conditions in the generic converge.

The Conditions. Let Card $'$ denote the class of all uncountable limit cardinals. A *condition* in P is a sequence $p = \langle (p_\alpha, p_\alpha^*) \mid \alpha \in$ Card $, \alpha \leq \alpha(p) \rangle$ where $\alpha(p) \in$ Card is not overlapped by T and:

(a) $p_{\alpha(p)} \in S_{\alpha(p)}$ and $p^*_{\alpha(p)} = \emptyset$.

(b) For $\alpha \in$ Card $\cap\, \alpha(p)$: $p(\alpha) = (p_\alpha, p^*_\alpha) \in R^{p_{\alpha^+}}$.

(c) For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$: $p \restriction \alpha$ codes p_α and belongs to $\mathcal{A}^{p_\alpha}$.
(d) (Restraint Requirement) For $\alpha \leq \alpha(p)$, α inaccessible in $\mathcal{A}^{p_\alpha}$: There exists a CUB $C \subseteq \alpha$, $C \in \mathcal{A}^{p_\alpha}$ such that $\beta \in C \to p^*_\beta = \emptyset$.
(e) (Preservation Requirement) Suppose that E belongs to T, $\alpha \leq \alpha(p)$ and α is overlapped by E.
(e0) p_α extends $E(p)_\alpha$.
(e1) If $|E(p)_\alpha| \leq \gamma < |p_\alpha|$ where for some $\xi \in \alpha^*_E$, γ belongs to b^ξ_E ($= b^\xi$ as defined in Ult_E), then $p_\alpha(\gamma^B) = 0$, unless $E(p)_{\alpha^+}(\xi) = 1$ and α^+ is *p-stable.*

We define *p-stability* as follows: An ordinal $\gamma \in [\alpha, \alpha^+)$ is *α-large* iff $\gamma \geq \alpha^+_E$ for each $E \in T$ overlapping α. p is *large up to* α iff $|p_\beta|$ is β-large for all $\beta \in \text{Card} \cap \alpha^+$. Then $\alpha \in \text{Card} \cap \alpha(p)^+$ is *p-stable* iff p is large up to α and $\kappa(E)$ is not Woodin in $\mathcal{A}^{p_{\kappa(E)}}$ for all $E \in T$ overlapping α.

Extension of conditions is defined as follows. An inaccessible cardinal $\alpha \leq \alpha(p)$ is *p-Woodin* iff it is Woodin in $\mathcal{A}^{p_\alpha}$. Then $p \leq q$ iff $\alpha(p) \geq \alpha(q)$ and for $\alpha \in \text{Card} \cap \alpha(q)^+$:
$(*)_0$ $|p_\alpha| \geq |q_\alpha|$, $p^*_\alpha \supseteq q^*_\alpha$.
$(*)_1$ $\gamma \in [\alpha, |q_\alpha|) \to p_\alpha(\gamma) = q_\alpha(\gamma)$, unless $\gamma < |E(p)_\alpha|$ for some $E \in T$ overlapping α.
$(*)_2$ $\gamma \in b^\eta$, $\eta \in q^{*,0}_\alpha$, α q-stable, $|q_\alpha| \leq \gamma < |p_\alpha|$, γ α-large $\to p_\alpha(\gamma^B) = 0$.
$(*)_3$ Suppose that α is inaccessible but not q-Woodin and q is large up to α. Then there exists a CUB $C \subseteq \alpha$ in $\mathcal{A}^{p_\alpha}$ such that $|p_\beta| = |q_\beta|$, $p^*_\beta = q^*_\beta$ for $\beta \in \bigcup\{(\bar\alpha, \bar\alpha^+_{D_\alpha}] \mid \bar\alpha \in C\}$.

Lemma 5 *Suppose that $\alpha \in \text{Card} \cap \alpha(q)^+$ is q-stable and p extends q. Then p_α extends q_α.*

Proof. It suffices to show that $E(p)_\alpha = E(q)_\alpha$ for all $E \in T$ overlapping α. Requirement $(*)_3$ from the definition of extension implies that $E(p)_\alpha$ and $E(q)_\alpha$ have the same length. So $E(p)_\alpha$, $E(q)_\alpha$ can only differ if $F(E(p))_\alpha$, $F(E(q))_\alpha$ are incompatible for some $F \in E[T]$ overlapping α. But by induction we may assume that $F(p)_\alpha = F(q)_\alpha$ for all $F \in T$ overlapping α which satisfy $\alpha(F) < \alpha(E)$. Therefore $F(E(p))_\alpha$, $F(E(q))_\alpha$ are compatible for all $F \in E[T]$ overlapping α, as $F(p)_\alpha$, $F(q)_\alpha$ extend $F(E(p))_\alpha$, $F(E(q))_\alpha$, respectively, and F belongs to T by the uniformity of T. $\square$

Lemma 6 *The ordering of conditions is transitive.*

Proof. Suppose that $p \leq q \leq r$. Then $(*)_0$ is clear for the pair p, r. Note that $p \leq q \to$ every q-stable cardinal is p-stable and $|E(p)_\alpha| \geq |E(q)_\alpha|$ whenever $\alpha \in$ Card $\cap\, \alpha(q)^+$ and $E \in T$ overlaps α, since $|p_\alpha| \geq |q_\alpha|$ for all $\alpha \in$ Card $\cap\, \alpha(q)^+$. Thus $(*)_1$ holds for p, r. Using Lemma 5, $q_\alpha^{*,0} \supseteq r_\alpha^{*,0}$ for r-stable α and therefore $(*)_2$ holds for p, r. Finally, $(*)_3$ holds for p, r since the intersection of CUB sets is CUB. □

To state the proper form of extendibility for P we must take into account requirement $(*)_3$ and therefore introduce the notion of a *p-witness*. This is a function w with the following properties:

1. The domain of w consists of all inaccessible $\alpha \leq \alpha(p)$ such that α is not p-Woodin and p is large up to α.
2. $w(\alpha)$ is a CUB subset of $\{\beta \in D_\alpha \mid \beta$ is not p-Woodin$\}$ for each $\alpha \in$ Dom w.
3. For all $\alpha \in$ Card $\cap\, \alpha(p)^+$, $w \restriction \alpha^+ \in \mathcal{A}^{p_\alpha}$.

The *support* of a p-witness w, written supp (w), is the union of all intervals $(\bar{\alpha}, \bar{\alpha}^+_{D_\alpha}]$, where $\bar{\alpha}$ belongs to $w(\alpha)$ and α is in the domain of w.

Lemma 7 *(Extendibility) Suppose that p belongs to P, $\beta \in$ Card $\cap\, \alpha(p)^+$ and $s \in S_\beta$ extends p_β. Also suppose that $|s|$ is β-large, $X \subseteq \beta$ belongs to $\mathcal{A}^s$, w is a p-witness and for $|p_\beta| \leq \gamma < |s|$:*

(a) If β is overlapped by $E \in T$ and γ belongs to b^ξ_E where $\xi \in \beta^*_E$ then $s(\gamma^B) = 0$, unless $E(p)_{\beta^+}(\xi) = 1$ and β^+ is p-stable.
(b) $\gamma \in b^\eta$, $\eta \in p^{*,0}_\beta$, β^+ p-stable, γ β-large $\to s(\gamma^B) = 0$.

Then there exists $q \leq p$ in P such that $|q_\beta| = |s|$, $X \cap \gamma \in \mathcal{A}^{q_\gamma}$ for all $\gamma \in$ Card $\cap\, \beta^+$ not in supp $(w \restriction \beta^+)$, q_β and s are the same above the maximum of $\{|E(q)_\beta| \mid E \in T$ overlaps $\beta\}$ and for all $\alpha \in$ Card $\cap\, (\beta, \alpha(p)]$, q_α and p_α are the same above the maximum of $\{|E(q)_\alpha| \mid E \in T$ overlaps $\alpha\}$.

Moreover we can require that q be large up to β.

Proof. By induction on $\beta \in$ Card $\cap\, \alpha(p)^+$. The result is clear if β equals ω, as ω is not overlapped in T and (b) guarantees that we can extend p_ω to s without violating $(*)_2$ from the definition of extension. If β is an uncountable successor cardinal then let $\bar{\beta}$ be the cardinal predecessor to β and choose $\bar{s} = p_{\bar{\beta}} * \vec{0} \in S_{\bar{\beta}}$ of $\bar{\beta}$-large length so that $X \cap \bar{\beta} \in \mathcal{A}^{\bar{s}}$. Apply induction to $p, \bar{s}, X \cap \bar{\beta}, w$ to obtain $\bar{q} \leq p$. Then obtain q from $\bar{q}$ by redefining $\bar{q}_\beta$

to be the same as s above the maximum of $\{|E(\bar{q})_\beta| \mid E \in T$ overlaps $\beta\}$. The hypotheses on s guarantee that the resulting q is the desired condition extending p.

Now suppose that β is an uncountable limit cardinal not overlapped by T. Let k be large enough so that $p \restriction \beta$, s, $X \cap \beta$, C_β (if β is singular in $\mathcal{A}^s$), D_β (if β is not Woodin in $\mathcal{A}^s$) and $w \restriction \beta^+$ belong to $\mathcal{A} = L^A_{\mu^s_k}$. Choose $Y \subseteq \beta$ such that Even $(Y) = \{\xi \mid 2\xi \in Y\}$ precodes s and Odd $(Y) = \{\xi \mid 2\xi + 1 \in Y\}$ is the Σ_1 theory of $\mathcal{A}$ with parameters from $\beta \cup \{s\}$, viewed as a subset of β. For $\gamma \in$ Card $\cap\, \beta^+$, let $\overline{\mathcal{A}}_\gamma$ be the transitive collapse of $H(\gamma) = \Sigma_1$ Hull of $\gamma \cup \{s\}$ in $\mathcal{A}$ and let $g(\gamma) = \gamma^+$ of $\overline{\mathcal{A}}_\gamma$. (If $\overline{\mathcal{A}}_\gamma \models \gamma^+$ does not exist, then $g(\gamma) = \mathrm{ORD}(\overline{\mathcal{A}}_\gamma)$. When $\gamma = \beta$, we have $\overline{\mathcal{A}}_\gamma = H(\gamma) = \mathcal{A}$.) Using Condensation (a), choose $\beta_0 < \beta$ large enough so that $\overline{\mathcal{A}}_\gamma$ is an initial segment of L^A for $\gamma \in C^{s,k}_\beta \cap (\beta_0, \beta]$. Also suppose that $p \restriction \beta, s, X \cap \beta, w \restriction \beta^+$ belong to $H(\beta_0)$ and if $C^{s,k}_\beta = C_\beta$ then $\beta_0 >$ ordertype C_β.

We first define $\bar{q}$, a preliminary version of q. Set $\bar{q}_\beta = s$. For $\gamma \in$ Card $\cap\, [\beta_0^+, \beta)$: If $C^{s,k}_\beta \neq C_\beta$ and $\gamma \in$ Lim $C^{s,k}_\beta$ then $\bar{q}_\gamma = s_\gamma$ where Even $(Y \cap \gamma)$ precodes $s_\gamma \in S_\gamma$; if $C^{s,k}_\beta = C_\beta$ and $\gamma \in$ Lim C_β then $\bar{q}_\gamma = p_\gamma * \vec{0}$ with $\vec{0}$ of length $g(\gamma)$; and if $\gamma \in B^{s,k}_\beta$ then $\bar{q}_\gamma = p_\gamma * \vec{0} * 1 * (Y \cap \gamma)^C$ where $\vec{0}$ has length $g(\gamma)+1$ (and $(Y \cap \gamma)^C$ has length γ). For $\gamma \in$ Card $\cap\, \alpha(p)^+$ not falling under the above cases, $\bar{q}_\gamma = p_\gamma$. Also set $\bar{q}^*_\gamma = p^*_\gamma$ for all $\gamma \in$ Card $\cap\, \alpha(p)^+$.

We claim that $\bar{q}$ obeys the requirements for being a condition, with the exception of the Preservation Requirement (e0). We need only check that $\bar{q} \restriction \gamma$ belongs to $\mathcal{A}^{\bar{q}_\gamma}$ and codes $\bar{q}_\gamma$ for $\gamma \in$ Card $' \cap \alpha(p)^+$. We may assume that γ belongs to Lim $C^{s,k}_\beta \cap [\beta_0^+, \beta]$. Note that $g \restriction \gamma$, $Y \cap \gamma$ and therefore $\bar{q} \restriction \gamma$ are definable over $\overline{\mathcal{A}}_\gamma$ for $\gamma \in$ Card $\cap\, \beta^+$, so for the first of these properties it suffices to show $\overline{\mathcal{A}}_\gamma \in \mathcal{A}^{\bar{q}_\gamma}$. But by choice of β_0, $\overline{\mathcal{A}}_\gamma$ is a proper initial segment of $\mathcal{A}^{g(\gamma)} = \mathcal{A}^{\bar{q}_\gamma}$. Thus we have established the first of these properties. For the second property, we must verify that there is $\eta_\gamma > |p_\gamma|$ and $k_\gamma \in \omega$ such that for some final segment B_γ of $B^{\eta_\gamma, k_\gamma}_\gamma$, $f^{p_\gamma}_{\bar{q}\restriction\gamma}$ is defined on B_γ, $f^{p_\gamma}_{\bar{q}\restriction\gamma} \restriction B_\gamma \in \mathcal{A}^{\eta_\gamma}$ and for some $X_\gamma \subseteq \gamma$ in $\mathcal{A}^{\eta_\gamma}$, $X_\delta = X_\gamma \cap \delta$ for $\delta \in B_\gamma$, where for $\delta < \gamma$, X_δ is defined by $\xi \in X_\delta$ iff $\bar{q}_\delta((f^{p_\gamma}_{\bar{q}\restriction\gamma}(\delta) + 1 + \xi)^C) = 1$. If $\gamma = \beta$ then we may take η_γ, k_γ, B_γ and X_γ to be $|s|$, k, $B^{s,k}_\beta - \beta_0^+$ and Y, respectively, and in this case Even(Y) precodes s, implying that $\bar{q} \restriction \beta$ codes s. Suppose that γ is less than β. If $C^{s,k}_\beta \neq C_\beta$ then we can similarly take $|s_\gamma|$,

k, $B_\gamma^{s_\gamma,k} - \beta_0^+$ and $Y \cap \gamma$, respectively, and in this case Even$(Y \cap \gamma)$ precodes s_γ, implying that $\bar{q} \restriction \gamma$ codes $s_\gamma = \bar{q}_\gamma$. Finally if $C_\beta^{s,k} = C_\beta$ note that γ is singular in $\mathcal{A}^{g(\gamma)}$ and therefore we can choose k' so that $C_\gamma^{g(\gamma),k'} = C_\gamma$; then we may take η_γ, k_γ, B_γ and X_γ to be $g(\gamma)$, k', $C_\gamma - \beta_0^+$ and $Y \cap \gamma$, respectively, and in this case Even$(Y \cap \gamma)$ does not precode an element of S_γ. It follows that $\bar{q} \restriction \gamma$ codes $p_\gamma * \vec{0}$, with $\vec{0}$ of length $g(\gamma) = g(\gamma) - |p_\gamma|$, as desired.

Let $B \subseteq \beta$ be the closure of $B_\beta^{s,k} \cap [\beta_0^+, \beta)$ (i.e., B is the union of $B_\beta^{s,k}$ and Lim $C_\beta^{s,k} \cap (\beta_0, \beta)$). To obtain the desired $q \leq p$, we inductively modify $\bar{q} \restriction \gamma^+$ for $\gamma \in B$ to $q \restriction \gamma^+$ such that $q(\gamma) = \bar{q}(\gamma)$ and $q \restriction \gamma^+ \cup p \restriction [\gamma^+, \alpha(p)])$ is a condition satisfying the *Growth Requirement up to* γ: For δ in Card $\cap \gamma^+$, $|q_\delta|$ is δ-large, and either δ belongs to supp $(w \restriction \gamma^+)$ or $X \cap \delta \in \mathcal{A}^{q_\delta}$. If $\gamma = \min B$ then we apply induction to $p, \bar{q}_\gamma, X \cap \gamma, w_0$, where $w_0(\alpha) = w(\alpha)$ for $\alpha \in$ Dom $w \cap \gamma^+$ and $w_0(\alpha) = w(\alpha) - \gamma^+$ for $\alpha \in$ Dom $w - \gamma^+$, to ensure the Growth Requirement up to γ. Suppose that γ is a successor element of B and γ_0 is its B-predecessor. It is possible that γ_0 is the critical point of an extender $E \in T$. E is unique and must satisfy $\alpha(E) < \gamma$. In this case we modify $\bar{q} \restriction (\gamma_0, \alpha(E)] = p \restriction (\gamma_0, \alpha(E)]$ to $q' \restriction (\gamma_0, \alpha(E)]$ by requiring q'_δ to extend $E(q \restriction \gamma_0)_\delta$ for $\delta \in$ Card $\cap (\gamma_0, \alpha(E)]$, thereby ensuring the Preservation Requirement (e0) with respect to E. As by induction our modified $q \restriction \gamma_0^+$ satisfies the Preservation Requirement with respect to all $E \in T$, it follows that $E(q \restriction \gamma_0^+)$ satisfies the Preservation Requirement with respect to all $F \in E[T]$, and therefore by the uniformity of T, with respect to all $F \in T$, $\alpha(F) < \alpha(E)$. As $\bar{q}$ agrees with p on the interval $(\gamma_0, \alpha(E)]$ and p satisfies the Preservation Requirement, it follows that Preservation Requirement (e0) will hold for $q \restriction \gamma_0^+ \cup q' \restriction (\gamma_0, \alpha(E)]$ with respect to all extenders in T. Preservation Requirement (e1) also holds for $q \restriction \gamma_0^+ \cup q' \restriction (\gamma_0, \alpha(E)]$ as it holds for $\bar{q}$, and the modifications for the purpose of ensuring Preservation Requirement (e0) do not affect Preservation Requirement (e1). Now apply induction to $q \restriction \gamma_0^+ \cup q' \restriction (\gamma_0, \alpha(E)] \cup p \restriction (\alpha(E), \alpha(p)]$, $\bar{q}_\gamma$, $X \cap \gamma$, w_0, where $w_0(\alpha) = w(\alpha)$ for $\alpha \in$ Dom $w \cap \gamma^+$ and $w_0(\alpha) = w(\alpha)$ for $\alpha \in$ Dom $w - \gamma^+$, to obtain the desired $q \restriction \gamma^+$ satisfying the Growth Requirement up to γ, without changing $q \restriction \gamma_0^+ \cup q' \restriction (\gamma_0, \alpha(E)] \cup p \restriction (\alpha(E), \alpha(p)]$ at a cardinal $\delta \in (\gamma_0, (\gamma_0)^+_{D_\beta}]$, if $\gamma_0 \in w(\beta)$ and $|p_{\bar{\delta}}|$ is $\bar{\delta}$-large for all $\bar{\delta} \in$ Card $\cap \delta^+$. Finally, if $\gamma \in$ Lim B and we have inductively modified $\bar{q} \restriction \delta^+$, $\delta \in B \cap \gamma$ in the L^A-least way to the desired $q \restriction \delta^+$, it follows that the resulting $q \restriction \gamma^+ = \bigcup\{q \restriction \delta^+ \mid \delta \in B \cap \gamma\} \bigcup \{\langle \gamma, \bar{q}(\gamma) \rangle\}$ is as desired, since the definition of $\bar{q}$

guarantees that $Y \cap \gamma$, and therefore the new $q \restriction \gamma$, belongs to $\mathcal{A}^{q_\gamma}$.

At the end of the above construction either we obtain a condition q or some $E \in T$ has critical point β; in the latter case we modify once more on Card $\cap (\beta, \alpha(E)]$ to ensure the Preservation Requirement. The resulting q is a condition such that $|q_\beta| = |s|$, $|q_\alpha|$ is α-large for all $\alpha \in$ Card $\cap \beta^+$, $X \cap \gamma \in \mathcal{A}^{q_\gamma}$ for all $\gamma \in$ Card $\cap \beta^+$ not in supp $(w \restriction \beta^+)$, q_β and s are the same above the maximum of $\{|E(q)_\beta| \mid E \in T \text{ overlaps } \beta\}$ and for all $\alpha \in$ Card $\cap (\beta, \alpha(p)]$, q_α and p_α are the same above the maximum of $\{|E(q)_\alpha| \mid E \in T \text{ overlaps } \alpha\}$.

We must verify that the extension $q \leq p$ obeys property $(*)_3$. If α is from the statement of $(*)_3$, we may assume that α belongs to Lim B. The desired property for the pair $p, \bar{q}$ is witnessed by the CUB set $B \cap \alpha$, as for $\delta \in B \cap \alpha$, $\delta^+_{D_\beta} = \delta^+_{D_\alpha}$ is less than δ^+_B, and therefore the extensions on B avoid the intervals $(\delta, \delta^+_{D_\alpha}]$, δ in $B \cap \alpha$. Then to verify the result when α belongs to Lim B for the pair p, q, note that α belongs to $w(\beta) \cup \{\beta\}$ and by construction $|p_\gamma| = |q_\gamma|$ for $\gamma \in (\bar{\alpha}, \bar{\alpha}^+_{D_\beta}]$, $\bar{\alpha} \in B \cap \alpha$, so $B \cap \alpha$ is again a witness to $(*)_3$.

Thus the only possible problem in verifying that q extends p is that as a result of $(*)_2$, the restraint p^*_γ may prevent us from making the extension from p_γ to q_γ when $q_\gamma = s_\gamma$ and Even $(Y \cap \gamma)$ precodes s_γ. However if there are unboundedly many such $\gamma < \beta$ then β is inaccessible in $\mathcal{A}^{p_\beta}$ and therefore by the Restraint Requirement, $p^*_\gamma = \emptyset$ for γ in a CUB subset of β in $\mathcal{A}^{p_\beta}$, which we may assume belongs to $\mathcal{A}$. Thus for sufficiently large γ such that $Y \cap \gamma$ precodes s_γ, γ belongs to C and hence $p^*_\gamma = \emptyset$. So $q \leq p$ on a final segment of Card $\cap \beta$, and by induction we may arrange that this holds on all of Card $\cap \beta$.

Finally, suppose that β is an uncountable limit cardinal overlapped by T. Let κ be the largest critical point of an extender in T overlapping β. By induction we can assume that p satisfies the Growth Requirement up to κ, without altering p_α above the maximum of $\{\alpha^+_E \mid E \in T \text{ overlaps } \alpha\}$ for $\alpha \in$ Card $\cap (\kappa, \alpha(p)]$. Now apply the argument from the previous case to extend p to q on Card $\cap [\kappa^+, \beta]$ (and on Card $\cap (\beta, \alpha(E)]$, if some $E \in T$ has critical point β) to ensure the Growth Requirement up to β as well as $|q_\beta| = |s|$ (with q_β the same as s above the maximum of $\{\beta^+_E \mid E \in T \text{ overlaps } \beta\}$ and q_α the same as p_α above the maximum of $\{\alpha^+_E \mid E \in T \text{ overlaps } \alpha\}$ for $\alpha \in$ Card , $\beta < \alpha \leq \alpha(p)$). $\square$

Lemma 8 *Suppose that G is P-generic and let G_ω denote $\bigcup\{p_\omega \mid p \in G\}$. Then G_ω is* not *set-generic over V.*

Proof. For each infinite cardinal α, G *converges on* $[\alpha, \alpha^+)$ in the sense that for some $p \in G$, every extension q of p satisfies $q_\alpha \supseteq p_\alpha$. This follows from Lemma 5, as only finitely many $E \in T$ overlap α and by Lemma 7 we can choose $p \in G$ so that p_β is β-large for each $\beta \in$ Card $\cap\, \alpha(p)^+$ and the critical point $\kappa(E)$ of each $E \in T$ overlapping α is not Woodin in $\mathcal{A}^{p_{\kappa(E)}}$. Let G_α denote $\bigcup\{p_\alpha \mid p \in G$ and α is p-stable$\}$. We claim that G_{α^+} is coded by G_α and for uncountable limit cardinals α, G_α is coded by $\bigcup\{G_\beta \mid \beta \in \text{Card} \cap \alpha\}$. The first statement follows immediately from Lemma 7. The second statement follows from Lemma 7 together with the fact that for uncountable limit cardinals α, the coding of p_α by $p \restriction \alpha$ takes place at cardinals in C^T_α and the collection of conditions $q \in P$ such that each $\beta \in C^T_\alpha$ is q-stable is dense in P. Thus G can be decoded from G_ω. As G_α adds an α^+-Cohen set to V, it follows that G_ω is not set-generic over V. $\Box$

To establish cofinality-preservation for P we must consider *nested witnesses.* A p-witness w is *nested* iff whenever $\bar{\alpha} \in w(\alpha)$, $\bar{\beta} \in w(\beta)$, $\alpha \leq \beta$ and $\bar{\alpha} \leq \bar{\beta} < \alpha$ then $w(\alpha) = w(\beta) \cap \alpha$.

Lemma 9 *For every condition p there exists a nested p-witness. Moreover, if w is a nested p-witness and q extends p, then there is a nested q-witness extending w.*

Proof. We begin with the first statement. For $\alpha \leq \alpha(p)$, a *(nested) p, α-witness* is a function satisfying the requirements of a nested p-witness, but only defined on cardinals $\leq \alpha$. We show that if $\alpha < \beta \leq \alpha(p)$ then each p, α-witness w can be extended to a p, β-witness w^*. This is proved by induction on β. We may assume that p is large up to β. If β is not the limit of inaccessibles then by induction we extend w up to the supremum γ of α and the inaccessibles less than β and then, if β is non p-Woodin and inaccessible, extend up to β itself with a witness w^* such that $w^*(\beta)$ only includes cardinals greater than γ. Now assume that β is a limit of inaccessibles. If β is singular then we can inductively choose end-extending p, γ-witnesses for $\gamma \in C_\beta$ above α and take the union. If β is inaccessible and p-Woodin then we similarly use a canonical CUB subset C of β consisting of p-Woodins. Finally, if β is non p-Woodin and inaccessible, then choose a CUB $D \subseteq D_\beta - \alpha$ such that

$\bar\beta \in D \to \bar\beta$ is not p-Woodin and $D \cap \beta \in \mathcal{A}^{p_\beta}$, using the inaccessibility of β. Then successively extend w to elements of D, modifying choices if necessary so that for $\bar\beta \in D$, the chosen witness between $\bar\beta$ and $\bar\beta^+_D$ only include cardinals strictly greater than $\bar\beta$. At inaccessible limit elements $\bar\beta$ of D, define $w^*(\bar\beta)$ to be $D \cap \bar\beta$. In this way we obtain the nestedness of the resulting witness w^*.

Now suppose that w is a nested p-witness, q extends p and we wish to define a nested q-witness w^* extending w. If every inaccessible $\alpha \le \alpha(q)$ which is not q-Woodin with q large up to α is already in the domain of w then we take $w^* = w$. Otherwise let α be the least exception. Sufficiently large elements of D_α are p-Woodin, using the fact that α is a p-Woodin inaccessible which is not Woodin. Thus sufficiently large elements of D_α do not belong to any $w(\beta)$. Also note that sufficiently large $\gamma < \alpha$ do not belong to $w(\beta)$ for any $\beta \ge \alpha$, because α is p-Woodin and w is nested. And for each $\bar\beta < \alpha$, the set of $\beta < \alpha$ such that $\bar\beta$ belongs to $w(\beta)$ is bounded in α, else by the nestedness of w, there would be a CUB subset of α consisting of non p-Woodins. Thus as $w \restriction \alpha$ belongs to $\mathcal{A}^{p_\alpha}$, it follows that sufficiently large elements $\bar\alpha$ of D_α are closure points of $w \restriction \alpha$, in the sense that for some fixed $\alpha_0 < \alpha$ (independent of $\bar\alpha$), if $\bar\beta$ belongs to $w(\beta) \cap (\alpha_0, \bar\alpha)$ for some β then β is less than $\bar\alpha$. We therefore achieve the nestedness of w^* up to α by choosing $w^*(\alpha)$ to be a CUB subset D of D_α with sufficiently large minimum such that $\bar\alpha \in D \to \bar\alpha$ is not q-Woodin and $D \cap \bar\alpha \in \mathcal{A}^{p_{\bar\alpha}}$. Finally, combine this argument with the argument used in the first part of this proof to show that for $\alpha < \beta \le \alpha(q)$, each q, α-witness can be extended to a q, β-witness, compatibly with w. □

Definition. Suppose that γ is an infinite successor cardinal and $D \subseteq P$ is open dense. A condition $p \in P$ *reduces* D *below* γ iff for every $q \le p$ there exists $r \le q$ such that r belongs to D, $\alpha(r) = \alpha(q)$ and $r(\alpha) = q(\alpha)$ for all $\alpha \in \text{Card} \cap [\gamma, \alpha(q)]$.

Lemma 10 *(Density Reduction) (a) If D_i is open and dense on P for each $i < \omega$ then for each $p \in P$ there is a $q \le p$ which belongs to each D_i.*
(b) If D_i is open and dense on P for each $i < \gamma$ where γ is an infinite successor cardinal then for each $p \in P$ there is a $q \le p$ which reduces each D_i below γ.
(c) If D_i is open and dense on P for each $i < \gamma$, where γ is inaccessible and

not Woodin, then for each $p \in P$ there are $q \leq p$ and a CUB $D \subseteq \gamma$ such that q reduces D_i below $(\bar{\gamma}^+_{D_\gamma})^+$ for $\bar{\gamma} \in D$ and $i < (\bar{\gamma}^+_{D_\gamma})^+$. (Note that D_γ is the canonical CUB subset of γ defined earlier, and is unrelated to the D_i's.)
(d) If D_i is open and dense on P for each $i < \gamma$ where γ is Woodin then for each $p \in P$ there are $q \leq p$ and a CUB $D \subseteq \gamma$ such that q reduces D_i below $\bar{\gamma}^+$ for $\bar{\gamma} \in D$ and $i < \bar{\gamma}^+$.

Proof. In the statement of this Lemma, we intend that the sequence of D_i's in each case be L^A-definable. Choose $n > 1$ so that this sequence is Σ_n definable over L^A and let θ be a cardinal of cofinality greater than γ (greater than ω in part (a)) such that L^A_θ is Σ_{n+1}-elementary in L^A. Let X be the Σ_n theory of $\langle L^A_\theta, y\rangle_{y \in L^A_\theta}$, viewed as a subset of θ. Assume first that $\{p\}$ is Σ_n-definable in L^A and the defining parameter for the sequence of D_i's is 0.

(a) Define a sequence of conditions $p^i \in P$ with associated nested p^i-witnesses w_i and w^*_i, $i \in \omega$ as follows:

1. $p^0 = 1^P$, $w_0 = w^*_0$ is any nested p-witness.
2. For $i \in \omega$, p^{i+1} is the L^A-least extension q of p^i belonging to D_i such that $L^A_{\alpha(q)}$ is Σ_n-elementary in L^A and $X \cap \gamma \in \mathcal{A}^{q_\gamma}$ for each $\gamma \in$ Card $\cap \alpha(q)^+$ not in supp (w^*_i). w_{i+1} is the L^A-least nested p^{i+1}-witness extending w_i and w^*_{i+1} is obtained from w_{i+1} by choosing $w^*_{i+1}(\alpha)$ to be a CUB subset C of $w_{i+1}(\alpha)$ with the property that $|p^{i+1}_\beta| = |p^i_\beta|$, ${p^{i+1}_\beta}^* = {p^i_\beta}^*$ for $\beta \in (\bar{\alpha}, \bar{\alpha}^+_{D_\alpha}]$ and $\bar{\alpha} \in C$, for each α in the domain of w_{i+1}.

We claim that the sequence of p^i's has a lower bound $q \in P$. Define q as follows: $\alpha(q) = \bigcup_i \ \alpha(p^i)$, $q_\beta = \bigcup_i \ p^i_\beta$ above $\max\{|E(q \restriction \kappa(E))_\beta| \mid E \in T$ overlaps $\beta\}$, q_β agrees with $E(q \restriction \kappa(E))_\beta$ below $|E(q \restriction \kappa(E))_\beta|$ when $E \in T$ overlaps β, $q^*_\beta = \bigcup_i \ {p^i_\beta}^*$ for $\beta \in$ Card $\cap\, \alpha(q)$ and $(q_{\alpha(q)}, q^*_{\alpha(q)}) = (\emptyset, \emptyset)$. We must verify that q_β belongs to S_β and $q \restriction \beta$ belongs to $\mathcal{A}^{q_\beta}$ for $\beta \in$ Card $\cap \alpha(q)$. Let $H(\beta)$ denote the Σ_{n+1} Skolem hull of β in $L^A_{\leq\alpha(q)}$ and $\overline{H}(\beta)$ its transitive collapse. By the definition of the p^i's, $|q_\beta|$ either is β^+ of $\overline{H}(\beta)$ or belongs to the support of w^*_i for sufficiently large i. In the former case, as $q \restriction \beta^+$ is definable over $\overline{H}(\beta)$, which by Condensation (b) is an initial segment of $\mathcal{A}^{q_\beta}$, it follows that q_β belongs to S_β and $q \restriction \beta$ belongs to $\mathcal{A}^{q_\beta}$. In the latter case, the nestedness of the w_i's implies that β belongs to a fixed left-open interval I contained in the support of w^*_i for sufficiently large i;

thus for some $i_0 \in \omega$, $|p^i_{\bar\beta}|$ is constant for $i \geq i_0$, not only for $\bar\beta = \beta$, but for all sufficiently large $\bar\beta < \beta$ (if β is a limit cardinal). Thus q_β belongs to S_β and $q \restriction \beta$ belongs to $\mathcal{A}^{q_\beta}$ as these properties hold for p^{i_0}.

The Preservation Requirement clearly holds for q, given the way q was defined and the fact that it holds for each p^i. The Restraint Requirement holds for q: Suppose that γ is inaccessible in $\mathcal{A}^{q_\gamma}$, $\gamma \leq \alpha(q)$ and for $i \in \omega$ let C^i be the least CUB subset of γ in $\mathcal{A}^{p^i_\gamma}$ such that ${p^i_{\bar\gamma}}^* = \emptyset$ for sufficiently large $\bar\gamma \in C^i$. Then $\bigcap\{C^i \mid i < \omega\}$ witnesses the Restraint Requirement for q at γ, either because the C^i's stabilise or because q_γ has length γ^+ of $\overline{H}(\gamma)$ and hence $\langle C^i \mid i \in \omega\rangle$ belongs to $\mathcal{A}^{q_\gamma}$. By a similar argument, $q \leq p$ satisfies $(*)_3$ from the definition of extension.

(b) By Lemma 7 we may assume that γ is p-stable. Let $\delta \in$ Card , $\gamma = \delta^+$. For any $r \leq p$ let $r \downarrow \gamma$ denote the function with domain Card $\cap\, \gamma$ defined by $(r \downarrow \gamma)(\bar\gamma) = r(\bar\gamma)$ for $\bar\gamma \in$ Card $\cap\, \delta$, $(r \downarrow \gamma)(\delta) = (r_\delta, \emptyset)$. Now let $\langle (D^*_i, \bar q_i) \mid i < \gamma\rangle$ be a list of all pairs $(D^*, \bar q)$ where $D^* = D_j$ for some $j < \gamma$ and $\bar q = r \downarrow \gamma$ for some $r \leq p$.

Define a sequence of conditions p^i with associated nested p^i-witnesses w_i and w^*_i, $i < \gamma$ as follows:

1. $p^0 = p$, $w_0 = w^*_0$ is any nested p-witness.
2. For $i < \gamma$, p^{i+1} is the L^A-least extension q of p^i such that $q \restriction \gamma = p \restriction \gamma$, for some $q^* \in D_i$, $q^* \downarrow \gamma = \bar q_i$, $\alpha(q^*) = \alpha(q)$, $q^* \restriction$ Card $\cap\, [\gamma, \alpha(q)] = q \restriction$ Card $\cap\, [\gamma, \alpha(q)]$, $L^A_{\alpha(q)}$ is Σ_n-elementary in L^A and $X \cap \mu \in \mathcal{A}^{q_\mu}$ for each $\mu \in$ Card $\cap\, [\gamma, \alpha(q)]$ not in supp (w^*_i). (If no such q exists, then set $p^{i+1} = p^i$.) w_{i+1} is the L^A-least nested p^{i+1}-witness extending w_i and w^*_{i+1} is obtained from w_{i+1} by choosing $w^*_{i+1}(\alpha)$ to be a CUB subset C of $w_{i+1}(\alpha)$ with the property that $|p^{i+1}_\beta| = |p^i_\beta|$, ${p^{i+1}_\beta}^* = {p^i_\beta}^*$ for $\beta \in (\bar\alpha, \bar\alpha^+_{D_\alpha}]$ and $\bar\alpha \in C$, for each α in the domain of w_{i+1}.
3. For limit $\lambda \leq \gamma$, p^λ is the condition q defined by: $\alpha(q) = \bigcup_{i<\lambda} \alpha(p^i)$, $q(\beta) = p(\beta)$ for $\beta \in$ Card $\cap\, \gamma$, $q_\beta = \bigcup_{i<\lambda} p^i_\beta$ above $\max\{|E(q \restriction \kappa(E))_\beta| \mid E \in T$ overlaps $\beta\}$ and q_β agrees with $E(q \restriction \kappa(E))_\beta$ below $|E(q \restriction \kappa(E))_\beta|$ when $E \in T$ overlaps β for $\beta \in$ Card $\cap\, [\gamma, \alpha(q))$, $q^*_\beta = \bigcup_{i<\lambda} {p^i_\beta}^*$ for $\beta \in$ Card $\cap\, [\gamma, \alpha(q))$ and $(q_{\alpha(q)}, q^*_{\alpha(q)}) = (\emptyset, \emptyset)$.

In 3. above, we must verify that q is a condition. First we show that q_β belongs to S_β and $q \restriction \beta$ belongs to $\mathcal{A}^{q_\beta}$ for $\beta \in$ Card $\cap\, [\gamma, \alpha(q))$. Let $H(\beta)$

denote the Σ_{n+1} Skolem hull of β in $L^A_{\alpha(q)}$ and $\overline{H}(\beta)$ its transitive collapse. By the definition of the p^i's, $|q_\beta|$ either is β^+ of $\overline{H}(\beta)$ or belongs to the support of w^*_i for sufficiently large i. In the former case, as $q \restriction \beta^+$ is definable over $\overline{H}(\beta)$, which by Condensation (b) is an initial segment of $\mathcal{A}^{q_\beta}$, it follows that q_β belongs to S_β and $q \restriction \beta$ belongs to $\mathcal{A}^{q_\beta}$. In the latter case, the nestedness of the w_i's implies that β belongs to a fixed left-open interval I contained in the support of w^*_i for sufficiently large i; thus for some $i_0 \in \omega$, $|p^i_{\bar{\beta}}|$ is constant for $i \geq i_0$, not only for $\bar{\beta} = \beta$, but for all sufficiently large $\bar{\beta} < \beta$ (if β is a limit cardinal). Thus q_β belongs to S_β and $q \restriction \beta$ belongs to $\mathcal{A}^{q_\beta}$ as these properties hold for p^{i_0}.

The Preservation Requirement clearly holds for q, given the way q was defined and the fact that it holds for each p^i. The Restraint Requirement holds for q: Suppose that μ is inaccessible in $\mathcal{A}^{q_\mu}$, $\mu \in \text{Card} \cap (\gamma, \alpha(q))$ and for $i < \lambda$ let C^i be the least CUB subset of μ in $\mathcal{A}^{p^i_\mu}$ such that ${p^i_{\bar{\mu}}}^* = \emptyset$ for sufficiently large $\bar{\mu} \in C^i$. Then $\bigcap\{C^i \mid i < \lambda\}$ witnesses the Restraint Requirement for q at μ, either because the C^i's stabilise or because q_μ has length μ^+ of $\overline{H}(\mu)$ and hence $\langle C^i \mid i < \lambda\rangle$ belongs to $\mathcal{A}^{q_\mu}$. By a similar argument, $q \leq p$ satisfies $(*)_3$ from the definition of extension.

Now note that q_γ reduces each D_i below γ because if $r \leq q$ then we may choose $s \leq r$ in D_i, and $j < \gamma$ such that $(D_i, s \downarrow \gamma)$ equals $(D^*_j, \bar{q}_j)$, in which case p^{j+1} is chosen so that for some s^*, $\alpha(s^*) = \alpha(p^{j+1})$, p^{j+1} agrees with s^* on Card $\cap[\gamma, \alpha(p^{j+1})]$; but then using the p-stability of γ, r has the extension $s^* \restriction \gamma \cup r \restriction [\gamma, \alpha(r)]$, which agrees with r on $[\gamma, \alpha(r)]$ and which belongs to D_i, as it extends s^*.

(c) Again by Lemma 7 we may assume that γ is p-stable. As a final segment of D_γ is contained in C^T, it follows that sufficiently large elements of D_γ are p-stable as well. Suppose that δ belongs to D_γ and all elements of D_γ above δ are p-stable. Then by the construction of case (b), we may extend p to q so that q reduces each D_i, $i < (\delta^+_{D_\gamma})^+$ below $(\delta^+_{D_\gamma})^+$ and $q \restriction (\delta^+_{D_\gamma})^+ = p \restriction (\delta^+_{D_\gamma})^+$. Note that by the definition of extension, there is a CUB $D \subseteq \gamma$ such that $|q_\beta| = |p_\beta|$, $q^*_\beta = p^*_\beta$ for $\beta \in \text{Card} \cap (\bar{\gamma}, \bar{\gamma}^+_{D_\gamma}]$, $\bar{\gamma} \in D$. By repeating this successively for each such δ, we obtain a γ-sequence of conditions p^i with associated CUB subsets of γ whose limit q reduces D_i below $(\bar{\gamma}^+_{D_\gamma})^+$ for $\bar{\gamma}$ in the diagonal intersection D of the associated CUB sets. Note that $q \leq p$ obeys $(*)_3$ from the definition of extension since $|q_\beta| = |p_\beta|$, $q^*_\beta = p^*_\beta$ for $\beta \in (\bar{\gamma}, \bar{\gamma}^+_D]$, $\bar{\gamma} \in D$.

(d) This is just like (c), except to each condition p^i we associate a CUB subset of γ consisting of cardinals which are p^i-Woodin, and for $\bar{\gamma}$ in the diagonal intersection of these sets, reduce D_i below $\bar{\gamma}^+$ for $i < \bar{\gamma}^+$.

This completes the proof of (a)-(d) when $\{p\}$ is Σ_n-definable in L^A and the defining parameter for the sequence of D_i's is 0. Now argue as follows: If the Lemma fails, then choose n so that it fails for some condition p and some Σ_n-definable sequence of D_i's. Let (p, x) be least so that the Lemma fails for p and some sequence of D_i's which is Σ_n-definable with parameter x. Then the pair (p, x) is Σ_m-definable for some $m > n$. For this m, $\{p\}$ is Σ_m-definable and the the Lemma fails for a sequence of D_i's which is Σ_m-definable with parameter 0. This contradicts what has been proven above. $\square$

Immediate consequences of this Lemma are that P preserves cofinalities as well as the axioms of ZFC, and every CUB subset of an inaccessible cardinal in a P-generic extension contains one in V (see Proposition 4.14 of [1]). If G is P-generic over V then by Lemma 8, $V[G] = V[G_\omega]$ where G_ω can be viewed as a subset of ω_1^V. Then by a simple ccc almost disjoint coding, G_ω can be further coded into $V[R]$ for some real R. As G_ω is not set-generic over V, neither is R.

Finally we show that the extenders E in T are preserved, i.e., that the canonical embedding $E^* : V \to \text{Ult}_E$ can be extended to an elementary embedding $V[G] \to \text{Ult}_E[G^*]$ for P-generic G. Thus we must define G^* which is P^*-generic over Ult_E, where $P^* = E^*[P]$, and which contains each condition $E^*(p)$, $p \in G$. By the Preservation Requirement, any two conditions of the form $E^*(p) \restriction [\gamma, \alpha(E^*(p))] \cup q \restriction \gamma$ are compatible for $p, q \in G$, $\gamma \in \text{Card} \cap \alpha(E)^+$, using the fact that when α is overlapped by E, α^*_E contains $E(p)^*_\alpha$. Let H^* denote the class of all such conditions. We claim that $G^* = \{q \in P^* \mid q \text{ is extended by some element of } H^*\}$ is the desired P^*-generic. Indeed suppose that $D^* \subseteq P^*$ is open dense, and is definable over Ult_E via some formula φ with parameter x. Then x can be written in the form $E^*(f)(a)$ where $f : L^A_\kappa \to L^A$, $\kappa = \text{crit } E$ and a is an element of $L^A_{\alpha(E)}$. Now enumerate the elements of L^A_κ in L^A-increasing order as a sequence $\langle b_i \mid i < \kappa \rangle$ and let D_i for $i < \kappa$ be defined in L^A by the formula φ, using parameter $f(b_i)$. We may assume that D_i is open dense on

P for each $i < \kappa$. By Density Reduction for P there exists $p \in G$ which reduces D_i below $(\bar{\kappa}^+_{D_\kappa})^+$ for each $i < (\bar{\kappa}^+_{D_\kappa})^+$, for CUB-many $\bar{\kappa} < \kappa$. Thus $E^*(p) \in H^*$ reduces D^*_i below $(\kappa^+_{E(D_\kappa)})^+$ for each $i < (\kappa^+_{E(D_\kappa)})^+$, where if $E(\langle b_i \mid i < \kappa\rangle) = \langle a_i \mid i < E(\kappa)\rangle$, D^*_i is defined in Ult_E via φ using the parameter $E^*(f)(a_i)$. But $a = a_i$ for some $i < \alpha(E)$ and therefore $E^*(p)$ reduces the original D^* below $(\kappa^+_{E(D_\kappa)})^+ = \alpha(E)$ for such an i. Then the genericity of G implies that $\{q \restriction \alpha(E) \mid q \in G\}$ generically codes $t = E^*(p)_{\alpha(E)}$ over $\mathcal{A}^t$ in the sense of Ult_E (using the fact that E belongs to $\mathcal{A}^\emptyset$; see Lemma 4.8 of [1]). Therefore H^* intersects D^*. We have shown that G^* intersects all Ult_E-definable open dense classes on P^*, and is therefore P^*-generic over Ult_E, as desired. $\square$

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